# RATIONAL DIOPHANTINE SEXTUPLES WITH STRONG PAIR

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ABSTRACT. A set of $m$ distinct nonzero rationals $\{a_1, a_2, \dots, a_m\}$ such that $a_i a_j + 1$ is a perfect square for all $1 \le i < j \le m$, is called a rational Diophantine $m$-tuple. If in addition, $a_i^2 + 1$ is a perfect square for $1 \le i \le m$, then we say the $m$-tuple is strong. In this paper, we construct infinite families of rational Diophantine sextuples containing a strong Diophantine pair.

## 1. Introduction

A Diophantine $m$-tuple is a set of $m$ distinct positive integers with the property that the product of any two of its distinct elements plus 1 is a square. Fermat found the first Diophantine quadruple in integers $\{1, 3, 8, 120\}$. If a set of $m$ nonzero rationals has the same property, then it is called a rational Diophantine $m$-tuple. If in addition, a rational Diophantine $m$-tuple has the property that the square of each element plus 1 is a square, we say that it is **strong**. The first example of a rational Diophantine quadruple was the set

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$$

found by Diophantus. Euler proved that the exist infinitely many rational Diophantine quintuples (see [15]), in particular he was able to extend the integer Diophantine quadruple found by Fermat, to the rational quintuple

$$\left\{1, 3, 8, 120, \frac{777480}{8288641}\right\}.$$

Stoll [17] recently showed that this extension is unique. Therefore, the Fermat set $\{1, 3, 8, 120\}$ cannot be extended to a rational Diophantine sextuple.

In 1969, using linear forms in logarithms of algebraic numbers and a reduction method based on continued fractions, Baker and Davenport [1] proved that if $d$ is a positive integer such that $\{1, 3, 8, d\}$ forms a Diophantine quadruple, then $d$ has to be 120. This result motivated the conjecture that there does not exist a Diophantine quintuples in integers. The conjecture has been proved recently by He, Togbé and Ziegler [14] (see also [5]).

In the other hand, it is not known how large can be a rational Diophantine tuple. In 1999, Gibbs found the first example of rational Diophantine sextuple [13]

$$\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}.$$

In 2017 Dujella, Kazalicki, Mikić and Szikszai [9] proved that there are infinitely many rational Diophantine triples that can be extended to a Diophantine sextuple in infinitely many ways, while Dujella and Kazalicki [8] (inspired by the work of Piezas [16]) described another construction of parametric families of rational Diophantine sextuples. Dujella, Kazalicki and Petričević [11] proved that there are infinitely many rational Diophantine sextuples such that denominators of all the

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elements (in the lowest terms) in the sextuples are perfect squares, and also proved [10] that there are infinitely many rational Diophantine sextuples containing two regular quadruples and one regular quintuple. No example of a rational Diophantine septuple is known. Lang's conjecture on varieties of general type implies that the number of elements in a rational Diophantine tuple is bounded by an absolute constant (for more details, see the introduction of [9]). For additional information on Diophantine $m$-tuples, refer to the survey article [6] and the book [7].

In this paper, we study rational Diophantine sextuples which contain a strong elements (i.e. the elements $a$ with the property that $a^2+1$ is a perfect square).

Denote by $C$ an affine curve given by the equation $p(u,v)=0$ where

$$\begin{aligned} p(u,v) = {} & 3u^4v^4 - 8u^4v^3 + 6u^4v^2 - u^4 \\ & - 8u^3v^4 + 4u^3v^3 - 8u^3v^2 + 12u^3v + 6u^2v^4 \\ & - 8u^2v^3 + 4u^2v^2 + 8u^2v + 6u^2 + 12uv^3 + 8uv^2 \\ & + 4uv + 8u - v^4 + 6v^2 + 8v + 3. \end{aligned}$$

The curve $C$ is birationally equivalent to the elliptic curve

$$E: \quad y^2 + xy + y = x^3 - 33x + 68.$$

Torsion subgroup of Mordell-Weil group of $E/\mathbb{Q}$ is generated by the point $[-1, 10]$ of order 6, while the free part of the group is generated by the point $[11/4, -25/8]$. In particular, $E$ has infinitely many rational points.

