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NONLINEAR HEISENBERG-ROBERTSON-SCHRODINGER UNCERTAINTY PRINCIPLE

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Abstract: We derive an uncertainty principle for Lipschitz maps acting on subsets of Banach spaces. We show that this nonlinear uncertainty principle reduces to the Heisenberg-Robertson-Schrodinger uncertainty principle for linear operators acting on Hilbert spaces.
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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and A be a possibly unbounded self-adjoint linear operator defined on domain $\mathcal{D}(A) \subseteq \mathcal{H}$. For $h \in \mathcal{D}(A)$ with $\|h\| = 1$, define the **uncertainty** of A at the point h as

$$\Delta_h(A) \coloneqq \|Ah - \langle Ah, h\rangle h\| = \sqrt{\|Ah\|^2 - \langle Ah, h\rangle^2}.$$

In 1929, Robertson [7] derived the following mathematical form of the uncertainty principle (also known as uncertainty relation) of Heisenberg derived in 1927 [4] (English translation of 1927 original article by Heisenberg). Recall that for two linear operators $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$, we define $[A, B] \coloneqq AB - BA$ and $\{A, B\} \coloneqq AB + BA$.

Theorem 1.1. *[1, 2, 4, 6, 7, 10] (**Heisenberg-Robertson Uncertainty Principle**) Let $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$ be self-adjoint operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have*

$$\frac{1}{2}\left(\Delta_h(A)^2 + \Delta_h(B)^2\right) \geq \frac{1}{4}\left(\Delta_h(A) + \Delta_h(B)\right)^2 \geq \Delta_h(A)\Delta_h(B) \geq \frac{1}{2}|\langle [A, B]h, h\rangle|. \tag{1}$$

In 1930, Schrodinger improved Inequality (1) [8] (English translation of 1930 original article by Schrodinger).

Theorem 1.2. *[8] (**Heisenberg-Robertson-Schrodinger Uncertainty Principle**) Let $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$ be self-adjoint operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have*

$$\Delta_h(A)\Delta_h(B) \geq |\langle Ah, Bh\rangle - \langle Ah, h\rangle\langle Bh, h\rangle| = \frac{\sqrt{|\langle [A, B]h, h\rangle|^2 + |\langle \{A, B\}h, h\rangle - 2\langle Ah, h\rangle\langle Bh, h\rangle|^2}}{2}.$$

Theorem 1.2 promotes the following question.

Question 1.3. *What is the nonlinear (even Banach space) version of Theorem 1.2?*

In this short note, we answer Question 1.3 by deriving an uncertainty principle for Lipschitz maps acting on subsets of Banach spaces. We note that there is a Banach space version of uncertainty principle by Goh and Goodman [3] which differs from the results in this paper. We also note that nonlinear Maccone-Pati uncertainty principle is derived in [5]. It is interesting to note that the uncertainty principle of Game theory, derived by Székely and Rizzo is nonlinear [9].

2. Nonlinear Heisenberg-Robertson-Schrodinger Uncertainty Principle

Let $\mathcal{X}$ be a Banach space. Recall that the collection of all Lipschitz functions $f : \mathcal{X} \to \mathbb{C}$ satisfying $f(0) = 0$, denoted by $\mathcal{X}^\#$ is a Banach space [11] w.r.t. the Lipschitz norm

$$\|f\|_{\operatorname{Lip}_0} \coloneqq \sup_{x,y\in\mathcal{X}, x\neq y} \frac{|f(x) - f(y)|}{\|x - y\|}.$$

Let $\mathcal{M} \subseteq \mathcal{X}$ be a subset such that $0 \in \mathcal{M}$. Let $A : \mathcal{M} \to \mathcal{X}$ be a Lipschitz map such that $A(0) = 0$. Given $x \in \mathcal{M}$ and $f \in \mathcal{X}^\#$ satisfying $f(x) = 1$, we define **two uncertainties** of A at $(x, f) \in \mathcal{M} \times \mathcal{X}^\#$ as

$$\Delta(A, x, f) \coloneqq \|Ax - f(Ax)x\|,$$
$$\nabla(f, A, x) \coloneqq \|fA - f(Ax)f\|_{\operatorname{Lip}_0}.$$

