## A CATEGORICAL FORMULATION OF KRAUS' PARADOX

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ABSTRACT. We give a categorical formulation of Kraus' "magic trick" for recovering information from truncated types. Rather than type theory, we work in Van den Berg-Moerdijk path categories with a univalent universe, and rather than propositional truncation we work with arbitrary cofibrations, which includes truncation as a special case. We show, using Kraus' argument that any cofibration with homogeneous domain is a monomorphism. We give some simple concrete examples in groupoids to illustrate the interaction between homogeneous types, cofibrations and univalent fibrations.

## 1. INTRODUCTION

Propositional truncation is one of the most basic and most important examples of higher induction types [Uni13, Section 3.7]. An *hProposition* is a type A such that any two elements of A are equal. We then define the propositional truncation of a type A to be a type ||A|| together with a map  $|-|: A \to ||A||$  which makes Ainto an hProposition in the "minimal way," in the sense that it satisfies a suitable induction principle. In [KECA17, Section 8.4] Kraus et al. gave a surprising proof that under certain conditions we can take a term of the form |a| for a : A and "extract" the term a back out. More formally, we have a family of dependent types  $z : ||A|| \vdash B(z)$  and a dependent function  $z : ||A|| \vdash b(z) : B(z)$  such that for a : A, B(|a|) is definitionally equal to A and b(|a|) is definitionally equal to a. This is surprising, since by definition any two elements of ||A|| are equal, so we wouldn't expect to be able to distinguish different elements of ||A|| to the extent that we can even find which element of A they came from.

In both the original presentation of [KECA17] and in our summary above, Kraus' paradox is described in purely type theoretic terms, using terminology such as "terms," "induction," "definitional equality," and "computation rules." This might mislead one into thinking that Kraus' paradox is an artefact of the technical aspects of type theory and has no deeper meaning or significance outside type theory. We remedy this with a purely categorical presentation of Kraus' argument. We work in the general setting of univalent Van den Berg-Moerdijk path categories [vdBM18], and show that any cofibration with homogeneous domain is a monomorphism.

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### 2. Path categories

We work in the general setting of path categories, as defined below.

**Definition 2.1** (Van den Berg-Moerdijk [vdBM18]). A path category is a category  $\mathbb{C}$  together with two classes of morphisms of  $\mathbb{C}$ ,  $\mathcal{F}$  whose elements we refer to as *fibrations* and  $\mathcal{W}$  whose elements we refer to as *weak equivalences*, satisfying the following conditions. Below and throughout the paper, we will refer to maps in the intersection of  $\mathcal{F}$  and  $\mathcal{W}$  as *trivial fibrations*.

- (1) Fibrations are closed under composition.
- (2) The pullback of a fibration along any map exists and is also a fibration.
- (3) The pullback of a trivial fibration along any map is a trivial fibration.
- (4) Weak equivalences satisfy 2-out-of-6: for all maps f : A → B, g : B → C and h : C → D, if both g ∘ f and h ∘ g are weak equivalences, then so are f, g, h and h ∘ g ∘ f.
- (5) Every isomorphism is a trivial fibration.
- (6) Every trivial fibration has a section.
- (7)  $\mathbb{C}$  has a terminal object, and for every object B, the unique map  $B \to 1$  is a fibration.
- (8) For any object B, the diagonal map  $B \to B \times B$  factors as  $B \xrightarrow{r} PB \xrightarrow{p} B \times B$ where r is a weak equivalence and p is a fibration. We refer to such a factorisation as a *path object* on B.

Given a path object  $B \xrightarrow{r} PB \xrightarrow{p} B \times B$ , we will write  $p_0$  for  $\pi_0 \circ p$  and  $p_1$  for  $\pi_1 \circ p$ . Note that we get the 3-for-2 property as a special case of the 2-out-of-6 property, i.e. for maps  $f: A \to B$  and  $g: B \to C$ , if any two of the maps f, g and  $g \circ f$  is a weak equivalence, then so is the third.

**Definition 2.2.** Let  $A, B \in \mathbb{C}$ , and let PB be a path object on B. For maps  $f, g: A \to B$ , a homotopy from f to g is a map  $H: A \to PB$  making the following diagram commute.



We say f and g are *homotopic* and write  $f \sim g$  if there is a homotopy from f to g.

We can think of the definition of path category as the minimal structure on a category necessary for the above notion of homotopy to be well behaved, in the sense that we have the theorem below.

**Theorem 2.3.** (1) The definition of homotopy is independent of the particular choice of path object on B.

- (2) Homotopy gives an equivalence relation on the set of maps  $A \to B$ .
- (3) If  $f \sim g$ , then  $f \circ h \sim g \circ h$  and  $k \circ f \sim k \circ g$ .

Proof. See [vdBM18, Theorem 2.14].

**Example 2.4.** Let  $\mathbb{C}$  be any category with finite limits. Then we can make  $\mathbb{C}$  a path category by taking all maps to be fibrations, and the weak equivalences to be

precisely the class of isomorphisms. We will refer to path categories of this form as *extensional*.

**Proposition 2.5.** The following are equivalent:

- (1)  $\mathbb{C}$  is extensional.
- (2) Two maps are homotopic iff they are equal.
- (3) Every weak equivalence is an isomorphism.

