The sup-completion of a Dedekind complete vector lattice II

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March 28, 2024

Abstract

We persist in our investigation of the sup-completion of a Dedekind complete Riesz space, extending to the broader context of Riesz spaces. some results initially obtained by Feng, Li, Shen, and also by Erdös, and Rényi.

1 Introduction

In this paper, we continue our investigation of the sup-completion of a Dedekind complete Riesz space started in [3]. We delve deeper into the decomposition of finite and infinite parts, initially introduced in [3], and further investigate the properties elucidated in that study. Within our work, we introduce a new concept that we call the 'star map' as a pivotal construct necessary for generalizing results from measure theory or classical stochastic theory to the domain of Riesz spaces. As we encounter instances where we

*The authors are members of the GOSAEF research group

†This document is the results of the research project funded by the National Science Foundation

seek to apply an inverse operation amidst dealing with non-invertible elements, we address this issue by introducing the notion of a 'partial inverse'. While briefly discussed in our previous work [3], this concept will be systematically explored here with comprehensive details. Consider a Dedekind complete Riesz space X with a weak order unit, denoted by e. Then the universal completion X^u of X has a natural structure of an f-algebra, where e serves as the identity element. Each element x in X functions as a weak unit within the band B_x generated by x in X^u. Consequently x has an inverse in that band, referred to as the partial inverse of x. If x is a positive element in the cone X^s_+, where X^s denotes the sup-completion of X, we denote by x^* the partial inverse of its finite part x^f. This partial inverse is also recently used by Roelands and Schwanke in [10] and they adopted the same notation. It is also used in [9] to develop a Hahn-Jordan theorem in Riesz spaces. Our motivation here is to get a Riesz space version of a result obtained by Feng, Li and Shen in [7]. A weaker form of this result was obtained earlier by Erdös and Rényi in [6] that allows to get a generalization of Borel Cantelli Lemma.

Let us give a brief outline of the content of the paper. Section 2 provides some preliminaries. Sections 3 and 4 are devoted to present new results concerning the sup-completion of a Dedekind complete Riesz space. In the first part we investigate finite and infinite parts. The second part deals with partial inverses of elements of X^s. We introduce that map $x \longmapsto x^*$ where x^* is the inverse of x^f in the band B_{x^f}. Then we prove under some conditions that if (x_α) converges to x in order then x^*_α converges in order to x^*. In the last section we apply our results to obtain a generalization of a theorem of Feng, Li and Shen to the setting of Riesz spaces. The reader is referred to [5] for the definition of the sup-completion, a fundamental concept in this paper, and to the papers [2] and [3] for more informations of that notion. All unexplained terminology and notation concerning Riesz spaces can be found in standard references [1], [12] and [11].

2 Preliminaries

We consider a Dedekind complete Riesz space X. We employ X^u to represent its universal completion, while its sup-completion is denoted by X^s. Recall that X^s is a lattice ordered cone that contains X, and which has a greatest element that we denote by ∞. If B is a band in X then its sup-completion B^s is contained in X^s (see [2, Theorem 6]) and its greatest element will be denoted by ∞_B. More about the space X^s can be found in [2, 3]. We denote by $\mathcal{B}(X)$ the Boolean algebra of projection bands in X. To a band $B \in \mathcal{B}(X)$ we associate the band projection P_B on B and we use the notation $P^d = I - P$ for any band projection P. We shorten P_{B_x} to P_x, with B_x denoting the principal band generated by x. It should be noted that this notion can be extended in a natural manner to elements in X^s. It was shown indeed in [2, Lemma 4] that if we define $\pi_x(a) = \sup_n (a \wedge nx)$ for a in X^s_+ and $\pi_x(a) = \pi_x(a^+) - \pi_x(a^-)$ for $a \in X$, then π_a is the band projection $P_{\pi_a(e)}$. We will simply write $P_x = P_{\pi_x(e)}$ and $B_x = R(P_x)$ the range of P_x. Notice that for every $x \in X^s$ we have $x + \infty = \infty$. In particular; if B is a band then for every $x \in B$, $x + \infty_B = \infty_B$. For $x \in X^s$ we can define its positive and negative parts as $x^+ = x \vee 0$ and $x^- = -(x \wedge 0)$. Then $x^+ - x^- = x$. (The formula $a \wedge b + a \vee b = a + b$ is still true in X^s). These parts can be characterized by the following property: if $x = a - b$ with $a, b \in X^s_+$ and $a \wedge b = 0$, then $a = x^+$ and $b = x^-$. Indeed we have

$$x^+ = x \vee 0 = (a - b) \vee 0 = a \vee b - b = a + b - b = a.$$

Now as $b \wedge a = b \wedge x^+ = 0$ the equality

$$P^d_{x^+} x = P^d_{x^+}\left(x^+ - x^-\right) = P^d_{x^+}(a - b)$$

gives $b = x^-$ as well. Recall that tow elements in X^s_+ are said to be disjoint and we write $x \perp y$ if $x \wedge y = 0$.

Lemma 1 *Let $(x_\alpha), (y_\alpha)$ be two nets in X^s_+ such that $(x_\alpha) \perp (y_\alpha)$. Then the following statements hold.*

(i) $\bigvee_\alpha (x_\alpha + y_\alpha) = \bigvee_\alpha x_\alpha + \bigvee_\alpha y_\alpha$ *and* $\bigwedge_\alpha (x_\alpha + y_\alpha) = \bigwedge_\alpha x_\alpha + \bigwedge_\alpha y_\alpha$;

(ii) $\limsup (x_\alpha + y_\alpha) = \limsup x_\alpha + \limsup y_\alpha$ *and* $\liminf (x_\alpha + y_\alpha) = \liminf x_\alpha + \liminf y_\alpha$.

Proof. Put $x = \bigvee_\alpha x_\alpha$ and $y = \bigvee_\alpha y_\alpha$. It follows from [3, Lemma 11.(iii)], that $\bigvee_\alpha x_\alpha \wedge \bigvee_\alpha y_\alpha = 0$ and hence $P_{x+y} = P_x + P_y$.

(i)The inequality $x + y \geq \bigvee_\alpha (x_\alpha + y_\alpha)$ is obvious. On the other hand we have $\bigvee_\alpha (x_\alpha + y_\alpha) \geq x$ and $\bigvee_\alpha (x_\alpha + y_\alpha) \geq y$, which gives

$$\bigvee_\alpha (x_\alpha + y_\alpha) \geq x \vee y = x + y,$$

where the last equality holds because $x \wedge y = 0$.

For the second part we have clearly

$$\bigwedge_\alpha (x_\alpha + y_\alpha) \geq \bigwedge_\alpha x_\alpha, \bigwedge_\alpha y_\alpha,$$

and then as $\bigwedge_\alpha x_\alpha$ and $\bigwedge_\alpha y_\alpha$ are disjoint we get

$$\bigwedge_\alpha (x_\alpha + y_\alpha) \geq \bigwedge_\alpha x_\alpha + \bigwedge_\alpha y_\alpha,$$

On the other hand we have

$$P_x \bigwedge_\alpha (x_\alpha + y_\alpha) \leq x_\beta,$$

for all β and then $P_x \bigwedge_\alpha (x_\alpha + y_\alpha) \leq \bigwedge_\alpha x_\alpha$. Similarly we get $P_y \bigwedge_\alpha (x_\alpha + y_\alpha) \leq \bigwedge_\alpha y_\alpha$ and so

$$\bigwedge_\alpha (x_\alpha + y_\alpha) = P_x \bigwedge_\alpha (x_\alpha + y_\alpha) + P_y \bigwedge_\alpha (x_\alpha + y_\alpha) \leq \bigwedge_\alpha x_\alpha + \bigwedge_\alpha y_\alpha,$$

which ends the proof of (i).

(ii) This follows easily from (i). ■

The above lemma is not valid if we have only $x_\alpha \perp y_\alpha$ for each α. Take, for example, $X = \mathbb{R}^2$, $x_1 = (1, 0) = y_2$ and $x_2 = (0, 1) = y_1$.

The following lemma gives another case when equalities in Lemma 1.(i) hold.

Lemma 2 *Let $(x_\alpha)_{\alpha \in A}$ and $(y_\alpha)_{\alpha \in A}$ be two decreasing nets in X_+^s then* $\inf(x_\alpha + y_\alpha) = \inf x_\alpha + \inf y_\alpha$.

Proof. We will make use of 9.(ii) where the equality is proved if one of the nets is constant. First observe that the inequality

$$\inf(x_\alpha + y_\alpha) \geq \inf(x_\alpha) + \inf(y_\alpha)$$

is quite obvious. Fix β in A. Then for any $\alpha \geq \beta$ we have

$$\begin{aligned} \inf_{\alpha \in A} (x_\alpha + y_\alpha) &= \inf_{\alpha \geq \beta} (x_\alpha + y_\alpha) \leq \inf_{\alpha \geq \beta} (x_\alpha + y_\beta) \\ &= \inf_{\alpha \geq \beta} x_\alpha + y_\beta = \inf_{\alpha \in A} x_\alpha + y_\beta. \end{aligned}$$

Hence

$$\inf(x_\alpha + y_\alpha) \le \inf_\beta (\inf_\alpha x_\alpha + y_\beta) = \inf(x_\alpha) + \inf(y_\beta).$$

This completes the proof. ■

Lemma 3 *Let $(x_\alpha)_{\alpha\in A}$ be a net in X_+^s and $(B_\alpha)_{\alpha\in A}$ a net in $\mathcal{B}(X)$ such that $x_\alpha \in B_\alpha^s$ for every $\alpha \in A$. Then $\sup x_\alpha \in (\sup B_\alpha)^s$, $\inf x_\alpha \in (\inf B_\alpha)^s$, $\limsup x_\alpha \in (\limsup B_\alpha)^s$ and $\liminf x_\alpha \in (\liminf B_\alpha)^s$.*

Proof. The statements are obvious if $x_\alpha \in X_+^u$ for every $\alpha \in A$. Now let y be fixed in X_+ and observe that $y \wedge x_\alpha \in B_\alpha$ for all α. So $y \wedge \sup x_\alpha = \sup (y \wedge x_\alpha) \in (\sup B_\alpha)^s$. As this happens for each $y \in X_+$ we get $\sup x_\alpha \in (\sup B_\alpha)^s$. The proof of the other results is similar. ■

Remark 4 *If $\{x_\alpha : \alpha \in A\}$ is a subset of X_+^s and $y \in X_+^s$ then $\sup_{\alpha\in A} y x_\alpha = y \sup_{\alpha\in A} x_\alpha$ holds in X_+^s. This follows from [3, Lemma 24] when A is finite and then holds for arbitrary subsets using [3, Lemma 23]. It should be noted that a similar formula for infimum fails in general (see Lemma 11 below).*

3 More about finite and infinite parts

We develop in this section some material concerning the space X^s, the sup-completion of X, that are needed to prove our results in Section 5. These results can be interesting in their own right.

