# On generalizations of Iwasawa's theorem \*

Jiangtao Shi\*\*, Fanjie Xu, Mengjiao Shan

School of Mathematics and Information Sciences, Yantai University, Yantai 264005, China

#### Abstract

Iwasawa's theorem indicates that a finite group G is supersolvable if and only if all maximal chains of the identity in G have the same length. As generalizations of Iwasawa's theorem, we provide some characterizations of the structure of a finite group G in which all maximal chains of every minimal subgroup have the same length. Moreover, let  $\delta(G)$  be the number of subgroups of G all of whose maximal chains in G do not have the same length, we prove that G is a non-solvable group with  $\delta(G) \leq 16$  if and only if  $G \cong A_5$ .

Keywords: supersolvable group; maximal chains; Iwasawa's theorem; solvable group; Fitting subgroup MSC(2020): 20D10

# 1 Introduction

In this paper all groups are considered to be finite. Let G be a group and H a subgroup of G. A maximal chain of H in G has length r means a chain of subgroups  $H = H_0 < H_1 < H_2 < \ldots < H_r = G$  such that  $H_i$  is maximal in  $H_{i+1}$  for every  $1 \le i \le r - 1$ .

Since every maximal subgroup of a supersolvable group G has prime index, all maximal chains of every subgroup in G have the same length. Moreover, the following proposition obviously holds for maximal chains of subgroups.

**Proposition 1.1** Let G be a group, H < K < G. If all maximal chains of H in G have the same length, then

- (1) all maximal chains of H in K have the same length;
- (2) all maximal chains of K in G have the same length.

**Remark 1.2** Let G be a group, H < K < G. If all maximal chains of K in G have the same length, we cannot ensure that all maximal chains of H in G have the same length. For example, let  $G = A_4 = \langle (123), (124) \rangle$ ,  $K = \langle (123) \rangle$  and H = 1. Since K is maximal in G, all maximal chains of K in G have the same length. For H, it is easy to see that H < K < G and  $H < \langle (12)(34) \rangle < \langle (12)(34), (13)(24) \rangle < G$  are two maximal chains of H in G having different length.

<sup>&</sup>lt;sup>\*</sup>This research was supported in part by Shandong Provincial Natural Science Foundation, China (ZR2017MA022) and NSFC (11761079).

 $<sup>^{**}\</sup>mathrm{Corresponding}$  author.

E-mail addresses: shijt2005@pku.org.cn (J. Shi), xufj2023@s.ytu.edu.cn (F. Xu), shanmj2023@s.ytu.edu.cn (M. Shan).

By Proposition 1.1, if all maximal chains of the identity in G have the same length, then all maximal chains of every subgroup in G have the same length. Using maximal chains of subgroups Iwasawa provided the following equivalent characterization of supersolvable groups.

**Theorem 1.3** [6, Theorem 10.3.5] A group G is supersolvable if and only if all maximal chains of the identity in G have the same length.

As a generalization of Iwasawa's theorem, considering maximal chains of some particular minimal subgroups, we have the following result whose proof is given in Section 3.

**Theorem 1.4** Let G be a group. If all maximal chains of every minimal subgroup of order both 2 and 3 in G have the same length, then G is solvable.

Furthermore, considering maximal chains of every minimal subgroup, we obtain the following result, the proof of which is given in Section 4.

**Theorem 1.5** Let G be a group and F(G) the Fitting subgroup of G. If all maximal chains of every minimal subgroup in G have the same length, then

(1) G is non-supersolvable if and only if  $G = F(G) \rtimes M$  and  $\Phi(G) = Z(G) = 1$ , where M acts faithfully on F(G), F(G) is the unique minimal normal subgroup of G and M is a supersolvable maximal subgroup of G,  $|F(G)| = p^n$ ,  $n \ge 2$ ;

(2) If G is non-supersolvable, then F(G) is a Sylow subgroup of G'. In particular, if  $Z(G') \neq 1$ , then F(G) = G'.

