

THE DENSITY OF BORROMEAN PRIMES

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Abstract

In this paper, we study an asymptotic distribution of sets of primes satisfying certain “linking conditions” in arithmetic topology, namely, conditions given by the Legendre and Rédei symbols among sets of primes. As our Main Theorem, we prove an asymptotic density formula for Borromean primes among all primes. For the proof, we use the effective Chebotarev density formula under the Generalized Riemann Hypothesis and explicit computations of discriminants of the number fields involved in Rédei’s extension.

Introduction

Borromean primes are triples of primes, which are defined as arithmetic analogues of Borromean rings, following the analogies between primes and knots in arithmetic topology ([Mor12]). The study of such triples of primes goes back to the work of Rédei in 1939. In [Réd39], Rédei attempted to generalize Gauss’s genus theory and introduced a certain triple symbol $[p_1, p_2, p_3]$ for primes p_1, p_2, p_3 . This symbol may be regarded as a triple generalization of the Legendre symbol $(\frac{p_1}{p_2})$. The Rédei symbol describes the decomposition law of p_3 in a certain dihedral extension over $\mathbb{Q}$ of degree 8, which is unramified outside p_1, p_2 (cf. [Ama14] for the uniqueness of such an extension). After a long silence, Morishita ([Mor00], [Mor02]) interpreted the Rédei symbol as an arithmetic analogue of Milnor’s triple linking number of a link ([Mil54]) in his study of arithmetic topology. Note that the Legendre symbol may be interpreted as an arithmetic analogue of Gauss’s linking number of two knots ([Mor12, Chapter 4]). Borromean primes are then defined as triples of primes p_1, p_2, p_3 satisfying

2020 Mathematics Subject Classification: 11R44, 11N45

Key words: Arithmetic Topology, Rédei extension, Borromean Primes, Refined Chebotarev Density Theorem

the following conditions:

$$p_i \equiv 1 \pmod{4} \ (i = 1, 2, 3), \quad \left(\frac{p_i}{p_j}\right) = 1 \ (1 \leq i \neq j \leq 3) \quad \text{and} \quad [p_1, p_2, p_3] = -1.$$

The purpose of this paper is to study the distribution of Borromean primes.

The study of asymptotic distribution of primes also goes back to Gauss, and it is viewed as an origin of the so called arithmetic statistics nowadays. Gauss predicted the following asymptotic formula

$$\pi(x) \sim \frac{x}{\log x},$$

where $f(x) \sim g(x)$ means that $f(x)/g(x) \rightarrow 1$ when $x \rightarrow +\infty$. This formula was proved independently by Hadamard and de la Vallée Poussin. In 19th century, the following formula was shown: for coprime integers $m(> 1)$ and a ,

$$\pi(x; a, m) \sim \frac{1}{\varphi(m)} \cdot \frac{x}{\log x}, \quad (0.1)$$

where $\varphi(m)$ is the Euler function. It was generalized to the following more general density theorem, known as the Chebotarev density theorem: Let M be a number field and M' be a finite Galois extension of M . For $\sigma \in \text{Gal}(M'/M)$ and any positive real number x , we define

$$\begin{aligned} & \pi_{M'/M}(x; \sigma) \\ &:= \# \left\{ \mathfrak{p} \in S_M^0 \mid \mathfrak{p} : \text{unramified in } M', N_M \mathfrak{p} < x, \left[\frac{M'/M}{\mathfrak{p}}\right] = C(\sigma) \right\}, \end{aligned}$$

where S_M^0 , $N_M \mathfrak{p}$, $\left[\frac{M'/M}{\mathfrak{p}}\right]$ and $C(\sigma)$ mean the set of prime ideals of M , the absolute norm of $\mathfrak{p} \in S_M^0$, the Artin symbol for $\mathfrak{p} \in S_M^0$ and the conjugacy class of σ in $\text{Gal}(M'/M)$, respectively. Then the Chebotarev density theorem asserts

$$\pi_{M'/M}(x; \sigma) \sim \frac{\#C(\sigma)}{\#G} \cdot \frac{x}{\log x}. \quad (0.2)$$

Note that Dirichlet's theorem (0.1) is interpreted as a special case of Chebotarev's theorem (0.2) in the m -th cyclotomic field $\mathbb{Q}(\zeta_m)$.

In this paper, we investigate the problem of asymptotic distribution of sets of primes satisfying certain “linking condition” in arithmetic topology, namely, conditions given by the Legendre symbols and the Rédei symbol among primes. This problem requires some refinements of the classical results mentioned above. Let $\pi_{\text{Borr}}(x)$ denote the number of Borromean primes $\{p_1, p_2, p_3\}$ with $p_i < x$ for $i = 1, 2, 3$. We prove the following asymptotic formula under the Generalized Riemann Hypothesis (abbreviated as GRH below).

Theorem (Theorem 3.1 below). *If GRH is true, we have*

$$\lim_{x \rightarrow +\infty} \frac{\pi_{\text{Borr}}(x)}{\#\{\{p_1, p_2, p_3\} \mid p_i < x \text{ (for } i = 1, 2, 3), p_i \neq p_j \text{ (} i \neq j)\}} = \frac{1}{128}.$$

We note that our theorem is not obtained by a straightforward application of the above classical density theorems and we need more elaborate analyses on the error term of the Chebotarev density theorem. For the proof, we employ the effective version of the Chebotarev density theorem under GRH ([GM19]) and, in order to apply the formula to our situation, we compute explicitly the discriminants of the fields involved in Rédei's extension.

This paper is organized as follows. In Section 1, we recall Rédei's extension, the Rédei triple symbols and Borromean primes. In addition, we evaluate the absolute discriminants of some extensions of Rédei's extension or its subextension. This value is necessary for the proof of our Main Theorem in Section 3. In Section 2, we calculate the density of pairs of distinct primes $\{p_1, p_2\}$ satisfying $p_1, p_2 \equiv 1 \pmod{4}$, $\left(\frac{p_1}{p_2}\right) = 1$ and $p_1, p_2 < x$ among all pairs of distinct primes $p_1, p_2 < x$ (Theorem 2.1). This value is also contributory to the proof of our theorem. In Section 3, we present our Main Theorem (Theorem 3.1) and prove it.

