A PROOF OF SYLVESTER'S THEOREM

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Abstract. We give a new elementary proof of existence and uniqueness of a solution to the Sylvester equation $AX - XB = Y$

1. Introduction

Given finite dimensional Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and operators $A \in \mathcal{L}(\mathcal{K})$, $B \in \mathcal{L}(\mathcal{H})$ and $Y \in \mathcal{L}(\mathcal{H}, K)$ the Sylvester equation asks for solutions $X \in \mathcal{L}(\mathcal{H}, K)$ to

$$AX - XB = Y \tag{1}$$

A particular case of interest is the Lyapunov equation

$$A^*X + XA = Y \tag{2}$$

which arises in stability theory (see [4]). Equation (1) was first studied by Sylvester in [6], who showed that it has a unique solution if $\sigma(A) \cap \sigma(B) = \emptyset$. This was generalized to infinite dimensions by Rosenblum in [5].

The purpose of this note is to give a short proof of Sylvester's theorem using elementary block matrix arguments. Other different proofs are given in [1, 2, 3]. A thorough survey on equation (1) can be found in [1].

2. Main result

Theorem 1. *Let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional Hilbert spaces and let $A \in \mathcal{L}(\mathcal{K})$ and $B \in \mathcal{L}(\mathcal{H})$ with $\sigma(A) \cap \sigma(B) = \emptyset$. Then for every $Y \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ there exists a unique $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $AX - XB = Y$.*

Proof. Consider the map $\Phi : \mathcal{L}(\mathcal{H}, \mathcal{K}) \to \mathcal{L}(\mathcal{H}, \mathcal{K})$ given by $\Phi(X) = AX - XB$. It suffices to show that Φ is injective. If $\ker \Phi$ contains an invertible X, we have

$$X^{-1}AX = B$$

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implying $\sigma(A) = \sigma(B)$, a contradiction. If not, we use a block matrix argument to reduce to this case. Let $X \in \ker \Phi$ such that $X \neq O$. Consider the direct sum decompositions

$$\mathcal{H} = (\ker X)^{\perp} \oplus \ker X$$

and

$$\mathcal{K} = \operatorname{im} X \oplus \ker X^*.$$

Note that $(\ker X)^{\perp} \neq \{0\}$. With respect to these decompositions, we have the block matrices

$$X = \begin{pmatrix} Y & O \\ O & O \end{pmatrix}$$

$$A = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

and

$$B = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}.$$

Observe that Y is invertible. The condition $AX = XB$ now yields

$$\begin{pmatrix} EY & O \\ GY & O \end{pmatrix} = \begin{pmatrix} YP & YQ \\ O & O \end{pmatrix}$$

implying

$$EY = YP, \tag{3}$$

$$G = O \text{ and } Q = O.$$

Now

$$A = \begin{pmatrix} E & F \\ O & H \end{pmatrix}$$

and

$$B = \begin{pmatrix} P & O \\ R & S \end{pmatrix}.$$

Thus, $\sigma(E) \subset \sigma(A)$ and $\sigma(P) = \sigma(P^t) \subset \sigma(B^t) = \sigma(B)$ which implies $\sigma(E) \cap \sigma(P) = \emptyset$. From (3), we have a contradiction due to the invertibility of Y. ∎

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