Define three parametric families

$$\begin{aligned} \mathcal{F}_1(u,v) &= \left[\tfrac{2u}{(u-1)(u+1)}, \quad \tfrac{2v}{(v-1)(v+1)}, \quad \tfrac{2(v-1)(v+1)(u-1)(u+1)}{(-v+uv-u-1)^2}\right], \\ \mathcal{F}_2(u,v) &= \left[\tfrac{2u}{(u-1)(u+1)}, \quad -\tfrac{2(u-v)(uv+1)}{(uv+v+1-u)(uv-v+u+1)}, \quad -\tfrac{2(uv-v+u+1)(u^3v-u^3-v-1)}{(u-1)(u+1)(uv+v+1-u)^2}\right], \\ \mathcal{F}_3(u,v) &= \left[-\tfrac{2v}{(v-1)(v+1)}, \quad -\tfrac{2(u-v)(uv+1)}{(uv+v+1-u)(uv-v+u+1)}, \quad \tfrac{2(uv+v+1-u)(v^3u-v^3-u-1)}{(uv-v+u+1)^2(v+1)(v-1)}\right]. \end{aligned}$$

By carefully selecting parameters $(u,v)$, we can utilize methods described in [9] to extend Diophantine triples to Diophantine sextuples, thus deriving our main result.

**Theorem 1.** *If $(u,v) \in C(\mathbb{Q})$, then each triple $\mathcal{F}_i(u,v)$ is a rational Diophantine triple (provided that all the elements are defined, distinct and nonzero), whose first two elements form a strong Diophantine pair. Moreover, each such $\mathcal{F}_i(u,v)$ can be extended to a rational Diophantine sextuple in infinitely many ways.*

**Remark 1.** Note that $\mathcal{F}_2(v,u) = -\mathcal{F}_3(u,v) = \mathcal{F}_3(-u,1/v)$ for all pairs $(u,v)$. Therefore, since the mappings $(u,v) \mapsto (v,u)$ and $(u,v) \mapsto (-u,1/v)$ are the automorphisms of the curve $\mathcal{C}$, the families $\mathcal{F}_2$ and $\mathcal{F}_3$ are parameterizing the same sets of triples.

As a corollary, we obtain the following result.

**Theorem 2.** *There are infinitely many rational Diophantine sextuples that contain a strong Diophantine pair.*

## 2. Induced elliptic curves and overview of [9]

To extend a rational Diophantine triple $\{a,b,c\}$ to a quadruple, we need to find $d \in \mathbb{Q}$ for which $ad+1, bd+1$ and $cd+1$ are perfect squares. Such $d$ naturally defines a rational point on the elliptic curve $y^2 = (ax+1)(bx+1)(cx+1)$ which is isomorphic (via transformation $x \mapsto x/abc, y \mapsto y/abc$) to the curve

$$E_{a,b,c}: \quad y^2 = (x+ab)(x+ac)(x+bc).$$

Conversely, the two descent argument implies that each $d$ is equal to $x(T+P)/abc$ for some $T \in 2E_{a,b,c}(\mathbb{Q})$ and $P = [0, abc] \in E_{a,b,c}(\mathbb{Q})$ (see Proposition 1 in [4]).

Besides the rational points of order 2,

$$T_1 = [-ab, 0], \quad T_2 = [-ac, 0], \quad T_3 = [-bc, 0],$$

we will also need rational point $S = [1, rst] \in E_{a,b,c}(\mathbb{Q})$, where $ab+1 = r^2$, $ac+1 = s^2$ and $bc+1 = t^2$, for some $r, s, t \in \mathbb{Q}$. Note that $S = 2R$, where $R = [rs+rt+st, (r+s)(r+t)(s+t)]$. In the case when $\{a, b\}$ is a strong pair, we have two more rational points

$$A = [a \cdot abc, abc \cdot rsu], \quad B = [b \cdot abc, abc \cdot rtv] \in E_{a,b,c}(\mathbb{Q}),$$

where $a^2+1 = u^2$ and $b^2+1 = v^2$ for some $u, v \in \mathbb{Q}$.