Notations

For integers x, a , and b , $x \equiv a \pmod{b}$ means that x is congruent to a modulo b . The symbol $\left(\frac{a}{p}\right)$ and $\left[\frac{L/K}{p}\right]$ stands for the Legendre symbol and the Artin symbol, respectively.

Let M be a number field and M' be a finite extension of M . We write as n_M the degree of the extension of M over $\mathbb{Q}$. We denote by Δ_M the absolute discriminant of M and by $\Delta_{M'/M}$ the relative discriminant of M' over M .

For any positive real number x , $\pi(x)$ stands for the number of primes less than x . For such x and any integers a, b , we denote by $\pi(x; a, b)$ the number of primes less than x and congruent to a modulo b .

Acknowledgements

We would like to express our sincere gratitude to our advisor, Professor Masanori Morishita, for his invaluable guidance and support throughout this study. We also extend our thanks to Professor Toshiki Matsusaka for providing an analytical method for computing density which is contained in Section 2. We also thank Deng Yuqi for useful discussions.

1 Rédei's Dihedral Extension, Rédei Symbol and Borromean Primes

In this section, we recall the Rédei symbol defined in [Réd39]. We construct an extension of $\mathbb{Q}$ known as Rédei's extension. This is a Galois extension of degree 8 over $\mathbb{Q}$, where the Galois group is isomorphic to the dihedral group D_4 with order 8 and only specific primes ramify. Then we define the Rédei Symbol and introduce the Borromean primes. In addition, we calculate the absolute discriminants of the fields obtained by adjoining $\sqrt{-1}$ to Rédei's extension or its subextension. These values are necessary later in the proof of our Main Theorem, in Section 3.

Let p_1, p_2 be distinct primes satisfying the conditions $p_1 \equiv p_2 \equiv 1 \pmod{4}$ and $\left(\frac{p_2}{p_1}\right) = 1$. In this case, the equation for X, Y, Z

$$X^2 - p_1 Y^2 - p_2 Z^2 = 0 \quad (1.1)$$

has non-trivial integer solutions (x, y, z) (i.e. $xyz \neq 0$) satisfying the conditions

$$y \equiv 0 \pmod{2}, \quad x - y \equiv 1 \pmod{4} \quad (1.2)$$

([Réd39]). We choose one of such solutions and write it as (x_0, y_0, z_0) . We define

$$\begin{aligned} \alpha_2 &= x_0 + y_0 \sqrt{p_1}, & \bar{\alpha}_2 &= x_0 - y_0 \sqrt{p_1}, \\ \alpha_1 &= \alpha_2 + \bar{\alpha}_2 + 2z_0 \sqrt{p_2}, & \bar{\alpha}_1 &= \alpha_2 + \bar{\alpha}_2 - 2z_0 \sqrt{p_2}, \\ k_1 &= \mathbb{Q}(\sqrt{p_1}), & k_2 &= \mathbb{Q}(\sqrt{p_2}) \end{aligned}$$

and $k = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha_2})$. We consider the following tower of field extensions:

$$\begin{array}{ccccccc} & & k & & & & \\ & \swarrow & | & \searrow & & \swarrow & \searrow \\ k_2(\sqrt{\alpha_1}) & & k_1 k_2 & & k_1(\sqrt{\alpha_2}) & & k_1(\sqrt{\alpha_2}) \\ & \swarrow & | & \searrow & & \swarrow & \searrow \\ & k_2 & \mathbb{Q}(\sqrt{p_1 p_2}) & k_1 & & & \\ & \swarrow & | & \searrow & & \swarrow & \searrow \\ & & \mathbb{Q} & & & & \end{array} \quad (1.3)$$

It can be seen that the extension $k/\mathbb{Q}$ in the tower (1.3) is a Galois extension whose Galois group is isomorphic to the dihedral group D_4 . The field k has the following uniqueness.

Proposition 1.1 ([Ama14, Theorem 2.1, p3]). *The field k can be defined independently from the choice of the solution (x, y, z) of the equation (1.1) satisfying conditions (1.2) and it is characterized by the following two properties:*

1. $k/\mathbb{Q}$ is a Galois extension with the Galois group being isomorphic to the dihedral group D_4 .
2. Primes ramified in $k/\mathbb{Q}$ are only p_1 and p_2 and both these ramification indices in $k/\mathbb{Q}$ are equal to 2.

For a prime p_3 different from p_1 and p_2 which satisfy the conditions

$$p_3 \equiv 1 \pmod{4} \quad \text{and} \quad \left(\frac{p_1}{p_3}\right) = \left(\frac{p_2}{p_3}\right) = 1,$$

we define the symbol $[p_1, p_2, p_3]$ as follows:

$$[p_1, p_2, p_3] = \begin{cases} 1 & (p_3 \text{ is completely decomposed in } k), \\ -1 & (p_3 \text{ is not completely decomposed in } k). \end{cases}$$

This definition is known to be well-defined regardless of the choice of α_2 ([Réd39]). The symbol $[p_1, p_2, p_3]$ is called *Rédei symbol*.

The Legendre symbol and the Rédei symbol can be regarded as arithmetic analogues of the linking number and Milnor's triple linking number, respectively, in view of the analogies between knots and primes ([Mor12, Chapter 4, Chapter 8]). Therefore following after the Borromean rings, *Borromean primes* are defined as triples of primes p_1, p_2, p_3 satisfying the following conditions:

$$p_i \equiv 1 \pmod{4} \quad (i = 1, 2, 3), \quad \left(\frac{p_i}{p_j}\right) = 1 \quad (1 \leq i \neq j \leq 3) \quad \text{and} \quad [p_1, p_2, p_3] = -1.$$

The Rédei symbol has the following reciprocity.

Proposition 1.2 ([Réd39, (37), p21], [Ama14, Theorem 3.2, p4]). *For any permutation $\sigma \in \mathfrak{S}_3$ (the symmetric group of degree 3), we have the following reciprocity property:*

$$[p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}] = [p_1, p_2, p_3].$$

Because of this proposition, it can be seen that the definition of Borromean primes is independent from the order of primes p_1, p_2, p_3 . In other words, the

definition is only due to the set of primes $\{p_1, p_2, p_3\}$ satisfying the above condition.

