REGULAR EXTREME SEMISIMPLE LIE ALGEBRAS

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Abstract. A subalgebra of a semisimple Lie algebra is *wide* if every simple module of the semisimple Lie algebra remains indecomposable when restricted to the subalgebra. A subalgebra is *narrow* if the restrictions of all non-trivial simple modules to the subalgebra have proper decompositions. A semisimple Lie algebra is *regular extreme* if any regular subalgebra of the semisimple Lie algebra is either narrow or wide. Douglas and Repka [DR24] previously showed that the simple Lie algebras of type A_n are regular extreme. In this article, we show that, in fact, all simple Lie algebras are regular extreme. Finally, we show that no non-simple, semisimple Lie algebra is regular extreme.

1. Introduction

A subalgebra of a semisimple Lie algebra is *wide* if every simple module of the semisimple Lie algebra remains indecomposable when restricted to the subalgebra. The term "wide" was coined by Panyushev in [Pan14], where he established conditions for large families of subalgebras of semisimple Lie algebras to be wide. Douglas and Repka extended this work, in part, by examining conditions that make an arbitrary regular subalgebra of a semisimple Lie algebra wide [DR24].

In [DR24], Douglas and Repka also initiated the study of narrow subalgebras, that is, subalgebras on which the restriction of any non-trivial simple module of the ambient semisimple Lie algebra has a proper decomposition. In this article, we investigate regular extreme semisimple

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Date: March 28, 2024.

2010 *Mathematics Subject Classification.* 17B05, 17B10, 17B20, 17B22, 17B30.

Key words and phrases. Narrow subalgebras, wide subalgebras, regular extreme semisimple Lie algebras, Dynkin diagrams.

Lie algebras. A semisimple Lie algebra is *regular extreme* if any regular subalgebra of the semisimple Lie algebra is either narrow or wide.

In [DR24], Douglas and Repka showed that the simple Lie algebras of type A_n are regular extreme using the bases of A_n-type modules created by Feigin, Fourier, and Littelmann [FFL11]. This property does not hold for non-regular subalgebras of A_n [DR11]; that is, a non-regular subalgebra of A_n may be neither narrow nor wide.

In the present article, we extend the examination of regular extreme semisimple Lie algebras. We show that all simple Lie algebras are regular extreme. Specifically, we establish that the simple Lie algebras of types B_n, C_n, D_n, E_6, E_7, E_8, F_4, and G_2 are regular extreme. Then, we prove that no non-simple, semisimple Lie algebra is regular extreme. In this paper, our proofs do not rely on specific bases of simple modules, even though this was the approach used with A_n in [DR24].

The article is organized as follows. Section 2 contains necessary background information. In Section 3, we prove that all simple Lie algebras are regular extreme. Then, in Section 4, we establish that no non-simple, semisimple Lie algebra is regular extreme.

Note that all Lie algebras and modules in this article are over the complex numbers, and finite-dimensional.

2. Background, terminology, and notation

In this section, we review background on semsimple Lie algebras and their modules; closed subsets of root systems and regular subalgebras; simple Lie algebras and Dynkin diagrams; and relevant previous results from [DR24, Pan14].

2.1. **Semisimple Lie algebras.**

Let $\mathfrak{g}$ denote a semisimple Lie algebra, and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$. The corresponding root system is denoted Φ, with Weyl group $\mathcal{W}$. For $\alpha \in \Phi$, $\mathfrak{g}_\alpha$ denotes the corresponding root space. The set of positive roots of Φ is denoted Φ^+, and $\Delta = \{\alpha_1, ..., \alpha_n\} \subseteq \Phi^+$ is a fixed base of Φ. The *rank* of $\mathfrak{g}$ is n, the number of simple roots in Δ.

If $\alpha \in \Phi$, we may fix a nonzero $e_\alpha \in \mathfrak{g}_\alpha$. Then there is a unique $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$, such that e_α, $e_{-\alpha}$, and $h_\alpha = [e_\alpha, e_{-\alpha}] \in \mathfrak{h}$ satisfy $[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha = 2e_\alpha$, and $[h_\alpha, e_{-\alpha}] = -2e_{-\alpha}$; thus e_α, $e_{-\alpha}$, and h_α span a subalgebra isomorphic to $\mathfrak{sl}_2$. We define $f_\alpha \coloneqq e_{-\alpha}$ for $\alpha \in \Phi^+$.