**Remark 1.6** If we assume that all maximal chains of every subgroup of G of order a square of a prime in G have the same length, we cannot ensure that G is solvable. For example, let  $G = A_5$ . It is easy to see that all maximal chains of subgroup of  $A_5$  of order 4 in  $A_5$  have the same length, but  $A_5$  is non-solvable.

**Remark 1.7** Assume that G is a solvable group, we cannot ensure that all maximal chains of every minimal subgroup in G have the same length. For example, let  $G = S_4 = \langle (12), (13), (14) \rangle$ . Take  $H = \langle (12) \rangle$  being a minimal subgroup of G. It is easy to see that  $H < S_3 = \langle (12), (13) \rangle < G$  and  $H < \langle (12), (34) \rangle < D_8 = \langle (1423), (34) \rangle < G$  are two maximal chains of H in G having different length.

**Remark 1.8** The group in Theorem 1.5 might not be supersolvable. For example, all maximal chains of every minimal subgroup in  $A_4$  have the same length, but  $A_4$  is non-supersolvable.

**Remark 1.9** In Theorem 1.5 even if assume that G has odd order, we cannot ensure that G is supersolvable. For example, let  $G = \langle a, b, c | a^5 = b^5 = c^3 = 1$ , [a, b] = 1,  $c^{-1}ac = (ab)^{-1}$ ,  $c^{-1}bc = a \rangle$ . By the definition of G, it is easy to see that all maximal chains of every minimal subgroup in G have the same length but G is non-supersolvable.

Note that two examples in Remarks 1.7 and 1.8 also indicate that even if all maximal chains of every minimal subgroup in G have the same length, all maximal chains of the identity in G might not have the same length.

**Remark 1.10** Suppose that G is a group satisfying all hypotheses in Theorem 1.5 (2), we cannot ensure that all maximal chains of every minimal subgroup in G have the same length. For example, let  $G = S_4 = \langle (12), (13), (14) \rangle$ , one has  $F(G) = K_4 = \langle (12)(34), (13)(24) \rangle$ . Take  $M = S_3 = \langle (12), (13) \rangle$ , then  $G = F(G) \rtimes M$ . Note that G satisfies all hypotheses in Theorem 1.5 (2), but all maximal chains of  $\langle (12) \rangle$  in G have different length.

Denote by  $\delta(G)$  the number of subgroups of G all of whose maximal chains in G do not have the same length. Observe that  $\delta(A_5) = 16$  including the identity and 15 subgroups of order 2. In this paper, we obtain the following result whose proof is given in Section 5.

**Theorem 1.11** Let G be a non-solvable group. Then  $\delta(G) \leq 16$  if and only if  $G \cong A_5$ .

In Section 6, we also discuss the cases that all maximal chains of some other special subgroups in G have the same length.

#### 2 Some necessary lemmas

**Lemma 2.1** Let G be a group. Then G is supersolvable if and only if all maximal chains of  $\Phi(G)$  in G have the same length.

**Proof.** We only need to prove the sufficiency part. Since all maximal chains of  $\Phi(G)$  in G have the same length, all maximal chains of the identity subgroup  $\overline{1} = \Phi(G)/\Phi(G)$  in  $G/\Phi(G)$  have the same length. By [6, Theorem 10.3.5],  $G/\Phi(G)$  is supersolvable. It follows that G is supersolvable.

**Lemma 2.2** [8] A non-abelian simple group all of whose proper subgroups are solvable is said to be a minimal simple group, there are five classes in all:

- (1)  $PSL_2(p)$ , where p > 3 and  $5 \nmid p^2 1$ ;
- (2)  $PSL_2(2^q)$ , where q is a prime;
- (3)  $PSL_2(3^q)$ , where q is an odd prime;
- (4)  $PSL_3(3);$
- (5)  $S_z(2^q)$ , where q is an odd prime.