*Proof.* For 1 implies 2, note that if every map is a fibration, then in particular for any object B, the diagonal map  $B \to B \times B$  is a fibration. It follows that taking PB = B gives a path object on B. However, this implies that for any object A, if two maps  $f, g: A \to B$  are homotopic, then they are equal.

For 2 implies 3, we recall that any weak equivalence is a homotopy equivalence by [vdBM18, Theorem 2.16], and so by 2 any homotopy equivalence is an isomorphism.

For 3 implies 1, we recall that any map  $f : A \to B$  factors as a weak equivalence followed by a fibration by [vdBM18, Proposition 2.3]. It follows that f is a fibration, since fibrations contain isomorphisms and are closed under composition.

Although the above example is simple, it is a useful one. When working with weak equivalences, it is often useful to think of them as maps that behave "like isomorphisms." In order to make this idea precise it is important that we can in fact view isomorphisms as a degenerate special case of weak equivalences. By placing categories with finite limits within the same general framework as other examples of path categories, such as Kan complexes in simplicial and cubical sets, we can easily compare them.

The example is also a useful one in type theory the same reason. Any locally cartesian closed category is a model of extensional type theory [See84] and also, since it has all finite limits an extensional path category. Many more recent examples of models of type theory coming from homotopy theory [HS96, KL21, AW09, CCHM18] very naturally give rise to path categories, and so we can use path categories as a common framework in which to compare the two different kinds of model.

**Example 2.6.** The category of fibrant objects in a model structure where every object is cofibrant is a path category [vdBM18, Section 2.1].

**Example 2.7.** As a special case of Example 2.6, the category of Kan complexes [GZ67] is a path category with fibrations given by Kan fibrations and weak equivalences by homotopy equivalences.

**Example 2.8.** The category of groupoids is a path category, taking fibrations to be isofibrations and weak equivalences to be equivalences. See e.g. [JT91] for more details. This is also a special case of Example 2.6.

**Example 2.9** (Gambino-Garner [GG08]). We can construct examples of path categories using type theory. We take  $\mathbb{C}$  to be the category of contexts, fibrations to be the closure of projections  $(\Gamma, x : A) \to \Gamma$  under retracts, and weak equivalences to be homotopy equivalences.

**Example 2.10** (Avigad-Kapulkin-Lumsdaine [AKL15, Theorem 3.2.5]). For any contextual category the underlying category of contexts can be viewed as a path category. This is a generalisation of Example2.9, by considering the syntactic contextual category of type theory.

In any path categories we can identify the important class of maps of *cofibrations*. We will define them via the general notion of lifting property below.

**Definition 2.11.** Given two maps  $m : A \to B$  and  $f : C \to D$ , we say m has the *left lifting property* against f and f has the *right lifting property* against m if for every commutative square as in the solid lines in the diagram below, there is a diagonal map, as in the dotted line below making two commutative triangles.



**Definition 2.12.** A *cofibration* is a map with the left lifting property against all trivial fibrations.

In keeping with spirit of path categories, we do not assume any "existence conditions" on the class of cofibrations, so in principle it could be the case that a path category has "very few" cofibrations. However, in any path category we can identify the class of cofibrations and talk about what properties elements of this class must satisfy.

Having said that, most of the particular examples of path categories that we consider here are the fibrant objects of a model structure, and as such every map can be factored as a cofibration followed by a trivial fibration.

**Example 2.13.** In extensional path categories any map is a cofibration, since any map has the left lifting property against isomorphisms. In fact the converse also holds: If every map is a cofibration, then every trivial fibration is an isomorphism, but every map factors as a section of a trivial fibration followed by a fibration [vdBM18, Proposition 2.3] and so also every map is a fibration.

**Example 2.14.** The cofibrations in groupoids are precisely the functors that are injective on objects.

**Example 2.15** (Lumsdaine [Lum11]). We consider the path category structure on the syntactic category of type theory (Example 2.9). Any point constructor of any higher inductive type is a cofibration in the category of contexts, as long as the induction principle satisfies judgemental computation rules on points.<sup>1</sup> In fact the judgemental computation rule precisely gives us commutativity of the upper triangle for the diagonal filler. As a special case of this we can consider the truncation map |-| for propositional truncation. We will use this idea to recover the original version of Kraus' paradox.

If we augment type theory with the mapping cylinder higher inductive type then we can additionally show the following.

- (1) Cofibrations, homotopy equivalences and fibrations give a model structure on the category of contexts in type theory.<sup>2</sup>
- (2) Every cofibration is a retract of a point constructor of a higher inductive type.

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 $<sup>^{1}</sup>$ Without a general formal definition of higher inductive types, Lumsdaine presented this result as an informal general principle that holds for all known examples of higher inductive types.

 $<sup>^{2}</sup>$ Lumsdaine refers to this as a pre-model-structure, since the category of contexts is not complete or cocomplete, and the choice of factorisation is not functorial, and these conditions are often assumed as part of the definition of model category.

We recall the following basic proposition for "realigning" maps using cofibrations.

**Proposition 2.16.** Suppose we are given a cofibration  $m : A \to B$  and maps  $f : A \to C$  and  $g : B \to C$  such that  $g \circ m \sim f$ 



Then we can find  $g': B \to C$  such that  $g' \circ m = f$ .

