A PRIME NUMBER "GAME OF LIFE": CAN $\lfloor y \cdot p\# \rfloor$ BE PRIME FOR ALL $p \geq 2$?

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Abstract. A new sequence in the spirit of the Mills primes is presented and its properties are investigated.

1. Introduction

In the 1940s, W. H. Mills [8] examined prime-representing functions, most notably he pointed out, based on a result of Ingham [5], that there exists a constant A such that $\lfloor A^{3^n} \rfloor$ is prime for all integers $n \geq 1$. Mills' ansatz reproduces only a very sparse subset of primes. There are several variations to obtain a larger subset of primes, e. g. by Elsholtz [2], who conjectures the existence of a constant A such that $\lfloor A^{(n+1)^2} \rfloor$ is prime for all $n \geq 1$; Kuipers [6], or Plouffe [12], who consider sequences of the form $\lfloor A^{c^n} \rfloor$ for c as small as 1.01.

In this paper, we give an even more slowly growing formula—in fact, for its kind as an asymptotically exponentially growing function, it exhibits the slowest possible growth rate while still producing prime numbers, and a comparatively large subset at that.

While large portions of this paper consist of artisan work with well-known tools of number theory, there may something to be gained in-between the lines, for example a way to calculate Chebyshev's $\theta(x)$ directly from known values of $Li(x) - \pi(x)$ which we haven't seen before in literature. There are also a couple of open problems which may or may not be worth investigating further.

2. Definitions

p, q	variables for prime numbers
p_n	the n^{th} prime; $p_n \in \{2, 3, 5, 7, 11, 13, 17 \ldots\}$
$\pi(x)$	the prime counting function; the number of primes $p_n \leq x$
$p\#$	the primorial function: $p\# = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot \ldots \cdot p$
	$1\# = 1$ by convention
$\Phi(n, p)$	number of integers $\leq n$ that are relatively prime to $p\#$
$\log x$	natural logarithm of x
$\lfloor x \rfloor$	the floor function; the largest integer $\leq x$
$[a, b[$	the open interval containing real numbers x such that $a \leq x < b$
$\lvert x \rvert$	absolute value of x; $\lvert x \rvert = -x$ for $x < 0$

3. Setting the stage

Lemma: Any real number y such that $q = \lfloor y \cdot p\# \rfloor$ is a prime number for every $p \geq 2$ is contained in the interval $[1, 1.5[$.

If $y < 1$, then $\lfloor y \cdot 2\# \rfloor < 2$, which is less than the first prime number.

For $y \geq 1.5$, $\lfloor y \cdot 2\# \rfloor$ must be an odd prime $2m - 1$ for some integer $m > 1$. Then $\lfloor y \cdot 3\# \rfloor$ is a number in the interval $[6m - 3, 6m - 1]$, only the latter of which can be prime because $6m - 3$ is divisible by 3 and $6m - 2$ is divisible by 2. Likewise, any integer in the successive intervals $[p\# \cdot m - p, p\# \cdot m - 1]$ is divisible by a small prime $\leq p$ except for $p\# \cdot m - 1$.

Hence there would have to be an integer m for which $p\# \cdot m - 1$ is always prime, i.e. the sequence $f(0) = m - 1$, $f(s+1) = (f(s) + 1) \cdot p_s - 1$ must only produce primes for all $s \geq 1$, which is practically impossible (though we have not seen this being explicitly disproven yet!).

It is possible, however, to find arbitrarily long finite sequences for some m:

Table 1. Smallest integers m for which $p\# \cdot m - 1$ is prime for $p \leq r$. The first unknown entry for $r = 47$ has $m > 1.6 \cdot 10^{12}$ and is expected to occur when $10^{14} < m < 10^{15}$.

r	m	r	m
5	2	23	384427
7	2	29	16114470
11	9	31	259959472
13	9	37	13543584514
17	9	41	100318016379
19	224719	43	100318016379

To find out about an admissible value for y, we have to look for primes q, where each q requires the primality of $\lfloor q/p\# \cdot r\# \rfloor$ for every $r \leq p$. In the following setup, Y will denote the interval in which any number y as described above can lie, setting $Y = [1, 1.5[$ for a start, and then successively sharpen the bounds where, for every p, $Y \cdot p\#$ must contain at least one prime to satisfy the conditions.

- Stage 1: $p = 2 \to 2 \cdot Y = [2, 3[\to \lfloor 2 \cdot Y \rfloor = 2$ – no further action required yet
- Stage 2: $p = 3 \to 6 \cdot Y = [6, 9[$ – the only prime in here is 7, so Y narrows down to $[\frac{7}{6}, \frac{8}{6}[$
- Stage 3: $p = 5 \to 30 \cdot Y = [35, 40[$ – one prime (37) in this interval, Y shrinks to $[\frac{37}{30}, \frac{38}{30}[$
- Stage 4: $p = 7 \to 210 \cdot Y = [259, 266[$ – once again, one prime in here and any possible y lies between $\frac{263}{210}$ and $\frac{264}{210}$
- Stage 5: $p = 11 \to 2310 \cdot Y = [2893, 2904[$ – now it's getting interesting, since there are two primes in that interval, 2897 and 2903. We will split Y into two separate intervals, $Y_1 = [\frac{2897}{2310}, \frac{2898}{2310}[$ and $Y_2 = [\frac{2903}{2310}, \frac{2904}{2310}[$ and proceed
- Stage 6: $p = 13 \to$ each Y_1 and Y_2 brings forth one prime, 37663 and 37747 respectively. Cutting back both Y's, then
- Stage 7: $p = 17 \to Y_1$ leads to 640279, whereas Y_2 gives 641701 and 641713, and Y_2 will be split into Y_2 and Y_3 in accordance with the primes
- Stage 8: $p = 19 \to 9699690 \cdot Y_1$ contains a prime triplet, $12165311 + d$ for $d = \{0, 2, 6\}$, while Y_2 and Y_3 both fail to produce any further prime
- etc.

The results can be put into a nice grid:

Table 2. This seems to be the new "game of life", a family tree of prime numbers. Prime triplets $q + \{0, 2, 6\}$ or $q + \{0, 4, 6\}$ like in stage 8 are rather rare in the process, one finds the next example at stage 152 ($p = 881$), and there are no prime quadruplets $q + \{0, 2, 6, 8\}$ in any stage ≤ 318.

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Will the evolution go on forever? The following table shows how many partial intervals (or primes, respectively) n there are after stage s, up to $s = 60$:

Table 3. Talking about a bottleneck: the only prime at stage 29 is 350842542483891235293716663559065020274899073 ($\approx 3.5 \cdot 10^{44}$).

s	p	n	s	p	n	s	p	n
1	2	1	21	73	5	41	179	18
2	3	1	22	79	4	42	181	17
3	5	1	23	83	6	43	191	14
4	7	1	24	89	3	44	193	24
5	11	2	25	97	2	45	197	24
6	13	2	26	101	1	46	199	28
7	17	3	27	103	3	47	211	30
8	19	3	28	107	1	48	223	36
9	23	4	29	109	1	49	227	49
10	29	6	30	113	3	50	229	44
11	31	4	31	127	2	51	233	52
12	37	5	32	131	5	52	239	53
13	41	5	33	137	6	53	241	55
14	43	9	34	139	12	54	251	67
15	47	11	35	149	21	55	257	69
16	53	10	36	151	19	56	263	72
17	59	12	37	157	15	57	269	81
18	61	8	38	163	16	58	271	79
19	67	6	39	167	24	59	277	85
20	71	11	40	173	18	60	281	83

Although the population increases considerably after stage 33, the data doesn't provide too much confidence that it will continue to do so. We have to take a closer look at why this sequence is so erratic.

4. Analyzing the game

The size of the primes at each stage s with the corresponding p, p_s being the s^{th} prime, is always about $1.25 \cdot p_s\#$. The probability of a random number of this size being prime is $1/(\log p_s\# + 0.23)$, or $1/(\theta(p_s) + 0.23)$ where $\theta(p)$ is the first Chebyshev function. For one prime q of the stage $s-1$ there has to be on average at least one prime in the interval $[q \cdot p_s, (q+1) \cdot p_s]$ to have a chance that the sequence keeps on producing ever more primes. More precisely, the probability of getting at least one prime out of the respective interval must be bigger than getting no prime at all.

$$\frac{\binom{p-1}{1}}{(\log q - 1)^1}(1 - \frac{1}{\log q})^{p-1} > \frac{\binom{p-1}{0}}{(\log q - 1)^0}(1 - \frac{1}{\log q})^{p-1} \text{ or simply } \frac{p-1}{\theta(p) - 0.77} > 1. \tag{1}$$

This is true for $p \geq 3$ if and only if $\theta(p) < p - 0.23$, or $\theta(p) < p$ for short, since the constant value $0.23 = \log y$ is negligible for large p.