Let X be a Dedekind complete Riesz space with weak order unit e. It was shown in [3] that every element $y \in X_+^s$ has a decomposition:

$$y = y^f + y^\infty \in X^s,$$

where y^∞ is the largest element in B^s for some band B in X and $y^f \in B^d$. It is easy to see that $x^\infty \le y^\infty$ whenever $x \le y$ in X_+^s, but it is not the case for the finite parts in general. Consider for example $x = (1,1) \le y = (1,\infty)$ in $(\mathbb{R}^2)^s$.

We would like to note this useful point for further reference.

Remark 5 *Elements of X_+^s of the form x^∞ are characterized by the following property:*

$$0 < a \leq x^\infty \Longrightarrow na \leq x^\infty \text{ for all } n \in \mathbb{N}.$$

Additionally, it is noteworthy to observe that if $P = P_B$ is a band projection such that $Px = \infty_B$ and $P^d x \in X^u$ then $Px = x^\infty$ and $P^d x = x^f$.

Lemma 6 *Let X be a Dedekind complete Riesz space and $x, y \in X_+^s$. Then the following statements hold.*

(i) $(x+y)^\infty = x^\infty + y^\infty$ *and* $(x+y)^f \leq x^f + y^f$ *with equality if x and y are disjoint.*

(ii) $(x \vee y)^\infty = x^\infty \vee y^\infty = x^\infty + y^\infty$ *and* $(x \vee y)^f = P_{y^\infty}^d x^f \vee P_{x^\infty}^d y^f \leq x^f \vee y^f$.

(iii) $(x \wedge y)^\infty = x^\infty \wedge y^\infty$ *and* $(x \wedge y)^f = x^f \wedge y^f + x^f \wedge y^\infty + x^\infty \wedge y^f$.

(iv) $(x.y)^\infty = x^f.y^\infty + x^\infty.y^f + x^\infty.y^\infty$ *and* $(xy)^f = x^f y^f$. *In particular, if $x \in X_+^u$ then $(xy)^f = xy^f$ and $(xy)^\infty = xy^\infty$.*

Proof. Let B be the band generated by $x^\infty + y^\infty$, so that $\infty_B = x^\infty + y^\infty = x^\infty \vee y^\infty$, and let P be the corresponding band projection.

(i) Clearly, $P^d(x+y) = P^d\left(x^f + y^f\right) \in X^u$, and then

$$P(x+y) \geq x^\infty + y^\infty = \infty_B.$$

So $P(x+y) = x^\infty + y^\infty = \infty_B$ and then $x^\infty + y^\infty = \infty_B = (x+y)^\infty$ and

$$(x+y)^f = P^d\left(x^f + y^f\right) = P_{y^\infty}^d x^f + P_{x^\infty}^d y^f \leq x^f + y^f.$$

(ii) Again as $\infty_B \geq P(x \vee y) \geq x^\infty \vee y^\infty = \infty_B$ we get

$$\infty_B = P(x \vee y) = x^\infty \vee y^\infty.$$

On the other hand

$$P^d(x \vee y) = P^d x \vee P^d y = P^d x^f \vee P^d y^f = P_{y^\infty}^d x^f \vee P_{x^\infty}^d y^f \in X^u.$$

This shows that

$$\infty_B = (x+y)^\infty = (x \vee y)^\infty, \text{ and } (x \vee y)^f = P_{y^\infty}^d x^f \vee P_{x^\infty}^d y^f.$$

If $x \perp y$ then $x^f + y^f \perp x^\infty + y^\infty$ and then

$$P^d\left(x^f + y^f\right) = x^f + y^f.$$

(iii) It follows from [3, Lemma 12] that

$$x \wedge y = x^\infty \wedge y^\infty + x^f \wedge y^\infty + x^\infty \wedge y^f + x^f \wedge y^f.$$

Considering $x^\infty \wedge y^\infty$ is infinite (unless zero) and $x^f \wedge y^\infty + x^\infty \wedge y^f + x^f \wedge y^f$ is finite, mutually disjoint, they are likely the infinite and finite parts of $x \wedge y$.

(iv) The proof is similar. ∎

Remark 7 *As mentioned earlier the map $x \longmapsto x^\infty$ is increasing on X_+^s, whereas the map $x \longmapsto x^f$ is not. However, there is an important case where the implication: $x \leq y \Longrightarrow x^f \leq y^f$ holds true. This occurs when the difference is finite: If $y = x + a$ with $a \in X_+^u$ and $x \leq y$, then $x^f \leq y^f$. Indeed we have*

$$x = y^\infty + y^f - a = y^\infty + y^f - P_{y^\infty} a - P_{y^\infty}^d a = y^\infty + y^f - P_{y^\infty}^d a.$$

But as $y^f - P_{y^\infty}^d a \in B_{y^\infty}^d$, we deduce from the uniqueness of the decomposition [3, Theorem 15] that $x^f = y^f - P_{y^\infty}^d a \leq y^f$.

Remark 8 *(i) It is well known that for every $x, y \in X_+$ we have $B_{xy} = B_{x \wedge y} = B_x \cap B_y$. This formula is still valid when $x, y \in X_+^s$. This can be shown by taking two nets (x_α) and (y_α) in X such that $x_\alpha \uparrow x$ and $y_\alpha \uparrow y$.*
(ii) It was shown in [3, Proposition 25] that if $x \in X_+^s$ and B is a projection band then $\infty_B.x = \infty_{P_B x} = \infty_{B \cap B_x}$. In particular, if $B \subseteq B_x$ we have $\infty_B.x = \infty_B$.

Proposition 9 *Let $(x_\alpha)_{\alpha \in A}$ be a net in X^s and let $y \in X^s$. Then the following statements hold.*

(i) $\sup(y + x_\alpha) = y + \sup x_\alpha$.

(ii) If (x_α) is order bounded from below in X^s, then $\inf\limits_\alpha (y + x_\alpha) = y + \inf\limits_\alpha x_\alpha$.

(iii) $\limsup\limits_\alpha (y + x_\alpha) = y + \limsup\limits_\alpha x_\alpha$.

(iv) If (x_α) is order bounded from below in X^s, then

$$\liminf_\alpha (y + x_\alpha) = y + \liminf_\alpha x_\alpha.$$

Proof. (i) This is a particular case of [3, Property (P8)].

(ii) The inequality $y + \inf_\alpha x_\alpha \leq \inf_\alpha (y + x_a)$ is obvious. For the converse assume first that $y \in X^u$. Then by the first inequality

$$-y + \inf_\alpha (y + x_\alpha) \leq \inf_\alpha x_\alpha,$$

and so

$$\inf_\alpha (y + x_\alpha) = y + \inf_\alpha x_\alpha.$$

This shows the result for this particular case. Moreover, as $(x_\alpha)_{\alpha \in A}$ is order bounded from below we can assume without loss of generality that $(x_\alpha)_{\alpha \in A}$ and y are in the positive cone X^s_+. We treat now the case $y = \infty_B$ for some band B. Let P denotes the corresponding band projection. Then from the inequality

$$\inf_\alpha (x_\alpha + \infty_B) \leq x_\beta + \infty_B, \qquad \beta \in A,$$

we deduce that

$$P^d \inf_\alpha (x_\alpha + \infty_B) \leq P^d x_\beta \leq x_\beta.$$

As this happens for every β we get

$$P^d \inf_\alpha (x_\alpha + \infty_B) \leq \inf_\alpha x_\alpha.$$

Now observe that

$$\inf_\alpha (x_\alpha + \infty_B) = P^d \inf_\alpha (x_\alpha + \infty_B) + P \inf_\alpha (x_\alpha + \infty_B) \leq \inf_\alpha x_\alpha + \infty_B,$$

which shows the second inequality. Finally the general case can be derived by employing the decomposition $y = y^f + y^\infty$ in the following way:

$$\inf_\alpha (y + x_\alpha) = y^\infty + \inf_\alpha \left(y^f + x_\alpha\right) = y^\infty + y^f + \inf_\alpha (x_\alpha) = y + \inf_\alpha x_\alpha.$$

(iii) and (iv) can be deduced easily from (i) and (ii). ∎

It follows from [3, Theorem 5] that if (x_α) and (y_α) are two nets in X^s_+ such that $x_\alpha \uparrow x$ and $y_\alpha \uparrow y$ in X^s then $(x_\alpha + y_\alpha)_\alpha \uparrow x + y$ (apply (i) to the map $X \times X \longrightarrow X$; $(x, y) \longmapsto x + y$).

Proposition 10 *Let (x_α) and (y_α) be two nets in X^s that are bounded from below. Then the following statements hold.*

(i) $\liminf x_\alpha + \liminf y_\alpha \leq \liminf (x_\alpha + y_\alpha) \leq \liminf x_\alpha + \limsup y_\alpha$

(ii) $\limsup (x_\alpha + y_\alpha) \leq \limsup x_\alpha + \limsup y_\alpha$.