*Proof.* Let  $h: A \to PC$  be the map witnessing the homotopy relation. In particular  $p_0 \circ h = g \circ m$  and  $p_1 \circ h = f$ . Note that we have the commutative square given by the solid lines in the diagram below.

$$\begin{array}{ccc} A & \stackrel{h}{\longrightarrow} PC \\ \downarrow^{m} & j & \downarrow^{p_{0}} \\ B & \stackrel{g}{\longrightarrow} C \end{array}$$

We take j to be the diagonal filler, as in the dotted arrow in the diagram. We can then define  $g' := p_1 \circ j$ , giving  $g' \circ m = p_1 \circ j \circ m = p_1 \circ h = f$ .

## 3. HPROPOSITIONS AND PROPOSITIONAL TRUNCATION

In many cases higher inductive types can be natural formulated in path categories. We give such a formulation for propositional truncation here. Our definition is based on the type theoretic definition of propositional truncation with a judgemental computation rule for the point constructor as in [KECA17, Section 8], since that is most relevant for our results here on Kraus' paradox. We point out that other, less strict versions of propositional truncation are also possible.

**Definition 3.1.** A fibration  $f : A \to B$  in a path category is an *hProposition* if the diagonal map  $A \to A \times_B A$  is a weak equivalence.

A propositional truncation of an arbitrary fibration  $f: A \to B$  is a factorisation of f as  $f = f' \circ i$  where f' is an hProposition and i has the left lifting property against all hPropositions.

**Proposition 3.2.** If a map  $i : A \to C$  has the left lifting property against all hPropositions, then it is a cofibration.

*Proof.* Note that it suffices to show that every trivial fibration  $g: D \to E$  is an hProposition. However, this is easy to check: since trivial fibrations are closed under pullback and composition, the map  $D \times_E D \to E$  is a trivial fibration, and so the diagonal  $D \to D \times_E D$  is a weak equivalence by 3-for-2.

**Example 3.3.** In any extensional path category, a map  $f : A \to B$  is an hProposition if and only if it is a monomorphism. Proposition truncations exists iff the underlying category is regular [AB04, Mai05]. We note that the truncation map  $A \to C$  can only be a monomorphism when f is already monic, and in this case the truncation map is an isomorphism.

**Example 3.4.** We can explicitly describe the hPropositions in groupoids as follows. Note that the map  $A \to A \times_B A$  is an equivalence iff each projection map  $A \times_B A \to A$  is a trivial fibration. However, this is the case iff for all objects a of A, the fibre  $f^{-1}(a)$  is a contractible category, which in groupoids precisely says that f is full and faithful.

A propositional truncation of f is a factorisation of f as a bijective on objects functor followed by a full and faithful functor, which exists by taking the groupoid with the same objects as A, and defining a morphism  $a \rightsquigarrow a'$  to be a morphism  $f(a) \rightsquigarrow f(a')$  in B.

**Example 3.5.** In the syntactic category of type theory, as in Example 2.9, we can use propositional truncation in type theory to construct propositional truncations in the category of contexts. For this to work, we need to formulate propositional truncation using an induction principle with a judgemental computation rule for induction as in [KECA17, Section 8]. To verify that truncation maps  $(\Gamma, A) \rightarrow$  $(\Gamma, ||A||)$  do have the left lifting property against all hPropositions, we first consider the special case where the hPropositions is a projection of the form  $(\Gamma, z : ||A||, B) \rightarrow$  $(\Gamma, ||A||)$  where  $\Gamma, z : ||A|| \vdash B$  Type, and the lower map in the lifting problem is the identity. Note that the induction term precisely gives a diagonal filler, with the upper triangle commuting using the judgemental computation rule.

$$(\Gamma, A) \xrightarrow{I} (\Gamma, z : ||A||, B)$$
$$|-| \downarrow \qquad \operatorname{ind}(B, f) \qquad \downarrow$$
$$(\Gamma, ||A||) = (\Gamma, ||A||)$$

Next, note that for arbitrary lifting problems against maps of the form  $(\Delta, B) \to \Delta$  we can pull back along the bottom map to obtain a lifting problem of the form above, as illustrated below.

$$\begin{array}{ccc} (\Gamma, A) & \longrightarrow (\Gamma, z : \|A\|, B[\sigma]) & \longrightarrow (\Delta, B) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ (\Gamma, \|A\|) & & & & \downarrow \\ \end{array}$$

Finally, note that every hProposition is a fibration, and so in particular a retract of a dependent projection by [GG08, Lemma 11]. Since maps with the right lifting property against a given map are closed under retract, we can deduce that truncation maps have the left lifting property against all hPropositions.

It is clear that the projection map  $(\Gamma, ||A||) \to \Gamma$  is an hProposition, and so we are done.

#### 4. Univalence in path categories

**Definition 4.1.** For a fixed fibration  $\dot{\mathcal{U}} \to \mathcal{U}$ , we say a fibration  $f : A \to B$  is  $\mathcal{U}$ -small if it is a pullback of  $\dot{\mathcal{U}} \to \mathcal{U}$  along some map  $B \to \mathcal{U}$ . We say an object A is  $\mathcal{U}$ -small if the unique map  $A \to 1$  is  $\mathcal{U}$ -small.

We will use the definition of univalence in path categories due to Van den Berg [vdB20, Definition 2.13]. For completeness, we spell out below the details of the definition and his proof that in type theory it is equivalent to the more usual one.