**Lemma 2.3** [1, Theorem 5.4] Let G be a group of order n. If (n, 15) = 1, then G is solvable.

**Lemma 2.4** [6, Theorem 10.4.2] Let G be a group having a nilpotent maximal subgroup of odd order. Then G is solvable.

## 3 Proof of Theorem 1.4

**Proof.** Let G be a counterexample of minimal order.

Note that if all maximal chains of every minimal subgroup in G have the same length, then all maximal chains of every non-trivial subgroup in G have the same length by Proposition 1.1.

For any maximal subgroup M of G, it is obvious that all maximal chains of every minimal subgroup of order both 2 and 3 in M have the same length by the hypothesis (Note that if there exists a maximal subgroup E of G such that  $2 \nmid |E|$  or  $3 \nmid |E|$ , then Enaturally satisfies that all maximal chains of every minimal subgroup of order both 2 and 3 in E have the same length.). By the minimality of G, M is solvable. It follows that G is a minimal non-solvable group and then  $G/\Phi(G)$  is a minimal non-abelian simple group.

Claim  $\Phi(G) = 1$ . Otherwise, assume  $\Phi(G) \neq 1$ . If  $2 \mid |\Phi(G)|$  or  $3 \mid |\Phi(G)|$ , one has that all maximal chains of  $\Phi(G)$  in G have the same length. By Lemma 2.1, G is supersolvable, a contradiction. If  $(6, |\Phi(G)|) = 1$ , then all maximal chains of every minimal subgroup of  $G/\Phi(G)$  of order both 2 and 3 in  $G/\Phi(G)$  have the same length. It follows that  $G/\Phi(G)$ is solvable by the minimality of G, which implies that G is solvable, also a contradiction.

Therefore,  $\Phi(G) = 1$ . One has that G is a minimal non-abelian simple group. By Lemma 2.2, we divide our analyses into the following five cases.

(1) Let  $G = PSL_2(p)$ , where p > 3,  $(5, p^2 - 1) = 1$  and  $|G| = \frac{p(p^2 - 1)}{2}$ .

Since G is non-solvable and  $(5, p^2 - 1) = 1$ , one has  $3 \mid p^2 - 1$  by Lemma 2.3.

First, suppose  $p^2 \equiv 1 \pmod{16}$ . Then G has a maximal subgroup  $M \cong S_4$  by [3]. It is easy to see that all maximal chains of subgroups of order 2 in  $S_4$  do not have the same length, which implies that all maximal chains of subgroups of order 2 in G do not have the same length, a contradiction.

Next, suppose  $p^2 \not\equiv 1 \pmod{16}$ . Then G has a maximal subgroup  $M \cong A_4$  by [3]. Consider a subgroup N of M of order 3, one has that N < M < G is a maximal chain of N in G. Let P be a Sylow 3-subgroup of G such that  $N \leq P$ . By Lemma 2.4, P is not a maximal subgroup of G. Assume  $|P| = 3^n$ , where  $n \geq 1$ .

If  $n \ge 2$ , it is easy to see that all maximal chains of N in G do not have the same length, a contradiction. Therefore, N = P is a Sylow 3-subgroup of G.

Since  $3 \mid p^2 - 1$ , one has  $3 \mid p + 1$  or  $3 \mid p - 1$ . It follows that  $3 \mid \frac{p+1}{2}$  or  $3 \mid \frac{p-1}{2}$ .

If  $3 \mid \frac{p+1}{2}$ , let *L* be a maximal subgroup of *G* that is isomorphic to a dihedral group of order  $2 \cdot \frac{p+1}{2}$ . By Sylow theorem, we can assume N < L. Since all maximal chains of *N* in *G* have the same length, one must have  $\frac{p+1}{2} = 3$ , which implies p = 5. Then  $G = PSL_2(5)$ . It is easy to see that all maximal chains of subgroups of order 2 in  $PSL_2(5)$  do not have the same length, a contradiction.