(iii) If $\lim y_\alpha$ *exists then* $\liminf (x_\alpha + y_\alpha) = \liminf x_\alpha + \lim y_\alpha$.

Proof. (i) We will make use of Lemma 9. We have for all $\beta \geq \theta$,

$$\begin{aligned} \inf_{\alpha\geq\beta} x_\alpha + \inf_{\alpha\geq\beta} y_\alpha \leq \inf_{\alpha\geq\beta}(x_\alpha + y_\alpha) &\leq \inf_{\alpha\geq\beta}\left(x_\alpha + \sup_{\alpha\geq\theta} y_\alpha\right) \\ &= \inf_{\alpha\geq\beta} x_\alpha + \sup_{\alpha\geq\theta} y_\alpha \leq \liminf x_\alpha + \sup_{\alpha\geq\theta} y_\alpha. \end{aligned}$$

Taking the supremum over β, we obtain

$$\liminf(x_\alpha) + \liminf(y_\alpha) \leq \liminf(x_\alpha + y_\alpha) \leq \liminf x_\alpha + \sup_{\alpha\geq\theta} y_\alpha.$$

Then taking the infimum over θ and using Proposition 9 we get the desired inequalities.

(ii) We have for each β,

$$\sup_{\alpha\geq\beta} (x_\alpha + y_\alpha) \leq \sup_{\alpha\geq\beta} x_\alpha + \sup_{\alpha\geq\beta} y_\alpha.$$

Then, taking the infimum over β an using Lemma 2, we get the desired inequality.

(iii) This is an easy consequence of (i). ■

Proposition 11 *Let (x_α) be a net in X_+^s, $u \in X_+^s$ and $B \in \mathcal{B}(X)$. Then the following statements hold.*

(i) If $u^\infty \in B^s_{\inf_\alpha x_\alpha}$ then

$$\inf_\alpha u x_\alpha = u \inf_\alpha x_\alpha.$$

In particular we have:

(a) If $B \subset B_{\inf_a x_\alpha}$ then $\inf_a(\infty_B x_\alpha) = \infty_B . \inf_a x_\alpha = \infty_B$.

(b) *If* $u \in X_+^u$ *then* $\inf_\alpha ux_\alpha = u \inf_\alpha x_\alpha$.

(ii) *If* $u^\infty \subset B^s_{\limsup x_\alpha}$ *then* $\limsup_\alpha(ux_\alpha) = u \limsup_\alpha (x_\alpha)$. *In particular, if* $B \subset B_{\limsup_\alpha(x_\alpha)}$ *then* $\limsup_\alpha(x_\alpha \infty_B) = \infty_B \limsup_\alpha (x_\alpha)$.

(iii) *If* $u^\infty \in B_{\liminf_\alpha x_\alpha}$ *then* $\liminf_\alpha(ux_\alpha) = u \liminf_\alpha x_\alpha$. *In particular, if* $B \subset B_{\liminf_\alpha x_\alpha}$ *then*

$$\liminf_\alpha(\infty_B x_\alpha) = \infty_B \liminf_\alpha (x_\alpha) = \infty_B.$$

Proof. (i)(a) Assume first that $B \subseteq B_{\inf_\alpha x_\alpha}$. Then $\infty_B \inf_\alpha x_\alpha = \infty_B = \infty_B x_\beta$ for each $\beta \in A$. Thus the formula

$$\inf_\alpha(x_\alpha \infty_B) = \infty_B \inf_\alpha x_\alpha = \infty_B$$

holds.

(b) Assume now that $u \in X_+^u$. The inequality $\inf_\alpha (ux_\alpha) \geq u \inf_\alpha x_\alpha$ is obvious. For the converse let $z \in X_+^u$ such that $z \leq \inf_\alpha(ux_\alpha)$. Then $z \in B_{ux_\alpha} \subset B_u = B_{u^*}$. Hence $u^*z \leq u^*ux_\alpha \leq x_\alpha$ for every α. It follows that $u^*z \leq \inf x_\alpha$ and then $z = u.u^*z \leq u \inf_\alpha x_\alpha$. From this we deduce the inequality $\inf_\alpha(ux_\alpha) \leq u \inf_\alpha x_\alpha$.

(c) The general case. Assume now that $u \in X_+^s$. Since $(u^f x_\alpha)_\alpha \perp (u^\infty x_\alpha)_\alpha$ it follows from Lemma 1 and cases (a) and (b) that

$$\begin{aligned}\inf_\alpha(ux_\alpha) &= \inf_\alpha(u^f x_\alpha) + \inf_\alpha(u^\infty x_\alpha)\\ &= u^f \inf_\alpha(x_\alpha) + u^\infty \inf_\alpha(x_\alpha) = u \inf_\alpha(x_\alpha),\end{aligned}$$

as required.

(ii) We have for each $\beta \in A$,

$$\sup_{\alpha \geq \beta}(x_\alpha \infty_B) = \infty_B . \sup_{\alpha \geq \beta}(x_\alpha) = \infty_B.$$

Since $B \subset B_{\limsup x_\alpha}$ it follows by (i) that

$$\begin{aligned}\infty_B \limsup_\alpha(x_\alpha) &= \infty_B \inf_\beta \left(\sup_{\alpha \geq \beta}(x_\alpha)\right) = \inf_\beta \left(\infty_B . \sup_{\alpha \geq \beta}(x_\alpha)\right)\\ &= \limsup_\alpha(x_\alpha . \infty_B) = \infty_B.\end{aligned}$$

The general case can be deduced in a similar way as in (i).

(iii) Assume first that $u = \infty_B$ for some $B \subseteq B^s_{\liminf_\alpha x_\alpha}$. One inequality is obvious as $\liminf(\infty_B x_\alpha) \leq \infty_B = \infty_B \liminf x_\alpha$. To prove the converse let us put $u_\beta = u \inf_{\alpha \geq \beta} x_\alpha$ for $\beta \in A$. Then $u_\beta \uparrow \infty_B$. So for every $\gamma \geq \beta$, $u_\beta \in B_{\inf_{\alpha \geq \gamma} x_\alpha}$. It follows in view of (i) that

$$\inf_{\alpha \geq \gamma} u_\beta x_\alpha = u_\beta \inf_{\alpha \geq \gamma} x_\alpha = u_\beta.$$

By taking the supremum over γ we get

$$\begin{aligned} u_\beta &= \liminf_\alpha u_\beta x_\alpha = \sup_\gamma \inf_{\alpha \geq \gamma} u_\beta x_\alpha = \sup_\beta u_\beta \inf_{\alpha \geq \beta} x_\alpha \\ &= u_\beta \sup_\beta \inf_{\alpha \geq \beta} x_\alpha = u_\beta \liminf_\alpha x_\alpha. \end{aligned}$$

Taking the supremum over β we get

$$\infty_B = \infty_B \liminf_\alpha x_\alpha = \sup_\beta \liminf_\alpha u_\beta x_\alpha \leq \liminf_\alpha \infty_B x_\alpha.$$

This proves (iii) in that special case. The general case can be deduced as in (i). ■

Remark 12 *In Proposition 11, the condition $B \subseteq B_{\inf_\alpha x_\alpha}$ can not be dropped as the following example can show. If $X = \mathbb{R}$, $u = \infty$ and $x_n = n^{-1}$, $n \geq 1$, then $\infty = \inf a x_n \neq u \inf x_n = 0$. But it is useful to note the following inequality $\inf_\alpha (u x_\alpha) \leq u^\infty + u^f \inf x_\alpha$.*

Lemma 13 *Let $(x_\alpha)_{\alpha \in A}$, $(y_\alpha)_{\alpha \in A}$ be two nets in X^s_+. Then*

$$\limsup_{\alpha \in A} (x_\alpha y_\alpha) \geq \limsup_{\alpha \in A} x_\alpha \liminf_{\alpha \in A} y_\alpha.$$

If, in addition, $\left(\sup_{\alpha \geq \beta} x_\alpha\right)^\infty \in B^s_{\inf_{\alpha \geq \beta} y_\alpha}$ for some β and $\left(\liminf_{\alpha \in A} y_\alpha\right)^\infty \in B^s_{\limsup x_\alpha}$ then

$$\liminf_\alpha (x_\alpha y_\alpha) \leq \limsup_\alpha x_\alpha . \liminf_\alpha y_\alpha.$$

Proof. Fix β, γ in A with $\beta \geq \gamma$. Then we have for each $\theta \geq \beta$,

$$\sup_{\alpha \geq \beta} (x_\alpha y_\alpha) \geq x_\theta y_\theta \geq x_\theta \inf_{\alpha \geq \beta} y_\alpha.$$

According to Remark 4 we have

$$\begin{aligned} \sup_{\alpha \geq \gamma} (x_\alpha y_\alpha) \geq \sup_{\alpha \geq \beta} (x_\alpha y_\alpha) \geq \sup_{\theta \geq \beta} \left(x_\theta \inf_{\alpha \geq \beta} y_\alpha \right) \\ = \sup_{\theta \geq \beta} x_\theta . \inf_{\alpha \geq \beta} y_\alpha \geq \limsup x_\alpha . \inf_{\alpha \geq \beta} y_\alpha . \end{aligned}$$

Taking the supremum over β we get

$$\sup_{\alpha \geq \gamma} (x_\alpha y_\alpha) \geq \sup_{\beta \geq \gamma} \left(\limsup x_\alpha . \inf_{\alpha \geq \beta} y_\alpha \right) = \limsup x_\alpha . \liminf y_\alpha .$$