We first recall the following fact.

**Lemma 4.2.** Given any choice of path object PU on U, we can choose a path object  $P\dot{U}$  on  $\dot{U}$  and fibration  $P\dot{U} \rightarrow PU$  such that the following diagram commutes, and

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the canonical map  $P\dot{\mathcal{U}} \rightarrow P\mathcal{U} \times_{\mathcal{U}\times\mathcal{U}} \dot{\mathcal{U}} \times \dot{\mathcal{U}}$  given by the right hand commutative square is a fibration.



Furthermore, if we are given two such path objects  $P\dot{\mathcal{U}}$  and  $P\dot{\mathcal{U}}'$ , then there is a weak equivalence  $P\dot{\mathcal{U}} \rightarrow P\dot{\mathcal{U}}'$  making the following triangles commute.



*Proof.* We first check that  $P\mathcal{U} \times_{\mathcal{U} \times \mathcal{U}} \dot{\mathcal{U}} \times \dot{\mathcal{U}}$  is a well defined object. Note that  $\dot{\mathcal{U}} \times \dot{\mathcal{U}} \to \mathcal{U} \times \mathcal{U}$  factors as  $\dot{\mathcal{U}} \times \dot{\mathcal{U}} \to \dot{\mathcal{U}} \times \mathcal{U} \to \mathcal{U} \times \mathcal{U}$  where both maps are pullbacks of  $\dot{\mathcal{U}} \to \mathcal{U}$ , and so fibrations. Hence  $\dot{\mathcal{U}} \times \dot{\mathcal{U}} \to \mathcal{U} \times \mathcal{U}$  is a fibration, and so its pullback along  $P\mathcal{U} \to \mathcal{U} \times \mathcal{U}$  exists.

We can factor the canonical map  $\dot{\mathcal{U}} \to P\mathcal{U} \times_{\mathcal{U}\times\mathcal{U}} \dot{\mathcal{U}} \times \dot{\mathcal{U}}$  as a weak equivalence followed by a fibration by [vdBM18, Proposition 2.3], and this precisely gives us the required structure by composing with the pullback projection maps.

For the second part we apply [vdBM18, Lemma 2.9] to the factorisation above and again compose with the pullback projection maps.  $\Box$ 

**Definition 4.3** (Van den Berg). A fibration  $\dot{\mathcal{U}} \to \mathcal{U}$  is *univalent* if it satisfies the following. Suppose that we are given  $\mathcal{U}$ -small maps  $A \to C$  and  $B \to C$ , witnessed by pullback squares as explicitly labelled below, together with an equivalence  $e : A \to B$  over C.



Then we have homotopies  $f \sim g$  and  $j \circ e \sim i$ , which are coherent, in the sense that if  $P\dot{\mathcal{U}}$  is chosen as in Lemma 4.2, the homotopies can be given by maps  $C \rightarrow P\mathcal{U}$  and  $A \rightarrow P\dot{\mathcal{U}}$  making the square on the right below commute.



**Proposition 4.4.** The definition of univalence is independent of the particular choice of path objects and map  $P\dot{\mathcal{U}} \rightarrow P\mathcal{U}$ .

*Proof.* First note that the choice of path object  $P\mathcal{U}$  makes no difference by the argument below [vdBM18, Corollary 2.10]. Now note that given a homotopy for the upper triangle given relative to a path object  $P\dot{\mathcal{U}}'$ , we can use the weak equivalence  $P\dot{\mathcal{U}} \rightarrow P\dot{\mathcal{U}}'$  given in the second part of Lemma 4.2 to construct a homotopy relative to  $P\dot{\mathcal{U}}$  which is still coherent.

**Theorem 4.5** (Van den Berg). Suppose that  $\mathbb{C}$  is a path category constructed from a model of type theory, as in Example 2.10, and that the model satisfies function extensionality. Then a fibration  $\dot{\mathcal{U}} \to \mathcal{U}$  satisfies univalence in the sense of Definition 4.3 iff there is a term witnessing that it is univalent according to the usual formulation in type theory.

*Proof.* Path categories arising from directly from models of type theory satisfy the additional conditions of *type theoretic fibration categories*, as formulated by Shulman in [Shu14]. In particular, this allows us to define the universal equivalence between pairs of types, as in [Shu14, Section 5]. This can be used to replace the quantification over all objects C in the definition of univalence with one particular C. Explicitly, in type theoretic notation, we take  $C := \sum_{A,B:\mathcal{U}} A \simeq B$  with A, B and e given by the appropriate projections.

Since the definition of univalence is independent of the choice of path object, we can make an explicit choice of path objects that is most convenient for the proof. We define both path objects using type theoretic notation, which can then be interpreted into the type theoretic fibration category, following Shulman. We take  $P\mathcal{U} := \sum_{A,B:\mathcal{U}} A = B$  and we define  $P\dot{\mathcal{U}}$  as below.

$$P\dot{\mathcal{U}} := \sum_{A,B:\mathcal{U}} \sum_{p:A=B} \sum_{a:A} \sum_{b:B} p_*(a) = b$$

We can now see that the homotopies precisely give us a term giving for all  $A, B : \mathcal{U}$  and all  $e : A \simeq B$  a choice of equality p : A = B together with an equality  $q(a) : p^*(a) = e(a)$  for each a : A. Note that this precisely says that for every such e, we have a path  $p : A \simeq B$  such that e is homotopic to transport along p. Assuming function extensionality this says that we can find p such that e is equal to transport along p, i.e. that idtoiso has a section. It is clear that this follows from type theoretic univalence, which states that idtoiso is an equivalence. However, the converse can also be proved in type theory, as shown by Orton and Pitts [OP19, Theorem 3.5],<sup>3</sup> which can then be interpreted into the model.