Furthermore, we consider the following sequence of field extensions.

$$\begin{array}{ccc}
 & k(\sqrt{-1}) & \\
 & | & \\
 k & & (k_1 k_2)(\sqrt{-1}) \\
 | & \swarrow & | \\
 k_1 k_2 & & \mathbb{Q}(\sqrt{-1}) \\
 | & \swarrow & \\
 \mathbb{Q} & &
 \end{array} \tag{1.4}$$

In this case, all the extensions that appear in (1.4) are finite Galois extensions. The following degrees of extensions are required for later computations in Section 3:

$$[k_1 k_2 : \mathbb{Q}] = 4, \quad [k : \mathbb{Q}] = 8, \quad [(k_1 k_2)(\sqrt{-1}) : \mathbb{Q}] = 8, \quad [k(\sqrt{-1}) : \mathbb{Q}] = 16.$$

The absolute discriminants of $(k_1 k_2)(\sqrt{-1})$ and $k(\sqrt{-1})$ will be used later to prove our Main Theorem. We state them here as the next proposition.

Proposition 1.3. *We have*

$$\Delta_{(k_1 k_2)(\sqrt{-1})} = 2^8 p_1^4 p_2^4 \quad \text{and} \quad \Delta_{k(\sqrt{-1})} = 2^{16} p_1^8 p_2^8.$$

Proof. Let K and L be number fields with $\text{GCD}(\Delta_K, \Delta_L) = 1$, $[K : \mathbb{Q}] = m$ and $[L : \mathbb{Q}] = n$. Then it follows that

$$\Delta_{KL} = \Delta_K^n \Delta_L^m \tag{1.5}$$

([Mol99, Theorem 5.13, p215]). Note that direct computation leads to

$$\Delta_{k_1} = p_1, \quad \Delta_{k_2} = p_2 \quad \text{and} \quad \Delta_{\mathbb{Q}(\sqrt{-1})} = -2^2.$$

Substituting k_1 and k_2 to K and L in the formula (1.5) respectively, we have $\Delta_{k_1 k_2} = p_1^2 p_2^2$. Hence we obtain

$$\Delta_{k_1 k_2(\sqrt{-1})} = (p_1^2 p_2^2)^2 \cdot (-2^2)^4 = 2^8 p_1^4 p_2^4$$

by substituting $k_1 k_2$ and $\mathbb{Q}(\sqrt{-1})$ to K and L in (1.5) respectively.

Next, we compute $\Delta_{k(\sqrt{-1})}$. In order to do so, we use the formula (1.5) again. First, we evaluate Δ_k by the following formula: When M is an intermediate field of L/K , it follows that

$$\Delta_{L/K} = N_{M/K}(\Delta_{L/M}) \cdot \Delta_{M/K}^{[L:M]} \quad (1.6)$$

([Mol99, Lemma 5.5, p198]) where $N_{M/K}(\Delta_{L/M})$ stands for the relative norm of $\Delta_{L/M}$ over K . Substituting $\mathbb{Q}, k_1$ and k to K, M and L respectively, we have

$$\Delta_{k/\mathbb{Q}} = N_{k_1/\mathbb{Q}}(\Delta_{k/k_1}) \cdot \Delta_{k_1/\mathbb{Q}}^4. \quad (1.7)$$

Since $\Delta_{k_1/\mathbb{Q}}$ is equal to the ideal (p_1) , computing Δ_{k/k_1} is sufficient to obtain $\Delta_{k/\mathbb{Q}}$. Here is a proposition with relative discriminants proved by Rédei.

Lemma 1.4 ([Réd39, p6]). *The following holds:*

- (1) $\Delta_{k/k_1} = \Delta_{k_1 k_2/k_1} N_{k_1/\mathbb{Q}}(\Delta_{k_1(\sqrt{\alpha_2})/k_1})$.
- (2) $N_{k_1/\mathbb{Q}}(\Delta_{k_1(\sqrt{\alpha_2})/k_1}) = (p_2)$.

Because of this Lemma, we only have to compute $\Delta_{k_1 k_2/k_1}$. Since it is easily shown that the only prime p_2 is ramified in $k_1 k_2/k_1$, $\Delta_{k_1 k_2/k_1}$ is equal to some power of ideal (p_2) in k_1 . Using the formula (1.6) with $K = \mathbb{Q}$, $M = k_1$ and $L = k_1 k_2$, we have $\Delta_{k_1 k_2/k_1} = (p_2)$. According to Lemma 1.4, we have

$$\Delta_{k/k_1} = (p_2) \cdot (p_2) = (p_2)^2.$$

Substituting this and $\Delta_{k_1/\mathbb{Q}} = (p_1)$ to (1.7), we obtain

$$\Delta_{k/\mathbb{Q}} = N_{k/\mathbb{Q}}((p_2)^2) \cdot (p_1)^4 = (p_1^4 p_2^4),$$

so the absolute discriminant Δ_k is equal to $p_1^4 p_2^4$. Since $p_1 \equiv p_2 \equiv 1 \pmod{4}$, Δ_k and $\Delta_{\mathbb{Q}(\sqrt{-1})}$ are relatively prime. Hence we have the following by the formula (1.5) with $K = k$ and $L = \mathbb{Q}(\sqrt{-1})$:

$$\Delta_{k(\sqrt{-1})} = (p_1^4 p_2^4)^2 \cdot (-2^2)^8 = 2^{16} p_1^8 p_2^8.$$

This completes the proof of Proposition 1.3. ■

2 The Density of Quadratic Residue Primes

In this section, we evaluate the density of pairs of primes each of which is congruent to 1 modulo 4 and quadratic residue modulo the other. This fact can be proved by using Theorem 3.2 in Section 3, but here we show it in another unconditional way. This way is due to T.Matsusaka.

In the following arguments, p_i means a prime for all i .