From this we derive the inequality

$$\limsup (x_\alpha y_\alpha) \geq \limsup x_\alpha . \liminf y_\alpha .$$

(ii) Assume now $\left(\sup_{\alpha \geq \beta} x_\alpha\right)^\infty \in B^s_{\inf\limits_{\alpha \geq \beta} y_\alpha}$ for some $\beta \in A$. Then

$$\left(\sup_{\alpha \geq \gamma} x_\alpha\right)^\infty \in B^s_{\inf\limits_{\alpha \geq \gamma} y_\alpha} \text{ for every } \gamma \geq \beta.$$

Now for $\theta \geq \gamma \geq \beta$ we have

$$\inf_{\alpha \geq \gamma} (x_\alpha y_\alpha) \leq y_\theta \sup_{\alpha \geq \gamma} x_\alpha .$$

It follows that

$$\inf_{\alpha \geq \gamma} (x_\alpha y_\alpha) \leq \inf_{\theta \geq \gamma} \left(y_\theta . \sup_{\alpha \geq \gamma} x_\alpha \right) = \inf_{\theta \geq \gamma} y_\theta . \sup_{\alpha \geq \gamma} x_\alpha \leq \liminf y_\alpha . \sup_{\alpha \geq \gamma} x_\alpha .$$

where we have used 11.(i) in the equality above. For a fixed γ we have for every $\delta \geq \gamma$,

$$\inf_{\alpha \geq \gamma} (x_\alpha y_\alpha) \leq \inf_{\alpha \geq \delta} (x_\alpha y_\alpha) \leq \liminf y_\alpha . \sup_{\alpha \geq \delta} x_\alpha .$$

Now taking the infimum over $\delta \geq \gamma$ and using again Lemma 11(i) we get

$$\inf_{\alpha \geq \gamma} (x_\alpha y_\alpha) \leq \inf_{\beta \geq \gamma} \left(\liminf y_\alpha . \sup_{\alpha \geq \beta} x_\alpha \right) = \liminf y_\alpha . \limsup x_\alpha .$$

as required. ■

We conclude this section with a brief discussion on Boolean algebras. Recall that a Boolean algebra is a distributive lattice $\mathcal{A}$ with smallest and largest elements that is complemented. The latter means that for every element $a \in \mathcal{A}$ there exists a (necessarily unique) element a' such that $a \wedge a' = 0$ and $a \vee a' = 1$, where 0 denotes the smallest element of $\mathcal{A}$ and 1 its largest one. The Boolean algebra $\mathcal{A}$ is said to be Dedekind complete if every nonempty subset has a supremum.

Consider a Dedekind complete Riesz space X with weak order unit e. Three crucial Boolean algebras in this context are isomorphic. The two first are familiar: the set $\mathcal{C}(e)$ consisting of all components of e, and the set of all band projections $\mathcal{B}(X)$. These are isomorphic through the mapping:

$$\mathcal{C}(e) \longrightarrow \mathcal{B}(X); \qquad u \longmapsto B_u .$$

It should be noted that this map preserves suprema and infima. Specifically, for any set $\{p_\alpha : \alpha \in A\}$ of components of e, $\sup B_{p_\alpha} = B_{\sup p_\alpha}$ and $\inf B_{p_\alpha} = B_{\inf p_\alpha}$. Observe that the first formula remains valid for general sets, the second, however, fails in general. The third noteworthy Boolean algebra of interest is similarly isomorphic to the aforementioàned ones. It is intricately associated to the space X^s as it is consisting of infinite parts of positive elements within X^s. Let us employ the following notation to represent it:

$$\infty(X) = \left\{ x^\infty : x \in X_+^s \right\} = \left\{ \infty_B : B \in \mathcal{B}(X) \right\} .$$

The following result tells us that $\infty(X)$ is isomorphic to $\mathcal{B}(X)$.

Proposition 14 *Let $(B_\alpha)_{\alpha \in A}$ be a net in $\mathcal{B}(X)$. The following hold.*

(i) $\inf\limits_\alpha \infty_{B_\alpha} = \infty_{\inf\limits_\alpha B_\alpha}$ *and* $\sup\limits_\alpha \infty_{B_\alpha} = \infty_{\sup\limits_\alpha B_\alpha}$.

(ii) $\liminf_{\alpha} (\infty_{B_\alpha}) = \infty_{\liminf_\alpha B_\alpha}$ *and* $\limsup_{\alpha} (\infty_{B_\alpha}) = \infty_{\limsup_\alpha B_\alpha}$.

(iii) The map $\phi : \mathcal{B}(X) \longrightarrow \{\infty_B : B \in \mathcal{B}(X)\}$; $B \longmapsto \infty_B$ *is an order continuous Boolean algebra isomorphism.*

Proof. (i) The inequality $\bigwedge_\alpha \infty_{B_\alpha} \geq \infty_{\wedge_\alpha B_\alpha}$ is evident. Conversely, if $z \in [\bigwedge_\alpha \infty_{B_\alpha}]^{\leq}$, then $z \leq \infty_{B_\alpha}$ for every α, so $z \in B_\alpha$ for every α, and consequently $z \in \bigwedge_\alpha B_\alpha$. Therefore $z \leq \infty_{\wedge_\alpha B_\alpha}$, establishing the desired inequality. For the second result, it is clear that if F is finite then

$$\sup_{\alpha \in F} \infty_{B_\alpha} = \infty_{\sup\limits_{\alpha \in F} B_\alpha}.$$

Now it is sufficient to observe that

$$\sup_{\alpha} \infty_{B_\alpha} = \sup_{F \text{ finite } \subseteq A} \sup_{\alpha \in F} \infty_{B_\alpha},$$

and

$$\infty_{\sup\limits_{\alpha} B_\alpha} = \sup_{F \text{ finite } \subseteq A} \infty_{\sup\limits_{\alpha \in F} B_\alpha}.$$

(ii) is an immediate consequence of (i).

(iii) The fact that ϕ is an isomorphism is quite clear, and since it respects infinite suprema and infima, it is order continuous by (i) and (ii). ■

4 The star map

According to [8, Theorem 5] any universally complete Riesz space X with weak unit e is von Neumann regular (that is, for every $a \in X$ there exists $b \in X$ such that $a = a^2 b$) and it is not difficult to deduce from this result that any order weak unit element is invertible (see for example [10, Remark 3.3]). Here, we present a concise proof of this result employing the concept of sup-completion.

Lemma 15 *Let X be a universally complete Riesz space with weak unit e, which is also an algebraic unit. Then every weak unit x has an inverse in X.*

Proof. Assume first that $x \in X_+$. We know by [12, Theorem 146.3] that $x + \dfrac{1}{n}e$ is invertible. Let y_n denotes its inverse. Then y_n is increasing and if we put $y = \sup y_n \in X_+^s$ we have

$$\left(x + \frac{1}{n}e\right) y_n = xy_n + \frac{1}{n}y = e. \tag{*}$$

In particular, $xy_n \leq e$. By taking the supremum over n we get $xy \leq e$. As x is a weak unit we get $y^\infty = 0$, that is, $y \in X$. Now taking the limit as $n \longrightarrow \infty$ in $(*)$ we obtain $xy = e$ and we are done.

For the general case we write $x = x^+ - x^-$. So $(x^+ - x^-)\, y = e$. Write $a = P_{x^+}y$ and $b = P_{x^-}y$. Then

$$e = \left(x^+ - x^-\right)(a+b) = ax^+ - bx^-.$$

So $x\,(a-b) = e$ and we are done. ■

For every $y \in X_+$ the band B_{y^f} in X^u generated by the finite part of y is a universally complete Riesz space with unit $p = P_{y^f}e$. The inverse of y^f in the band B_{y^f} will be denoted by y^*. Thus we have the following:

$$yy^* = y^*y = e_{y^f} = e_{B_{y^f}} := P_{y^f}e.$$

In particular $B_{y^*} = B_{y^f} \subseteq B_y$.

We list now some useful properties of the map $x \longmapsto x^*$ defined from X^s to X^u.

Proposition 16 *Let X be a Dedekind complete Riesz space and $x, y \in X^s$. Then the following hold.*

(i) $0^* = \infty_B^* = 0$ *for every band in X.*

(ii) $(\lambda x)^* = \lambda^{-1}x^*$ *for every real $\lambda \neq 0$.*

(iii) If $x \perp y$, then $x^* \perp y^*$ and $(x+y)^* = x^* + y^*$. In particular, $|x|^* = (x^+)^* + (x^-)^*$ and $(Qx)^* = Qx^*$ for every band projection Q.

(iv) $(xy)^* = x^*y^* \in B_{x^*y^*} = B_{x^* \wedge y^*}$.

(v) $(x^p)^* = (x^*)^p$ *for $x \in X_+^s$ and $p > 0$.*

(vi) If $0 \le x \le y$ and P is a band projection such that $P \le P_x$. Then $Py^* \le Px^*$. In particular $P_x y^* \le x^*$.