The main result of [9] states that if $\{a, b, c\}$ is a rational Diophantine triple such that the point $S$ on induced elliptic curve $E_{a,b,c}$ has order 3, then for each integer $n$

$$\left\{a, b, c, \frac{x([2n+1]P)}{abc}, \frac{x([2n+1]P+S)}{abc}, \frac{x([2n+1]P-S)}{abc}\right\}$$

is a rational Diophantine sextuple. Moreover, Lemma 1 in [9] shows that the order of $S$ is 3 if and only if $S(a, b, c) = 0$ where

$$S(a,b,c) = 3+4(ab+ac+bc)+6abc(a+b+c)-(abc)^2(-12+a^2+b^2+c^2-2ab-2ac-2bc).$$

Thus we are led to the following question.

**Question 1.** *Are there infinitely many rational Diophantine triples* $\{a, b, c\}$ *for which* $a^2+1$ *and* $b^2+1$ *are perfect squares and* $S(a, b, c) = 0$*? We refer to such triples as* **special**.

For an affirmative answer to this question, one would need to find a curve of genus zero or one (with infinitely many rational points) on the surface of the general type, which is a 32-cover of the surface $S(a, b, c) = 0$. This surface is defined by the condition that $ab+1$, $ac+1$, $bc+1$, $a^2+1$, and $b^2+1$ are perfect squares. In general, this is a difficult problem, so we sought inspiration from experimental data.

## 3. Experiments and regularity

Our key insight came from examining numerical examples of special Diophantine triples

{30464/2223, 22815/5168, 361/7956},
{30464/2223, 4807/31824, 10881/1292},
{-22815/5168, 4807/31824, -8092/2223}.

To understand these examples, it is necessary to introduce the concept of regularity (see [10, 12]).

**Definition 1.** The quadruple $(a, b, c, d) \in \mathbb{Q}^4$ is called **regular** if $r_4(a, b, c, d) = 0$ where

$$r_4(a, b, c, d) = (a+b-c-d)^2 - 4(ab+1)(cd+1).$$

Similarly, the quintuple $(a, b, c, d, e)$ is regular if $r_5(a, b, c, d, e) = 0$ where

$$r_5(a,b,c,d,e) = (abcde+2abc+a+b+c-d-e)^2-4(ab+1)(ac+1)(bc+1)(de+1).$$

Note that polynomials $r_4$ and $r_5$ are symmetric.

In the examples above, we noticed that for the first triple $\{a, b, c\}$ the (improper) quintuple $\{a, a, b, b, c\}$ is regular, i.e. $r_5(a, a, b, b, c) = 0$. Similarly, for the second and third triple the (improper) quadruple $\{a, b, b, c\}$ is regular, i.e. $r_4(a, b, b, c) =$

0. Furthermore, the elliptic curves associated to these Diophantine triples are isomorphic to each other.

These regularity conditions can be restated in the context of the arithmetic of the elliptic curve $E_{a,b,c}$.

**Proposition 3.** *Let $\{a,b,c\}$ be a rational Diophantine triple containing a strong pair $\{a,b\}$. Let $A$, $B$, $P$, and $S$ be points in $E_{a,b,c}(\mathbb{Q})$ as defined in Section 2. We have that*

a) $r_4(a,a,b,c)=0$ *if and only if* $A=\pm P\pm S$ *for some choice of signs,*
b) $r_5(a,a,b,b,c)=0$ *if and only if* $A\pm B\pm S=\mathcal{O}$ *for some choice of signs.*

*Proof.* It is known (see Section 3.1 of [7]) that for a Diophantine triple $\{a,b,c\}$, $r_4(a,b,c,d)=0$ if and only if $d=x(P\pm S)$, or equivalently $D=\pm P\pm S$ for some choice of signs, where $D\in E_{a,b,c}(\mathbb{Q})$ and $x(D)=d$. Similarly, for a Diophantine quintuple $\{a,b,c,d\}$, $r_5(a,b,c,d,e)=0$ if and only if $e=x(D\pm S)$ or equivalently $E=\pm D\pm S$ for some choice of signs, where $E\in E_{a,b,c}(\mathbb{Q})$ and $x(E)=e$.