Theorem 2.1. *We have the following asymptotic formula:*

$$\lim_{x \rightarrow +\infty} \frac{\#\{(p_1, p_2) \mid p_1, p_2 < x, p_1 \equiv p_2 \equiv 1(4), (\frac{p_1}{p_2}) = 1\}}{\pi(x)^2} = \frac{1}{8}.$$

Proof (T. Matsusaka). We have

$$\begin{aligned} & \#\{(p_1, p_2) \mid p_1, p_2 < x, p_1 \equiv p_2 \equiv 1(4), (\frac{p_1}{p_2}) = 1\} \\ &= \frac{1}{2} \sum_{\substack{p_1, p_2 < x \\ p_1, p_2 \equiv 1(4)}} \left(1 + (\frac{p_1}{p_2})\right) - \frac{1}{2} \sum_{\substack{p < x \\ p \equiv 1(4)}} 1 \\ &= \frac{1}{2} \pi(x; 1, 4)^2 + \frac{1}{2} \sum_{\substack{p_1, p_2 < x \\ p_1, p_2 \equiv 1(4)}} (\frac{p_1}{p_2}) - \frac{1}{2} \pi(x; 1, 4). \end{aligned} \tag{2.1}$$

We evaluate the second term in the last RHS. We define

$$\begin{aligned} a_m &= \begin{cases} 1 & (\text{if } m : \text{prime, } m \equiv 1(4)), \\ 0 & (\text{otherwise}) \end{cases} \\ \text{and } b_m &= \begin{cases} \sum_{\substack{p_1 < x \\ p_1 \equiv 1(4)}} (\frac{p_1}{m}) & (\text{if } m : \text{square-free, odd}), \\ 0 & (\text{otherwise}), \end{cases} \end{aligned}$$

where $(\frac{n}{m})$ stands for the Jacobi symbol in the definition of b_m . Then we can write the second term in the last RHS of (2.1) as

$$E(x) := \sum_{\substack{p_2 < x \\ p_2 \equiv 1(4)}} \sum_{\substack{p_1 < x \\ p_1 \equiv 1(4)}} (\frac{p_1}{p_2}) = \sum_{m < x} a_m b_m.$$

By the Cauchy-Schwarz's inequality, we get

$$\begin{aligned}
E(x)^2 &= \left(\sum_{m < x} a_m b_m \right)^2 \leq \left(\sum_{m < x} a_m^2 \right) \left(\sum_{m < x} b_m^2 \right) \\
&= \pi(x; 1, 4) \sum_{m < x}^* \left(\sum_{\substack{p_1 < x \\ p_1 \equiv 1(4)}} \left(\frac{p_1}{m} \right) \right)^2 \quad (2.2) \\
&= \pi(x; 1, 4) \sum_{m < x}^* \left| \sum_{n < x}^* a_n \cdot \left(\frac{n}{m} \right) \right|^2.
\end{aligned}$$

where $\sum^*$ indicates summation over positive odd square-free integers only. We now employ the following theorem.

Theorem 2.2 ([HB95, Theorem 1], [IK04, Theorem 7.20]). *Let M, N be positive integers, and let $a_1, \dots, a_n$ be arbitrary complex numbers. Then*

$$\sum_{m \leq M}^* \left| \sum_{n \leq N}^* a_n \left(\frac{n}{m} \right) \right|^2 \leq C_\varepsilon (MN)^\varepsilon (M + N) \sum_{n \leq N}^* |a_n|^2$$

holds for any $\varepsilon > 0$, where C_ε is constant depends only on ε .

According to this theorem with $M = N = \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the largest integer not exceeding x , we have for any real number $\varepsilon > 0$

$$\begin{aligned}
E(x)^2 &\leq C_\varepsilon \pi(x; 1, 4) \lfloor x \rfloor^{1+2\varepsilon} \sum_{n \leq x}^* |a_n|^2 \\
&\leq C_\varepsilon \pi(x; 1, 4)^2 x^{1+2\varepsilon}.
\end{aligned}$$

Taking the square root of the both sides of this inequality, we obtain

$$E(x) \leq \sqrt{C_\varepsilon} \cdot \pi(x; 1, 4) x^{\frac{1}{2} + \varepsilon}.$$

Because of $\pi(x; 1, 4) \sim \frac{x}{2 \log x}$ and Prime Number Theorem, we see that

$$\lim_{x \rightarrow +\infty} \frac{E(x)}{\pi(x)^2} = 0.$$

when we take ε less than $\frac{1}{2}$. Therefore from the formula (2.1), the following holds:

$$\lim_{x \rightarrow +\infty} \frac{\#\{(p_1, p_2) \mid p_1, p_2 < x, p_1 \equiv p_2 \equiv 1(4), (\frac{p_2}{p_1}) = 1\}}{\pi(x)^2} = \frac{1}{8}.$$

This is the desired result for Theorem 2.1. ■

3 Main Theorem and Proof

In the following argument, p, p_i are primes for all i . We henceforth abbreviate the condition $p_i < x$ for $i = 1, 2, 3$ as $p_i < x$ and similarly for other conditions with p_i .

Let M be a number field and M' be a finite Galois extension of M . Recall that for $\sigma \in \text{Gal}(M'/M)$ and any positive real number x , we define

$$\begin{aligned} & \pi_{M'/M}(x; \sigma) \\ &:= \# \left\{ \mathfrak{p} \in S_M^0 \mid \mathfrak{p} : \text{unramified in } M', N_M \mathfrak{p} < x, [\frac{M'/M}{\mathfrak{p}}] = C(\sigma) \right\}, \end{aligned}$$

where S_M^0 , $N_M \mathfrak{p}$, $[\frac{M'/M}{\mathfrak{p}}]$ and $C(\sigma)$ mean the set of prime ideals of M , the absolute norm of $\mathfrak{p} \in S_M^0$, the Artin symbol for $\mathfrak{p} \in S_M^0$ and the conjugacy class of σ in $\text{Gal}(M'/M)$, respectively.

In this section, we prove our Main Theorem under GRH.

Theorem 3.1 (Main Theorem). *If GRH is true, we have*

$$\lim_{x \rightarrow +\infty} \frac{\pi_{\text{Borr}}(x)}{\#\{\{p_1, p_2, p_3\} \mid p_i < x, p_i \neq p_j\}} = \frac{1}{128}. \quad (3.1)$$

Recall that triples of primes $\{p_1, p_2, p_3\}$ are called Borromean primes when $p_i < x$, $p_i \equiv 1(4)$, $(\frac{p_i}{p_j}) = 1$ and $[p_1, p_2, p_3] = -1$ and $\pi_{\text{Borr}}(x)$ stands for the number of sets of Borromean primes $\{p_1, p_2, p_3\}$ with $p_i < x$.