Proof. We will show only the last property. Multiply the inequality $x \le y$ by $y^* x^*$, yields

$$P_{x^*} y^* \le P_{y^*} x^* \le x^*.$$

Furthermore, as $x \le y$, we have $x^\infty \perp y^f$, implying $P_{x^\infty} y^* = 0$. Now, given that $P \le P_x = P_{x^*} + P_{x^\infty}$, we deduce that

$$Py^* \le P_{x^*} y^* + P_{x^\infty} y^* = P_{x^*} y^*,$$

and then

$$Py^* \le PP_{x^*} y^* \le PP_{y^*} x^* \le Px^*.$$

Thus, the desired inequality is established. ■

Lemma 17 *Let $(x_\alpha)_{\alpha \in A}$ be a net in X_+^u and $x \in X^s$ such that $x_\alpha \uparrow x$, then $x_\alpha^* \xrightarrow{o} x^*$ in X^u.*

Proof. It is enough to prove the following inequalities:

$$\limsup x_\alpha^* \le x^* \le \liminf x_\alpha^*.$$

According to Lemma 16 we infer from the inequalities $x_\alpha \le x$ (for $\alpha \in A$) that

$$P_{x_\alpha} x^* \le x_\alpha^*.$$

Since the net (x_α) is increasing, this implies that

$$P_x x^* = x^* \le \liminf x_\alpha^*. \tag{1}$$

On the other hand we have by Lemma 16.(iv) $P_{x_\beta} x_\alpha^* \le x_\beta^*$ for all $\alpha \ge \beta$. So

$$\limsup_\alpha P_{x_\beta} x_\alpha^* = P_{x_\beta} \limsup_\alpha x_\alpha^* \le x_\beta^*.$$

Multiplying the above inequality by x_β, we obtain

$$x_\beta \limsup x_\alpha^* \le x_\beta x_\beta^* \le e.$$

Taking the supremum over β yields

$$x \limsup x_\alpha^* \leq e. \tag{2}$$

In particular, $x^\infty \limsup x_\alpha^* \leq e$, implying $x^\infty \perp \limsup x_\alpha^*$. As $\limsup x_\alpha^*$ belongs to B_x^s, we have

$$\limsup x_\alpha^* = P_{x^\infty} \limsup x_\alpha^* + P_{x^f} \limsup x_\alpha^* = P_{x^*} \limsup x_\alpha^*.$$

Thus (2) implies that

$$\limsup x_\alpha^* = P_{x^*} \limsup x_\alpha^* = x^* x \limsup x_\alpha^* \leq x^*. \tag{3}$$

Now, combining (1) and (3), we conclude that $x_\alpha^* \xrightarrow{o} x^*$ in X^u. ■

Proposition 18 *Let $(x_\alpha)_{\alpha\in A}$ and $(y_\alpha)_{\alpha\in A}$ be two sequences in X_+^u such that $x_\alpha \uparrow x$ and $y_\alpha \uparrow y$ in X^s and $x_\alpha^2 \leq y_\alpha$ for all α. Then $x_\alpha y_\alpha^* \xrightarrow{o} x^f y^*$ in X^u.*

Proof. According to Lemma 17, we have that

$$y_\alpha^* \xrightarrow{o} y^* \text{ in } X^u. \tag{4}$$

Moreover, from the inequality $x_\alpha^2 \leq y_\alpha$ it follows that

$$x_\alpha y_\alpha^* = e_{x_\alpha} x_\alpha y_\alpha^* \leq x_\alpha^*.$$

Notably, the sequence $(x_\alpha y_\alpha^*)$ is order bounded in X^u. Now as $x_\alpha \uparrow x$ we have $e_{x^*} x_\alpha \uparrow x^f \in X^u$ and thus

$$P_{x^*} x_\alpha \xrightarrow{o} x^f \text{ in } X^u. \tag{5}$$

As the product is order continuous in X^u we derive from 4 and 5 that $P_{x^*}(x_\alpha y_\alpha^*) \xrightarrow{o} x^f y^*$ in X^u. In particular

$$\begin{aligned} x^f y^* &= \limsup \left(P_{x^*}(x_\alpha y_\alpha^*)\right) = P_{x^*} \limsup (x_\alpha y_\alpha^*) \\ &= \liminf \left(P_{x^*}(x_\alpha y_\alpha^*)\right) = P_{x^*} \liminf (x_\alpha y_\alpha^*). \end{aligned} \tag{6}$$

To conclude our proof observe that both $\limsup (x_\alpha y_\alpha^*)$ and $\liminf (x_\alpha y_\alpha^*)$ belong to the band B_{x^*} in X^u, enabling us to simplify (6) by removing P_{x^*} from the right side members. ■

Lemma 19 *Let $(x_\alpha)_{\alpha\in A}$ be an order bounded net in X_+^u and $u \in X_+^u$. Then the following statements are valid.*

(i) $\inf_\alpha (ux_\alpha) = u \inf x_\alpha$ *and* $\sup_\alpha (ux_\alpha) = u \sup x_\alpha$.

(ii) $\liminf_\alpha (ux_\alpha) = u \liminf_\alpha x_\alpha$ *and* $\limsup_\alpha (ux_\alpha) = u \limsup_\alpha x_\alpha$.

Proof. (i) The inequality $\inf_\alpha (ux_\alpha) \geq u \inf x_\alpha$ is obvious. To establish the reverse inequality, let $z \in X$ be a positive lower bound of $\{ux_\alpha : \alpha \in A\}$. It is enough to show that $z \leq u \inf x_\alpha$. Observe that $u^*z \leq u^*ux_\alpha \leq x_\alpha$ and then $u^*z \leq \inf x_\alpha$. But as z belongs to the band B_u we have $z = uu^*z \leq u \inf x_\alpha$. This proves the first assertion; the proof of the second assertion follows a similar line of reasoning.

(ii) This is an immediate consequence of (i). ■

Lemma 20 *Let (x_α) and (y_α) be two order bounded nets in X_+^u. Then the following inequalities hold.*

$$\begin{aligned}\liminf x_\alpha . \liminf y_\alpha \leq \liminf (x_\alpha y_\alpha) &\leq \liminf x_\alpha . \limsup y_\alpha \\ &\leq \limsup (x_\alpha y_\alpha) \leq \limsup x_\alpha . \limsup y_\alpha .\end{aligned}$$

Proof. (i) For any γ large enough and $\beta \geq \gamma$ we have

$$\sup_{\alpha\geq\beta} (x_\alpha y_\alpha) \leq \sup_{\alpha\geq\beta} x_\alpha . \sup_{\alpha\geq\gamma} y_\alpha .$$

So taking the infimum over β it follows from Lemma 19 that

$$\limsup (x_\alpha y_\alpha) \leq \limsup x_\alpha . \sup_{\alpha\geq\gamma} y_\alpha .$$

Hence, taking the infimum over γ, we obtain

$$\limsup (x_\alpha y_\alpha) \leq \limsup (x_\alpha) \limsup (y_\alpha) .$$

(ii) For $\beta \geq \gamma$ we have

$$\inf_{\alpha\geq\beta} (x_\alpha y_\alpha) \geq \inf_{\alpha\geq\beta} x_\alpha . \inf_{\alpha\geq\gamma} y_\alpha .$$

So taking the supremum over β it follows from Lemma 19 that

$$\liminf (x_\alpha y_\alpha) \geq \liminf x_\alpha . \inf_{\alpha\geq\gamma} y_\alpha .$$

Hence, taking the supremum over γ, we obtain

$$\liminf (x_\alpha y_\alpha) \geq \liminf x_\alpha . \liminf y_\alpha .$$

The two other inequalities are similar. ■

Lemma 21 *Let (x_α) and (y_α) be two order bounded sequences in X_+^u. If $x_\alpha \xrightarrow{o} x$ then $\limsup (x_\alpha y_\alpha) = x \limsup y_\alpha$ and $\liminf (x_\alpha y_\alpha) = x \liminf y_\alpha$.*

Proof. By Lemma 20 we have

$$\begin{aligned} x \liminf y_\alpha &= \liminf x_\alpha . \liminf y_\alpha \leq \liminf x_\alpha y_\alpha \\ &\leq \limsup x_\alpha . \liminf y_\alpha = x \liminf y_\alpha \end{aligned}$$

Hence $\liminf (x_\alpha y_\alpha) = x \liminf y_\alpha$. The second equality follows similarly. ■

We will prove now a multiplicative decomposition property in X_+^s.

Proposition 22 *Let X be a Dedekind complete Riesz space and $x, y, z \in X_+^s$. If $x \leq yz$ then there exists a decomposition $x = ab$ of x with $0 \leq a \leq y$ and $0 \leq b \leq z$.*

Proof. By restricting ourselves to the band B_{y+z} we may assume that X has a weak unit $e > 0$. Assume first that y is finite. Then $x = y.y^* x$ is a suitable decomposition. Indeed as $x \leq yz$ we have $x \in B_y$, and so $P_y x = x$. Moreover $y^* x \leq y^* yz \leq z$. If y is infinite, then a suitable decomposition could be $x = x\left(z^* + e_{z^\infty}\right) . \left(z^f + e_{z^\infty}\right)$.

General case. Write $x = Px + P^d x$ where $P = P_{y^f}$, and observe that $Px \leq Pyz = y^f Pz$ and $P^d x \leq y^\infty P^d z$. Using the previous two cases we can write $Px = ab$ and $P^d x = a^d b^d$, with $a, b \in B_{y^f}^s$, $a^d, b^d \in \left(B_{y^f}^d\right)^s$ with $0 \leq a \leq Py$, $0 \leq b \leq Pz$, $0 \leq a^d \leq P^d y$, and $0 \leq b^d \leq P^d z$. It is easily seen now that $x = \left(a + a^d\right) . \left(b + b^d\right)$ is a suitable decomposition of x. ■

5 Applications

In this section, we consider a Riesz conditional triple (X, e, T) , unless explicitely stated otherwise. Here X is a Dedekind complete Riesz space with order weak unit e, and T a conditional expectation operator on X with $Te = e$.

Definition 23 *Let X be an Archimedean Riesz space with weak order unit e. Let $M = (m_{ij})_{1\leq i,j\leq n}$ be an n-matrix with entries in X. We say that M is positive semi-definite if M is symmetric and $x^T Mx \in X_+$ for every $x \in (X_e)^n$, where X_e is the ideal generated by e and $x^T = (x_1, ..., x_n)$ is the transpose of x. This means that $\sum\limits_{i,j} m_{i,j}x_i x_j \geq 0$ for all $x_1, ..., x_n \in X_e$.*

Remark 24 *It is clear that if M is a positive semi-definite matrix then $x^T Mx \in X_+$ for every $x \in (X)^n$. Moreover, in order to prove that a matrix $M = (m_{i,j})$ is positive semi-definite it is enough to check the positivity of $x^T Mx$ for elements x in $(\mathcal{R}(T))^n$. Indeed if $x \in X_e^n$ and V the f-subalgebra of X^u generated by the set of the coefficients of M and of x. Then for every $\omega \in H_m(V)$ and $\lambda = (\omega(x_j)e, ..., \omega(x_j)e)^T$ we have*

$$\omega\left(x^T Mx\right) = \omega\left(\lambda^T M\lambda\right) \geq 0,$$

as $\lambda \in \mathcal{R}(T)^n$ and ω is positive. Thus M is positive semi-definte.

