

# Quantifying separability in RAAGs via representations

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## Abstract

We answer the question in [8] and prove the following statement. Let  $L$  be a RAAG,  $H$  a word quasiconvex subgroup of  $L$ , then there is a finite dimensional representation of  $L$  that separates the subgroup  $H$  in the induced Zariski topology. As a corollary, we establish a polynomial upper bound on the size of the quotients used to separate  $H$  in  $L$ . This implies the same statement for a virtually special group  $L$  and, in particular, a fundamental groups of a hyperbolic 3-manifold.

## 1 Introduction

A subgroup  $H < G$  is *separable* if for any  $g \in G - H$  there exist a homomorphism  $\phi : G \rightarrow K$ , where  $K$  is finite and  $\phi(g) \notin \phi(H)$ . Alternatively,  $H = \bigcap_{H \leq L \leq G, [G:L] < \infty} L$ . Residual finiteness means that the trivial subgroup  $1 < G$  is separable. It was shown in [6, Theorem F] that every word quasiconvex subgroup of a finitely generated right-angled Artin group (RAAG) is a virtual retract, and hence is separable. If  $B$  is a virtually special compact cube complex such that  $\pi_1(B)$  is word-hyperbolic, then every quasiconvex subgroup of  $\pi_1(B)$  is separable [7]. For both these cases we will quantify separability. Namely, we answer the question in [8] and prove the following statement. Let  $L$  be a RAAG, if  $H$  is a cubically convex-cocompact subgroup of  $L$ , then there is a finite dimensional representation of  $L$  that separates the subgroup  $H$  in the induced Zariski topology. As a corollary, we establish a polynomial upper bound on the size of the quotients used to separate  $H$  in  $L$ . This implies the same statement for a virtually special group  $L$  and, in particular, a fundamental groups of hyperbolic 3-manifold.

**Definition 1.** A subset  $S$  of a geodesic metric space  $X$  is  $K$ -quasiconvex if for every geodesic  $\gamma$  in  $X$  whose endpoints lie in  $S$ , the  $K$ -neighborhood of  $S$  contains  $\gamma$ . We say that  $S$  is convex if it is 0-quasiconvex. A subcomplex  $Y$

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of a CAT(0) cube complex  $X$  is convex provided  $Y$  is connected and for each vertex  $v$  of  $Y$  the link of  $Y$  at  $v$  is a full subcomplex of the link of  $X$  at  $v$ . The combinatorial convex hull of a subcomplex  $Y \subset X$  is the intersection of all convex subcomplexes containing  $Y$ .

A group  $H$  acting on a geodesic metric space  $X$  is quasiconvex if the orbit  $Hx$  is a  $K$ -quasiconvex subspace of  $X$  for some  $K > 0$  and some  $x \in X$ . If  $H$  preserves a convex closed subset  $C$  and is cocompact on  $C$  we say that  $H$  is convex cocompact. Convexity clearly implies quasiconvexity.

We will use the previous notions in the following context: either  $X$  is a CAT(0) cube complex equipped with its CAT(0) metric, or  $X$  is the set of vertices of a cube complex equipped with the combinatorial distance (here a geodesic is the sequence of vertices of a combinatorial geodesic of the 1-skeleton). In the first case we say that  $H$  is CAT(0) quasiconvex, in the second that  $H$  is combinatorially quasiconvex (or word quasiconvex). For special cube complexes (in particular, Salvetti complexes) these notions coincide.

Convex cocompact subgroups of RAAGs are virtual retracts. A subgroup  $H$  of a RAAG is word quasi-convex if and only if it is convex cocompact [6]. For hyperbolic groups all definitions of quasiconvexity coincide and do not depend on a generating set.

A local isometry  $\phi : Y \rightarrow X$  of cube complexes is a locally injective combinatorial map with the property that, if  $e_1, \dots, e_n$  are 1-cubes (edges) of  $Y$  all incident to a 0-cube (vertex)  $y$ , and the (necessarily distinct) 1-cubes  $\phi(e_1), \dots, \phi(e_n)$  all lie in a common  $n$ -cube  $c$  (containing  $\phi(y)$ ), then  $e_1, \dots, e_n$  span an  $n$ -cube  $c_0$  in  $Y$  with  $\phi(c_0) = c$ . If  $\phi : Y \rightarrow X$  is a local isometry and  $X$  is nonpositively-curved, then  $Y$  is as well. Moreover,  $\phi$  lifts to an embedding  $\tilde{\phi} : \tilde{Y} \rightarrow \tilde{X}$  of universal covers, and  $\tilde{Y}$  is convex in  $\tilde{X}$ .