We may naturally associate $\mathfrak{h}$ with its dual space $\mathfrak{h}^*$ via the Killing form κ. Specifically, $\alpha \in \mathfrak{h}^*$ corresponds to the unique element $t_\alpha \in \mathfrak{h}$ such that $\alpha(h) = \kappa(t_\alpha, h)$, for all $h \in \mathfrak{h}$. We have the nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ given by $(\alpha, \beta) := \kappa(t_\alpha, t_\beta)$, and we may define $\langle \alpha, \beta \rangle := \frac{2(\alpha,\beta)}{(\beta,\beta)} = \alpha(h_\beta)$, where $h_\beta := \frac{2t_\beta}{\kappa(t_\beta,t_\beta)} = \frac{2t_\beta}{(\beta,\beta)}$.

A root system Φ is *irreducible* if it cannot be partitioned into two proper, orthogonal subsets. If Φ is irreducible, then $\mathfrak{g}$ is a simple Lie algebra.

The set of weights relative to the root system Φ is denoted Λ, and Λ^+ is the set of all dominant weights with respect to Δ. Let $\lambda_1, ..., \lambda_n$ be the *fundamental dominant weights* (relative to Δ). A dominant weight $\lambda \in \Lambda^+$ may be written as $\lambda = m_1\lambda_1 + \cdots + m_n\lambda_n$, where m_i is a nonnegative integer for each i. For each dominant weight λ, $V(\lambda)$ is the simple $\mathfrak{g}$-module of highest weight λ. Fix a highest weight vector $v_\lambda \in V(\lambda)$, unique up to scalar multiple.

For an arbitrary $\mathfrak{g}$-module V, let $\Pi(V)$ be the set of weights of V. Given $\mu \in \Pi(V)$, let $V_\mu = \{v \in V \mid h \cdot v = \mu(h)v, \text{ for all } h \in \mathfrak{h}\}$. Then, V decomposes into weight spaces

$$V = \bigoplus_{\mu \in \Pi(V)} V_\mu. \tag{1}$$

2.2. **Closed subsets of root systems and regular subalgebras.** A subset T of the root system Φ is *closed* if for any $x, y \in T$, $x+y \in \Phi$ implies $x+y \in T$. Any closed set T is a disjoint union of its *symmetric* component $T^r = \{\alpha \in T | -\alpha \in T\}$, and its *special* component $T^u = \{\alpha \in T | -\alpha \notin T\}$. Let $S \subseteq \Phi$, then the *closure* of S, denoted $[S]$, is the smallest closed subset of Φ containing S.

Let $T \subseteq \Phi$ be a closed subset. Let $\mathfrak{t}$ be a subspace of $\mathfrak{h}$ containing $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ for each $\alpha \in T^r$. Then

$$\mathfrak{s}_{T,\mathfrak{t}} = \mathfrak{t} \oplus \bigoplus_{\alpha \in T} \mathfrak{g}_\alpha \tag{2}$$

is a regular subalgebra of $\mathfrak{g}$. Moreover, all regular subalgebras of $\mathfrak{g}$ normalized by $\mathfrak{h}$ arise in this manner. Further, any regular subalgebra of $\mathfrak{g}$ is conjugate under the adjoint group of $\mathfrak{g}$ to a regular subalgebra normalized by $\mathfrak{h}$. Hence, we'll assume that any regular subalgebra is normalized by $\mathfrak{h}$, and thus in the form of Eq. (2).