For a matrix M in $\mathbf{M}_n(X)$ let $\Gamma(M)$ denote the sum of all entries of M, that is, $\Gamma(M) = \sum\limits_{1\leq i,j\leq n} m_{ij}$.

Lemma 25 *Let $(x_n)_{n\in\mathbb{N}}$ be a positive sequence in X_+, where X is a Dedekind complete Riesz space with weak order unit e, and let $R_n = \sum\limits_{k=n}^{\infty} x_k \in X_+^s$, for $n \in \mathbb{N}$. Then $R_1^\infty = (\inf R_n)^\infty$.*

Proof. Clearly $R_1^\infty \geq (\inf R_n)^\infty$ and by [3, Proposition 21],

$$R_1^\infty = R_n^\infty + \left(\sum_{k=1}^{n-1} x_k\right)^\infty = R_n^\infty.$$

Assume that $u \in B_{R_1^\infty}^+$. Then $R_n \geq tu$ for all real $t \geq 0$. So $\inf\limits_n R_n \geq tu$ for all real $t \geq 0$. This shows that $u \in B_{\left(\inf\limits_n R_n\right)^\infty}$. We deduce from this that $B_{R_1^\infty} \subseteq B_{\left(\inf\limits_n R_n\right)^\infty}$ and so $\left(\inf\limits_n R_n\right)^\infty \geq R_1^\infty$. ■

We aim to generalize to the setting of Riesz space the following result due to Feng and Shen [7].

Theorem 26 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* (A_n) *a sequence of events and* (w_n) *a sequence of positive reals. If* $\sum_{n=1}^{\infty} w_n \mathbb{P}(A_n) = \infty$*, then*

$$\mathbb{P}(\limsup A_n) \geq \limsup_n \frac{\left(\sum_{i=1}^n w_i \mathbb{P}(A_i)\right)^2}{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \mathbb{P}(A_i A_j)}.$$

Our inspiration for the proof derives from [7]. But the proof here is more technical.

For a subalgebra Y of X^u we will use $H_m(Y)$ to denote the set of all Riesz and algebra homomorphism from Y to $\mathbb{R}$. We recall that if Y is generated by a countable set the $H_m(Y)$ separates points of Y and we have, in particular, for $y \in Y$, the equivalence

$$y \geq 0 \Leftrightarrow \varphi(y) \geq 0 \text{ for all } \varphi \in H_m(V).$$

Lemma 27 *Let* $M = (m_{ij}) \in \mathbf{M}_n(X^u)$ *be a positive semi-definite matrix,* V *a unital subalgebra of* X^u *containing the entries of* M *and* $\varphi : V \longrightarrow \mathbb{R}$ *a positive algebra homomorphism. Then the real matrix* $\varphi(M) := (\varphi(m_{ij}))$ *is positive semi-definite.*

Proof. Let $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$ and $x = (\lambda_1 e, \lambda_2 e, ..., \lambda_n e)$. An easy computation gives $\varphi\left(xMx^T\right) = \lambda^T \varphi(M) \lambda$, and the result follows. ■

Lemma 28 *Let* $n, k_1, ..., k_n$ *be integers, with* $k = k_1 + ... + k_n$ *and* $A_{i,j} \in \mathbf{M}_{k_i,k_j}(X_+)$.

1. *If* $M = (A_{ij})_{1 \leq i,j \leq n} \in \mathbf{M}_k(X)$ *is positive semi-definite then the matrix* $S = (\Gamma(A_{ij}))$ *is so.*

2. *If* $M = (m_{ij})$ *is positive semi-definite then* $\det M \geq 0$.

Proof. (i) Let $x \in X^n$, then $x^T S x = v^T M v \geq 0$ with $v = (x_1, x_1, ..., x_1, x_2, ..., x_2, ..., x_n, ..., x_n)^T$.

(ii) Let V be the subalgebra of X^u generated by the entries of M and e. For every $\varphi \in H_m(V)$ the matrix $\varphi(M)$ is positive semi-definite by Lemma 27. So its determinant is positive. This shows that

$$\varphi(\det M) = \det \varphi(M) \geq 0.$$

As this holds for every $\varphi \in H_m(V)$ we deduce that $\det M \geq 0$. ■

Lemma 29 *Given a partition of an $(m+n)\times(m+n)$ symmetric matrix $M=(m_{i,j})\in\mathcal{M}_{m+n}(X)$:*

$$M=\left(\begin{array}{cc} A & C \\ {}^{t}C & B \end{array}\right),$$

where $A\in\mathbf{M}_m(X)$, $B\in\mathbf{M}_n(X)$ and $C\in\mathbf{M}_{m,n}(X)$. If M is positive semi-definite, then $\Gamma(C)^2\leq\Gamma(A).\Gamma(B)$.

Proof. This follows from Lemma 28 and the fact that $\Gamma(A).\Gamma(B)-\Gamma(C)^2=\det\left(\begin{array}{cc}\Gamma(A) & \Gamma(C)\\ \Gamma\left(C^T\right) & \Gamma(B)\end{array}\right)$. ■

Lemma 30 *Let $\{Q_i\}_{i=1}^n$ be a sequence of band projections on X. Then the matrix $M=(TQ_iQ_je)_{1\leq i,j\leq n}$ is positive semi-definite.*

Proof. Let $u=(u_1,u_2,...,u_n)\in\mathcal{R}(T)^n$. Then using the average property of T we have

$$\begin{aligned} uMu^T &= \sum_{i,j}u_iu_jTQ_iQ_je=\sum_{i,j}T\left(u_iu_jQ_iQ_je\right)\\ &= T\sum_{i,j}u_iu_jQ_iQ_je=T\left(\sum_{i=1}^{n}u_iQ_ie\right)^2\geq 0. \end{aligned}$$

which proves the desired result. ■

Consider now a sequence (v_n) in $\mathcal{R}(T)_+$ and a sequence (Q_n) in $\mathcal{B}(X)$. For any $1\leq q\leq n\leq\infty$, let us define the following

$K_{q,n}:=\sum_{i=q}^{n}v_iTQ_ie,$

$R_{q,n}:=\sum_{i=q}^{n}v_iv_1TQ_iQ_1e,$

$S_{q,n}:=\sum_{q\leq i,j\leq n}v_iv_jTQ_iQ_je,$

$R_{q,n}(j):=\sum_{i=q}^{n}v_jv_iTQ_iQ_je$, for $1\leq j\leq n$.

These notations will be utilized in subsequent discussions, notably in our main result, Theorem 34.

Lemma 31 *With the aforementioned notations, the following relationship holds true:*

(i) If $1 \leq q \leq n \leq \infty$ then $K_{q,n}^2 \leq S_{q,n}$.

(ii) For all $1 \leq p \leq q < \infty$ we have

$$S_{q,n}^* S_{p,n} \xrightarrow{o} e_{S_{q,\infty}} + S_{q,\infty}^* \left(S_{p,q-1} + 2\textstyle\sum_{j=p}^{q-1} R_{q,\infty}^f (j) \right) \text{ in } X^u \text{ as } n \longrightarrow \infty$$

In particular, if the finite part $S_{q,\infty}^f$ of $S_{q,\infty}$ is null, we get

$$S_{q,n}^* S_{p,n} \xrightarrow{o} e_{S_{q,\infty}} \text{ as } n \longrightarrow \infty.$$

Proof. (i) Put $x = \sum\limits_{i=q}^{n} v_i Q_i e$. Then by the averaging property of T we have $Tx = \sum\limits_{i=q}^{n} T(v_i Q_i e) = \sum\limits_{i=q}^{n} v_i T Q_i e$. Hence it follows from Cauchy-Schwarz Inequality or Lyapunov inequality [4] that

$$K_{q,n}^2 = (Tx)^2 \leq Tx^2 = S_{q,n},$$

(ii) Write $S_{p,n} = S_{p,q-1} + S_{q,n} + 2\sum\limits_{j=p}^{q-1} R_{q,n}(j)$. According to Lemma 17, we have $S_{q,n}^* \xrightarrow{o} S_{q,\infty}^*$ in X^u. Moreover, we have clearly

$$S_{q,n}^* S_{q,n} = P_{S_{q,n}} e \uparrow P_{S_{q,\infty}} e, \tag{7}$$

and

$$S_{q,n}^* S_{p,q-1} \xrightarrow{o} S_{q,\infty}^* S_{p,q-1} \text{ in } X^u. \tag{8}$$

Observe on the other hand that

$$R_{q,n}^2 (j) \leq v_j^2 \left(\sum_{i=q}^{n} v_i T Q_i e \right)^2 = v_j^2 K_{q,n}^2 \leq v_j^2 S_{q,n}.$$

It follows from Lemma 18 that

$$\left(v_j^2 S_{q,n} \right)^* R_{q,n}(j) \xrightarrow{o} \left(v_j^* \right)^2 S_{q,\infty}^* R_{q,\infty}^f (j) \text{ in } X^u.$$

As $S_{q,\infty}^* R_{q,\infty}^f (j)$ belongs to the band B_{v_j} we deduce that

$$S_{q,n}^* R_{q,n}(j) = v_j^2 \left(v_j^2 S_{q,n} \right)^* R_{q,n}(j) \xrightarrow{o} S_{q,\infty}^* R_{q,\infty}(j) \text{ in } X^u.$$