We will prove Main Theorem by the following 3 steps:

1. The LHS of (3.1) is transformed to (3.3) below in order to use the refined Chebotarev's density theorem (Theorem 3.2 below).
2. Some limits are derived by the refined Chebotarev (Proposition 3.3 below).

3. Main Theorem is proved by using the proposition derived in the second step.

Proof of Theorem 3.1.

(step 1): Now, we calculate the LHS of (3.1). The conditions for p_1, p_2, p_3

$$p_i < x, \quad p_i \neq p_j$$

are symmetric with p_1, p_2, p_3 , that is, the same holds, even if indices are permuted. Then we have

$$\#\{\{p_1, p_2, p_3\} \mid p_i < x, p_i \neq p_j\} = \frac{1}{3!} \#\{(p_1, p_2, p_3) \mid p_i < x, p_i \neq p_j\}$$

where $(*, *, *)$ denotes a vector with three components. Since the condition $[p_1, p_2, p_3] = -1$ is symmetric by Proposition 1.2, the conditions

$$p_i < x, p_i \equiv 1 \pmod{4}, \left(\frac{p_j}{p_i}\right) = 1, [p_1, p_2, p_3] = -1$$

are also symmetric. Then

$$\begin{aligned} \pi_{\text{Borr}}(x) &= \#\{\{p_1, p_2, p_3\} \mid p_i < x, p_i \equiv 1 \pmod{4}, \left(\frac{p_j}{p_i}\right) = 1, [p_1, p_2, p_3] = -1\} \\ &= \frac{1}{3!} \#\{(p_1, p_2, p_3) \mid p_i < x, p_i \equiv 1 \pmod{4}, \left(\frac{p_j}{p_i}\right) = 1, [p_1, p_2, p_3] = -1\} \end{aligned}$$

holds. Hence we derive

$$\begin{aligned} &\frac{\pi_{\text{Borr}}(x)}{\#\{\{p_1, p_2, p_3\} \mid p_i < x, p_i \neq p_j\}} \\ &= \frac{\#\{(p_1, p_2, p_3) \mid p_i < x, p_i \equiv 1 \pmod{4}, \left(\frac{p_j}{p_i}\right) = 1, [p_1, p_2, p_3] = -1\}}{\#\{(p_1, p_2, p_3) \mid p_i < x, p_i \neq p_j\}}. \end{aligned} \tag{3.2}$$

Since direct computation yields

$$\begin{aligned} \pi(x)^3 &= \#\{(p_1, p_2, p_3) \mid p_i < x\} \\ &= \#\{(p_1, p_2, p_3) \mid p_i < x, p_i \neq p_j\} + 3\pi(x)^2 - 2\pi(x), \end{aligned}$$

we have

$$\lim_{x \rightarrow +\infty} \frac{\#\{(p_1, p_2, p_3) \mid p_i < x, p_i \neq p_j\}}{\pi(x)^3} = \lim_{x \rightarrow +\infty} \left(1 - \frac{3}{\pi(x)} - \frac{2}{\pi(x)^2}\right) = 1.$$

By this identity, the following holds:

$$\begin{aligned}
& \lim_{x \rightarrow +\infty} \frac{\#\{(p_1, p_2, p_3) \mid p_i < x, p_i \equiv 1 \pmod{4}, (\frac{p_i}{p_i}) = 1, [p_1, p_2, p_3] = -1\}}{\#\{(p_1, p_2, p_3) \mid p_i < x, p_i \neq p_j\}} \\
&= \lim_{x \rightarrow +\infty} \frac{\#\{(p_1, p_2, p_3) \mid p_i < x, p_i \equiv 1 \pmod{4}, (\frac{p_i}{p_i}) = 1, [p_1, p_2, p_3] = -1\}}{\pi(x)^3} \\
&\quad \times \lim_{x \rightarrow +\infty} \frac{\pi(x)^3}{\#\{(p_1, p_2, p_3) \mid p_i < x, p_i \neq p_j\}} \\
&= \lim_{x \rightarrow +\infty} \frac{\#\{(p_1, p_2, p_3) \mid p_i < x, p_i \equiv 1 \pmod{4}, (\frac{p_i}{p_i}) = 1, [p_1, p_2, p_3] = -1\}}{\pi(x)^3}.
\end{aligned}$$

According to this and (3.2), it is sufficient to prove Main Theorem that we evaluate the value

$$\lim_{x \rightarrow +\infty} \frac{\#\{(p_1, p_2, p_3) \mid p_i < x, p_i \equiv 1 \pmod{4}, (\frac{p_i}{p_i}) = 1, [p_1, p_2, p_3] = -1\}}{\pi(x)^3}.$$

There is an identity

$$\begin{aligned}
& \#\{(p_1, p_2, p_3) \mid p_i < x, p_i \equiv 1 \pmod{4}, (\frac{p_i}{p_i}) = 1, [p_1, p_2, p_3] = -1\} \\
&= \sum_{\substack{p_1, p_2 < x \\ p_1 \equiv p_2 \equiv 1 \pmod{4}, (\frac{p_1}{p_2}) = 1}} \#\{p \mid p < x, p \equiv 1 \pmod{4}, (\frac{p_1}{p}) = (\frac{p_2}{p}) = 1, [p_1, p_2, p] = -1\},
\end{aligned}$$

which is easily derived by the quadratic reciprocity law. Because of this identity, we have