Both claims follow when we apply these results to $E_{a,b,c}$ and points $D=A$ and $E=B$.

□

## 4. Proof of Theorem 1

To construct family $\mathcal{F}_1$, we proceed as follows. Set $a=\frac{2u}{u^2-1}$ and $b=\frac{2v}{v^2-1}$ to ensure that $a^2+1$ and $b^2+1$ are perfect squares. If we substitute these values in

$$r_5(a,a,b,b,c)=(abc)^2-2ac^2b-4ac+c^2-4cb-4$$

the resulting expression factors as $r_5(a,a,b,b,c)=q_1q_2$ where

$$q_1=u^2v^2c+2ucv^2+2cvu^2+cv^2-2cv+c-2uc+cu^2+2-2v^2-2u^2+2u^2v^2,$$

$$q_2=cv^2-2ucv^2+2cv+u^2v^2c-2cvu^2+cu^2+2uc+c-2+2v^2-2u^2v^2+2u^2.$$

Solving for $c$ in $q_2=0$ we obtain two solution one of which is

$$c=\frac{2(u^2v^2-u^2-v^2+1)}{(-v+uv-u-1)^2}.$$

If we substitute all this in $S(a,b,c)=0$, the expression factors as $s_1s_2s_3$ where

$$\begin{aligned}s_1&=1+8vu^4-8u^3v^2-8v^3u^2+4vu^3+8uv^2-8v^3+8vu^2\\&\quad+8uv^4+12v^3u^3+4uv^3+12uv-4u^2v^2-6u^2v^4+u^4v^4\\&\quad-6u^4v^2-6u^2-6v^2-3v^4-3u^4-8u^3,\\s_2&=3-8u^3v^2+8u-8v^3u^2+12vu^3+8v-8v^3u^4\\&\quad-8u^3v^4+8uv^2+8vu^2+4v^3u^3+12uv^3+4uv\\&\quad+4u^2v^2+6u^2v^4+3u^4v^4+6u^4v^2+6u^2+6v^2-v^4-u^4,\\s_3&=(uv+v-u+1)^2(-v+uv+u+1)^2.\end{aligned}$$

Note that factor $s_2$ is equal to $p(u,v)$ from the definition of curve $C:p(u,v)=0$, thus given a rational point $(u,v)$ on $C$, we obtain the triple $\mathcal{F}_1(u,v)$ from the introduction. The curve defined by $s_1=0$ is isomorphic to $C$.

It remains to show that $\{a,b,c\}$ is a Diophantine triple (note that a priori we only know that $a^2+1$ and $b^2+1$ are perfect squares). To this end, it is important to notice that for regular quintuple $\{a,b,c,d,e\}$, not necessary Diophantine, we have that $(ab+1)(ac+1)(bc+1)(de+1)$ is a perfect square for every permutation of elements (since polynomial $r_5(a,b,c,d,e)$ is symmetric). In particular, the regularity of $\{a,a,b,b,c\}$ implies that $a^2+1, b^2+1, ac+1$ and $bc+1$ represent the same class modulo squares (i.e. they are equal in $\mathbb{Q}^\times/{\mathbb{Q}^\times}^2$). Since by construction $a^2+1$ is a perfect square, it remains to prove that $ab+1$ is a perfect square.

Let $t(u,v)$ denote the product of the denominator and numerator of $ab+1$. Thus, we have

$$\begin{aligned} t(u,v) = {} & u^4v^4 - 2u^4v^2 + u^4 + 4u^3v^3 - 4u^3v - 2u^2v^4 \\ & + 4u^2v^2 - 2u^2 - 4uv^3 + 4uv + v^4 - 2v^2 + 1. \end{aligned}$$

It is straightforward to verify that

$$p(u,v) + t(u,v) = (uv+1)^2(uv-u-v-1)^2,$$

hence $t(u,v)$ is a perfect square (as is $ab+1$) whenever $p(u,v)=0$. Consequently, the conclusion of Theorem 1 for $\mathcal{F}_1(u,v)$ follows.