As $R^{\infty}_{q,\infty}(j) \leq S^{\infty}_{q,\infty} \perp S^{*}_{q,\infty}$, we have

$$S^{*}_{q,\infty} R_{q,\infty}(j) = S^{*}_{q,\infty} R^{f}_{q,\infty}(j).$$

Hence,

$$S^{*}_{q,n} \sum_{j=p}^{q-1} R_{q,n}(j) \xrightarrow{o} S^{*}_{q,\infty} \sum_{j=p}^{q-1} R^{f}_{q,\infty}(j) \text{ in } X^{u}. \tag{9}$$

The required result follows by combining (7), (8), and (9). ■

Lemma 32 *Let X be a Dedekind complete Riesz space with weak order unit e, and let Y be a regular Dedekind complete Riesz subspace of X with $e \in Y$. Then $x^{f}, x^{\infty} \in Y^{s}$ for every element $x \in Y^{s}_{+}$. In particular this can be applied to $Y = \mathcal{R}(T)$, the range of a conditional expectation operator T.*

Proof. Let x be an element in Y^{s}_{+}. By [3, Lemma 12] we have

$$x \wedge ne = x^{f} \wedge ne + x^{\infty} \wedge ne \in Y.$$

It follows that

$$x \wedge (n+1) e - x \wedge ne = x^{f} \wedge (n+1) e - x^{f} \wedge ne + P_{x^{\infty}} e \in Y.$$

Taking the order limit we get $P_{x^{\infty}} e \in Y$. So

$$x^{f} \wedge ne = x \wedge ne - ne_{x^{\infty}} \in Y.$$

Hence

$$x^{f} = \sup_{n} \left(x^{f} \wedge ne \right) \in Y.$$

Now we get $x^{\infty} = x - x^{f} \in Y^{s}$, as required. ■

Remark. Let (x_n) and (y_n) be two bounded sequences in X^{u}_{+}. It is easy to see that if $x_n \xrightarrow{o} x$ then

$$\limsup (x_n y_n) = x \limsup y_n.$$

This will be used in the next result.

Proposition 33 *Using the same notations as previously and let B be the band generated by $K^{\infty}_{1,\infty}$ and P the corresponding band projection. Then the following assertions are valid.*

(i) $R_{1,\infty}^{\infty} \leq K_{1,\infty}^{\infty} \leq S_{1,\infty}^{\infty}$.

(ii) *The sequences* $\left(K_{q,\infty}^{f}\right)$ *and* $\left(R_{q,\infty}^{f}\right)$ *are decreasing.*

(iii) *For all integer* q *we have* $K_{q,\infty}^{\infty} = K_{1,\infty}^{\infty}$ *and* $R_{q,\infty}^{\infty} = R_{1,\infty}^{\infty}$.

(iv) *For* $V_n := P_{K_{q,n}} S_{1,n}^{*} K_{1,n}^{2}$ *we have:*

$$\begin{aligned}\limsup_n V_n &= P_{K_{q,\infty}} \limsup_n \left(S_{1,n}^{*} K_{1,n}^{2}\right) \\ &= \left(P_{K_{q,\infty}^{f}} S_{q,\infty}^{*} S_{1,q-1} + P_{K_{q,\infty}} e + 2P_{K_{q,\infty}^{f}} S_{q,\infty}^{*} \textstyle\sum_{j=1}^{q-1} R_{q,\infty}^{f}(j)\right)^{*} \\ &\quad \left(K_{1,q-1} K_{q,\infty}^{*} + P_{K_{q,\infty}} e\right)^{2} \limsup_n \left[S_{q,n}^{*} K_{q,n}^{2}\right].\end{aligned}$$

(v) *We have the following equality*

$$\limsup_n \left(P S_{1,n}^{*} K_{1,n}^{2}\right) = \limsup_n \left(P S_{q,n}^{*} K_{q,n}^{2}\right).$$

In particular, if $B = X$*, then* $\limsup_n \left(S_{q,n}^{*} K_{q,n}^{2}\right)$ *is independent of* q.

The last two points in Lemma 33 provide a generalization of [7, Proposition 6].

Proof. (i) By Lemma 31 $K_{1,n}^{2} \leq S_{1,n}$ and then $K_{1,\infty}^{2} \leq S_{1,\infty}$, which yields $K_{1,\infty}^{\infty} = \left(K_{1,\infty}^{2}\right)^{\infty} \leq S_{1,\infty}^{\infty}$. For the second inequality observe that

$$R_{1,n} = v_1 \sum_{k=1}^{n} v_k T Q_1 Q_k e \leq v_1 K_{1,n},$$

and we deduce as above that $R_{1,\infty}^{\infty} \leq K_{1,\infty}^{\infty}$.

(ii) This follows from Remark 7.

(iii) This follows immediately from the equality $(x+y)^{\infty} = x^{\infty} + y^{\infty}$ which holds for all elements $x, y \in X_{+}^{s}$. [3, Proposition 21].

(iv) In view of Lemma 31.(i) we have $S_{1,n}^{*}.K_{1,n}^{2} \leq e$. Hence the first equality follows from the remark preceding Proposition 33 and the fact that $P_{K_{q,n}} e \uparrow P_{K_{q,\infty}} e$ as $n \longrightarrow \infty$. For the second we will use the decomposition

$$V_n := P_{K_{q,n}} S_{1,n}^{*} K_{1;n}^{2} = P_{K_{q,n}} S_{q,n}^{*} K_{q,n}^{2}.A_n.M_n^{*},$$

where

$$A_n = \left(K_{1,q-1} K_{q,n}^{*} + P_{K_{q,n}} e\right)^{2},$$

and

$$M_n = P_{K_{q,n}} \left(S_{1,q-1} S_{q,n}^* + e + 2S_{q,n}^* \sum_{j=1}^{q-1} R_{q,n}(j) \right).$$

According to Lemma 31, the inequality $K_{q,n}^2 \leq S_{q,n}$ implies that $K_{q,n}^2 S_{q,n}^* \leq P_{S_{q,n}} e$. Hence the sequence $\left(K_{q,n}^2 S_{q,n}^*\right)$ is order bounded in X^u. We have

$$P_{K_{q,n}} K_{1,n}^2 = K_{q,n}^2 \left(K_{1,q-1} K_{q,n}^* + P_{K_{q,n}} e\right)^2 = K_{q,n}^2 A_n,$$

with

$$A_n = \left(K_{1,q-1} K_{q,n}^* + P_{K_{q,n}} e\right)^2 \xrightarrow{o} \left(K_{1,q-1} K_{q,\infty}^* + P_{K_{q,\infty}} e\right)^2 \text{ in } X^u.$$

Next, we observe that

$$\begin{aligned} P_{K_{q,n}} S_{1,n}^* &= P_{K_{q,n}} \left(S_{1,q-1} + S_{q,n} + 2\sum_{j=1}^{q-1} R_{q,n}(j) \right)^* \\ &= P_{K_{q,n}} S_{q,n}^* \left(S_{q,n}^* S_{1,q-1} + P_{S_{q,n}} e + 2S_{q,n}^* \sum_{j=1}^{q-1} R_{q,n}(j) \right)^* \\ &= P_{K_{q,n}} S_{q,n}^* M_n^*. \end{aligned}$$

with

$$M_n = P_{K_{q,n}} \left(S_{1,q-1} S_{q,n}^* + e + 2S_{q,n}^* \sum_{j=1}^{q-1} R_{q,n}(j) \right).$$

Now in the proof of Lemma 31 it has been shown that

$$S_{q,n}^* R_{q,n}(j) \xrightarrow{o} S_{q,\infty}^* R_{q,\infty}(j) \text{ in } X^u.$$

So we have

$$\begin{aligned} M_n \xrightarrow{o} M &:= P_{K_{q,\infty}} \left(S_{1,q-1} S_{q,\infty}^* + e + 2S_{q,\infty}^* \sum_{j=1}^{q-1} R_{q,\infty}(j) \right) \text{ in } X^u, \\ &= P_{K_{q,\infty}^f} S_{1,q-1} S_{q,\infty}^* + P_{K_{q,\infty}} e + 2P_{K_{q,\infty}^f} S_{q,\infty}^* \sum_{j=1}^{q-1} R_{q,\infty}^f(j) \end{aligned}$$

where we use in the last equality the fact that $R_{q,\infty}^{\infty}(j) \perp S_{q,\infty}^*$ $(1 \leq j \leq q-1)$ and that $K_{q,\infty}^{\infty} \perp S_{q,\infty}^*$. This shows, in particular, that (M_n) is order bounded in X^u. On the other hand we have by definition of M_n that $M_n, M_n^* \in B_{K_{q,n}} \subset B_{S_{q,n}}$ and

$$S_{q,n}^* M_n^* = P_{K_{q,n}} S_{q,n}^* M_n^* = P_{K_{q,n}} S_{1,n}^* \leq S_{1,n}^*.$$

Hence

$$M_n^* \leq S_{q,n} S_{1,n}^* = S_{1,n} S_{1,n}^* \leq e.$$

This shows that (M_n^*) is order bounded in X^u. Now using Lemma 21 we get

$$P_M e = P_{\liminf M_n} e \leq \liminf P_{M_n} e = \liminf (M_n M_n^*) = M \liminf M_n^*.$$

Thus $P_M e \leq M \liminf M_n^*$ and then

$$M^* \leq \liminf M_n^*. \tag{10}$$

Similarly we have

$$M \limsup M_n^* = \limsup (M_n M_n^*) = \limsup e_{M_n} \leq e.$$

Thus $P_M \limsup M_n^* \leq M^*$. Now as (M_n^*) is contained in the band $B_{K_{q,\infty}}$ and M is a weak order unit of that band; the last inequality becomes

$$\limsup M_n^* \leq M^*. \tag{11}$$

We deduce from (10) and (11) that $M_n^* \xrightarrow{o} M^*$ in X^u. Observe finally that each of the three sequences $\left(e_{K_{q,n}} S_{q,n}^* K_{q,n}^2\right)$, (A_n) and (M_n) is bounded in X^u, which enables us to use Lemma 21 and conclude that

$$\begin{aligned}
\limsup_n \left(P_{K_{q,n}} S_{1,n}^* K_{1,n}^2\right) &= \limsup_n \left[\left(K_{q,n}^2 S_{q,n}^*\right) M_n^* A_n\right] \\
&= \limsup_n \left[K_{q,n}^2 S_{q,n}^*\right] . \lim M_n^* . \lim_n A_n \\
&= \limsup_n \left[K_{q,n}^2 S_{q,n}^*\right] . M^* \left(K_{1,q-1} \left(K_{q,\infty}^f\right)^* + e_{K_{q,\infty}}\right)^2
\end{aligned}$$

in X^u. This completes the proof of (iv).