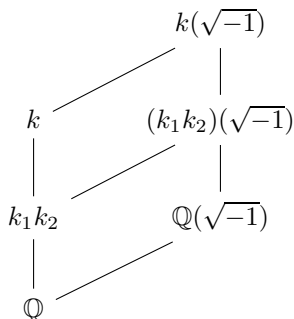
$$\begin{aligned}
& \frac{\#\{(p_1, p_2, p_3) \mid p_i < x, p_i \equiv 1 \pmod{4}, (\frac{p_i}{p_i}) = 1, [p_1, p_2, p_3] = -1\}}{\pi(x)^3} \\
&= \frac{1}{\pi(x)^2} \sum_{\substack{p_1, p_2 < x \\ p_1 \equiv p_2 \equiv 1 \pmod{4}, (\frac{p_1}{p_2}) = 1}} \frac{\#\{p \mid p < x, p \equiv 1 \pmod{4}, (\frac{p_1}{p}) = (\frac{p_2}{p}) = 1, [p_1, p_2, p] = -1\}}{\pi(x)}.
\end{aligned} \tag{3.3}$$

(Step 2): we evaluate the following limit by applying Theorem 3.2 below:

$$\lim_{x \rightarrow +\infty} \frac{\#\{p \mid p < x, p \equiv 1 \pmod{4}, (\frac{p_1}{p}) = (\frac{p_2}{p}) = 1, [p_1, p_2, p] = -1\}}{\pi(x)}, \tag{3.4}$$

where p_1, p_2 are distinct primes, $p_1 \equiv p_2 \equiv 1 \pmod{4}$ and $(\frac{p_2}{p_1}) = 1$. In order to do so, we replace the condition of the numerator of (3.4) with conditions written by the Artin symbol.

Here is the tower of extensions of fields (1.4)



in Section 1. Note that k is unramified over $\mathbb{Q}$ outside primes p_1 and p_2 . It is easy to see that

$$\begin{aligned}
 p \equiv 1 \pmod{4} & \iff p \text{ is completely decomposed in } \mathbb{Q}(\sqrt{-1}), \\
 (\frac{p_1}{p}) = (\frac{p_2}{p}) = 1 & \iff p \text{ is completely decomposed in } k_1 k_2.
 \end{aligned}$$

According to these facts and definition of Borromean primes, it follows that

$$\begin{aligned}
 [p_1, p_2, p] &= -1 \\
 \iff p &\text{ is completely decomposed in } (k_1 k_2)(\sqrt{-1}) \text{ and not in } k(\sqrt{-1}).
 \end{aligned}$$

For any positive real number x , we define sets of primes $S(x)$, $S'(x)$, $S''(x)$ by

$$S(x) = \{p \mid p < x, p \text{ is completely decomposed in } k(\sqrt{-1})\},$$

$S'(x) = \{p \mid p < x, p \text{ is not completely decomposed and unramified in } k(\sqrt{-1})\}$
and

$$S''(x) = \{p \mid p < x, p \text{ is completely decomposed in } (k_1 k_2)(\sqrt{-1})\}.$$

Then we have

$$\begin{aligned}
 \#\{p \mid p < x, p \equiv 1 \pmod{4}, (\frac{p_1}{p}) = (\frac{p_2}{p}) = 1, [p_1, p_2, p] = -1\} \\
 = \#(S'(x) \cap S''(x)),
 \end{aligned} \tag{3.5}$$

$S(x) \subset S''(x)$ and if $x > p_1, p_2$, it follows that

$$\{p \mid p < x\} = S(x) \sqcup S'(x) \sqcup \{p_1, p_2\}$$

where the operator $\sqcup$ means a disjoint union of sets. Considering an intersection with $S''(x)$ in both sides of the last identity, we see that

$$S''(x) = S(x) \sqcup (S'(x) \cap S''(x)).$$

Since $S(x)$ and $S'(x) \cap S''(x)$ are disjoint, on the number of elements we obtain

$$\#(S'(x) \cap S''(x)) = \#S''(x) - \#S(x).$$

Substituting this to (3.5), the following identity holds:

$$\#\{p \mid p < x, p \equiv 1 \pmod{4}, \left(\frac{p_1}{p}\right) = \left(\frac{p_2}{p}\right) = 1, [p_1, p_2, p] = -1\} = \#S''(x) - \#S(x).$$

By the definition of the Artin symbol, it follows for an unramified prime p and a number field M that

$$p \text{ is completely decomposed in } M \iff \left[\frac{M/\mathbb{Q}}{p} \right] = C(\text{id}) = \{\text{id}\}.$$

This equivalence leads to the following identities:

$$\#S(x) = \# \left\{ p \mid p < x, \left[\frac{k(\sqrt{-1})/\mathbb{Q}}{p} \right] = \{\text{id}\} \right\} = \pi_{k(\sqrt{-1})/\mathbb{Q}}(x; \text{id})$$

and

$$\#S''(x) = \# \left\{ p \mid p < x, \left[\frac{(k_1 k_2)(\sqrt{-1})/\mathbb{Q}}{p} \right] = \{\text{id}\} \right\} = \pi_{(k_1 k_2)(\sqrt{-1})/\mathbb{Q}}(x; \text{id}).$$

Hence we obtain

$$\begin{aligned} & \frac{\#\{p \mid p < x, p \equiv 1 \pmod{4}, \left(\frac{p_1}{p}\right) = \left(\frac{p_2}{p}\right) = 1, [p_1, p_2, p] = -1\}}{\pi(x)} \\ &= \frac{\pi_{(k_1 k_2)(\sqrt{-1})/\mathbb{Q}}(x; \text{id})}{\pi(x)} - \frac{\pi_{k(\sqrt{-1})/\mathbb{Q}}(x; \text{id})}{\pi(x)}. \end{aligned} \tag{3.6}$$

Now, we evaluate the limit of RHS of (3.6) and the speed of its convergence when $x \rightarrow +\infty$ using the following theorem which evaluate the error term of Chebotarev's density theorem for the natural density more precisely under GRH.

Theorem 3.2 ([GM19, Corollary 1.2.]). *Suppose that GRH is true. Let M be a number field, M' be a finite Galois extension of M and G be the Galois group $\text{Gal}(M'/M)$. For any real number $x \geq 2$,*

$$\begin{aligned} & \left| \pi_{M'/M}(x; \sigma) - \frac{\#C(\sigma)}{\#G} \int_2^x \frac{du}{\log u} \right| \\ & \leq \frac{\#C(\sigma)}{\#G} \cdot \sqrt{x} \left[\left(\frac{1}{2\pi} + \frac{3}{\log x} \right) \log \Delta_{M'} + \left(\frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} \right) n_{M'} \right] \end{aligned}$$

holds.