The curve given by the equation $s_1(u,v)=0$ is isomorphic to the curve $\mathcal{C}$ via the mapping $\sigma:(u,v)\mapsto(\frac{1}{u},-v)$. Since $\sigma(a)=-a$, $\sigma(b)=-b$, and $\sigma(c)=-c$, we observe that employing a parametrization by the equation $s_1(u,v)=0$ yields the same family of triples. Similarly, since the surface $q_1(u,v,c)=0$ is isomorphic to the surface $q_2(u,v,c)=0$ via the mapping $(u,v,c)\mapsto(-u,-v,c)$, it follows that we do not get anything new by employing parametrization for $c$ given by condition $q_1=0$. It is straightforward to verify that the condition $s_3(u,v)=0$ leads to triples with repeated elements. Thus, we conclude that every special rational Diophantine triple $\{a,b,c\}$ satisfying $r_5(a,a,b,b,c)=0$ belongs to the family $\mathcal{F}_1$.

Similarly, to obtain the family $\mathcal{F}_2(u,v)$ in the regularity condition

$$r_4(a,a,b,c) = -4 - 4ab + b^2 - 4ac - 2bc - 4a^2bc + c^2 = 0, \tag{1}$$

we substitute $a=\frac{2u}{u^2-1}$ and $b=\frac{2v}{v^2-1}$, yielding the condition $r_1r_2=0$ where

$$r_1 = -2 - c + 2cu + 2u^2 - cu^2 - 2v - 4uv - 2u^2v + 2v^2 + cv^2 - 2cuv^2 - 2u^2v^2 + cu^2v^2,$$

$$r_2 = 2 - c - 2cu - 2u^2 - cu^2 - 2v + 4uv - 2u^2v - 2v^2 + cv^2 + 2cuv^2 + 2u^2v^2 + cu^2v^2.$$

By solving for $c$ in the equation $r_1(u,v,c)=0$ and substituting the result into $S(a,b,c)$, we obtain $S(a,b,c)=t_1t_2t_3=0$, where

$$\begin{aligned} t_1 &= (1+u-v+uv)^2(1-u+v+uv)^2, \\ t_2 &= -3 + 8u - 6u^2 + u^4 - 16v + 4uv + 16u^2v - 4u^3v - 10v^2 \\ &\quad - 48uv^2 - 4u^2v^2 - 2u^4v^2 + 16v^3 - 4uv^3 - 16u^2v^3 \\ &\quad + 4u^3v^3 - 3v^4 + 8uv^4 - 6u^2v^4 + u^4v^4, \\ t_3 &= -1 + 6u^2 - 8u^3 + 3u^4 - 4uv + 16u^2v + 4u^3v - 16u^4v \\ &\quad + 2v^2 + 4u^2v^2 + 48u^3v^2 + 10u^4v^2 + 4uv^3 - 16u^2v^3 \\ &\quad - 4u^3v^3 + 16u^4v^3 - v^4 + 6u^2v^4 - 8u^3v^4 + 3u^4v^4. \end{aligned}$$

In this manner, we obtain a triple $a(u,v), b(u,v), c(u,v)$ parametrized by points $(u,v)$ on the curve $\mathcal{D}: t_3(u,v)=0$. Note that the curve $\mathcal{D}$ is isomorphic to $\mathcal{C}$ through the mapping $\alpha:\mathcal{C}\to\mathcal{D}$, defined as $(u,v)\mapsto\left(\frac{-1+uv}{u+v},-v\right)$. By precomposing the above parametrization with the map $\alpha$, we obtain the family $\mathcal{F}_2$.

It remains to show that $\mathcal{F}_2(u,v)$ is Diophantine triple. In general, the regularity condition $r_4(a,b,c,d)=0$ implies that $(ab+1)(cd+1)$ is a perfect square for all permutation of elements, as $r_4$ is symmetric polynomial. Thus, after combining the condition $r_4(a,a,b,c)=0$ with the requirement that $a^2+1$ is a perfect square, the remaining task is to establish that $ab+1$ (or equivalently $ac+1$) is also a perfect square. This is accomplished similarly to the case of the family $\mathcal{F}_1$. Similarly to before, we deduce that any special rational Diophantine triple $\{a,b,c\}$ satisfying $r_4(a,a,b,c)=0$ belongs to the family $\mathcal{F}_2$.