Now $\theta(p)$ is in fact smaller than p most of the time — but only slightly so. And indeed $p-\theta(p)$ behaves just like the mercurial $Li(p)-\pi(p)$, along with the infinitude of sign changes (see Littlewood [7]):

Using the simple sum [1]

$$Li^*(p) = \sum_{x=2}^{p} \frac{1}{\log x}\ , \tag{2}$$

then

$$p - \theta(p) = (\log p)(Li^*(p) - \pi(p)) + 1 - \sum_{x=2}^{p-1} \left(\log(\frac{x+1}{x})(Li^*(x) - \pi(x)) \right). \tag{3}$$

This formula makes it also clear that $p-\theta(p)$ changes sign quite a while before $Li(p)-\pi(p)$ acquires negative values. For the sake of a ballpark figure: assuming $Li(x)-\pi(x)$ is on average close enough to $\frac{\sqrt{x}}{\log x}$, then $p-\theta(p)$ drops below zero by the time $Li(p)-\pi(p) < \frac{2\sqrt{p}+O(p^{1/3})}{(\log p)(\log p-2)}$. It can be expected that this happens for the first time close to the first $Li(x)-\pi(x)$ crossover near $1.4 \cdot 10^{316}$ (Bays and Hudson [1], more extensive calculations by Saouter and Demichel [16]). The results of Platt and Trudgian [11] also confirm this.

Empirical data and heuristical reasoning suggests that $p - \theta(p)$ can usually be found in the vicinity of $\sqrt{p}$. The bias is given by Riemann's prime counting formula [14]

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(Li(x^{1/n}) - \sum_{\rho} Li(x^{\rho/n}) + \int_{x^{1/n}}^{\infty} \frac{\mathrm{d}u}{u(u^2-1)\log u} - \log(2) \right). \tag{4}$$

If the Riemann Hypothesis (RH) is true, $Li(p) - \pi(p)$ oscillates along $\frac{\sqrt{p}}{\log p}$ with a magnitude of at most $O(\sqrt{p}\log p)$ (von Koch [17]), and so $p - \theta(p)$ oscillates along $\sqrt{p}$ with a magnitude of at most $O(\sqrt{p}\log^2 p)$. Ingham [4] also proved this bias for sufficiently large p.

On a large scale, the number of intervals should increase slowly and asymptotically with $p - \theta(p)$, thus it may also decrease whenever $\theta(p) > p$. Yet the actual development strongly depends on the local irregularities of the distribution of prime numbers, so one can only assign the probability that the initial conditions are adequate.

Admittedly, (1) wasn't quite accurate. The number of primes in an interval immediately after a semiprime as it is the case, especially in this short type of interval, is on average a

[1] As opposed to the "European convention" $Li(x) = \int_2^p \frac{\mathrm{d}x}{\log x}$ and the "American convention" $Li(x) = \int_0^p \frac{\mathrm{d}x}{\log x}$ which each differ from said sum by less than 1 for $p \geq 12.0050107...$ (European) or $p \geq 2$ (American).

bit smaller than for a randomly chosen interval. The following table illustrates this sort of difference for a small interval of ten numbers:

Table 4. For a random interval start and interval length 10, the average of numbers that are coprime to $2310 = 11\#$ is above two, while for an interval of the same length after a prime number (or any number coprime to $11\#$), significantly fewer than two such numbers are to be expected. Hardy and Littlewood [3] laid a solid foundation regarding this issue.

number	divisible by 2	3	5	7	11	not divisible	interval sum
random +[1..10]	50%	33%	20%	14%	9%	21%	**2.08**
prime +1	100%	50%	25%	17%	10%	0%	0
prime +2	0%	50%	25%	17%	10%	28%	0.28
prime +3	100%	0%	25%	17%	10%	0%	0.28
prime +4	0%	50%	25%	17%	10%	28%	0.56
prime +5	100%	50%	0%	17%	10%	0%	0.56
prime +6	0%	0%	25%	17%	10%	56%	1.13
prime +7	100%	50%	25%	0%	10%	0%	1.13
prime +8	0%	50%	25%	17%	10%	28%	1.41
prime +9	100%	0%	25%	17%	10%	0%	1.41
prime +10	0%	50%	0%	17%	10%	38%	**1.78**

Ultimately, the number of numbers that are relatively prime to $k\#$ in the intervals in question $[q \cdot p + 1, (q+1) \cdot p - 1]$ — denoted by $\Phi(p-1, k)$ — is

$$\prod_{\substack{u=3\\ u\,prime}}^{k} (1 - \frac{1}{u-1}) \cdot \Bigg(\lfloor \frac{p-1}{2} \rfloor + \sum_{\substack{v=3\\ v\,prime}}^{\min(\frac{p-1}{2},k)} \frac{\lfloor \frac{p-1}{2v} \rfloor}{v-2} + \sum_{\substack{v_1=3\\ v_1\,prime}}^{\min(\sqrt{\frac{p-1}{2}},k)} \sum_{\substack{v_2=v_1+2\\ v_2\,prime}}^{\min(\frac{p-1}{2v_1},k)} \frac{\lfloor \frac{p-1}{2v_1v_2} \rfloor}{(v_1-2)(v_2-2)}$$
$$+ \sum_{\substack{v_1=3\\ v_1\,prime}}^{\min(\sqrt[3]{\frac{p-1}{2}},k)} \sum_{\substack{v_2=v_1+2\\ v_2\,prime}}^{\min(\sqrt{\frac{p-1}{2v_1}},k)} \sum_{\substack{v_3=v_2+2\\ v_3\,prime}}^{\min(\frac{p-1}{2v_1v_2},k)} \frac{\lfloor \frac{p-1}{2v_1v_2} \rfloor}{(v_1-2)(v_2-2)(v_3-2)} + \ldots \Bigg). \tag{5}$$

Yet disregarding that p itself doesn't appear as a factor in said interval, this particular formula is only valid for $2 < k < p$. In contrast, for a random interval R, $\Phi_R(p-1,k) = (p-1) \cdot W(k) + O(1)$, where

$$W(k) = \prod_{\substack{u=2\\ u\,prime}}^{k} \Big(1 - \frac{1}{u}\Big). \tag{6}$$

If $\Phi(p-1,k)$ is then divided by $W(k) \cdot \frac{p-2}{p-1}$ (as p itself doesn't appear as a factor in the intervals we're looking at), then the result is an "adjusted" interval length. For this, k must be appropriately large to attain the desired value ($k > \log p$, say).

There is a special connection between said adjusted interval length and the twin prime constant $C_2 = 0.6601618\ldots$: for $k \to \infty$, the adjusted interval length $\Phi(p-1,k)/W(k)$, on the basis of a prime number preceding the interval, can be expressed as $C_2 \cdot (p-1+a)$, with $a \in \mathbb{Q}$ being defined by

$$a = 2 \cdot \sum_{x=1}^{\frac{p-1}{2}} \Bigg(\prod_{\substack{r:\\ \text{every distinct}\\ \text{odd prime}\\ \text{factor of } x}} \frac{r-1}{r-2} - 1 \Bigg). \tag{7}$$

Some values of a include

TABLE 5. The value of $\frac{a}{p-1}$ is asymptotic to $\frac{1}{C_2} - 1 - O(\frac{\log p}{2p \cdot C_2})$.

$p-1$	a	$p-1$	a
2	0	20	116/15
4	0	30	6866/495
6	2	40	141274/8415
8	2	50	1329632/58905
10	8/3	100	129132288244/2731483755
12	14/3	150	123443421975532168/1666745013838905

As $k \to \infty$, the adjusted interval length (for our case with the semiprimes) is — apart from $p = 3, 7$, and 13 — always a bit smaller than $p - 1$. Dividing it again by $\log(q \cdot p)$ and we arrive at the expected average number of primes in one interval. The ratio of the adjusted interval length vs. $p-1$, which is equivalent to $\frac{\Phi(p-1,k)}{W(k) \cdot (p-2)}$ for appropriately large k, will be denoted below by $\psi(p)$.

Here we were focusing on heuristical aspects. For further reading about the structural analysis of the evolutional nature of the sequence, and potential vistas, we refer the reader to Santana [15] who gives an introduction on the notion of evolution on sets.

5. Proposing practical predictions

Using these heuristics, which are so far in very good agreement with the factual data, we can start to calculate the probabilities for the game to continue.

For example, there are 594 primes at stage 100, where $p = 541$ and every prime $q = \lfloor p\# \cdot Y \rfloor$ has 220 decimal digits. Each q then has a certain chance to yield n primes in the following stages:

Table 6. Only the fittest will survive.

s	p	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n>7$
101	547	34.52%	36.75%	19.53%	6.90%	1.83%	0.39%	0.07%	0.01%	0.00%
102	557	49.78%	19.46%	14.17%	8.22%	4.33%	2.16%	1.03%	0.47%	0.36%
103	563	58.57%	12.14%	10.10%	7.05%	4.63%	2.95%	1.83%	1.12%	1.62%
104	569	64.34%	8.33%	7.46%	5.75%	4.22%	3.03%	2.14%	1.50%	3.22%
105	571	68.45%	6.11%	5.72%	4.68%	3.68%	2.84%	2.17%	1.64%	4.71%
...	...	...	...	...	...	...	...	...	...	...
110	601	78.69%	2.06%	2.14%	1.99%	1.80%	1.62%	1.44%	1.27%	8.99%
150	863	90.06%	0.10%	0.11%	0.11%	0.11%	0.11%	0.11%	0.11%	9.20%
200	1223	91.20%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	8.69%
1000	7919	91.48%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	8.52%
2000	17389	91.48%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	8.52%

The column for $n=0$ is of importance to find out about the overall probability. (Once an interval is devoid of primes, it never comes back!) Within the first few thousand values of s, this value approaches $\omega \approx 91.476\%$. This gives the overall probability of $\omega^{594} \approx 1.039 \cdot 10^{-23}$ that the sequence fails at this point.