**Remark 4.6.** It as an open problem to determine whether or not this formulation of univalence implies function extensionality. If it does, then the assumption of function extensionality can be dropped from Theorem 4.5.

**Definition 4.7.** We say a path category is *univalent* if for every fibration  $f : A \to B$  there is a univalent fibration  $\dot{\mathcal{U}} \to \mathcal{U}$  for which f is  $\mathcal{U}$ -small:



<sup>&</sup>lt;sup>3</sup>This can be seen as an instance of very general constructions in type theory. See e.g. [Rij23, Exercise 11.8(d)] and [Esc19, Section 3.27].

We will make heavy use of the following basic fact.

**Proposition 4.8.** Let  $\dot{\mathcal{U}} \to \mathcal{U}$  be any fibration. Every  $\mathcal{U}$ -small object A is a subobject of  $\dot{\mathcal{U}}$ , witnessed by a monomorphism  $\iota_A : A \to \dot{\mathcal{U}}$ .

*Proof.* By the definition of  $\mathcal{U}$ -small object, we have a pullback diagram as below.



The bottom map is a monomorphism, since its domain is the terminal object. The top map is then a pullback of a monomorphism, and so also a monomorphism. We take  $\iota_A$  to be the top map.

**Remark 4.9.** Although we think of  $\mathcal{U} \to \mathcal{U}$  as acting like a universe that contains every type belonging to the fibrations  $A \to B$ , this isn't really accurate. We only required univalence, and no other closure conditions, such as  $\Sigma$  or  $\Pi$  types or that the universe contains any other types such as 0 and 1.

In particular, if  $\mathbb{C}$  is a preorder then every fibration is univalent since we only require that certain diagrams either commute or commute up to homotopy, and in preorders every diagram commutes. Hence every preorder path category is univalent. By applying this to the extensional path category on a lattice, we can see that univalence is not strong enough to show that coproducts are disjoint.

**Proposition 4.10.** A path category is both extensional and univalent if and only if it is a bounded  $\land$ -semi lattice.

*Proof.* As noted in the above remark, to satisfy univalence we only require that certain diagrams commute or commute up to homotopy, and so it is vacuously true for preorders. A preorder has all finite limits iff it is a bounded  $\wedge$ -semi lattice. Therefore we confirm that a bounded  $\wedge$ -semi lattice with any path category structure is univalent, in particular the extensional path category.

For the converse, assume that a path category  $\mathbb{C}$  is both univalent and extensional. For any  $B \in \mathbb{C}$ , we consider the automorphism  $\tau : B \times B \to B$  that swaps the two components of the product. In extensional path categories, two maps are homotopic iff they are equal. Hence univalence gives us a universe  $\dot{\mathcal{U}} \to \mathcal{U}$  containing B such that  $\iota_{B \times B} \circ \tau = \iota_B$ . Since  $\iota_B$  is a monomorphism, we deduce that  $\tau = 1_{B \times B}$ . However, we can now deduce that any two maps  $A \rightrightarrows B$  are equal.  $\Box$ 

**Remark 4.11.** The equivalence extension property [Sat17, Awo23] is another categorical formulation of univalence. It follows from the existence of a univalent universe, and is a useful concept to consider when constructing a univalent universe. However, viewed strictly as a property of a path category it is much weaker than univalence. In particular every extensional path category satisfies the equivalence extension property.

For this paper we don't require the full univalence condition, but only the following weaker version.

**Proposition 4.12.** Suppose  $\dot{\mathcal{U}} \to \mathcal{U}$  is univalent. If we are given two  $\mathcal{U}$ -small fibrations  $f: A \to C$  and  $g: B \to C$ , witnessed by pullback squares with top maps i and j and a weak equivalence e from A to B over C, as in (1) in the definition of univalence, then  $j \circ e \sim i$ .

*Proof.* This is one of the homotopies in the definition of univalence, so we simply forget the other homotopy.  $\Box$ 

## 5. KRAUS' PARADOX

We now have enough general theory to give the categorical version of Kraus' paradox. We first formulate the definition of transitive type in path categories as follows.

**Definition 5.1.** For a fibration  $\dot{\mathcal{U}} \to \mathcal{U}$ , a  $\mathcal{U}$ -small object  $(A, a_0)$  is  $\mathcal{U}$ -homogeneous if the maps  $\iota_A \circ \pi_i : A \times A \to \dot{\mathcal{U}}$  are homotopic, as illustrated below.



We now give the main theorem, corresponding to [KECA17, Theorem 8.12].

**Theorem 5.2.** Suppose that a  $\mathcal{U}$ -small object A is  $\mathcal{U}$ -homogeneous. Let  $m : A \to B$  be a cofibration and  $s : B \to A$  any map. Then m is a monomorphism.