First, we focus on $\frac{\pi_{(k_1 k_2)(\sqrt{-1})/\mathbb{Q}}(x; \text{id})}{\pi(x)}$, the first term of the RHS of (3.6). Recall

$$[(k_1 k_2)(\sqrt{-1}) : \mathbb{Q}] = 8 \quad \text{and} \quad \Delta_{(k_1 k_2)(\sqrt{-1})} = 2^8 p_1^4 p_2^4$$

(see Proposition 1.3 in Section 1). Applying Theorem 3.2 with $M = \mathbb{Q}$, $M' = (k_1 k_2)(\sqrt{-1})$ and $\sigma = \text{id}$, we have for any primes $p_1, p_2 < x$

$$\begin{aligned} & \left| \pi_{(k_1 k_2)(\sqrt{-1})/\mathbb{Q}}(x; \text{id}) - \frac{1}{8} \int_2^x \frac{du}{\log u} \right| \\ & \leq \frac{1}{8} \sqrt{x} \left[\left(\frac{1}{2\pi} + \frac{3}{\log x} \right) \log(2^8 p_1^4 p_2^4) + \left(\frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} \right) \cdot 8 \right] \\ & < \sqrt{x} \left[\left(\frac{1}{2\pi} + \frac{3}{\log x} \right) (\log 2 + \log x) + \left(\frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} \right) \right]. \end{aligned}$$

Dividing the both sides by $\pi(x)$, we see that

$$\begin{aligned} & \frac{1}{\pi(x)} \cdot \left| \pi_{(k_1 k_2)(\sqrt{-1})/\mathbb{Q}}(x; \text{id}) - \frac{1}{8} \int_2^x \frac{du}{\log u} \right| \\ & < \frac{\sqrt{x}}{\pi(x)} \left[\left(\frac{1}{2\pi} + \frac{3}{\log x} \right) (\log 2 + \log x) + \left(\frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} \right) \right] \\ & = \frac{\frac{x}{\log x}}{\pi(x)} \cdot \frac{(\log x)^2}{\sqrt{x}} \left[\left(\frac{1}{2\pi} + \frac{3}{\log x} \right) \left(\frac{\log 2}{\log x} + 1 \right) + \left(\frac{1}{8\pi} + \frac{1}{4\pi \log x} + \frac{6}{(\log x)^2} \right) \right]. \end{aligned}$$

Because of Prime Number Theorem and $\lim_{x \rightarrow +\infty} \frac{(\log x)^2}{\sqrt{x}} = 0$, there is some positive real number α for any real $\varepsilon > 0$ such that for any $x > \alpha$ and any primes $p_1, p_2 < x$

$$\left| \frac{\pi_{(k_1 k_2)(\sqrt{-1})/\mathbb{Q}}(x; \text{id})}{\pi(x)} - \frac{1}{8 \cdot \pi(x)} \int_2^x \frac{du}{\log u} \right| < \frac{1}{4} \varepsilon.$$

Therefore some positive real number α exists for any real $\varepsilon > 0$ such that for any $x > \alpha$ and any primes $p_1, p_2 < x$,

$$\begin{aligned} & \left| \frac{\pi_{(k_1 k_2)(\sqrt{-1})/\mathbb{Q}}(x; \text{id})}{\pi(x)} - \frac{1}{8} \right| \\ & \leq \left| \frac{\pi_{(k_1 k_2)(\sqrt{-1})/\mathbb{Q}}(x; \text{id})}{\pi(x)} - \frac{1}{8 \cdot \pi(x)} \int_2^x \frac{du}{\log u} \right| + \frac{1}{8} \left| \frac{1}{\pi(x)} \int_2^x \frac{du}{\log u} - 1 \right| \\ & < \frac{1}{4}\varepsilon + \frac{1}{8} \left| \frac{1}{\pi(x)} \int_2^x \frac{du}{\log u} - 1 \right|, \end{aligned}$$

where the first inequality is led by the triangle inequality. Because of Prime Number Theorem, there is some positive real number α' such that for any $x > \alpha'$,

$$\left| \frac{1}{\pi(x)} \int_2^x \frac{du}{\log u} - 1 \right| < 2\varepsilon.$$

Hence we obtain for any $x > \max\{\alpha, \alpha'\}$

$$\left| \frac{\pi_{(k_1 k_2)(\sqrt{-1})/\mathbb{Q}}(x; \text{id})}{\pi(x)} - \frac{1}{8} \right| < \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon = \frac{1}{2}\varepsilon.$$

Furthermore, we also showed

$$[k(\sqrt{-1}) : \mathbb{Q}] = 16 \quad \text{and} \quad \Delta_{k(\sqrt{-1})/\mathbb{Q}} = 2^{16} p_1^8 p_2^8$$

(see Proposition 1.3 in Section 1). In the manner quite similar to the above argument, we can have the same result for the second term of RHS of (3.6). In other words, we can see that there is some real $\alpha > 0$ for any real $\varepsilon > 0$ such that for any $x > \alpha$ and any primes $p_1, p_2 < x$ satisfying $p_1 \equiv p_2 \equiv 1 \pmod{4}$ and $\left(\frac{p_2}{p_1}\right) = 1$

$$\left| \frac{\pi_{k(\sqrt{-1})/\mathbb{Q}}(x; \text{id})}{\pi(x)} - \frac{1}{16} \right| < \frac{1}{2}\varepsilon.$$

Consequently, we can choose some real $\alpha > 0$ for any real $\varepsilon > 0$ such that for any $x > \alpha$ and any primes $p_1, p_2 < x$ satisfying $p_1 \equiv p_2 \equiv 1 \pmod{4}$ and $\left(\frac{p_2}{p_1}\right) = 1$,

$$\left| \frac{\pi_{(k_1 k_2)(\sqrt{-1})/\mathbb{Q}}(x; \text{id})}{\pi(x)} - \frac{1}{8} \right| < \frac{1}{2}\varepsilon \quad \text{and} \quad \left| \frac{\pi_{k(\sqrt{-1})/\mathbb{Q}}(x; \text{id})}{\pi(x)} - \frac{1}{16} \right| < \frac{1}{2}\varepsilon.$$

Then we have the following because of (3.6) and the triangle inequality:

$$\begin{aligned} & \left| \frac{\#\{p \mid p < x, p \equiv 1 \pmod{4}, (\frac{p_1}{p}) = (\frac{p_2}{p}) = 1, [p_1, p_2, p] = -1\}}{\pi(x)} - \frac{1}{16} \right| \\ & \leq \left| \frac{\pi_{(k_1 k_2)(\sqrt{-1})/\mathbb{Q}}(x; \text{id})}{\pi(x)} - \frac{1}{8} \right| + \left| \frac{\pi_{k(\sqrt{-1})/\mathbb{Q}}(x; \text{id})}{\pi(x)} - \frac{1}{16} \right| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

We show this fact as the next proposition.