**Theorem 1.** *Let  $L$  be a RAAG. If  $H$  is a word quasiconvex subgroup of  $L$ , then there is a faithful representation  $\rho_H : L \rightarrow GL(V)$  such that  $\overline{\rho_H(H)} \cap \rho_H(L) = \rho_H(H)$ , where  $\overline{\rho_H(H)}$  is the Zariski closure of  $\rho_H(H)$ .*

**Theorem 2.** *Let  $L$  be a virtually special group. If  $H$  is a word quasiconvex subgroup of  $L$ , then there is a faithful representation  $\rho_H : L \rightarrow GL(V)$  such that  $\overline{\rho_H(H)} \cap \rho_H(L) = \rho_H(H)$ , where  $\overline{\rho_H(H)}$  is the Zariski closure of  $\rho_H(H)$ .*

**Theorem 3.** *Let  $L$  be a hyperbolic virtually special group. If  $H$  is a quasiconvex subgroup of  $L$ , then there is a faithful representation  $\rho_H : L \rightarrow GL(V)$  such that  $\overline{\rho_H(H)} \cap \rho_H(L) = \rho_H(H)$ , where  $\overline{\rho_H(H)}$  is the Zariski closure of  $\rho_H(H)$ .*

**Corollary 4.** *Let  $L$  and  $H$  be as in the theorems above. Then there exists a constant  $N > 0$  such that for each  $g \in L - H$ , there exist a finite group  $Q$  and a homomorphism  $\varphi : L \rightarrow Q$  such that  $\varphi(g) \notin \varphi(H)$  and  $|Q| \leq \|g\|_S^N$ . If  $K = H \ker \varphi$ , then  $K$  is a finite-index subgroup of  $L$  whose index is at most  $|Q| \leq \|g\|_S^N$  with  $H \leq K$  and  $g \notin K$ . Moreover, the index of the normal core of the subgroup  $K$  is bounded above by  $|Q|$ .*

The groups covered by this corollary include fundamental groups of hyperbolic 3-manifolds,  $C'(1/6)$  small cancellation groups and, therefore, random groups for density less than  $1/12$ .

Our Theorems and Corollary 4 generalize results for free groups, surface groups from [8] and for limit groups [4]. We use [8] to deduce Corollary 4 from the theorems. Corollary 4 establishes polynomial bounds on the size of the normal core of the finite index subgroup used in separating  $g$  from  $H$ . The constant  $N$  explicitly depends on the subgroup  $H$  and the dimension of  $V$  in Theorem 1. For a general finite index subgroup, the upper bound for the index of the normal core is factorial in the index of the subgroup. It is for this reason that we include the statement about the normal core of  $K$  at the end of the corollary.

Recently, several effective separability results have been established; see [1]-[8], [9]-[11], [12]-[16]. Most relevant here are papers [8], [5]. The methods used in [5] give linear bounds in terms of the word length of  $|g|$  on the index of the subgroup used in the separation but do not produce polynomial bounds for the normal core of that finite index subgroup.

## 2 Fundamental group of the canonical completion

**Proposition 5.** [2, Theorem 2.6] *Let  $\Gamma$  be a finite graph,  $S(\Gamma)$  Salvetti complex,  $Y$  a compact cube complex that has a local isometric embedding in  $S(\Gamma)$  with respect to  $Cat(0)$  metric (So  $\pi_1(Y) \leq \pi_1(S(\Gamma))$ ). Then  $Y$  can be embedded in  $C(Y)$  that is a finite cover of  $S(\Gamma)$  by canonically completing the 1-skeleton and then adding cubes everywhere where there is a boundary of a cube.*

*Proof.* We define the *canonical completion* from [2] because we need this construction to analyse fundamental groups of  $Y$  and  $C(Y)$ . Since  $Y$  has a local isometric embedding in  $S(\Gamma)$ , each edge in  $Y$  is naturally equipped with a direction and labelling is induced by a generator of  $\pi_1(S(\Gamma))$ . Each vertex  $v$  of  $Y$  either has an edge  $x$  incoming and outgoing, or does not have  $x$  adjacent to it at all, or has  $x$  either incoming or outgoing. For each label  $x$  we consider connected paths with all the edges labelled by  $x$ . If such a path  $p$  is a cycle, we leave it alone. If the initial vertex  $v$  of the path  $p$  has valency 1 with respect to  $x$ , then the final vertex  $u$  has valency 1 as well because of  $y$  being isometrically embedded in  $S(\Gamma)$ . In this case we add an edge labelled by  $x$  with the initial vertex  $u$  and the final vertex  $v$ . If a vertex  $w$  does not have edges with label  $x$  adjacent to it, we add a loop with the label  $x$  to the vertex  $w$ . Now we add cubes everywhere where there is a boundary of a cube. See the example in Fig 1, we basically borrowed it from N. Lazarovich's talk in Montreal. □

This proposition implies the following.