*Proof.* We first observe that the map  $\iota_A \circ s \circ m : A \to \dot{\mathcal{U}}$  is homotopic to  $\iota_A$ . To see this, note that it follows from Theorem 2.3 and the  $\mathcal{U}$ -homogeneity of A by considering the following diagram.



However, we now note that we have the following homotopy commutative triangle.

$$A \xrightarrow{\iota_A} \dot{\mathcal{U}}$$

$$\downarrow \sim \qquad \swarrow_{\iota_A}$$

$$\downarrow m \qquad A$$

$$\downarrow \qquad \swarrow_{s}$$

$$B$$

Hence by Proposition 2.16 we can find  $j: B \to \dot{\mathcal{U}}$  such that  $j \circ m$  is strictly equal to the monomorphism  $\iota_A$ . Hence m must also be a monomorphism.

So far we have not used univalence. However, we do need it in order to get non trivial examples of  $\mathcal{U}$ -homogeneous types. We show how to obtain  $\mathcal{U}$ -homogeneous types from the more natural notion of homogeneity below.

**Definition 5.3.** A type A is homogeneous if there is a weak equivalence  $e : A \times A \times A \to A \times A \times A$  over  $A \times A$  such that for the two sections  $s_0 := \langle \pi_0, \pi_1, \pi_0 \rangle$ and  $s_1 := \langle \pi_0, \pi_1, \pi_1 \rangle$ , we have  $\pi_2 \circ e \circ s_0 \sim \pi_2 \circ s_1$ .



**Lemma 5.4.** Suppose that A is  $\mathcal{U}$ -small and homogeneous and that  $\mathcal{U}$  is univalent. Then A is  $\mathcal{U}$ -homogeneous.

*Proof.* We note that we have the following pullback diagram.

$$\begin{array}{c} A \times A \times A \xrightarrow{\pi_2} A \xrightarrow{\iota_A} \dot{\mathcal{U}} \\ \downarrow & \downarrow & \downarrow \\ A \times A \xrightarrow{} 1 \xrightarrow{} U \end{array}$$

Hence applying Proposition 4.12 with the equivalence e given by homogeneity, we get the following diagram commuting up to homotopy.

$$\begin{array}{c} A \times A \times A \xrightarrow{\pi_2} A \\ \downarrow^e & \sim & \dot{\mathcal{U}} \\ \downarrow^e & \sim & \dot{\mathcal{U}} \\ A \times A \times A \xrightarrow{\pi_2} A \end{array}$$

We now use Theorem 2.3 to paste the above diagram to the homotopy commutative diagrams given by the remaining parts of the definition of homogeneity, as illustrated below.



**Corollary 5.5.** If A is homogeneous and U-small for some univalent fibration  $\dot{\mathcal{U}} \to \mathcal{U}, m : A \to B$  is a cofibration and there is some map  $B \to A$ , then m is a monomorphism.

*Proof.* We apply Lemma 5.4 and Theorem 5.2.  $\Box$ 

**Corollary 5.6.** If  $\mathbb{C}$  is univalent,  $m : A \to B$  is a cofibration, A is homogeneous and there is any map at all  $s : B \to A$ , then m is a monomorphism.

*Proof.* Since  $\mathbb{C}$  is univalent, A is  $\mathcal{U}$ -small for some univalent  $\mathcal{U}$ . Hence we can apply Corollary 5.5.

**Corollary 5.7.** If  $\mathbb{C}$  satisfies univalence and A is an object that is homogeneous and has a point  $a_0: 1 \to A$ , then any cofibration  $A \to B$  is a monomorphism.

*Proof.* We define  $s: B \to A$  to be  $a_0 \circ B$  and apply Corollary 5.6.

univalence and that whenever 
$$m: A$$
 –

**Corollary 5.8.** Suppose that  $\mathbb{C}$  satisfies u  $\rightarrow B$ is a cofibration, so is  $C \times m : C \times A \to C \times B$  for any object C. Then any cofibration with homogeneous domain is a monomorphism.

*Proof.* Suppose we are given two maps  $f, g: C \to A$  such that  $m \circ f = m \circ g$ . We take the full subcategory of  $\mathbb{C}/C$  consisting of fibrations. We note that this is a path category and that  $C \times \dot{\mathcal{U}} \to C \times \mathcal{U}$  is univalent. Also note that  $C \times A \to C$  is pointed, e.g. by  $\langle 1_C, f \rangle$ . Hence we can apply Corollary 5.7. 

## 6. Some worked examples in groupoids

For this section we work in the category of groupoids as in Example 2.8, and we will see a few simple examples to illustrate the interaction between homogeneous types, cofibrations and univalence. We point out that many of the ideas that appear here can be thought of as simplified special cases of group theoretic constructions in homotopy type theory [BvDR18].

For the first example, note that using the explicit description of cofibration as maps that are injective on objects, it is easy to see that every cofibration with discrete domain is monic. However, assuming the law of excluded middle, we can also view it as a special case of Kraus' paradox. Using excluded middle, we can show that every discrete groupoid is homogeneous. However, this allows us to apply Corollary 5.5 together with the fact that any discrete fibration is  $\mathcal{U}$ -small for a univalent universe  $\mathcal{U}$  [HS96, Section 5.4], to give another proof that such cofibrations must be monic.