A Lie algebra may be either semisimple, solvable, or Levi decomposable (Levi's Theorem [[ŠW14], Chapter II, Section 2]). A regular semisimple subalgebra is given by $\mathfrak{s}_{T,\mathfrak{t}}$ for some symmetric closed subset T (non-empty) of Φ, and the subalgebra $\mathfrak{t}$ of $\mathfrak{h}$ generated by $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ for all $\alpha \in T \cap \Phi^+$. A regular solvable subalgebra is given by a subalgebra of the form $\mathfrak{s}_{T,\mathfrak{t}}$, where T is a special closed subset of Φ (possibly empty), and $\mathfrak{t}$ a subalgebra of $\mathfrak{h}$. A regular Levi decomposable subalgebra is a non-semisimple regular Lie algebra $\mathfrak{s}_{T,\mathfrak{t}}$, such that T is a closed subset, T^r is the corresponding (non-empty) symmetric closed subset, T^u is the corresponding special closed subset of Φ, and $\mathfrak{t}$ is a subalgebra of $\mathfrak{h}$ containing $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ for each $\alpha \in T^r$.

Lemmas 2.1, and 2.2; and Proposition 2.3 below are relevant results for closed subsets of root systems and regular subalgebras.

Lemma 2.1. [DR24] *Let S be a closed subset of Φ. Suppose β_1, β_2,...,$\beta_k \in S$, and $\beta_1 + \beta_2 + \cdots + \beta_k \in \Phi$. Then $\beta_1 + \beta_2 + \cdots + \beta_k \in S$.*

Lemma 2.2. [DR24] *Let T be a closed subset of Φ. Then, $[T \cup -T]$ is a symmetric closed subset of Φ containing T.*

In the following proposition, G is the adjoint group of $\mathfrak{g}$. The Weyl group $\mathcal{W}$ of Φ naturally acts on the dual space $\mathfrak{h}^*$. Further, by identifying $\mathfrak{h}$ and $\mathfrak{h}^*$ via the Killing form, the Weyl group also acts on $\mathfrak{h}$.

Proposition 2.3. *[* [DdG21]*, Proposition 5.1] The regular subalgebras $\mathfrak{s}_{T_1,\mathfrak{t}_1}$ and $\mathfrak{s}_{T_2,\mathfrak{t}_2}$ are conjugate under G if and only if there is a $w \in \mathcal{W}$ with $w(T_1) = w(T_2)$ and $w(\mathfrak{t}_1) = w(\mathfrak{t}_2)$.*

2.3. **Simple Lie algebras and Dynkin diagrams.** As mentioned previously, simple Lie algebras correspond to irreducible root systems. Semisimple Lie algebras are the direct sums of simple Lie algebras.

Given an irreducible root system Φ with base $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, we associate a *Dynkin diagram*, which is a graph having a vertex corresponding to each simple root. Between vertices α_i and α_j, we place $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges. It can be shown that between two vertices there are either one, two, three, or no edges [Hum72]. Whenever a double or triple edge occurs, an arrow is added pointing to the shorter of the two roots.

The Dynkin diagrams for all simple Lie algebras are provided in Figure 1. Namely, the figure contains the Dynkin diagrams of the simple Lie algebras of types A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, and G_2. Note that the subscripts of the Lie algebra types denote the rank of the simple Lie algebra.

A *simple root path* in a Dynkin diagram of an irreducible root system is a sequence of distinct simple roots $(\beta_1, \beta_2, ..., \beta_k)$ such that the vertices associated with the simple roots β_i and β_{i+1}, for all i with $1 \leq i \leq k-1$, are connected by at least one edge.

Note that in a simple root path $(\beta_1, \beta_2, ..., \beta_k)$, we have $\langle \beta_i, \beta_{i+1} \rangle < 0$ for all i with $1 \leq i \leq k-1$ since, by definition, the simple roots β_i and β_{i+1} correspond to vertices in the Dynkin diagram connected by at least one edge, and the fact that $\langle \alpha, \gamma \rangle \leq 0$ for all simple roots α and γ [[Hum72], Lemma 10.1]. Note also that $\langle \beta_i, \beta_j \rangle = 0$ for all i and j with $|i-j| > 1$, since, in this case, the vertices corresponding to β_i