Generally the value $\omega(s)$ can be defined for every $s \geq 1$ as the probability that a prime q at stage s will not cause an infinite sequence of primes in the further process.

For convenience, a recursive formula can be considered where, with $\psi(p) = \frac{\Phi(p-1,k)}{W(k)\cdot(p-2)}$ from above,

$$\omega(s-1) = \left(1 - \frac{\psi(p)}{\log q}\right)^{p-1} \sum_{j=0}^{p-1} \frac{\binom{p-1}{j}\omega(s)^j}{(\frac{\log q}{\psi(p)} - 1)^j} \tag{8}$$

and a starting point $\omega(t) = 1 - 1/\sqrt{2p_t}$ for some t can be set to quickly compute the values for all $s < t$. Varying $\omega(t)$ between 0 and $1-\varepsilon$ for some $\varepsilon > 0$ leads to a lower and upper bound on $\omega(s)$. It may be useful that if $\omega(t) = \omega(s) + \delta$ then $\omega(t+1) \sim \omega(s+1) + \delta \cdot (1 - \omega(s+1))/2$.

Again, we should bear in mind that these primes are not as flexible as (8) takes them to be: the probability of obtaining n primes in the following step is zero when n exceeds the maximal possible number of primes in the given interval (e.g. $n > 97$ for $p - 1 = 546$) as fixed in the specific rules for the k-tuple conjecture. To take account of that — at least to some degree —, $W(k)$ from (6) is deployed again such that $k\# < p$. For $p = 547$, we set $k = 7$ and $W(k) = 48/210 = 1/4.375$. As p gets larger than $11\# = 2310$, $W(k)$ can be changed to $480/2310 = 1/4.8125$ and so on.

$$\omega(s-1,k) = \left(1 - \frac{\psi(p)}{W(k)\log q}\right)^{W(k)\cdot(p-1)} \sum_{j=0}^{W(k)\cdot(p-1)} \frac{\binom{W(k)\cdot(p-1)}{j}\omega(s,k)^j}{(\frac{W(k)\log q}{\psi(p)} - 1)^j} \tag{9}$$

This refined calculation reveals a slightly smaller value for $\omega(100, 7)$ (compared to $\omega(100)$), namely 91.435%. The entire probability that the sequence fails at this point drops by 24% to $7.937 \cdot 10^{-24}$. To show how vague these percentages are (especially at this rather early stage of computation), the actual portion of primes that don't survive after stage 100 — at least 547 out of 594 — is $\geq 92.088\%$; to the 594^{th} power, this is over 68 times more than the value from the “refined” calculation! In other terms, 47 surviving primes as opposed to a predicted number of 51, from this point of view the deviation is not that bad (and it gets better for larger p).

$\omega(100) = \omega(100, 1) = 91.4760031\%$
$\omega(100, 2) = 91.4637729\%$
$\omega(100, 3) = 91.4515074\%$
$\omega(100, 5) = 91.4422850\%$
$\omega(100, 7) = 91.4345844\%$

It is difficult to pin down exact values there, but the refined calculation should at least give some indication on how large the error might be.

Not too surprisingly, $\omega(s)$ is largely situated in the neighborhood of $1 - 1/\sqrt{2p_s}$. This can be verified heuristically as follows: Proceed as to be seen in table 6, starting with stage s. Consider regular intervals with evenly distributed primes q, the size of the intervals being $z \cdot \log(q)$ for a chosen $z \geq 1$, so for lim $q \to \infty$, the probability that one interval doesn't contain a prime at stage $s + 1$ is e^{-z}. The probability to get 0 primes after t steps at stage $s + t$ is $f(s + t)$, where $f(s) = 0$ and recursively $f(x + 1) = e^{z\cdot(f(x)-1)}$. As $\lim z \to 1$ for s fixed, $x = 1 - 2(z - 1) + O((z - 1)^2)$, which can be approximated by $1 - 1/\sqrt{2p_s}$ for $z = p/\theta(p) \approx 1 + 1/\sqrt{p}$ (on average).

What is more, for $\theta(p) = p + m \cdot \sqrt{p}$, where $m = -1 + O(log^2 p)$ (under the RH),

$$\omega(s+1) - \omega(s) \sim (\omega(s) - 1)\left(\frac{m}{\sqrt{p}} + \frac{1}{p} + \frac{1 - \omega(s)}{2} + O\left(\frac{\log p}{\sqrt{p^3}}\right) + O\left(\frac{1 - \omega(s)}{\sqrt{p}}\right)\right) \quad (10)$$

with error terms depending on either p or $\omega(s)$. As long as $m \approx -1$ as is expected on average in the long run, if $\omega(s)$ is chosen a bit larger than $1 - 1/\sqrt{2p_s}$, the values for the subsequent $\omega(s + x)$ by the recursion formula above quickly head toward 1, so the first term in the parenthesis above, $m/\sqrt{p}$, becomes the most significant. If then at some point $Li(p) \approx \pi(p)$, thus $m \approx 0$, while $\omega(s)$ is very close to one, then the second term $1/p$ becomes the most significant, but doesn't nearly have the (then opposite) impact on $\omega(s)$ as in the region where $m \approx -1$. Now what does that say? If $\omega(s)$ stays close to 1, the initially chosen value is larger than the actual value, which is good because the actual value — the probability that the sequence terminates — is desired to be as small as possible. An m-value of $+1$ would adequately counter the effect of the expected average value -1, then again that should happen just as often as $m \leq -3$. It should be noted that there is yet no known effective region where $|m+1| \geq 2$, making it a candidate for a future endeavor. [2]

Once the number of primes n — the number of possibilities for y — reaches a "stable" value, one may give a rough estimate as to how the sequence continues. For instance, with $p = 1153$ we have $n = 19690$. In the next stage, with $p = 1163$, one is likely to find approximately $n = 19690 \cdot \psi(p) \cdot (p-1)/(\theta(p) + 0.23) \approx 20298$ primes.
Most of the time, the actual value lies between $n - \sqrt{n}$ and $n + \sqrt{n}$, and, by standard deviation measures, more than 99.7% of the time it should be well within $n \pm 3\sqrt{n}$. But even considering a negative deviation of $4\sqrt{n}$: $-4 \cdot \sqrt{20298} = -569.88...$ is still outnumbered by the predicted growth $20298 - 19690 = 608$, corroborating that the initial assumption holds with a very high probability if the distribution is normal (a sufficient but not particularly necessary condition).

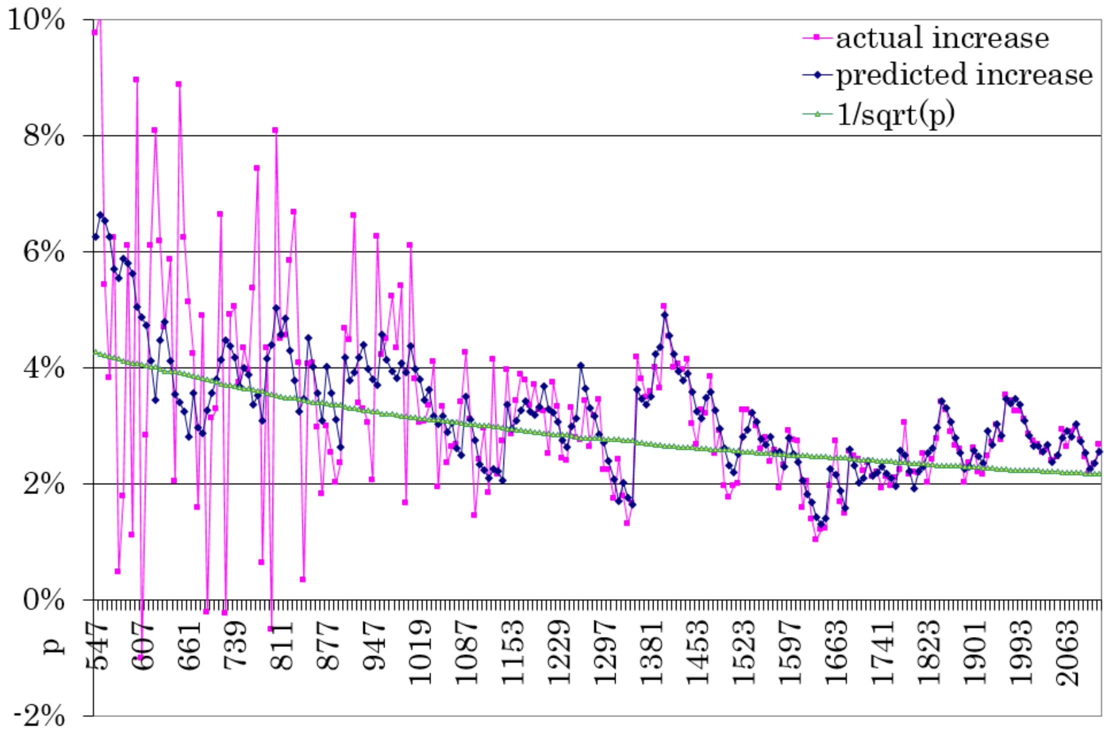

Figure 1. Comparison. With the prime number output becoming more generous, the forecast is increasingly accurate.