For the remaining examples, we recall that any group G can be viewed as a one object groupoid BG, and that this defines an embedding from groups to groupoids. We write the object of BG as \*.

## **Proposition 6.1.** A map $BG \rightarrow ||BG||$ is monic iff G is trivial.

*Proof.* In an hProposition, any two paths between two objects must be equal. In particular for  $g \in G$ , we must have  $|g| = 1_{|*|}$ , and so if |-| is monic, we can deduce  $g = 1_*$ . 

**Theorem 6.2.** If G is nontrivial and abelian, then BG is not  $\mathcal{U}$ -small for any univalent fibration  $\mathcal{U}$ .

*Proof.* By Proposition 6.1 and the assumption G is nontrivial, the truncation  $BG \rightarrow Proof$ . ||BG|| cannot be monic. Hence by Corollary 5.5, to show BG is not U-small for univalent  $\mathcal{U}$  it suffices to show that BG is homogeneous. Using the embedding from groups to groupoids, we can instead show the corresponding property of the group G. That is, we need an automorphism  $\theta: G \times G \times G \to G \times G \times G$  such that  $\pi_2 \circ \theta \circ s_0 \sim \pi_2 \circ \theta \circ s_1$ . We will in fact ensure that  $\theta \circ s_0 = \theta \circ s_1$ . We define  $\theta$  as follows.

$$\theta(g,h,k) := (g,h,hg^{-1}k)$$

We check that this is a group isomorphism. It is clear that  $\theta$  preserves the identity. Since G is abelian, we can see that  $\theta$  preserves group multiplication, and so is a homomorphism. It is also clear that is has an inverse defined similarly, and so is

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an isomorphism. However, we have now checked that BG is homogeneous and so we can apply Corollary 5.5.

On the other hand, there are examples of nonabelian G that are pullbacks of a univalent fibration. In particular we have the following.

**Theorem 6.3.** If G has trivial centre and every automorphism of G is an inner isomorphism (i.e. of the form  $\lambda x.h^{-1}xh$  for some h), then  $BG \to 1$  is univalent.

*Proof.* First note that if  $A \to B$  is a pullback of  $BG \to 1$ , then we have a canonical isomorphism  $A \cong B \times BG$ , and that equivalences over A are precisely automorphisms of  $A \times BG$  over A. Each automorphism of  $A \times BG$  over A is determined by a homomorphism  $\theta : A \times BG \to A$ .

Note that for every object a of A,  $\theta$  restricts to an automorphism  $\theta(1_a, -)$  of G. By assumption, this must be an inner automorphism. That is, for all objects a of A there must exist  $h_a$  such that  $\theta(1_a, g) = h_a^{-1}gh_a$ .

Now for any objects a, a' of A, any path  $p : a \rightsquigarrow a'$  and any element g of G we have the following.

$$h_{a'}\theta(p,1_{*})h_{a}^{-1}g = h_{a'}\theta(p,1_{*})h_{a}^{-1}h_{a}\theta(1_{a},g)h_{a}^{-1}$$
  
=  $h_{a'}\theta(p,g)h_{a}^{-1}$   
=  $h_{a'}\theta(1_{a'},g)h_{a'}^{-1}h_{a'}\theta(p,1_{*})h_{a}^{-1}$   
=  $qh_{a'}\theta(p,1_{*})h_{a}^{-1}$ 

We deduce that  $h_{a'}\theta(p, 1_*)h_a^{-1}$  belongs to the centre of G, and so  $h_{a'}\theta(p, 1_*)h_a^{-1} = 1_G$ , since G has trivial centre. Therefore  $h_{a'}\theta(p, 1_*) = h_a$ .

We now verify that h is a natural transformation from  $\theta : A \times BG \to BG$  to  $\pi_1 : A \times BG \to BG$ . Every morphism in  $A \times BG$  is of the form (p,g) where  $p : a \rightsquigarrow a'$  in A and  $g \in G$ . We then calculate as follows.

$$\begin{aligned} h_{a'}\theta(p,g) &= h_{a'}\theta(p,1_*)\theta(1_a,g) \\ &= h_a\theta(1_a,g) \\ &= gh_a \end{aligned}$$

**Example 6.4.**  $S_3$  is univalent as an object in groupoids.

**Remark 6.5.** The above proof is partially based on an example due to Mike Shulman of a "self referential" univalent fibration in homotopy type theory, i.e. a univalent family of types  $b : B \vdash A : U$  such that  $B \simeq A(b_0)$  for some  $b_0 : B$ . He observed one can show internally in HoTT that the classifying space BG has trivial automorphism group when G is a group with trivial centre and trivial outer automorphism group, and used this to construct the example as follows.  $S_n$  satisfies the conditions for  $n \neq 2, 6$ . We take B to be n copies of  $BS_n$ , and define the type A to be equal to B on points, with its value on paths determined by the action of  $S_n$  on n.

#### 7. Conclusion

7.1. Variations and generalisations. Recently David Wärn [Wär24] has noticed a variation on Kraus' paradox that works as follows. We will present it for all cofibrations, although following Kraus, the argument was originally phrased using propositional truncation.