(v) This is an immediate consequence of (iv) by applying P to (iv) and using the equality $K_{1,\infty}^\infty = K_{q,\infty}^\infty$ proved in (iii). ∎

We reach now the central result of this section.

Theorem 34 *Let (X,e,T) be a conditional Riesz triple, (v_n) a sequence in $\mathcal{R}(T)_+$ and (Q_n) a sequence of band projections. Let P be the projection on the band generated by the infinite part of $\sum_{i=1}^{\infty} v_i T Q_i e$. Then ($Q_{v_n} = P_{v_n} Q_n e$.)*

$$TP \limsup_n Q_{v_n} e \geq \limsup_n \left(P\left(S_{1,n}^* K_{1,n}^2\right)\right).$$

In particular, if $\left(\sum_{i=1}^{\infty} v_i T Q_i e\right)^f = 0$ then

$$T\left(\limsup_n Q_{v_n} e\right) \geq \limsup_n \left(S_{1,n}^* . K_{1,n}^2\right).$$

Proof. (i) Let B be the band associated to P. By Lemma 32 $\infty_B \in \mathcal{R}(T)^s$ and by [3, Proposition 7] we get $P_B T = T P_B$. Put $Q_{v_i} = Q_i P_{v_i}$. Then $c := \bigvee_{i=q}^{n} Q_{v_i} e$ is a component of e and it follows from Cauchy-Schwarz inequality that

$$K_{q,n}^2 = \left(T\left(c.\sum_{i=q}^{n} v_i Q_i e\right)\right)^2 \leq Tc.S_{q,n}.$$

So as T commutes with P, we obtain

$$PK_{q,n}^2 \leq T(Pc).P(S_{q,n}) = T\bigvee_{i=q}^{n} PQ_{v_n} e.P(S_{q,n}),$$

which implies that

$$\begin{aligned} P\left(K_{q,n}^2 S_{q,n}^*\right) = PK_{q,n}^2 . PS_{q,n}^* &\leq T\left(\bigvee_{i=q}^{n} PQ_{v_n} e\right).PS_{q,n}.PS_{q,n}^* \\ &= T\left(\bigvee_{i=q}^{n} PQ_{v_n} e\right), \end{aligned}$$

where the last equality holds because $T\left(\bigvee_{i=q}^{n} PQ_{v_n} e\right)$ belongs to the band $B \cap B_{S_{q,n}}$ and $PS_{q,n}.PS_{q,n}^*$ is the component of e on this band. It follows that

$$\begin{aligned} a := T\left(\limsup_n PQ_{v_n} e\right) &= \lim_q \sup_{n \geq q} T\left(\bigvee_{i=q}^{n} PQ_{v_i} e\right) \\ &\geq \lim_q \limsup_n P\left(S_{q,n}^* . K_{q,n}^2\right) = \limsup_n P\left(S_{1,n}^* . K_{1,n}^2\right), \end{aligned}$$

which proves the desired inequality. Here the last equality follows from Lemma 31(v). In the case $\left(\sum_{i=1}^{\infty} v_i T Q_i e\right)^f = 0$ we get $K_{1,n} \in B$ for every n, and then $P\left(S_{1,n}^* K_{1,n}^2\right) = S_{1,n}^* . K_{1,n}^2$. It follows from the first case that

$$T\left(\limsup_n Q_{v_n} e\right) \geq \limsup_n \left(S_{1,n}^* . K_{1,n}^2\right).$$

This completes the proof. ■

Applying Theorem 34 to $v_n = (Tq_n)^*$ with $q_i = Q_i e$, we obtain the following result which is a generalization of [7, Corollary 2].

Corollary 35 *Under the hypothesis of Theorem 34 with* $q_i = Q_i e$ *we have*

$$T\left(\limsup_n P_{Tq_n} q_n\right) \geq \limsup_n \left(\sum_{1 \leq i,j \leq n} (Tq_i Tq_j)^* T(q_i q_j)\right)^* . \left(\sum_{1 \leq i \leq n} e_{Tq_i}\right)^2 .$$

Theorem 34 allows to get a generalization of Borel-Cantelli Lemma proved by the author [3, Theorem 31].

Corollary 36 *Let* $(P_n)_{n \geq 1}$ *be a sequence of parwise* T*-independent band projections on* X*. If* B *is a band on* X^u *such that*

$$\sum_{n=1}^{\infty} T P_n e = \infty_B + u \qquad \text{with } u \in B^d,$$

then P_B *commutes with* T *and*

$$P_B = \limsup_n P_n.$$

Proof. We know that P_B commutes with T (see the proof of Theorem 34). Now observe that

$$\sum_{n=1}^{\infty} T P_B^d P_n e = P_B^d \sum_{n=1}^{\infty} T P_n e = u \in X^u.$$

Hence, accorfing to [3, Lemma 26], $P_B^d \limsup Q_n = 0$, which implies the inequality

$$\limsup P_n \leq P_B.$$

The reverse inequality will follow from Theorem 34 Indeed if we apply the theorem with $v_n = e$ we get

$$T \limsup_n Q_n = T P_B \limsup_n Q_n e \geq \limsup_n \left(P_B \left(S_{1,n}^* K_{1,n}^2\right)\right) \qquad \textbf{(A1)}$$

Moreover, we have by T-independance,

$$S_{1,n} \leq K_{1,n}^2 + K_{1,n},$$

which yields

$$P_{S_{1,n}} = S_{1,n}^* S_{1,n} \leq S_{1,n}^* K_{1,n}^2 + S_{1,n}^* K_{1,n},$$

and then

$$P_B P_{S_{1,n}} \leq P_B S_{1,n}^* K_{1,n}^2 + P_B S_{1,n}^* K_{1,n}.$$

Taking the $\limsup$ over n in this last inequality gives

$$P_B P_{S_{1,\infty}} \leq \limsup \left(P_B S_{1,n}^* K_{1,n}^2\right) + \limsup \left(P_B S_{1,n}^* K_{1,n}\right).$$

Since $\infty_B = K_{1,\infty}^{\infty} \leq S_{1,\infty}^{\infty}$ we have $P_B P_{S_{1,\infty}} = P_B P_{S_{1,\infty}^{\infty}} = P_B$. So

$$P_B P_{S_{1,\infty}} = P_B P_{S_{1,\infty}^f} + P_B P_{S_{1,\infty}^{\infty}} = P_B P_{S_{1,\infty}^{\infty}} = P_B. \tag{12}$$

Now, $K_{1,n}$ and $S_{1,n}$ are increasing with $K_{1,n}^2 \leq S_{1,n}$, we obtain thanks to Proposition 18,

$$S_{1,n}^* K_{1,n} \xrightarrow{o} S_{1,\infty}^* K_{1,\infty}^f, \tag{13}$$

which gives

$$P_B S_{1,n}^* K_{1,n} \xrightarrow{o} P_B S_{1,\infty}^* K_{1,\infty}^f = 0.$$

Combining (12) and (13) we derive that

$$\begin{aligned} T P_B e = P_B e = P_B P_{S_{1,\infty}} e &\leq \limsup \left(P_B S_{1,n}^* K_{1,n}^2\right) \\ &\leq T P_B \limsup_n P_n e. \end{aligned}$$

Applying Theorem 34 gives

$$T P_B e \leq T \limsup_n P_n e.$$

Therefore, since T is strictly positive we conclude that

$$P_B e = \limsup_n P_n e.$$

his complete the proof. ∎

References

[1] C.D. Aliprantis and O. Burkinshaw, *Positive Operators*, Springer, 2006.

[2] Y. Azouzi, Completeness for vector lattices, J. Math. Anal. Appl. 472 (2019) 216-230.

[3] Y. Azouzi and Y. Nasri, The sup-completion of a Dedekind complete vector lattice, J. Math. Anal. Appl. 506 (2022), no. 2, Paper No. 125651, 20.

[4] Y. Azouzi, M. Trabelsi, *L_p-spaces with respect to conditional expectation on RS,* J. Math. Anal. Appl., 447(2017), 798–816.

[5] K. Donner, Extension of Positive Operators and Korovkin Theorems, Springer (1982).

[6] P. Erdös, A. Rényi, On Cantor's series with convergent $\sum 1/q_n$, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 2 (1959) 93–109.

[7] C. Feng, L. Li, J. Shen, On the Borel–Cantelli lemma and its generalization, 2009.

[8] C.B. Huijsmans, B. de Pagter, On von Neumann regular f-algebras, Order 2 (1986) 403–408.

[9] A. Kalauch, W. Kuo, B. A. Watson, A Hahn-Jordan decomposition and Riesz-Frechet representation theorem in Riesz spaces, arXiv:2209.00715 [math.FA]

[10] M. Roelands and C. Schwanke, Series and power series on universally complete complex vector lattices, J. Math. Anal. Appl. 473 (2019), no. 2, 680–694.

[11] A. C Zaanen. *Introduction to operator theory in Riesz spaces*. Springer Science & Business Media, 2012.

[12] A. C. Zaanen, Riesz spaces II, North-Holland, Amsterdam, 1983.