[2] Myerscough [9] tabulates the logarithmic density for the case $m > 1$ as $1.603 \cdot 10^{-95}$.

Expanding this supposition, we can say the sequence is “stable” when $n > p$, where the average absolute growth rate $n/\sqrt{p}$ is bigger than the majority of the local deviations of order $\sqrt{n}$. This critical boundary is firmly overstepped at $s = 98$: $p = 521$ and $n = 559$ (compared to $s = 97$ with $p = 509$ and $n = 490$).

Looking at predicted vs. actual increase in terms of standard deviations, so far the sequence behaves pretty “normal”:

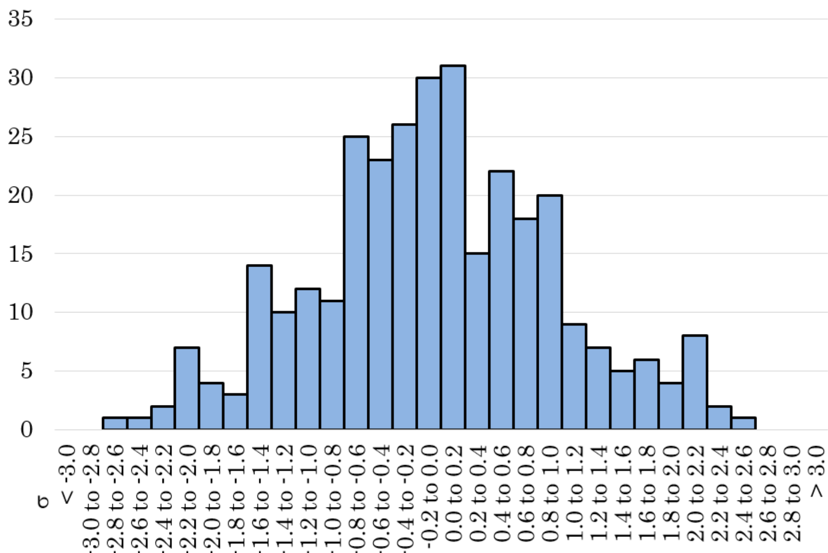

FIGURE 2. Histogram of the standard deviations from the predicted number of primes at each stage $s = 99 \ldots 251$. 70.6% of the values are within 1σ, 92.2% within 2σ, and all within 3σ. There are 78 negative deviations vs. 75 positive ones.

Long-term predictions imply that the number of primes may surpass 10^6 at $\approx$ stage 339 ($p = 2281$), 10^9 at $\approx$ stage 702 ($p = 5297$), 10^{12} at $\approx$ stage 1210 ($p = 9811$), 10^{100} at $\approx$ stage 118948 ($p = 1568341$), and may have reached $n \approx 10^{10^{155}}$ by the time that $\theta(p) > p$ where we expect to see the next slight decline of the sequence ever since stage 139 (see table 7).

The following seemingly unrelated approximation formula for n can be used in the medium term:

$$n \approx \frac{\sqrt{e^{\sqrt{3s}}}}{8} \tag{11}$$

which is surprisingly accurate in spite of its simplicity in the narrow sense that it is off by less than 20, with one little exception, for $s \leq 61$, off by at most 20% for $98 \leq s \leq 114$, and off by at most 15.4% for $115 \leq s \leq 318$ — compared to more elaborate predictions, the latter may hold even for a couple more $s \leq 400$. Furthermore, (11) may be off by a factor of no more than 2 for $66 \leq s \leq 1000+$. In light of the time it would take to meticulously calculate n for all $s \leq 1000$, the exact crossover on the large side of the inequality will always remain a mystery, thus calling into question any more daring predictions regarding (11).

6. The status quo, pt. I: risk of failure

At stage 318 ($p = 2111$) there are 592642 probable primes, and $\omega(318) \approx 95.408884\%$. $0.95408884^{592642} \approx 2.99 \cdot 10^{-12097}$ — that is, for the time being, the conjectured probability that the game fails for some $s > 318$.

Up to that point, 23036547 numbers have not yet been tested for certified primality, the smallest of those being in the ballpark of $2.38 \cdot 10^{247}$. No counterexample is known for the combination of a Baillie-Pomerance-Selfridge-Wagstaff pseudoprimality test and a strong Lucas test which were carried out on the probable primes. [3] However if there is one among all the established PRPs, it could take effect on at most 61642 PRPs (the most prolific branch at $p = 601$). $0.95408884^{592642-61642} \approx 4.59 \cdot 10^{-10839}$, i.e. $1.54 \cdot 10^{1258}$ times larger. It is yet unclear how much this will affect the above evaluation, especially by taking a look at Pomerance [13] who gives a heuristic argument that the number of counterexamples for a BPSW primality test up to x is $\gg x^{1-\varepsilon}$ for some $\varepsilon > 0$ and sufficiently large x. In either case, said ratio can be systematically reduced by checking the primality of the prolific branches in the sequence.

7. The status quo, pt. II: closing in on y

Now, what about the initially proposed values for y? The lower and upper bounds are easily obtained by picking the smallest prime $q_{s,1}$ and the largest prime $q_{s,n}$ of the last computed stage and dividing by $p\#$: $y_{\min} = q_{s,1}/p\#$ and $y_{\max} = (q_{s,n} + 1)/p\#$. The calculations so far demonstrate that any y lies in a very small range between

$$1.2541961015780119362776795549142134237798692180426221958327225546088646994287514475$$

and

$$1.2541961015780119362776795549142134237798692180426221958327225546088646994290445894.$$

In particular, $y_{\max} - y_{\min} = 2.931419... \cdot 10^{-76}$.

Although it is not dead certain that such a y exists, if it actually does, then there should be infinitely many of those. Otherwise, we'd have to assume a unique solution where the probability of getting one prime for $p \to \infty$ — the column for $n = 1$ in Table 6 — is

[3] including some trial division, additionally decreasing the chances that a pseudoprime slipped through

greater than zero. But if this is the case, then at every stage a factor of approximately $1/e$ falls back on the column for $n = 0$ which then would converge to 1, challenging our assumption that there is a unique solution. It would be nice, however, to have a rigorous proof.

8. Ancestors and survivors

Having calculated all primes up to a certain stage, we can trace back these primes for previous stages and see which of those have “survived” in the long run. All probable primes from stage 318 originated from one prime at stage 43. Recall that there were a total of 14 primes at stage 43, so the other 13 “petered out” along the way, including one branch that originated in stage 37 tenaciously keeping up until it succumbed at stage 95.

The first split then appears to be at stage 44. From there on, at least two values of y may satisfy the hitherto harsh conditions. And suddenly there’s a lot more to come:

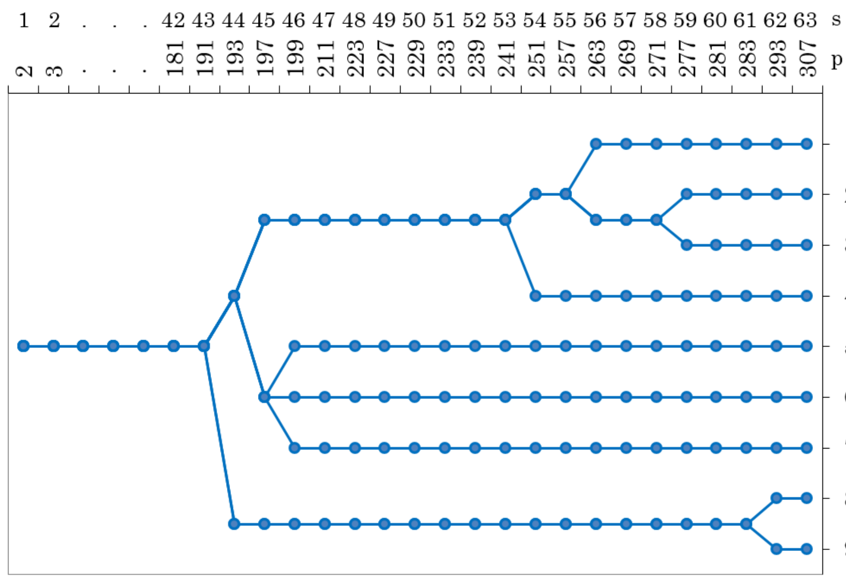

Figure 3. Number of possible values for y (for all we know). From $s = 95$ onward, all primes q emerge from one common ancestor at $s = 43$, namely 1292942159746921794791923187781692727375711831825607985864285936838920812561. Naturally, the branches in the picture are not equally strong. For $s = 318$ ($p = 2111$), the following number of probable primes divide up on the nine given groups: 8080 + 7132 + 48289 + 90644 + 100189 + 74398 + 10066 + 56007 + 197837.