The statement for the family $\mathcal{F}_3$ follows from the observation that $\mathcal{F}_2(v,u) = \mathcal{F}_3(-u, 1/v)$ as noted in Remark 1. It follows from a discusion in Section 2 that each of the triples from these families can be extended in infinitely many ways to a Diophantine sextuple.

It is intriguing that triples satisfying different regularity conditions are parameterized by the same curve. This implies that there could be a direct relationship between these families.

The observation that elliptic curves associated with the triples $\mathcal{F}_i(u,v)$, for $i = 1,2,3$, are isomorphic to each other provides an answer to this question.

## 5. Diophantine triples with isomorphic elliptic curves

Let $\{a,b,c\}$ be a rational Diophantine triple for which $S \in E_{a,b,c}(\mathbb{Q})$ has order 3 (i.e. $S(a,b,c) = 0$), and let $W \in E_{a,b,c}(\mathbb{Q})$, $W \neq \pm S$ and $2W \neq \mathcal{O}$, be such that $1 - x(W)$ is a perfect square. Write $1 - x(W) = k^2$ for some $k \in \mathbb{Q}^\times$. We can choose the sign of $k$ such that it is equal to the sign of $y(W)$. Consider the change of variable and its inverse

$$(x,y) \mapsto \left(\frac{x}{k^2} + 1 - \frac{1}{k^2}, \frac{y}{k^3}\right), \quad (X,Y) \mapsto \left(k^2X + 1 - k^2, k^3Y\right),$$

which defines an isomorphism $\phi_W : E_{a,b,c} \to \tilde{E}$ where $\tilde{E} : Y^2 = (X+A)(X+B)(X+C)$ for some distinct $A,B,C \in \mathbb{Q}$. Note that $X(\phi_W(W)) = 0$, thus $ABC$ is a perfect square and $\frac{AB}{C} = c'^2$, $\frac{AC}{B} = b'^2$ and $\frac{BC}{A} = a'^2$ for some $a',b',c' \in \mathbb{Q}^\times$. We can choose signs of $a',b'$ and $c'$ such that $a'b' = C$, $a'c' = B$ and $b'c' = A$. It follows that $\tilde{E} = E_{a',b',c'}$. Since $X(\phi_W(S)) = 1$, and $\phi_W(S) \in 2E_{a',b',c'}(\mathbb{Q})$ (since $S \in 2E_{a,b,c}(\mathbb{Q})$ and $\phi_W$ is a group isomorphism), we have that that $1+A, 1+B$ and $1+C$ are perfect squares. Elements $a',b'$ and $c'$ are non-zero and distinct since $A,B$ and $C$ are non-zero and distinct, therefore $\{a',b',c'\}$ is a rational Diophantine triple. Moreover, since $\phi_W(S) = \pm S'$, it follows that $S'$ has order 3, thus $S(a',b',c') = 0$.

Conversely, let $\{a',b',c'\}$ be a rational Diophantine triple for which $S(a',b',c') = 0$ and let $\phi : E_{a,b,c} \to E_{a',b',c'}$ be an isomorphism. Denote by $W = \phi^{-1}(P')$, where $P' \in E_{a',b',c'}(\mathbb{Q})$ with $X(P') = 0$. Since $\phi^{-1}(X,Y) = (u^2X + v, u^3Y)$ for some $u,v \in \mathbb{Q}$, it follows from $\phi^{-1}(S') = \pm S$ that $u^2 + v = 1$. Since $x(W) = v$, it follows that $1 - x(W)$ is a perfect square, and $\phi = \phi_{\pm W}$. Thus, we proved the following proposition.