If  $3 \mid \frac{p-1}{2}$ , let R be a maximal subgroup of G that is a dihedral group of order  $2 \cdot \frac{p-1}{2}$ . By Sylow theorem, we can assume N < R. Since all maximal chains of N in G have the same length, one must have  $\frac{p-1}{2} = 3$ , which implies p = 7, this contradicts  $p^2 \not\equiv 1 \pmod{16}$ .

(2) Let  $G = PSL_2(2^q)$ , where q is a prime.

By [3], it is easy to see that G has two distinct maximal subgroups  $M_1$  and  $M_2$  such that  $M_1$  is a dihedral group of order  $2 \cdot (2^q - 1)$  and  $M_2$  is the normalizer of the Sylow 2-subgroup of G of order  $2^q \cdot (2^q - 1)$ , where subgroups of  $M_1$  and  $M_2$  of order  $2^q - 1$  are cyclic. Let  $N_1$  be a subgroup of  $M_1$  of order 2, and let P be a Sylow 2-subgroup of G such that  $P < M_2$ . By Sylow theorem, we can assume  $N_1 \leq P$ . Since  $q \geq 2$ , it is easy to see that all maximal chains of  $N_1$  in G do not have the same length, a contradiction.

(3) Let  $G = PSL_2(3^q)$ , where q is an odd prime.

By [3], G has the following two distinct maximal subgroups:  $M_1 \cong A_4$  and  $M_2$  is the normalizer of the Sylow 3-subgroup of G of order  $3^q \cdot \frac{(3^q-1)}{2}$ , where subgroups of  $M_2$  of order  $\frac{3^q-1}{2}$  are cyclic. Let  $N_1$  be a subgroup of  $M_1$  of order 3 and let Q be a Sylow 3-subgroup of G such that  $Q < M_2$ . By Sylow theorem, we can assume  $N_1 \leq Q$ . Since  $q \geq 3$ , it is easy to see that all maximal chains of  $N_1$  in G do not have the same length, a contradiction.

(4) Let  $G = PSL_3(3)$ .

By [3], G has a maximal subgroup  $M \cong S_4$ . It is obvious that there exists a subgroup S of M of order 2 such that all maximal chains of S in M do not have the same length, which implies that all maximal chains of S in G do not have the same length, a contradiction.

(5) Let  $G = S_z(2^q)$ , where q is an odd prime.

By [7], G has two distinct maximal subgroups:  $M_1$  is the normalizer of the Sylow 2subgroup of order  $2^{2q} \cdot (2^q - 1)$ ,  $M_2$  is a dihedral group of order  $2 \cdot (2^q - 1)$ , where subgroups of  $M_1$  and  $M_2$  of order  $2^q - 1$  are cyclic. Let  $N_2$  be a subgroup of  $M_2$  of order 2 and let P be a Sylow 2-subgroup of G such that  $P < M_1$ . By Sylow theorem, we can assume  $N_2 \leq P$ . Note that  $q \geq 3$  and then  $2q \geq 6$ . It is easy to see that all maximal chains of  $N_2$ in G do not have the same length, a contradiction.

By above arguments, the counterexample of minimal order does not exist and so G is solvable.

### 4 Proof of Theorem 1.5

**Lemma 4.1** Let G be a group in which all maximal chains of every minimal subgroup have the same length, F(G) be a Fitting subgroup of G. Then G is non-supersolvable if and only if  $G = F(G) \rtimes M$  and  $\Phi(G) = Z(G) = 1$ , where M acts faithfully on F(G), F(G)is the unique minimal normal subgroup of G and M is a supersolvable maximal subgroup of G,  $|F(G)| = p^n$ ,  $n \ge 2$ .

**Proof.** Since any minimal normal subgroup of a supersolvable group has prime order, the sufficiency part holds.

In the following we prove the necessity part.

Claim  $\Phi(G) = 1$ . Otherwise, assume  $\Phi(G) \neq 1$ . Then all maximal chains of  $\Phi(G)$  in G have the same length. One has that G is supersolvable by Lemma 2.1, a contradiction. Hence  $\Phi(G) = 1$ .