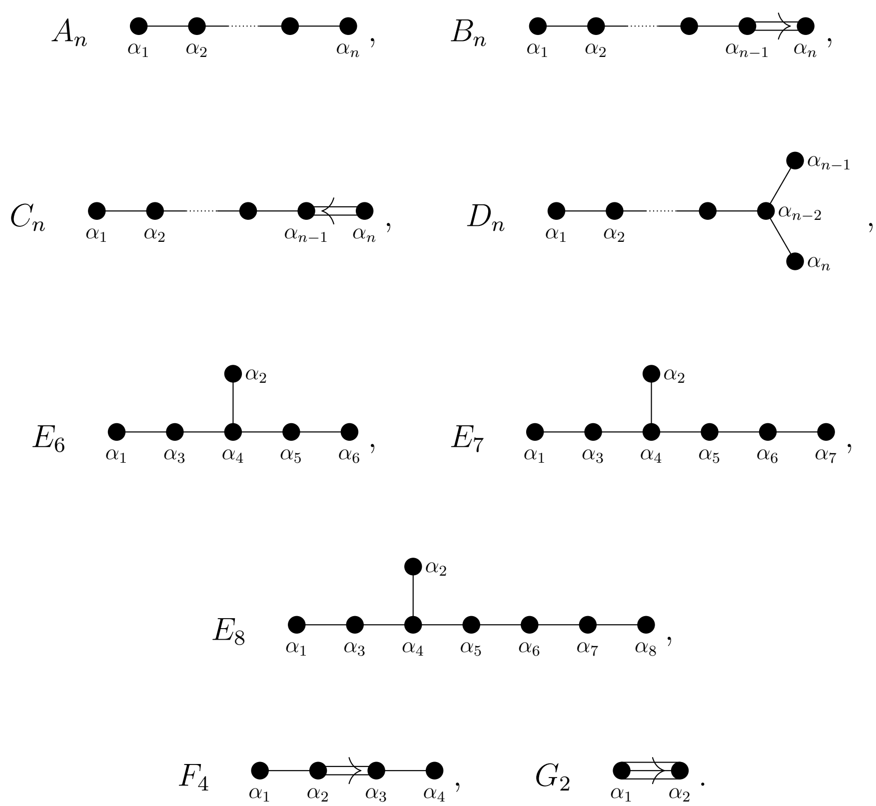

FIGURE 1. Dynkin diagrams of the simple Lie algebras.

and β_j are not connected by an edge. We illustrate a simple root path of E_8 in Figure 2.

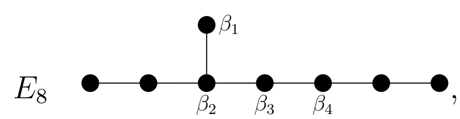

FIGURE 2. A simple root path $(\beta_1, \beta_2, \beta_3, \beta_4)$ of E_8.

Lemma 2.4. *Let $(\beta_1, \beta_2, ..., \beta_k)$ be a simple root path in a Dynkin diagram for an irreducible root system Φ. Then, $\beta_i + \beta_{i+1} + \cdots + \beta_j \in \Phi$ for all i and j such that $1 \leq i \leq j \leq k$.*

Proof. For i and j such that $1 \leq i \leq j \leq k$, consider $\beta_i + \beta_{i+1} + \cdots + \beta_j$. We have $\langle \beta_p, \beta_q \rangle = 0$ for p, and q with $|p - q| > 1$, as the vertices β_p and β_q are not connected by an edge. Since β_i and β_{i+1} are connected by an edge for each i with $1 \leq i \leq k-1$, then $\langle \beta_i, \beta_{i+1} \rangle < 0$. This implies that $\beta_i + \beta_{i+1} \in \Phi$ [[Hum72], Lemma 9.4].