**Proposition 4.** *Let $\{a,b,c\}$ be a rational Diophantine triple such that $S(a,b,c) = 0$, $E_{a,b,c}$ the corresponding elliptic curve and $W \in E_{a,b,c}(\mathbb{Q})$, $6W \neq \mathcal{O}$, a point for which $1 - x(W)$ is a perfect square. Then $\phi_W$ defines an isomorphism between $E_{a,b,c}$ and $E_{a',b',c'}$, where $\{a',b',c'\}$ is a rational Diophantine triple, determined up to the sign, for which $S(a',b',c') = 0$. Furthermore, every rational Diophantine triple $\{a',b',c'\}$ with the property that $S(a',b',c') = 0$ and $E_{a',b',c'} \cong E_{a,b,c}$ can be obtained in this manner.*

**Remark 2.** The condition $1 - x(W) = k^2$ is a perfect square defines a curve

$$y^2 = (1 - k^2 + ab)(1 - k^2 + ac)(1 - k^2 + bc).$$

If $rst \neq 0$ (or equivalently, if $S$ is not a point of order 2), this curve has genus two. Consequently, in our situation, only a finite number of points $W \in E_{a,b,c}(\mathbb{Q})$ satisfy the required property. The point $P = [0, abc]$ induces the identity map.

For specificity, we will select elements $a', b'$, and $c'$ such that $\phi_W([-ab, 0]) = [-a'b', 0]$, $\phi_W([-ac, 0]) = [-a'c', 0]$, and $\phi_W([-bc, 0]) = [-b'c', 0]$. Note that the triple $\{a',b',c'\}$ is determined only up to the sign.

## 6. Another view on families $\mathcal{F}_i$

We start with elements of the family $\mathcal{F}_1$. Let $\{a,b,c\}$ be a special rational Diophantine triple ($a^2+1$ and $b^2+1$ are perfect squares and $S(a,b,c)=0$) for which $r_5(a,a,b,b,c)=0$ (i.e. $(a,a,b,b,c)$ is a regular quintuple). Let $A,B\in E_{a,b,c}(\mathbb{Q})$ for which $x(A)=a\cdot abc$ and $x(B)=b\cdot abc$ (these points are rational since $\{a,b\}$ is a strong pair). Proposition 3 implies that the regularity condition is equivalent to $A\pm B\pm S=\mathcal{O}$ for some choice of sign. We can choose $A,B$ and $S$ so that $A+B+S=\mathcal{O}$ (recall that $S$ is a point of order 3 with $x(S)=1$). Let $W_1=A+T_3$ and $W_2=B+T_2$, where $T_2=[-ac,0]$ and $T_3=[-bc,0]$ are the points of order 2.

It follows from the following result (Proposition 4 in [9]) that $1-x(W_1)$ and $1-x(W_2)$ are perfect squares.

**Proposition 5.** *Let $Q$, $T$ and $[0,\alpha]$ be three rational points on an elliptic curve $\mathcal{E}$ over $\mathbb{Q}$ given by the equation $y^2=f(x)$, where $f$ is a monic polynomial of degree 3. Assume that $\mathcal{O}\notin\{Q,T,Q+T\}$. Then*

$$x(Q)x(T)x(Q+T)+\alpha^2$$

*is a perfect square.*

Indeed, for $\mathcal{E}=E_{a,b,c}$ we have that

$$x(W_1)x(T_3)x(A)+(abc)^2=x(W_1)(-bc)a\cdot abc+(abc)^2=(abc)^2(1-x(W_1))$$

is a perfect square. Similarly, we obtain that $1-x(W_2)$ is a perfect square.

Let $\phi_{W_1}:E_{a,b,c}\to E_{a',b',c'}$ be an isomorphism from Proposition 4 associated to the point $W_1$. The following proposition implies that a rational Diophantine triple $\{a',b',c'\}$ is special, satisfying the regularity condition (1), and thus belongs to the $\mathcal{F}_2$ family.