Claim Z(G) = 1. Otherwise, assume  $Z(G) \neq 1$ . Then all maximal chains of Z(G) in G have the same length. For the quotient group G/Z(G), since all maximal chains of the identity subgroup Z(G)/Z(G) in G/Z(G) have the same length, G/Z(G) is supersolvable by [6, Theorem 10.3.5], which implies that G is supersolvable, a contradiction. Therefore, Z(G) = 1.

Note that G is solvable by Theorem 1.4. Let N be a minimal normal subgroup of G, where  $|N| = p^n$ ,  $n \ge 1$ . Then all maximal chains of N in G have the same length. Consider the quotient group G/N. Since all maximal chains of the identity subgroup N/N in G/Nhave the same length, G/N is supersolvable by [6, Theorem 10.3.5]. If G has at least two distinct minimal normal subgroups, let  $N_0$  be another minimal normal subgroup of G such that  $N_0 \ne N$ , then  $N \cap N_0 = 1$ . Arguing as above, one has that  $G/N_0$  is supersolvable. It follows that  $G \cong G/(N \cap N_0)$  is isomorphic to a subgroup of  $G/N \times G/N_0$  and then G is supersolvable, a contradiction. Therefore, N is the unique minimal normal subgroup of G.

It follows that F(G) = N since  $\Phi(G) = 1$  by [5, Chapter III, Theorem 4.5]. So  $F(G) \nleq \Phi(G)$ . There exists a maximal subgroup M of G such that  $F(G) \nleq M$ . One has G = F(G)M. Since G is solvable, F(G) is an elementary abelian group. One has  $F(G) \cap M \trianglelefteq F(G)$ . Moreover, as  $F(G) \cap M \trianglelefteq M$ , one has  $F(G) \cap M \trianglelefteq G$ . Note that  $F(G) \cap M < F(G)$ , then  $F(G) \cap M = 1$  by the minimality of F(G). Therefore,  $G = F(G) \rtimes M$ . It implies that  $M \cong G/F(G)$  is supersolvable.

Since G is solvable, one has  $C_G(F(G)) \leq F(G)$  by [5, Chapter III, Theorem 4.2]. Moreover, as  $F(G) \leq C_G(F(G))$ , it follows that  $C_G(F(G)) = F(G)$ , which implies that M acts faithfully on F(G).

Note that G/F(G) is supersolvable and G is non-supersolvable. Therefore,  $|F(G)| = p^n$ ,  $n \ge 2$ .

**Lemma 4.2** Let G be a group in which all maximal chains of every minimal subgroup have the same length. If G is non-supersolvable, then F(G) is a Sylow subgroup of G'. In particular, if  $Z(G') \neq 1$ , then F(G) = G'.

**Proof.** Since G is non-supersolvable, one has that G/F(G) is supersolvable by Lemma 4.1, where  $|F(G)| = p^n$ ,  $n \ge 2$ . By [4, Theorem 10.5.4], (G/F(G))' = G'F(G)/F(G) is nilpotent. It is obvious that  $G' \ne 1$ . Since F(G) is the unique minimal normal subgroup of G,  $F(G) \le G'$ , which implies that G'/F(G) is nilpotent.

Let  $P \in \operatorname{Syl}_p(G')$ . Then  $F(G) \leq P$ . If F(G) < P, then  $1 < P/F(G) \operatorname{char} G'/F(G) \leq G/F(G)$  since G'/F(G) is nilpotent. One has  $P \leq G$ , which implies that  $P \leq F(G)$ , a contradiction. Hence  $F(G) = P \in \operatorname{Syl}_p(G')$ .

Assume  $Z(G') \neq 1$ . Since Z(G') char  $G' \trianglelefteq G$ ,  $Z(G') \trianglelefteq G$ . One has Z(G') = F(G). Then G'/Z(G') = G'/F(G) is nilpotent. It follows that G' is nilpotent and then  $G' \leq F(G)$ . Moreover, as  $F(G) \leq G'$ , one has F(G) = G'.