Proposition 3.3. *There exists some real $\alpha > 0$ for any real $\varepsilon > 0$ such that for any $x > \alpha$ and any primes $p_1, p_2 < x$ satisfying $p_1 \equiv p_2 \equiv 1 \pmod{4}$ and $(\frac{p_2}{p_1}) = 1$,*

$$\left| \frac{\#\{p \mid p < x, p \equiv 1 \pmod{4}, (\frac{p_1}{p}) = (\frac{p_2}{p}) = 1, [p_1, p_2, p] = -1\}}{\pi(x)} - \frac{1}{16} \right| < \varepsilon.$$

(Step 3): Now, we prove Main Theorem by Proposition 3.3. Here is the identity (3.3) again:

$$\begin{aligned} & \frac{\#\{(p_1, p_2, p_3) \mid p_i < x, p_i \equiv 1 \pmod{4}, (\frac{p_j}{p_i}) = 1, [p_1, p_2, p_3] = -1\}}{\pi(x)^3} \\ & = \frac{1}{\pi(x)^2} \sum_{\substack{p_1, p_2 < x \\ p_1 \equiv p_2 \equiv 1 \pmod{4}, (\frac{p_1}{p_2}) = 1}} \frac{\#\{p \mid p < x, p \equiv 1 \pmod{4}, (\frac{p_1}{p}) = (\frac{p_2}{p}) = 1, [p_1, p_2, p] = -1\}}{\pi(x)}. \end{aligned}$$

Let $\rho_{p_1, p_2}(x) = \#\{p \mid p < x, p \equiv 1 \pmod{4}, (\frac{p_1}{p}) = (\frac{p_2}{p}) = 1, [p_1, p_2, p] = -1\}$ and $(\dagger)$ denote the conditions

$$p_1, p_2 < x, p_1 \equiv p_2 \equiv 1 \pmod{4}, (\frac{p_1}{p_2}) = 1.$$

Take any real $\varepsilon > 0$. According to Proposition 3.3, we can choose some real $\alpha > 0$ such that for any $x > \alpha$ and any primes p_1, p_2 satisfying $(\dagger)$,

$$\left| \frac{\rho_{p_1, p_2}(x)}{\pi(x)} - \frac{1}{16} \right| < \varepsilon. \quad (3.7)$$

According to the inequality (3.7), we obtain for any real number $x > \alpha$

$$\begin{aligned}
& \left| \frac{1}{\pi(x)^2} \sum_{(p_1, p_2) \in \{(p_1, p_2) \mid (\dagger)\}} \frac{\rho_{p_1, p_2}(x)}{\pi(x)} - \frac{1}{16} \cdot \frac{\#\{(p_1, p_2) \mid (\dagger)\}}{\pi(x)^2} \right| \\
&= \left| \frac{1}{\pi(x)^2} \sum_{(p_1, p_2) \in \{(p_1, p_2) \mid (\dagger)\}} \left(\frac{\rho_{p_1, p_2}(x)}{\pi(x)} - \frac{1}{16} \right) \right| \\
&\leq \frac{1}{\pi(x)^2} \sum_{(p_1, p_2) \in \{(p_1, p_2) \mid (\dagger)\}} \left| \frac{\rho_{p_1, p_2}(x)}{\pi(x)} - \frac{1}{16} \right| < \frac{\#\{(p_1, p_2) \mid (\dagger)\}}{\pi(x)^2} \cdot \varepsilon
\end{aligned}$$

where the second inequality is derived from the triangle inequality. Since $\frac{\#\{(p_1, p_2) \mid (\dagger)\}}{\pi(x)^2}$ converge on $\frac{1}{8} < 1$ when $x \rightarrow +\infty$ because of Theorem 2.1, there exists some real $\alpha' > 0$ such that for any $x > \alpha'$ we have $\frac{\#\{(p_1, p_2) \mid (\dagger)\}}{\pi(x)^2} < 1$. Therefore when $x > \max\{\alpha, \alpha'\}$,

$$\left| \frac{1}{\pi(x)^2} \sum_{(p_1, p_2) \in \{(p_1, p_2) \mid (\dagger)\}} \frac{\rho_{p_1, p_2}(x)}{\pi(x)} - \frac{1}{16} \cdot \frac{\#\{(p_1, p_2) \mid (\dagger)\}}{\pi(x)^2} \right| < \varepsilon$$

holds. By arbitrariness of $\varepsilon > 0$, we see that

$$\lim_{x \rightarrow +\infty} \frac{1}{\pi(x)^2} \sum_{(p_1, p_2) \in \{(p_1, p_2) \mid (\dagger)\}} \frac{\rho_{p_1, p_2}(x)}{\pi(x)} = \lim_{x \rightarrow +\infty} \left(\frac{1}{16} \cdot \frac{\#\{(p_1, p_2) \mid (\dagger)\}}{\pi(x)^2} \right).$$

According to Theorem 2.1, we obtain

$$\lim_{x \rightarrow +\infty} \frac{1}{\pi(x)^2} \sum_{(p_1, p_2) \in \{(p_1, p_2) \mid (\dagger)\}} \frac{\rho_{p_1, p_2}(x)}{\pi(x)} = \frac{1}{16} \cdot \frac{1}{8} = \frac{1}{128}.$$

This completes the proof of Theorem 3.1, which is Main Theorem. ■

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