If one was to specify these values as y_0, y_1 and so on to attain fixed constants for the various possibilities, one might set y_0 as the minimum value y_{min} as described above, and label each of the consecutive numbers y_1, y_2 ... as the other smallest possible value(s) after a precalculated split, when it is "safe enough to assume" that those branches are stable.

So $y_0 = y_{min}$ (see above), and then
$y_1 = y_0 + 2.93141872600279176982 43\ldots \cdot 10^{-76}$
(from the first split at stage 44, $\lfloor y_0 \cdot 191\#\rfloor \cdot 193 + 30$ and $+88$, $y_1 \approx y_0 + 58/193\#$)
$y_2 = y_0 + 1.92780965670634129381 36\ldots \cdot 10^{-79}$
$y_3 = y_2 + 1.18449604078070062135 69\ldots \cdot 10^{-80}$
$y_4 = y_2 + 1.54068271936773581231 85\ldots \cdot 10^{-80}$
$y_5 = y_0 + 1.56197860641990499760 25\ldots \cdot 10^{-100}$
$y_6 = y_0 + 1.47644351193235934011 52\ldots \cdot 10^{-104}$
$y_7 = y_6 + 1.42105146030419738099 66\ldots \cdot 10^{-111}$
$y_8 = y_1 + 2.55038711040362235565 38\ldots \cdot 10^{-119}$
...

The value y_6 is one that may be doubted more than y_0 ... y_5 since it derives from the weakest branch of Figure 3 with only 7132 primes. But since $7132 > 2111$ (thus a stable branch), it can be expected to remain valid.

The stability criterion as described above, $\lim_{sup} n/p < 1$ for finite branches, may even be strengthened to $\lim_{sup} n/(\frac{1}{2} \cdot \sqrt{p} \cdot \log p) < 1 + \varepsilon$ and still hold for the entirety of the sequence for relatively small ε. Even if this is not true for all $\varepsilon > 0$ (and under certain conditions — looking at runs of consecutive non-surviving primes of a given stage — can be anticipated to fail), in practice, it will be a safe bet that no-one will explicitly find a counterexample for branches with more than $\frac{1}{2} \cdot \sqrt{p} \cdot \log p$ primes. There might be a bound $\lim_{sup} n/(\frac{1}{2} \cdot \sqrt{p} \cdot \log p) \leq c$ for finite branches for some constant c, and thus far $c > 0.75478$ (177^{th} branch/prime of stage 111 with 64 primes at stage 123 but dissipating at stage 316). So two more questions are probably to remain unanswered: Is there such a bound, maybe with $c > 1$? If yes, what is the heuristically/actually largest c? If no, is there any other bound?

The bifurcation tree as depicted in Figure 3 gets quite impressive when expanded. For example, tracing back the primes from stage 318 to stage 100 crystallizes out 47 primes (compared to originally 594) with an eventful history.

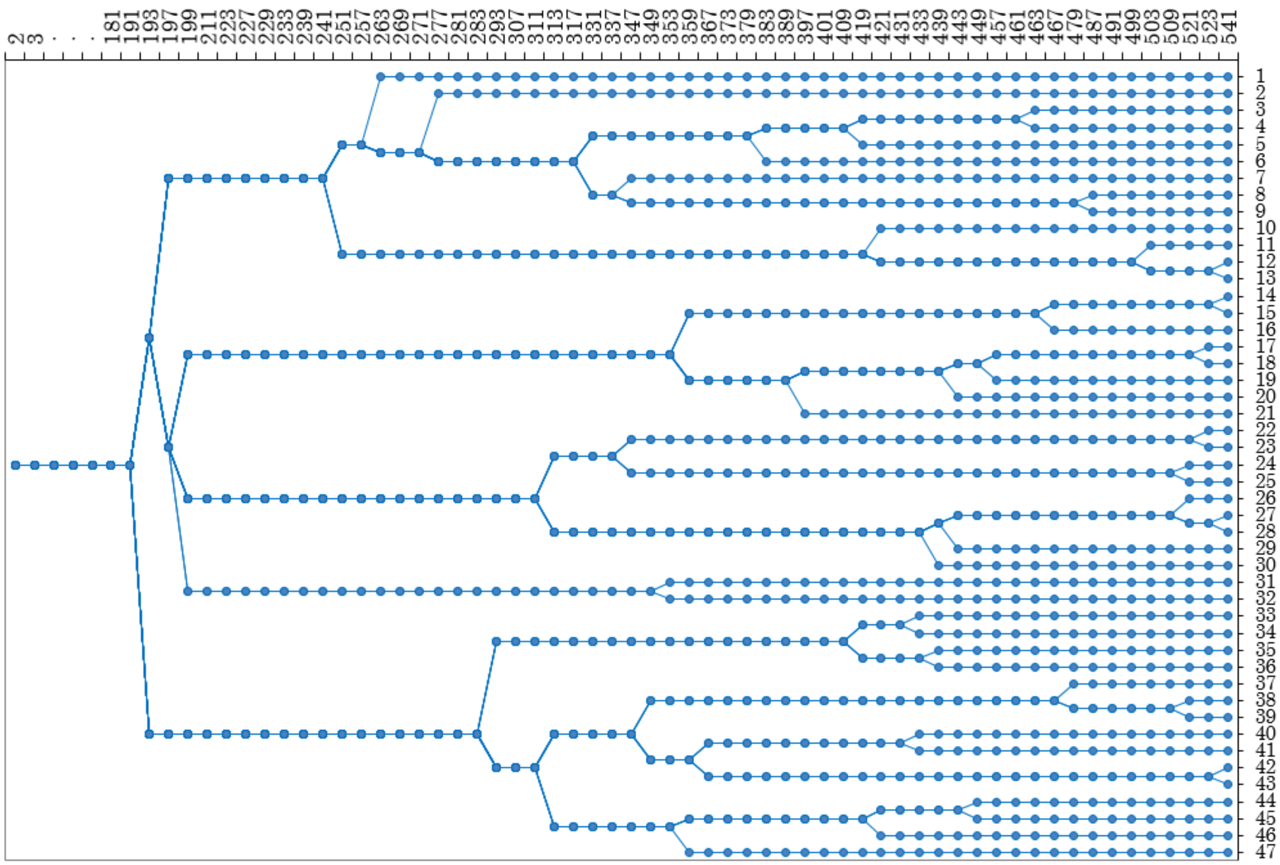

Figure 4. Alleged number of possibilities up to stage 100.

It is interesting to note the threefold split at stage 46, $p = 199$. If all of these primes persist an infinite run, this still is the only point for $s \leq 100$ where more than two primes out of a single prime from the previous stage will simultaneously continue the race. But how probable is this now? We have to consider the possible number of primes arising from one prime of the previous stage and the possibility that at least three of them become stable sequences, including the permutations if more than three primes show up simultaneously in the current stage:

$$\sum_{x=3}^{\infty} \frac{\binom{p-1}{x}}{(\log q - 1)^x} (1 - \frac{1}{\log q})^{p-1} (1 - \omega(s-1))^x \binom{x}{3} \tag{12}$$

For $s = 101$, this value is 0.0043%, thus $\approx 2.76\%$ for all 594 primes of stage 100.

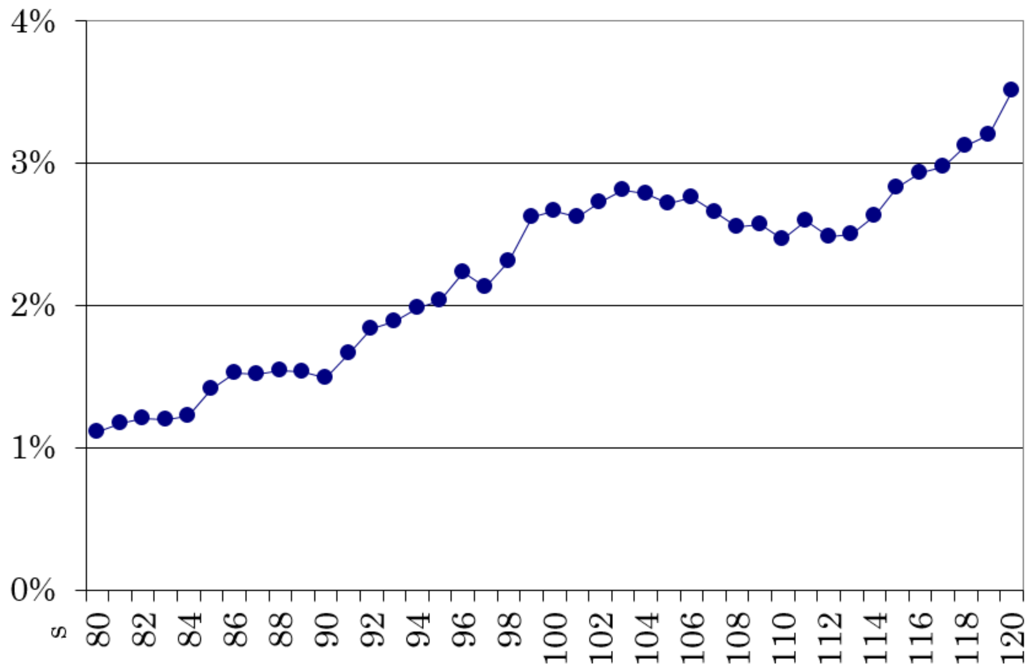

FIGURE 5. The value is growing in the process (e.g. about 52% for $s = 252$), so we can soon expect another threefold split. In fact, the next one appears to be in the upper region of $p = 613$, $s = 112$.