**Proposition 6.** *We have that $a'^2=a^2$ and $b'=\frac{x(\phi_{W_1}(B+T_3))}{a'b'c'}$.*

*Proof.* It is easy to check that $x(W_1)=1-k^2$, where $k^2=\frac{(ab+1)(ac+1)}{a^2+1}$. Hence

$$\begin{aligned}\phi_{W_1}([-ab,0])&=\left[-\frac{a(a-c)}{ac+1},0\right],\\ \phi_{W_1}([-ac,0])&=\left[-\frac{a(a-b)}{ab+1},0\right],\\ \phi_{W_1}([-bc,0])&=\left[-\frac{(a-b)(a-c)}{(ab+1)(ac+1)},0\right].\end{aligned}$$

Since $-a'^2=\frac{x(\phi_{W_1}([-ab,0]))x(\phi_{W_1}([-ac,0]))}{x(\phi_{W_1}([-bc,0]))}$, it follows that $a'^2=a^2$. The second statement follows from direct computation in MAGMA. □

It follows that $\{a',b'\}$ is a strong pair since $a'^2+1=a^2+1$ is a perfect square, and $b'^2+1$ is a perfect square since the point $B'=\phi_{W_1}(B+T_3)$, with $x(B')=b'\cdot a'b'c'$ is rational. Moreover,

$$\begin{aligned}\mathcal{O}&=\phi_{W_1}(A+B+S)\\ &=\phi_{W_1}(A+T_3)+\phi_{W_1}(B+T_3)+\phi_{W_1}(S)\\ &=P'+B'+S',\end{aligned}$$

which, according to Proposition 3, implies the regularity condition $r_4(a',b',b',c')=0$.

More precisely, through direct computation, we derive the following proposition.

**Proposition 7.** *Let $(u_0, v_0) \in \mathcal{C}(\mathbb{Q})$ be a rational point on the curve $\mathcal{C}$, $[a, b, c] = \mathcal{F}_1(u_0, v_0)$ the corresponding Diophantine triple, and $W_1, W_2 \in E_{a,b,c}(\mathbb{Q})$ points defined as above. The triples associated to points $W_1$ and $W_2$ by Proposition 4 are equal to $\mathcal{F}_2(u_0, v_0)$ and $\mathcal{F}_3(u_0, v_0)$ respectively.*

*Similarly, if $[a, b, c] = \mathcal{F}_2(u_0, v_0)$ then the triples associated to points $W_1$ and $W_2$ are equal to $\mathcal{F}_1(u_0, v_0)$ and $\mathcal{F}_3(u_0, v_0)$ respectively, and if $[a, b, c] = \mathcal{F}_3(u_0, v_0)$ then the triples associated to points $W_1$ and $W_2$ are equal to $\mathcal{F}_1(u_0, v_0)$ and $\mathcal{F}_2(u_0, v_0)$ respectively.*

**Example.** We now go back to our starting numerical examples from Section 3. Consider first a special rational Diophantine triple $\{a, b, c\}$ where $a = 30464/2223$, $b = 22815/5168$ and $c = 361/7956$. Note that $\{a, b, c\} = \mathcal{F}_1(u_0, v_0)$, where $(u_0, v_0) = (-119/128, -135/169)$ is a rational point on the curve $\mathcal{C}$. Consider the rational points

$$A = [250880/6669, 94938136300/252028179],$$

$$B = [266175/21964, 18177179755/170264928],$$

on $E_{a,b,c}$ which correspond to the strong elements $a$ and $b$. Let $S = [1, -3307949/302328]$ be a point of order 3. The regularity condition $r_5(a, a, b, b, c) = 0$ is then equivalent to $A+B+S = \mathcal{O}$. Let $W_1 = A+[-bc, 0] = [19824/42025, -726438832196/108524729625]$ and $W_2 = B + [-ac, 0] = [-64155/24649, 29291888395/1764671208]$. When we apply Proposition 4 to the points $W_1$ and $W_2$ (recall that $1 - x(W_1)$ and $1 - x(W_2)$ are perfect squares), using the isomorphisms $\phi_{W_1}$ and $\phi_{W_2}$ respectively, we obtain triples $\mathcal{F}_2(u_0, v_0) = \{\frac{30464}{2223}, \frac{4807}{31824}, \frac{10881}{1292}\}$ and $\mathcal{F}_3(u_0, v_0) = \{\frac{-22815}{5168}, \frac{4807}{31824}, \frac{-8092}{2223}\}$ from our introductory example.

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