Let  $m: A \to B$  be a cofibration with A a small type. Working internally in type theory, for each element a of A we have a contractible type  $C(a) := \sum_{a':A} a' = a$ , with the centre of contraction given explicitly as (a, refl). The special case of univalence for contractible types tells us that the type of all small contractible types is contractible. Hence we can lift to obtain for each b: B a type C(b) together with a proof that C(b) is contractible. In particular this includes a witness that C(b) in inhabited, say c(b): C(b). The upper triangle in the lifting diagram tells us that for a: A, C(m(a)) is definitionally equal to  $\sum_{a':A} a' = a$  and c(m(a)) is definitionally equal to (a, refl). Hence we if apply the first projection we get  $\pi_0(c(m(a))) \equiv a$ .

In some ways this argument is more general than Kraus': it only requires the special case of univalence for contractible types, and does not require A to be homogeneous or inhabited. However, in at least one way it is, unfortunately, less general. In formal treatments of type theory the projection terms  $\pi_0$  include type annotations that explicitly describe the types used to construct the  $\Sigma$  type. In particular, the variable a occurs in the term  $\pi_0$  we used above implicitly as a part of the type annotation. In Wärn's formalisation in the Agda proof assistant the type annotation appears as an implicit parameter to the projection function that is inferred automatically by Agda. This issue prevents us from adapting the argument to arbitrary path categories.

For a specific counterexample, we can consider the extensional path category of **Set**, and make the following observation, based on the cardinal model of type theory due to Bauer and Winterhalter [Win20, Section 8.3]. We say a fibration is *semi-univalent* if in Definition 4.3 we only have the "lower homotopy"  $f \sim g$ .<sup>4</sup> Note that in type theory semi-univalence implies propositional extensionality, and so also univalence for contractible types.

# **Proposition 7.1.** Assuming the axiom of choice, the extensional path category on **Set** is semi-univalent.

*Proof.* Given a map  $f : A \to B$ , we take  $\mathcal{U}$  to be the quotient of B identifying elements if their fibres have the same cardinality. The quotient map has a section  $s : \mathcal{U} \to B$  by the axiom of choice, and pulling back f along s gives the required universe.

The author expects nonetheless that a semantic version of Wärn's argument is possible. Note that if we use the "standard" interpretation of  $\Sigma$ -types in set theory as sets of ordered pairs we can define the first projection "uniformly." E.g. Using Kuratowski ordered pairs, we always define  $\pi_0(z)$  to be  $\bigcup \bigcap z$  independent of which particular  $\Sigma$ -type we are considering. Moreover, for the more standard definition of universe in **Set** as  $V_{\kappa}$  for inaccessible  $\kappa$ , we can restrict this uniform projection to an operation on small sets. However, we leave it for future work to find a natural formulation of this property for universes in path categories, to study which examples of path categories it applies to, and to show that under those conditions univalence for contractible types implies that every cofibration is a monomorphism.

The variation of Kraus' paradox due to the author in [Swa18, Section 6] is another example where the requirement of homogeneous domain is dropped, but which turns out to be less general in other ways. For one thing, it requires the very

<sup>&</sup>lt;sup>4</sup>This is a reformulation of the propositional version of the *isomorphism reflection principle* from [Win20, Section 8.3].

limiting condition that the cofibration  $m : A \to B$  is also an embedding in type theory. In addition it is very specific to the syntactic categories from type theory, since the arguments makes use of raw terms. As such it is unlikely to generalise to arbitrary path categories.

However, the current version of Kraus' paradox is likely not the most powerful. One can naturally ask the following questions, which are all, to the author's knowledge, open problems:

- (1) Can we say any more about cofibrations with homogeneous domain, other than that they are monomorphisms?
- (2) Can replace the requirement of homogeneity with something weaker? What can we still say if the domain has a rich supply of automorphisms, but not quite enough to get homogeneity?
- (3) Is there a better formulation of Kraus' paradox for higher inductive types with recursive point constructors, such as *W*-types and nullification [RSS20]?

7.2. Is Kraus' paradox surprising? We can now give another perspective on why Kraus' paradox is a surprising result.

In path categories we have two different notions of injective map. As in any category we can think of monomorphisms as injective maps, following our intuition from many natural examples of categories. The second, arguably more correct definition is maps that are "homotopy monomorphic," i.e. those that factor as a weak equivalence followed by an hProposition. Certainly the latter agrees with the internal statement in type theory that the map is injective when we work in models of type theory.

In extensional path categories the two definitions are equivalent, so there is no ambiguity. For non extensional path categories, we no longer expect the two classes to coincide. For models coming from homotopy theory we have the intuition that propositional truncation no longer identifies points, but instead "adds new paths." From this perspective, it is not surprising that there are at least some examples of path categories where the truncation maps are always monomorphisms, even though truncations are almost never homotopy monomorphisms. In Cisinski model structures, for example, a map is a cofibration if and only if it is monic, by definition [Cis02, Definition 2.12].

However, just as for extensional path categories, there's no reason that our intuitions about one class of model should apply to every model. It is easy to think of the choice of definition for cofibration as an "implementation detail" that is somewhat independent of the logical principles holding in a model. Hence one might naïvely still expect some flexibility here, with path categories intermediate between extensional ones and "homotopical" ones, where univalence holds, but truncation maps do not have to be monic. Kraus' paradox tells us that at least for homogeneous types this is wrong again: we have to take truncation maps with homogeneous domain to be monomorphisms if we want univalence to hold.

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