Substituting 3 for k in (12), it can handily be used for the probability of a k-fold split for any $k \geq 2$.

The possibly last stage without any split, that is without a temporarily increasing number of surviving primes and hence admissible values for y, is found at $s = 103$ ($p = 563$).

9. Broader utilization

It's not without a little luck that the sequence as described exists. If we look for y' such that $y' \cdot p\#$ rounded to the nearest integer is prime, we wouldn't get far: at stage 23, one prime would be left: $\lfloor y' \cdot 83\# + \frac{1}{2} \rfloor = q'_{23,1} = 20678031399936908333235632776 4879$. There's no prime within $q'_{23,1} \cdot 89 \pm 44$, so that sequence would be finite (or "mortal") — even though the primes are smaller compared to $y = 1.2541961...$ ($\#q'_s = \{2, 2, 2, 2, 1, 1, 2, 3, 6, 4, 1, 3, 5, 4, 2, 2, 2, 4, 5, 5, 2, 2, 1, 0.\}$ — and the ratio $\frac{y}{y'} = 1.6198...$ is close to the golden ratio:)

And for those who can't get enough sequences that grow like $y \cdot p\#$, here is a definition for "semi-p#Y-sequences": find y such that $\lfloor y \cdot p\# \rfloor$ is a prime number for every $p \geq 3$. The only difference here is that $\lfloor 2y \rfloor$ is allowed to be an even number other than 2 (maybe even a semiprime, depending on the taste about restrictions, though it's hard to find such an example like $\lfloor y \rfloor = 210457743323$ — with 1124 primes at $s = 331$). This leaves a lot of sequences on the verge of extinction for quite a while during the calculations. Some of them should survive an infinite run:
$0 < y < 1$: this gives a sequence which is stronger than the original p#Y-sequence
($1 < y < 2$: original sequence)
$25 < y < 26$: stronger than the original sequence up to stage 166, weaker after stage 168
$1411 < y < 1412$: a weak sequence with only 883 primes at $s = 168$, $p = 997$
$3432 < y < 3433$: a very weak sequence, only 168 primes at $s = 168$, $p = 997$
$13948 < y < 13949$: another weak sequence
$201420 < y < 201421$: another one which has somewhat good chances to survive
$6007103 < y < 6007104$: rather weak
$25510020 < y < 25510021$: just a bit stronger than $\lfloor y \rfloor = 3432$ until $s = 167$, $p = 991$

... Can you find others? Not like these, they're only finite (successive records):
$2 < y < 3$: extinct at stage 4
$3 < y < 4$: extinct at stage 7
$5 < y < 6$: extinct at stage 11
$16 < y < 17$: extinct at stage 14
$978 < y < 979$: extinct at stage 17
$6640 < y < 6641$: extinct at stage 23
$11456 < y < 11457$: extinct at stage 35
$160563 < y < 160564$: extinct at stage 38
$283257 < y < 283258$: extinct at stage 68
$1117230 < y < 1117231$: extinct at stage 78
$1594501 < y < 1594502$: extinct at stage 86
$55990660 < y < 55990661$: extinct at stage 106
$108286142 < y < 108286143$: extinct at stage 114

The number of y's with different integer part that result in infinite semi-sequences should also be infinite.

10. Appendix A. More on the evolution

Table 7. Data for $s \leq 318$, using BPSW-pseudoprimality for the primes q for $s \geq 110$. $n*$ stands for the number of surviving branches, backtracked from the primes of stage 318. This number seems to be settled for $s \leq 130$ but will be subject to change for $s \geq 131$ by the time the calculation is taken further since many of them will get cancelled out, so this column only gives a momentary picture (as well as an upper bound). Nevertheless, the rate of decrease for larger s may be of interest in itself.

s	p	n	s	p	n	$n*$	s	p	n	$n*$	s	p	n	$n*$
1	2	1	31	127	2	1	61	283	93	8	91	467	413	35
2	3	1	32	131	5	1	62	293	81	9	92	479	426	36
3	5	1	33	137	6	1	63	307	81	9	93	487	446	37
4	7	1	34	139	12	1	64	311	67	9	94	491	454	37
5	11	2	35	149	21	1	65	313	66	11	95	499	491	37
6	13	2	36	151	19	1	66	317	74	11	96	503	456	38
7	17	3	37	157	15	1	67	331	91	12	97	509	490	38
8	19	3	38	163	16	1	68	337	88	12	98	521	559	41
9	23	4	39	167	24	1	69	347	90	14	99	523	573	43
10	29	6	40	173	18	1	70	349	95	15	100	541	594	47
11	31	4	41	179	18	1	71	353	102	16	101	547	652	47
12	37	5	42	181	17	1	72	359	126	18	102	557	718	50
13	41	5	43	191	14	1	73	367	152	19	103	563	757	50
14	43	9	44	193	24	2	74	373	154	19	104	569	786	53
15	47	11	45	197	24	3	75	379	166	19	105	571	835	55
16	53	10	46	199	28	5	76	383	187	20	106	577	839	57
17	59	12	47	211	30	5	77	389	214	20	107	587	854	59
18	61	8	48	223	36	5	78	397	206	21	108	593	906	63
19	67	6	49	227	49	5	79	401	201	21	109	599	916	65
20	71	11	50	229	44	5	80	409	220	21	110	601	998	69
21	73	5	51	233	52	5	81	419	241	23	111	607	988	71
22	79	4	52	239	53	5	82	421	249	25	112	613	1016	74
23	83	6	53	241	55	5	83	431	269	25	113	617	1078	78
24	89	3	54	251	67	6	84	433	320	27	114	619	1165	81
25	97	2	55	257	69	6	85	439	354	29	115	631	1237	84
26	101	1	56	263	72	7	86	443	354	31	116	641	1295	86
27	103	3	57	269	81	7	87	449	365	32	117	643	1371	90
28	107	1	58	271	79	7	88	457	369	33	118	647	1399	93
29	109	1	59	277	85	8	89	461	358	33	119	653	1523	97
30	113	3	60	281	83	8	90	463	387	34	120	659	1618	101

Stage 139 is the last known point where the number of primes decreases. (Interestingly enough, just before that happens, we have $n = 4p$, which is the only known — but certainly not the only — example where n is a multiple of p.)

s	p	n	$n*$	s	p	n	$n*$	s	p	n	$n*$
121	661	1701	105	156	911	5971	389	191	1153	19690	1147
122	673	1773	108	157	919	6366	404	192	1163	20364	1179
123	677	1801	116	158	929	6582	414	193	1171	21154	1216
124	683	1889	121	159	937	6799	433	194	1181	21954	1262
125	691	1885	127	160	941	7006	455	195	1187	22688	1287
126	701	1944	132	161	947	7151	463	196	1193	23528	1333
127	709	2008	138	162	953	7599	474	197	1201	24295	1363
128	719	2141	144	163	967	7920	494	198	1213	25085	1401
129	727	2136	150	164	971	8276	511	199	1217	25717	1442
130	733	2241	158	165	977	8708	524	200	1223	26680	1492
131	739	2354	165	166	983	9086	535	201	1229	27569	1527
132	743	2442	177	167	991	9577	553	202	1231	28242	1572
133	751	2548	179	168	997	9736	570	203	1237	28917	1614
134	757	2649	184	169	1009	10329	586	204	1249	29871	1659
135	761	2791	193	170	1013	10722	594	205	1259	30707	1708
136	769	2998	200	171	1019	11048	611	206	1277	31555	1754
137	773	3017	207	172	1021	11387	639	207	1279	32633	1808
138	787	3148	214	173	1031	11769	669	208	1283	33494	1871
139	797	3132	224	174	1033	12251	685	209	1289	34556	1921
140	809	3385	231	175	1039	12490	711	210	1291	35744	1981
141	811	3537	245	176	1049	12905	733	211	1297	36546	2034
142	821	3698	250	177	1051	13209	750	212	1301	37363	2087
143	823	3914	261	178	1061	13556	775	213	1303	38018	2150
144	827	4175	266	179	1063	13918	797	214	1307	38936	2218
145	829	4345	273	180	1069	14393	822	215	1319	39633	2303
146	839	4360	284	181	1087	15006	852	216	1321	40152	2375
147	853	4537	292	182	1091	15472	874	217	1327	40822	2456
148	857	4722	304	183	1093	15695	908	218	1361	42524	2541
149	859	4862	308	184	1097	16075	937	219	1367	44138	2626
150	863	4951	320	185	1103	16548	971	220	1373	45674	2704
151	877	5099	332	186	1109	16852	1008	221	1381	47308	2791
152	881	5228	345	187	1117	17548	1033	222	1399	49203	2876
153	883	5334	357	188	1123	17926	1064	223	1409	50999	2966
154	887	5460	364	189	1129	18415	1090	224	1423	53578	3047
155	907	5715	379	190	1151	19145	1124	225	1427	56009	3139