Theorem 2.1. ***(Nonlinear Heisenberg-Robertson-Schrodinger Uncertainty Principle)*** *Let $\mathcal{X}$ be a Banach space and $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ be subsets such that $0 \in \mathcal{M} \cap \mathcal{N}$. Let $A : \mathcal{M} \to \mathcal{X}$, $B : \mathcal{N} \to \mathcal{X}$ be Lipschitz maps such that $A(0) = B(0) = 0$. Then for all $x \in \mathcal{M} \cap \mathcal{N}$ and $f \in \mathcal{X}^\#$ satisfying $f(x) = 1$, we have*

$$\begin{aligned}\frac{1}{2}\left(\nabla(f, A, x)^2 + \Delta(B, x, f)^2\right) &\geq \frac{1}{4}\left(\nabla(f, A, x) + \Delta(B, x, f)\right)^2\\ &\geq \nabla(f, A, x)\Delta(B, x, f) \geq |f(ABx) - f(Ax)f(Bx)|.\end{aligned}$$

Proof.

$$\begin{aligned}\nabla(f, A, x)\Delta(B, x, f) &= \|fA - f(Ax)f\|_{\operatorname{Lip}_0}\|Bx - f(Bx)x\|\\ &\geq |[fA - f(Ax)f][Bx - f(Bx)x]|\\ &= |f(ABx) - f(Bx)f(Ax) - f(Ax)f(Bx) + f(Ax)f(Bx)f(x)|\\ &= |f(ABx) - f(Bx)f(Ax) - f(Ax)f(Bx) + f(Ax)f(Bx) \cdot 1|\\ &= |f(ABx) - f(Ax)f(Bx)|.\end{aligned}$$

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Corollary 2.2. *Let $\mathcal{X}$ be a Banach space and $\mathcal{M}, \mathcal{N} \subseteq \mathcal{X}$ be subsets such that $0 \in \mathcal{M} \cap \mathcal{N}$. Let $A : \mathcal{M} \to \mathcal{X}$, $B : \mathcal{N} \to \mathcal{X}$ be Lipschitz maps such that $A(0) = B(0) = 0$. Then for all $x \in \mathcal{M} \cap \mathcal{N}$ and $f \in \mathcal{X}^\#$ satisfying $f(x) = 1$, we have*

$$\nabla(f, A, x)\Delta(B, x, f) + \|fB\|_{Lip_0}\Delta(A, x, f) \geq |f([A, B]x)|.$$

Proof. Note that

$$\begin{aligned}\nabla(f,A,x)\Delta(B,x,f) &\geq |f(ABx)-f(Ax)f(Bx)|\\ &= |f(ABx-BAx)+f(BAx)-f(Ax)f(Bx)|\\ &= |f([A,B]x)+(fB)[Ax-f(Ax)x]|\\ &\geq |f([A,B]x)|-|(fB)[Ax-f(Ax)x]|\\ &\geq |f([A,B]x)|-\|fB\|_{\mathrm{Lip}_0}\|Ax-f(Ax)x\|\\ &= |f([A,B]x)|-\|fB\|_{\mathrm{Lip}_0}\Delta(A,x,f).\end{aligned}$$

□

Corollary 2.3. *Let $\mathcal{X}$ be a Banach space and $\mathcal{M},\mathcal{N}\subseteq\mathcal{X}$ be subsets such that $0\in\mathcal{M}\cap\mathcal{N}$. Let $A:\mathcal{M}\to\mathcal{X}$, $B:\mathcal{N}\to\mathcal{X}$ be Lipschitz maps such that $A(0)=B(0)=0$. Then for all $x\in\mathcal{M}\cap\mathcal{N}$ and $f\in\mathcal{X}^{\#}$ satisfying $f(x)=1$, we have*

$$\nabla(f,A,x)\Delta(B,x,f)+\|fB\|_{Lip_0}\Delta(A,x,-f)\geq |f(\{A,B\}x)|.$$

Proof. Note that

$$\begin{aligned}\nabla(f,A,x)\Delta(B,x,f) &\geq |f(ABx)-f(Ax)f(Bx)|\\ &= |f(ABx+BAx)-f(BAx)-f(Ax)f(Bx)|\\ &= |f(\{A,B\}x)-(fB)[Ax+f(Ax)x]|\\ &\geq |f(\{A,B\}x)|-|(fB)[Ax+f(Ax)x]|\\ &\geq |f(\{A,B\}x)|-\|fB\|_{\mathrm{Lip}_0}\|Ax-(-f)(Ax)x\|\\ &= |f(\{A,B\}x)|-\|fB\|_{\mathrm{Lip}_0}\Delta(A,x,-f).\end{aligned}$$