Next, $\langle \beta_i+\beta_{i+1}, \beta_{i+2}\rangle = \langle \beta_i, \beta_{i+2}\rangle + \langle \beta_{i+1}, \beta_{i+2}\rangle < 0$. Once again this implies that $\beta_i + \beta_{i+1} + \beta_{i+2} \in \Phi$. Continuing in this manner, we have $\beta_i + \beta_{i+1} + \cdots + \beta_j \in \Phi$, as required. □

2.4. **Previous results.** Finally, we record relevant results from [DR24, Pan14] that will be used in the present article. First, however, we need two additional definitions. The first definition is for a λ-wide subalgebra. Suppose $V(\lambda)$ is the simple $\mathfrak{g}$-module of highest weight λ. A subalgebra is λ-*wide* if the simple module of $\mathfrak{g}$ of highest weight λ remains indecomposable when restricted to the subalgebra.

Let T be a closed subset of Φ. Then, define an $\mathfrak{s}_{[T\cup -T],\mathfrak{h}}$-submodule of $V(\lambda)$:

(3) $[T \cup -T] \cdot \lambda \coloneqq \mathrm{Span}\big\{e_{-\beta_1} \cdots e_{-\beta_m} \cdot v_\lambda \mid -\beta_j \in [T \cup -T] \cap \Phi^-\big\}$,

where $j = 1, ..., m$, and such that $-\beta_1, ...,$ and $-\beta_m$ are not necessarily distinct. Note that we include $v_\lambda \in [T \cup -T] \cdot \lambda$. In addition, note that if $e_{-\beta_1} \cdots e_{-\beta_m} \cdot v_\lambda \neq 0$, then $\lambda - \beta_1 - \cdots - \beta_m \in \Pi(V(\lambda))$.

We are now ready to present the important results from [DR24, Pan14] that establish necessary and sufficient conditions for a regular subalgebra to be λ-wide, and wide, respectively.

Theorem 2.5. [DR24] *Let T be a closed subset of Φ, $\mathfrak{t}$ a subalgebra of $\mathfrak{h}$ containing $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ for each $\alpha \in T^r$, and $\lambda \in \Lambda^+$. Then, $\mathfrak{s}_{T,\mathfrak{t}}$ is λ-wide if and only if $[T \cup -T] \cdot \lambda = V(\lambda)$.*

Corollary 2.6. [DR24, Pan14] *Let T be a closed subset of Φ, and $\mathfrak{t}$ a subalgebra of $\mathfrak{h}$ containing $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ for each $\alpha \in T^r$. Then, $\mathfrak{s}_{T,\mathfrak{t}}$ is wide if and only if $[T \cup -T] = \Phi$.*

Note that necessary and sufficient conditions for essentially all regular solvable subalgebras to be wide was established in [Pan14]. The result was extended to all regular subalgebras in [DR24].

3. Regular extreme simple Lie algebras

In this section, we prove that all simple Lie algebras are regular extreme. We begin with a lemma pertaining to simple root paths and elements of simple modules.

Lemma 3.1. *Let $(\beta_1, ..., \beta_k)$ be a simple root path of a Dynkin diagram for an irreducible root system, with associated simple Lie algebra $\mathfrak{g}$. Further, let $V(\lambda)$ be the simple $\mathfrak{g}$-module of highest weight λ, and highest weight vector v_λ. If $f_{\beta_1} v_\lambda \neq 0$, then $f_{\beta_1+\cdots+\beta_k} v_\lambda \neq 0$.*

Proof. Since $f_{\beta_1} v_\lambda \neq 0$, and necessarily $e_{\beta_1} v_\lambda = 0$, we have that $\langle \lambda, \beta_1\rangle > 0$ by $\mathfrak{sl}_2$-theory. By way of contradiction, suppose that $f_{\beta_1+\cdots+\beta_k} v_\lambda = 0$.