s	p	n	$n*$	s	p	n	$n*$	s	p	n	$n*$
226	1429	58245	3213	261	1663	141747	8687	296	1949	323493	35595
227	1433	60614	3340	262	1667	144129	8937	297	1951	332369	37696
228	1439	63012	3442	263	1669	146278	9203	298	1973	344077	40005
229	1447	65622	3520	264	1693	150011	9484	299	1979	355840	42551
230	1451	67609	3633	265	1697	153719	9799	300	1987	367371	45250
231	1453	69411	3747	266	1699	157438	10097	301	1993	379303	48293
232	1459	71677	3844	267	1709	160935	10460	302	1997	391133	51726
233	1471	73978	3960	268	1721	164713	10819	303	1999	402306	55407
234	1481	76816	4078	269	1723	168209	11187	304	2003	413306	59585
235	1483	78753	4173	270	1733	171914	11552	305	2011	424348	64450
236	1487	81051	4292	271	1741	175208	11966	306	2017	435084	69845
237	1489	82647	4411	272	1747	178924	12392	307	2027	446383	76167
238	1493	84106	4551	273	1753	182442	12861	308	2029	457067	83297
239	1499	85756	4684	274	1759	186286	13363	309	2039	468311	91883
240	1511	87469	4811	275	1777	190466	13889	310	2053	482068	102217
241	1523	90333	4958	276	1783	196294	14432	311	2063	494789	114639
242	1531	93284	5084	277	1787	200525	14980	312	2069	509095	130467
243	1543	96279	5233	278	1789	204944	15548	313	2081	524247	150449
244	1549	99150	5376	279	1801	209420	16163	314	2083	538695	177304
245	1553	101723	5539	280	1811	214672	16789	315	2087	551951	214976
246	1559	104560	5675	281	1823	218990	17483	316	2089	564045	271897
247	1567	107041	5832	282	1831	224272	18194	317	2099	577252	370885
248	1571	109795	5999	283	1847	230486	18982	318	2111	592642	592642
249	1579	111912	6155	284	1861	238339	19837				
250	1583	114508	6341	285	1867	246127	20704				
251	1597	117842	6528	286	1871	253236	21653				
252	1601	121087	6716	287	1873	259946	22631				
253	1607	124407	6901	288	1877	266707	23698				
254	1609	126382	7113	289	1879	272114	24815				
255	1613	128949	7305	290	1889	278512	26035				
256	1619	130732	7493	291	1901	285821	27356				
257	1621	132088	7720	292	1907	292121	28714				
258	1627	133678	7965	293	1913	298404	30241				
259	1637	135308	8188	294	1931	305779	31927				
260	1657	137970	8439	295	1933	314060	33679				

The calculation up to the point given above takes about five months with Pari/GP [10] on a single CPU core of a 2020 state-of-the-art PC with the following program, which in the given form starts at $s = 29$ and keeps track of the numbers as `a` $= q_{s,1}$ and a memory friendly vector `d` comprising only the consecutive differences between the numbers of a given stage where `d[1]` $= 0$ and, for $i > 1$, `d[i]` $= q_{s,i} - q_{s,i-1}$: [4]

```
{
a=350842542483891235293716663559065020274899073; d=[0];

s=0; b=a;
while(b>1, s++; b\=prime(s));
i=#d; gettime();
while(s<318,
   s++; p=prime(s); o=a*p; c=d;
   e=99+floor(i*(1+2/sqrt(p))); d=vector(e);
   m=i; i=0;
   for(j=1, m,
      print1("interval "j"/"m,Strchr(13));
      o+=c[j]*p; v=vector(p);
      forprime(b=3, p-2, r=b-lift(Mod(o,b)); forstep(l=r, p, b, v[l]=1));
      forprime(b=p+2, p^2, r=b-lift(Mod(o,b)); if(r<p, v[r]=1));
      forstep(r=2, p-1, 2,
         if(!v[r],
            q=o+r;
            if(ispseudoprime(q),
               i++; if(i>1, d[i]=q-z, a=q); z=q;
               print1("interval "j"/"m" - "i" primes found",Strchr(13))
            )
         )
      )
   );
   g=floor(gettime()/1000); x="[CPU time: ";
   f=floor(g/3600); if(f,x=Str(x,f"h "));
   f=floor(g/60); if(f,x=Str(x,f%60"m "));
   x=Str(x,g%60"s] "); t=Str("Stage "s); print(t" (p="p"): "i" primes "x);
   t=Str("p#Y "t".txt"); write(t,"a="a"; d="vecextract(d,Str("1.."i)))
)
}
```

Be sure to set `allocatemem(10^8)` or higher (and watch for the second "while" condition) when aiming for $s > 318$.

[4] `a` and `d` can also be read in from a previously calculated output file

11. Appendix B. "Mille"stone

The first titanic p#Y-prime ($s = 350$, $p = 2357$, primality proved over all stages): $q_{350,1} =$
2975888029446690045674329889350866418922182539508083466546067996664167034032353861106265874760710002626916716792245132115751760492949133469320566753486211657967302638955616142480561239809500345286868322115000144352486048339582098192783731615508169499622598927362686160017313404427512143294259970499306893769171468799563044449516983850885392460815319763768434329781194956097848054398473868538461756097734062947802989545222272955432147884759489532153762581009047589114806476317194896907515989326012264806400258114511922593654143140279086686012569157673559904417606633172278402597099368063119065089573166748626176375557296805505524999457560919427667284621117997598039465130796334009757240891094211460440403405892625061675846984519517870896194511937137877212361818965759263390653624063721048744786408768337842010153020573502872211833734417328472053391417872840348972391626846327447542268725657336629787247428617722407540470581500497332256038129464337230773818385423630283496477698386126623480539308646019
It's a nice fact that 2357# of all primorials has exactly 1000 digits.

The ostensibly first titanic p#Y-prime that survives in the long run, the 16^{th} smallest of them all at $p = 2357$, spanning its branch beyond 13147# ($s > 1563$), is $q_{350,1}+$
64889002924617557159295368172225527028589573701517325174945374247864241568387545962282013199022969351372420058933779808194771388980505362882448043357530075872493531089513889637480426037406091083077809690,
which leads to $y_{min} \geq$
1.254196101578011936277679554914213423779869218042622195832722554608864699428751447513231696736473312007130293138358294105190550680714644545957773479897214724735498508703837740455386485449104321045698006250714051461327976060192217343991366694102316850262706569419874198220202069730042807050891212595258018556113036107026149400651452048529072340778028008544316734128253583224021277855951903443706113250453367301336848062710640304848112964348021638784247997789152654859485938006014693824809800408752716535619977599286973472398147892837251062676723990765128719603302423561269079491606830908042627055234080081589099112095862083987783158432601041511728803871417616354122955584764825564835336501561572838431790634989265724764141451742040724856932927017872006666654124586130154464923232827694115726870974699096120144117434056054291626301005003851057094254938947162455644528349430263680568551398090174046059310097968393716013493042619454507538255565123810953362966551870774863385941160720014897114954349144158...

The first 129 primes of stage 428 ($p = 2969$) are known to be "mortal": $q_{428,130}$ is the first that may survive an infinite run, which is a currently known maximum. It takes on average about $\sqrt{p}/2$ primes to find a surviving prime. An error term for this approximation, assumed to be of order of at most $O(\sqrt{p}/\log p)$, would probably be desireable.