□

Corollary 2.4. ***(Functional Heisenberg-Robertson-Schrodinger Uncertainty Principle)*** *Let $\mathcal{X}$ be a Banach space with dual $\mathcal{X}^*$, $A:\mathcal{D}(A)\to\mathcal{X}$ and $B:\mathcal{D}(B)\to\mathcal{X}$ be linear operators. Then for all $x\in\mathcal{D}(AB)\cap\mathcal{D}(BA)$ and $f\in\mathcal{X}^*$ satisfying $f(x)=1$, we have*

$$\begin{aligned}\frac{1}{2}\left(\nabla(f,A,x)^2+\Delta(B,x,f)^2\right) &\geq \frac{1}{4}\left(\nabla(f,A,x)+\Delta(B,x,f)\right)^2\\ &\geq \nabla(f,A,x)\Delta(B,x,f)\geq |f(ABx)-f(Ax)f(Bx)|.\end{aligned}$$

Corollary 2.5. *Let $\mathcal{X}$ be a Banach space, $A:\mathcal{D}(A)\to\mathcal{X}$ and $B:\mathcal{D}(B)\to\mathcal{X}$ be linear operators. Then for all $x\in\mathcal{D}(AB)\cap\mathcal{D}(BA)$ and $f\in\mathcal{X}^*$ satisfying $f(x)=1$, we have*

$$\nabla(f,A,x)\Delta(B,x,f)+\|fB\|\Delta(A,x,f)\geq |f([A,B]x)|.$$

Corollary 2.6. *Let $\mathcal{X}$ be a Banach space, $A:\mathcal{D}(A)\to\mathcal{X}$ and $B:\mathcal{D}(B)\to\mathcal{X}$ be linear operators. Then for all $x\in\mathcal{D}(AB)\cap\mathcal{D}(BA)$ and $f\in\mathcal{X}^*$ satisfying $f(x)=1$, we have*

$$\nabla(f,A,x)\Delta(B,x,f)+\|fB\|\Delta(A,-x,f)=\nabla(f,A,x)\Delta(B,x,f)+\|fB\|\Delta(A,x,-f)\geq |f(\{A,B\}x)|.$$

Corollary 2.7. *Theorem 1.2 follows from Theorem 2.1.*

Proof. Let $\mathcal{H}$ be a complex Hilbert space. $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$ be self-adjoint operators. Let $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$. Define $\mathcal{X} := \mathcal{H}$, $\mathcal{M} := \mathcal{D}(A)$, $\mathcal{N} := \mathcal{D}(B)$, $x := h$ and

$$f : \mathcal{H} \ni u \mapsto f(u) := \langle u, h\rangle \in \mathbb{C}.$$

Then

$$\Delta(B, x, f) = \Delta(B, h, f) = \|Bh - f(Bh)h\| = \|Bh - \langle Bh, h\rangle h\| = \Delta_h(B),$$

$$\begin{aligned}\nabla(f, A, x) = \nabla(f, A, h) &= \|fA - f(Ah)f\| = \|fA - \langle Ah, h\rangle f\| \\ &= \sup_{u\in\mathcal{H}, \|u\|\leq 1} |f(Au) - \langle Ah, h\rangle f(u)| = \sup_{u\in\mathcal{H}, \|u\|\leq 1} |\langle Au, h\rangle - \langle Ah, h\rangle\langle u, h\rangle| \\ &= \sup_{u\in\mathcal{H}, \|u\|\leq 1} |\langle u, Ah\rangle - \langle Ah, h\rangle\langle u, h\rangle| = \sup_{u\in\mathcal{H}, \|u\|\leq 1} |\langle u, Ah - \langle Ah, h\rangle h\rangle| \\ &= \|Ah - \langle Ah, h\rangle h\| = \Delta_h(A)\end{aligned}$$

and

$$\begin{aligned}|f(ABx) - f(Ax)f(Bx)| &= |f(ABh) - f(Ah)f(Bh)| = |\langle ABh, h\rangle - \langle Ah, h\rangle\langle Bh, h\rangle| \\ &= |\langle Bh, Ah\rangle - \langle Ah, h\rangle\langle Bh, h\rangle| = |\langle Ah, Bh\rangle - \langle Ah, h\rangle\langle Bh, h\rangle|.\end{aligned}$$

□

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