Since, again, we necessarily have $e_{\beta_1+\cdots+\beta_k}v_\lambda = 0$, then $\langle \lambda, \beta_1 + \cdots + \beta_k\rangle = 0$. This implies $(\lambda, \beta_1 + \cdots + \beta_k) = 0$, and

$$(\lambda, \beta_1) = -(\lambda, \beta_2 + \cdots + \beta_k). \tag{4}$$

Note that $\beta_2 + \cdots + \beta_k \in \Phi$ by Lemma 2.4. And since $\langle \lambda, \beta_1\rangle > 0$, we have

$$\langle \lambda, \beta_2 + \cdots + \beta_k\rangle < 0, \tag{5}$$

which is impossible since λ is the highest weight. Hence, it must be the case that $f_{\beta_1+\cdots+\beta_k}v_\lambda \neq 0$, as required. $\square$

Theorem 3.2. *The simple Lie algebras are regular extreme.*

Proof. Let $\mathfrak{s}_{T,\mathfrak{t}}$ be a regular subalgebra of the simple Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{s}_{T,\mathfrak{t}}$ is not wide. We must show that $\mathfrak{s}_{T,\mathfrak{t}}$ is narrow. That is, given an arbitrary simple $\mathfrak{g}$-module $V(\lambda)$ with $\lambda \neq 0$, we'll show that $V(\lambda)$ has a non-trivial $\mathfrak{s}_{T,\mathfrak{t}}$-decomposition. Since $\mathfrak{s}_{T,\mathfrak{t}}$ is not wide, $[T \cup -T] \subsetneq \Phi$ by Corollary 2.6. This implies that $\Delta \setminus [T \cup -T]$ is not empty.

Let $\lambda = m_1\lambda_1 + \cdots + m_l\lambda_l + \cdots + m_n\lambda_n \neq 0$ (l fixed and $1 \leq l \leq n$) be a dominant weight, with $m_l > 0$. Then, $f_{\alpha_l}v_\lambda \neq 0$. We proceed in cases to show that $V(\lambda)$ has a non-trivial decomposition with respect to $\mathfrak{s}_{T,\mathfrak{t}}$.

Case 1. $\alpha_l \notin [T \cup -T]$: Since $f_{\alpha_l}v_\lambda \neq 0$, then $\lambda - \alpha_l \in \Pi(V(\lambda))$. We claim that $\lambda - \alpha_l \notin \Pi([T \cup -T] \cdot \lambda)$. By way of contradiction, suppose that $\lambda - \alpha_l \in \Pi([T \cup -T] \cdot \lambda)$. Then $\lambda - \alpha_l = \lambda - \gamma_1 - \cdots - \gamma_p$ for some $-\gamma_1, \ldots, -\gamma_p \in [T \cup -T] \cap \Phi^-$. Therefore $\alpha_l = \gamma_1 + \cdots + \gamma_p \in \Phi$, which is thus an element of $[T \cup -T]$ by Lemma 2.1, a contradiction.

Hence, it must be the case that $\lambda - \alpha_l \notin \Pi([T \cup -T] \cdot \lambda)$. Since $\lambda - \alpha_l \notin \Pi([T \cup -T] \cdot \lambda)$ and $\lambda - \alpha_l \in \Pi(V(\lambda))$, Theorem 2.5 implies that $V(\lambda)$ has a non-trivial $\mathfrak{s}_{T,\mathfrak{t}}$-decomposition.

Case 2. $\alpha_l \in [T \cup -T]$: Then, there exists a simple root path $(\beta_1, \ldots, \beta_k)$ such that $\alpha_l = \beta_1$, and such that $\beta_i \in [T \cup -T]$ if $1 \leq i \leq k-1$, and $\beta_k \in \Delta \setminus [T \cup -T]$. Such a simple root path exists since $\Delta \setminus [T \cup -T]$ is not empty, and any two vertices in Dynkin diagram of an irreducible root system may be joined by such a path. Hence, by Lemma 3.1,

$$f_{\beta_1+\cdots+\beta_k}v_\lambda \neq 0, \tag{6}$$

since $f_{\beta_1}v_\lambda = f_{\alpha_l}v_\lambda \neq 0$. This implies that $\lambda - \beta_1 - \cdots - \beta_{k-1} - \beta_k \in \Pi(V(\lambda))$. We claim that $\lambda - \beta_1 - \cdots - \beta_{k-1} - \beta_k \notin \Pi([T \cup -T] \cdot \lambda)$. By way of contradiction, suppose that $\lambda - \beta_1 - \cdots - \beta_{k-1} - \beta_k \in \Pi([T \cup -T] \cdot \lambda)$.