12. Appendix C. Sequence tuples (multiple primes in one interval)

First twin: (1^{st} prime of stage 4)*11+ {4, 10}
First triplet: (1^{st} prime of $s = 7$)*19+ {10, 12, 16}
First quadruplet: (2^{nd} prime of $s = 9$)*29+ {6, 14, 24, 26}
First quintuplet: (18^{th} prime of $s = 46$)*211+ {42, 110, 140, 144, 194}
First sextuplet: (69^{th} prime of $s = 69$)*349+ {18, 40, 234, 262, 292, 298}
First septuplet: (238^{th} prime of $s = 84$)*439+ {34, 228, 252, 282, 364, 378, 382}
First octuplet: (5687^{th} prime of $s = 159$)*941+ {170, 234, 294, 462, 696, 740, 752, 812}
First nonuplet: (270^{th} prime of $s = 181$)*1091+ {18, 46, 60, 166, 180, 312, 850, 1062, 1080}
First decuplet: (193057^{th} prime of $s = 289$)*1889 + {220, 238, 378, 624, 934, 1048, 1414, 1612, 1678, 1750}

Higher-order descendants (first instances of number of "grandchildren"):
3 descendants of order 2: (1^{st} prime of stage 6)*17*19+ {162, 164, 168}
4 descendants: (1^{st} prime of $s = 7$)*19*23+ {246, 284, 384, 386}
5 descendants: (5^{th} prime of $s = 21$)*79*83+ {3504, 3520, 5190, 5200, 5224}
6 descendants: (4^{th} prime of $s = 12$)*43*47+ {102, 108, 582, 598, 1428, 1450}
7 descendants: (7^{th} prime of $s = 34$)*149*151+ {754, ..., 12484}
8 descendants: (30^{th} prime of $s = 55$)*263*269+ {7686, ..., 40560}
9 descendants: (10^{th} prime of $s = 59$)*281*283+ {11388, ..., 59016}
10 descendants: (16^{th} prime of $s = 47$)*223*227+ {13650, ..., 38340}
11 descendants: (42^{nd} prime of $s = 67$)*337*347+ {24432, ..., 116892}
12 descendants: (27^{th} prime of $s = 55$)*263*269+ {45742, ..., 68590}
13 descendants: (3053^{rd} prime of $s = 138$)*797*809+ {195918, ..., 603780}
14 descendants: (623^{rd} prime of $s = 101$)*557*563+ {94812, ..., 261562}
15 descendants: (32^{nd} prime of $s = 72$)*367*373+ {2424, ..., 134612}
16 descendants: (45231^{st} prime of $s = 224$)*1427*1429+ {74772, ..., 1944498}
17 descendants: (2030^{th} prime of $s = 176$)*1051*1061+ {74496, ..., 1083072}

The like-minded reader with enough time on their hands might be encouraged to find successive maxima of descendants of higher order. To start, here's 4 descendants of order 3: (1^{st} prime of stage 6)*17*19*23+ {3742, 3780, 3880, 3882}

13. Appendix D. Relative strength of split intervals

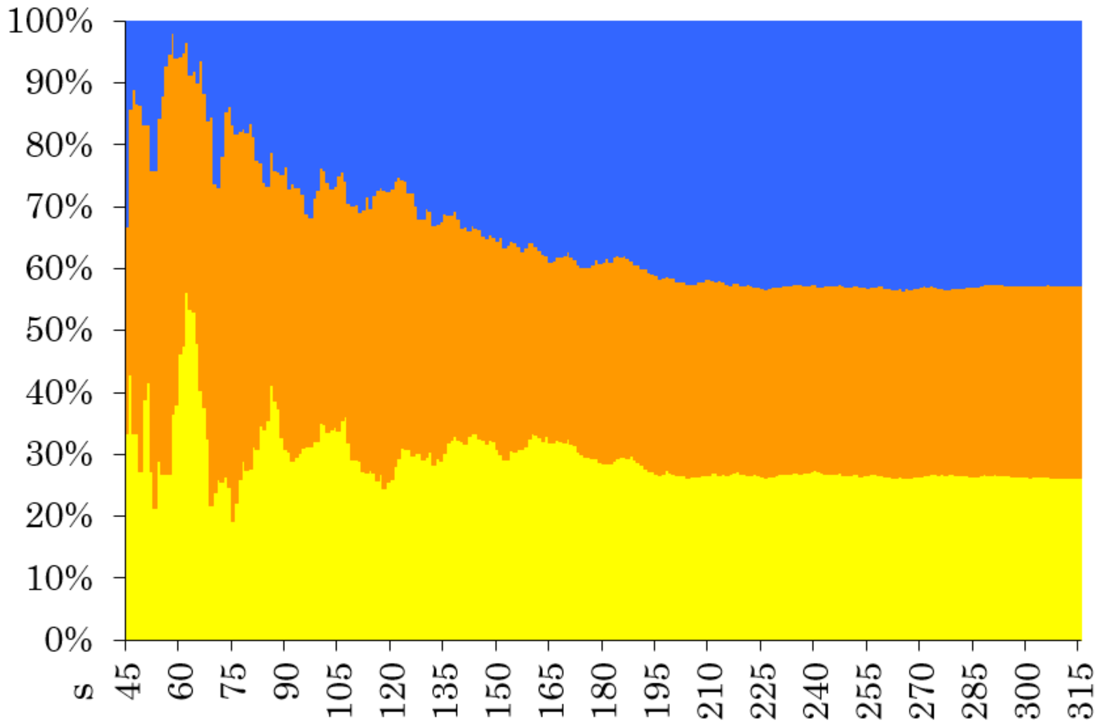

Figure 6. A nice and colorful edit showing the strength of the first three split intervals, starting at stage 45. The 592642 probable primes at stage 318 are here divided into three groups (154145 + 184653 + 253844) with the respective relative strength 26.01%, 31.16%, and 42.83%. These values may (hopefully) stabilize while the number of primes in the sequence continues to grow. Giving the exact percentages as s goes to infinity will be rather difficult.

14. Appendix E. Plouffe, revisited

As a final tidbit of numerical info, inspired by Plouffe's work mentioned in the introduction, we give a constant A (to 1029 places after the decimal point) such that $\lfloor A^{1.001^n} \rfloor$ should give an infinite sequence of primes, for integer $n \geq 0$:

$A = 10^{3875} -$ 9840440.013269045402699925892590230034000537023543058287831784100957
276330984498140336774521413625639037370179638123065989644921831945828284474797146
214314469465943653321876212421217357462243438103894210660374222702124802145689793
662592212239695486359151401226648141442872210231891779023391189805301737658502412
189281932141114052020391866640443434821779837126175086985360497145764015143114697
436609454475477982979703079133332121473465799127802190473213691503591043455180286
502985352685402044844822486579962417722857237662143660192493843854713085865446265
026287773151577320009216914265110643337790882410603483306933710003501741787035605
297213202524866787149107203669351182374236682986886760407222192912257430323369703
769706007229005048714432375371543157174405792236627110947314049230460840453363435
080982900105891219946283712237437066606879315185506382375652998319912097105539735
706790752939943728420192710162069587039287823862485223550170317760285385396844814
481707064610195932400139047255566802623529101069361896409968571400144014982338...

We would be happy to learn about a smaller known value for A.

15. Acknowledgements

Many thanks to Charles R. Greathouse IV for his support when we first presented this sequence at `Mersenneforum.org` back in 2008. We are also deeply grateful to Alexei Kourbatov for helpful suggestions on the paper.

References

[1] C. Bays, R. H. Hudson, "A new bound for the smallest x with $\pi(x) > Li(x)$", *Math. Comp.* **69:231** (2000), 1285–1296.
[2] C. Elsholtz, "Unconditional prime-representing functions, following Mills", *The American Mathematical Monthly*, **127** (2020), 639–642. https://arxiv.org/abs/2004.01285
[3] G. H. Hardy and J. E. Littlewood, "Some Problems of 'Partitio Numerorum.' III. On the Expression of a Number as a Sum of Primes", *Acta Math.* **44** (1923), 1–70.
[4] A. E. Ingham, "The distribution of prime numbers", *Cambridge University Press, New York* (1932), 105–106.
[5] A. E. Ingham, "On the difference between consecutive primes", *Quart. J. Math. Oxford Ser. vol. 8* (1937), 255–266.
[6] L. Kuipers, "Prime-representing functions", *Indag. Math.* **12** (1950), 57–58.
[7] J. E. Littlewood, "Sur la distribution des nombres premiers", *Comptes Rendus.* **158** (1914), 1869–1872.
[8] W. H. Mills, "A prime-representing function", *Bull. Am. Math. Soc.* **53** (1947), 604.
[9] C. Myerscough, "Application of an accurate remainder term in the calculation of Residue Class Distributions", available at https://personal.math.ubc.ca/~gerg/teaching/592-Fall2018/papers/arXiv.Myerscough.pdf
[10] PARI/GP, The Pari Group ©2003-2024, Univ. Bordeaux, available at https://pari.math.u-bordeaux.fr
[11] D. Platt, T. Trudgian, "On the first sign change of $\theta(x) - x$", arXiv [math.NT]: https://arxiv.org/abs/1407.1914.
[12] S. Plouffe, "A set of formulas for primes", arXiv [math.NT]: https://arxiv.org/abs/1901.01849.
[13] C. Pomerance, "Are there counter-examples to the Baillie-PSW primality test?", *Department of Mathematics, University of Georgia, Athens, GA 30602 U.S.A.* (1984).
[14] G. F. B. Riemann, "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse", *Monatsbericht Preussische Akademische Wissenschaft* (1859), 671–680.
[15] E. Santana, "Evolution on sets", arXiv [math.GM]: https://arxiv.org/abs/2302.12767.
[16] Y. Saouter, P. Demichel, "A sharp region where $\pi(x) - li(x)$ is positive", *Mathematics of computation, S0025-5718(10)02351-3* (2010).
[17] H. von Koch, "Sur la distribution des nombres premiers", *Acta Math.* **24** (1901), 159–182.

Keywords: prime number sequence, prime generating formula, primorial function, Chebyshev's bias.

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