**Proof of Theorem 1.5.** Combine Lemmas 4.1 and 4.2 together, we complete the proof of Theorem 1.5.  $\hfill \Box$ 

### 5 Proof of Theorem 1.11

**Lemma 5.1** Let G be a group. If  $\delta(G) < 16$ , then G is solvable.

**Proof.** Let G be a counterexample of minimal order.

For any proper subgroup M of G, if there exists a subgroup H of M such that all maximal chains of H in M do not have the same length, then all maximal chains of H in G do not have the same length. Therefore,  $\delta(M) \leq \delta(G) < 16$ . By the minimality of G, M is solvable. It follows that G is a minimal non-solvable group and then  $G/\Phi(G)$  is a minimal non-abelian simple group.

If  $\Phi(G) \neq 1$ , since  $\delta(G/\Phi(G)) \leq \delta(G) < 16$ ,  $G/\Phi(G)$  is solvable by the minimality of G. It implies that G is solvable, a contradiction. Therefore,  $\Phi(G) = 1$  and then G is a minimal non-abelian simple group.

Note that if all maximal chains of the subgroup H in G do not have the same length, then for any  $g \in G$ , all maximal chains of  $H^g$  in G do not have the same length, too.

By Lemma 2.2, we divide our analyses into the following five cases.

(1) Let  $G = PSL_2(p)$ , where p > 3,  $(5, p^2 - 1) = 1$  and  $|G| = \frac{p(p^2 - 1)}{2}$ .

If  $p^2 \not\equiv 1 \pmod{16}$ . Arguing as in proof of Theorem 1.4, one has  $3 \mid \frac{p+1}{2}$  or  $3 \mid \frac{p-1}{2}$ .

Case(i): Assume  $3 \mid \frac{p+1}{2}$ . When p = 5, then  $G = PSL_2(5) \cong A_5$ . However,  $\delta(A_5) = 16$ , a contradiction. When  $p \ge 7$ , G has a subgroup H of order 3 all of whose maximal chains in G do not have the same length. Observe that  $|N_G(H)| = 2 \cdot \frac{p+1}{2} = p + 1$ . Then  $\delta(G) \ge |G: N_G(H)| = \frac{\frac{p(p^2-1)}{2}}{p+1} = \frac{p(p-1)}{2} \ge 21 > 16$ , a contradiction. Case (*ii*): Assume  $3 \mid \frac{p-1}{2}$ . As  $p^2 \ne 1 \pmod{16}$ , one has p > 7. Then G has a subgroup

Case (*ii*): Assume  $3 \mid \frac{p-1}{2}$ . As  $p^2 \not\equiv 1 \pmod{16}$ , one has p > 7. Then G has a subgroup K of order 3 all of whose maximal chains in G do not have the same length. Observe that  $|N_G(K)| = 2 \cdot \frac{p-1}{2} = p - 1$ . Then  $\delta(G) \geq |G: N_G(K)| = \frac{\frac{p(p^2-1)}{2}}{p-1} = \frac{p(p+1)}{2} > 28 > 16$ , also a contradiction.

Next assume  $p^2 \equiv 1 \pmod{16}$ . Then  $p \ge 7$ .

If p = 7, then  $G = PSL_2(7)$ . It is easy to see that G has a subgroup H of order 2 all of whose maximal chains in G do not have the same length. By [2], all subgroups of  $PSL_2(7)$ of order 2 are conjugate in G. Then  $\delta(G) \ge |G: N_G(H)| = 21 > 16$ , a contradiction.