Then, $\lambda - \beta_1 - \cdots - \beta_{k-1} - \beta_k = \lambda - \gamma_1 - \cdots - \gamma_p$ for some $-\gamma_1, \ldots, -\gamma_p \in [T \cup -T] \cap \Phi^-$. Hence $\beta_k = \gamma_1 + \cdots + \gamma_p - \beta_1 - \cdots - \beta_{k-1} \in$

Φ, which is thus an element of $[T \cup -T]$ by Lemma 2.1 (note that $\gamma_1, ..., \gamma_p, -\beta_1, .., -\beta_{k-1} \in [T \cup -T]$), a contradiction.

Hence, it must be the case that $\lambda - \beta_1 - \cdots - \beta_{k-1} - \beta_k \notin \Pi([T \cup -T] \cdot \lambda)$. Since $\lambda - \beta_1 - \cdots - \beta_{k-1} - \beta_k \notin \Pi([T \cup -T] \cdot \lambda)$ and $\lambda - \beta_1 - \cdots - \beta_{k-1} - \beta_k \in \Pi(V(\lambda))$, Theorem 2.5 implies that $V(\lambda)$ has a non-trivial $\mathfrak{s}_{T,\mathfrak{t}}$-decomposition.

With cases 1 and 2, we've established that if $\mathfrak{s}_{T,\mathfrak{t}}$ is not wide, then $\mathfrak{s}_{T,\mathfrak{t}}$ is narrow, as required. □

4. Non-simple, semisimple Lie algebras

In the final section, we show that no non-simple, semisimple Lie algebra is regular extreme.

Theorem 4.1. *No non-simple, semisimple Lie algebra is regular extreme.*

Proof. Let $\mathfrak{g} = \oplus_{i=1}^{k} \mathfrak{g}^i$ be a non-simple, semisimple Lie algebras, with $\mathfrak{g}^1$, $\mathfrak{g}^2$,..., and $\mathfrak{g}^k$ simple Lie algebras, and $k > 1$. To show that $\mathfrak{g}$ is not regular extreme, it suffices to identify a regular subalgebra of $\mathfrak{g}$ which is neither narrow nor wide. The regular subalgebra of $\mathfrak{g}$ that we'll show is neither narrow nor wide is the regular simple subalgebra $\mathfrak{g}^1 \subset \mathfrak{g}$.

Let $V^1(\lambda)$ be the simple $\mathfrak{g}^1$-module of highest weight $\lambda \neq 0$. Let $V^i(0)$ be the trivial $\mathfrak{g}^i$-module for $i = 1, ..., k$. Then

$$V := V^1(\lambda) \otimes V^2(0) \otimes \cdots \otimes V^k(0) \tag{7}$$

is a non-trivial simple $\mathfrak{g}$-module. Since $V|_{\mathfrak{g}^1} \cong V^1(\lambda)$, then the simple $\mathfrak{g}$-module V is $\mathfrak{g}^1$-indecomposable. Hence, the regular simple subalgebra $\mathfrak{g}^1$ is not narrow.

We now show that $\mathfrak{g}^1$ is also not wide. Towards this end, let $V^2(\eta)$ be the simple $\mathfrak{g}^2$-module with highest weight $\eta \neq 0$. Then

$$W := V^1(0) \otimes V^2(\eta) \otimes V^3(0) \otimes \cdots \otimes V^k(0) \tag{8}$$

is a non-trivial simple $\mathfrak{g}$-module. However, W has a non-trivial $\mathfrak{g}^1$-decomposition. In particular, let u^i be the single basis element of $V^i(0)$, and $u_1^2, ..., u_m^2$ be a basis of $V^2(\eta)$, so that $m > 1$. Then, the following is a non-trivial $\mathfrak{g}^1$-decomposition of W:

$$W|_{\mathfrak{g}^1} = \oplus_{i=1}^{m} \mathbb{C}\{u^1 \otimes u_i^2 \otimes u^3 \otimes \cdots \otimes u^k\}. \tag{9}$$

Hence, $\mathfrak{g}^1$ is also not wide, as required. □

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