If p > 7, then  $p \ge 17$ . By [3], G has the following four distinct maximal subgroups:  $M_1 \cong S_4, M_2$  is a dihedral group of order  $2 \cdot \frac{p+1}{2}, M_3$  is a dihedral group of order  $2 \cdot \frac{p-1}{2}$  and  $M_4$  is the normalizer of the Sylow *p*-subgroup of *G* of order  $\frac{p(p-1)}{2}$ . It is easy to see that *G* has a subgroup *H* of order 2 all of whose maximal chains in *G* do not have the same length, and only one of the following three cases might be true by the hypothesis:  $(a) |N_G(H)| \leq |S_4| = 24$ ;  $(b) |N_G(H)| \leq 2 \cdot \frac{p+1}{2} = p+1$ ;  $(c) |N_G(H)| \leq 2 \cdot \frac{p-1}{2} = p-1$ . Note that  $p \geq 17$ . For case (a), one has  $\delta(G) \geq |G : N_G(H)| \geq \frac{p(p^2-1)}{24} \geq 102 > 16$ , a contradiction. For case (b), one has  $\delta(G) \geq |G : N_G(H)| \geq \frac{p(p^2-1)}{2} \geq 136 > 16$ , a contradiction. For case (c), one has  $\delta(G) \geq |G : N_G(H)| \geq \frac{p(p^2-1)}{2} \geq 153 > 16$ , also a contradiction.

(2) Let  $G = PSL_2(2^q)$ , where q is a prime and  $|G| = 2^q(2^{2q} - 1)$ .

Arguing as in proof of Theorem 1.4, G has a subgroup H of order 2 all of whose maximal chains in G do not have the same length and  $|N_G(H)| = 2^q$ . Then  $\delta(G) \ge |G|$ :  $N_G(H)| + 1 = \frac{2^q(2^{2q}-1)}{2^q} + 1 = 2^{2q} - 1 + 1 \ge 16$ , a contradiction.

(3) Let  $G = PSL_2(3^q)$ , where q is an odd prime number and  $|G| = \frac{3^q(3^{2q}-1)}{2}$ .

Arguing as in proof of Theorem 1.4, G has a subgroup H of order 3 all of whose maximal chains in G do not have the same length and  $|N_G(H)| \leq \frac{3^{q}(3^q-1)}{2}$ . Then  $\delta(G) \geq |G: N_G(H)| \geq \frac{(\frac{3^q}{2}(3^q-1))}{(\frac{3^q}{2}(3^q-1))} = 3^q + 1 \geq 28 > 16$ , a contradiction. (4) Let  $G = PSL_3(3)$ .

Arguing as in proof of Theorem 1.4, G has a subgroup H of order 2 all of whose maximal chains in G do not have the same length. By [2], all subgroups of G of order 2 are conjugate in G. Then  $\delta(G) \geq |G: N_G(H)| = 117 > 16$ , a contradiction.

(5) Let  $G = S_z(2^q)$ , where q is an odd prime and  $|G| = (2^{2q} + 1)2^{2q}(2^q - 1)$ .

Arguing as in proof of Theorem 1.4, G has a subgroup H of order 2 all of whose maximal chains in G do not have the same length and  $|N_G(H)| \leq 2^{2q}(2^q - 1)$ . Then  $\delta(G) \geq |G: N_G(H)| \geq \frac{(2^{2q}+1)2^{2q}(2^q-1)}{2^{2q}(2^q-1)} = 2^{2q} + 1 \geq 65 > 16$ , also a contradiction.

All above arguments show that the counterexample of minimal order does not exist and so G is solvable.

#### **Lemma 5.2** Let G be a non-solvable group. Then $\delta(G) = 16$ if and only if $G \cong A_5$ .

**Proof.** We only need to prove the necessity part.

Since G is non-solvable, there exists a subgroup M of G such that M is a minimal non-solvable group. Then  $M/\Phi(M)$  is a minimal non-abelian simple group. Note that  $\delta(M/\Phi(M)) \leq \delta(M) \leq \delta(G) = 16$ , one must have  $\delta(M/\Phi(M)) = \delta(M) = \delta(G) = 16$  by Lemma 5.1. Then arguing as in proof of Lemma 5.1, one has  $M/\Phi(M) \cong PSL_2(5) \cong PSL_2(4) \cong A_5$ .

Claim  $\Phi(M) = 1$ . Otherwise, assume  $\Phi(M) \neq 1$ . Since all maximal chains of the identity in G do not have the same length, one has  $\delta(G) \geq \delta(M/\Phi(M))+1$ , a contradiction. Therefore,  $\Phi(M) = 1$ .

It follows that  $M \cong A_5$ . If M < G. Since  $\delta(M) = \delta(G) = 16$ , one has that M is normal in G. Let L be a subgroup of G such that M is maximal in L. Then  $L/M \cong Z_p$ for some prime p. One gets  $L \cong S_5$  or  $A_5 \times Z_p$ . However, it is easy to see that  $\delta(S_5) > 16$ and  $\delta(A_5 \times Z_p) > 16$ , a contradiction. Therefore,  $G = M \cong A_5$ .

**Proof of Theorem 1.11.** Combine Lemmas 5.1 and 5.2 together, we complete the proof of Theorem 1.11.  $\Box$ 

#### 6 Remarks

**Remark 6.1** A group G might not be solvable if G has at least one minimal subgroup H all of whose maximal chains in G do not have the same length. For example, let  $G = SL_2(5)$ . It is obvious that all maximal chains of every minimal subgroup of G of odd order in G have the same length. However, observe that  $Z(G) = Z_2 < Z_4 < Q_8 < SL_2(3) < G$  and  $Z(G) = Z_2 < Z_6 < SL_2(3) < G$  are two maximal chains of Z(G) in G having different length and Z(G) is the unique subgroup of G of order 2.

**Remark 6.2** A group G might not be solvable if all maximal chains of any non-trivial subgroups of G except minimal subgroups of order p for a fixed prime divisor p of |G| have the same length. For example, it is easy to see that all maximal chains of non-trivial subgroups of  $A_5$  except minimal subgroups of order 2 in  $A_5$  have the same length, but  $A_5$  is non-solvable.

**Remark 6.3** If all maximal chains of every second maximal subgroup of a group G in G have the same length, then G might not be solvable. For example, let  $G = PSL_2(17)$ . By [3], it is easy to see that all maximal chains of every second maximal subgroup of  $PSL_2(17)$  in  $PSL_2(17)$  have the same length, but  $PSL_2(17)$  is non-solvable.

**Remark 6.4** Assume that G is solvable. If all maximal chains of every second maximal subgroup of G in G have the same length, then G might not be supersolvable. For example, let  $G = GL_2(3)$ . Let H be any subgroup of G, then  $H \in \{1, \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, Q_8, SD_{16}, SL_2(3), GL_2(3)\}$ , where  $SD_{16} = \langle a, b \mid a^8 = b^2 = 1, b^{-1}ab = a^3 \rangle$ . It is easy to see that all maximal chains of every second maximal subgroup of  $GL_2(3)$  in  $GL_2(3)$  have the same length, but  $GL_2(3)$  is non-supersolvable.

#### References

[1] Z. Chen, Inner-Outer  $\Sigma$ -Groups and Minimal Non- $\Sigma$ -Groups (Chinese), Southwest China Normal University Press, Chongqing, 1988.

- [2] J.H. Conway, R.T. Curtis, S.P. Norton, et al, Atlas of Finite Groups, Clarendon Press, Oxford, 1985.
- [3] L.E. Dickson, Linear Groups with an Exposition of the Galois Field Theory, Leipzig, Teubner, 1901.
- [4] M. Hall, The Theory of Groups, The Macmillan Company XIII, New York, 1964.
- [5] B. Huppert, Endliche Gruppen I, Spring-Verlag, New York, 1967.
- [6] D.J.S. Robinson, A Course in the Theory of Groups (Second Edition), Springer-Verlag, New york, 1996.
- [7] M. Suzuki, On a class of doubly transitive groups, Ann. Math. 75 (1962) 105-145.
- [8] J.G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable, I, Bull. Amer. Math. Soc. 74 (1968) 383-437.