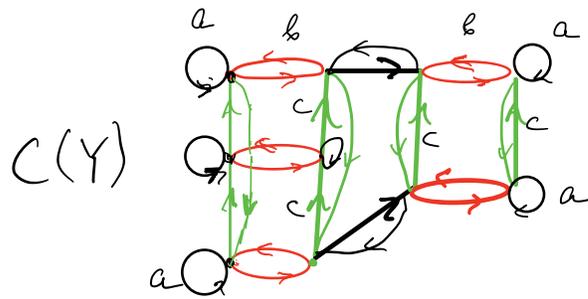
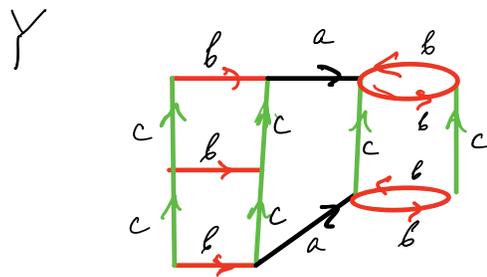


Figure 1: Example

**Proposition 6.** *Let  $L$  be a RAAG,  $H$  a word quasiconvex subgroup. Then there exists a finite index subgroup  $K$  obtained from  $H$  by adding some conjugates of powers of standard generators of  $L$ . Algebraically,  $K$  is obtained from  $H$  by a series of the following operations: 1) free product with infinite cyclic group, 2) HNN-extension such that associated subgroups are the same and with identical isomorphism.*

*Proof.* Let  $L = A_\Gamma$  be a RAAG,  $S_\Gamma$  its Salvetti complex. It is explained in the proof of Corollary B in [5] that  $H = \pi_1(Y)$ , where  $Y$  is a compact connected cube complex, based at a 0-cube  $x$ , with a based local isometry  $f : Y \rightarrow S_\Gamma$ .

When constructing the canonical completion  $C(Y)$  in the proof of Proposition 5 we add edges that create loops that correspond to conjugates of  $x^k$ , where  $x$  is a generator of  $L$ .

Now we will prove the second statement of the Proposition by induction on the number of steps needed to complete the one-skeleton of  $Y$ . We will only work with 2-skeletons since they determine fundamental groups. We take an incomplete vertex  $v$  of  $Y$ . A step is the following transformation.

Case 1.  $v$  has valency 0 in  $x$ . Then we add a loop labelled by  $x$  in  $v$  and to all vertices connected to  $v$  by a path with label that commutes with  $x$ .

Case 2. Vertex  $v$  has valency 1 in  $x$ . Suppose  $v$  has an outgoing edge labelled by  $x$ . Let  $u$  be the endpoint of the path from  $v$  with edges labelled by  $x$ . Then we connect  $u$  with  $v$  by an edge labelled by  $x$  and do the same in all vertices connected to  $v$  by a path with label that commutes with  $x$ . We can do this because  $f$  is a local isometry.

Case 3. Vertex  $v$  has valency 2 in  $x$ . If there is no loop labelled by  $x^k$  starting at  $v$ , we complete the path labelled by a power of  $x$  and passing through  $v$ . We also do the same in all vertices connected to  $v$  by a path with label that commutes with  $x$ . We can do this because  $f$  is a local isometry.

Case 4. If  $\deg_x(v) = 2$  and there is a loop labelled by  $x^k$  starting at  $v$ , then we do nothing.

By [2, Lemma 2.7], if in the obtained extension of  $Y$  (denoted  $Y_1$ ) we have a lift of the boundary of 2-cube, then this lift is a closed path. To complete the step, we fill in newly obtained lifts of boundaries of 2-cubes by 2-cubes. Denote the obtained complex by  $Y_2$ .

In case 4  $\pi_1(Y) = \pi_1(Y_2)$ . In all other cases we added an extra generator  $t = hx^k h^{-1}$ , where  $h \in A(\Gamma)$ .

Now we claim that in Cases 1-3  $\pi_1(Y_2)$  is an HNN-extension of  $\pi_1(Y)$  with stable letter  $t$ . We can move the base point  $v$  to  $hv$ , then  $t = x^k$ . Both  $\pi_1(Y), \pi_1(Y_2)$  have a local isometry into  $S(\Gamma)$ . Suppose there is a relation in  $\pi_1(Y_2)$

$$x^{\pm k} g_1 x^{\pm k} g_2 \dots g_t = 1,$$

where  $g_1, \dots, g_t \in \pi_1(Y)$ . We will show by induction on  $t$  that it follows from relations  $[x^k, g] = 1$ , where  $g \in \pi_1(Y)$  and  $g$  does not contain  $x$ . For some  $i$  it should be  $[g_i, x] = 1$ . Then  $g_i = x^{k_1} g$ , where  $g \in A(\Gamma)$ ,  $[g, x] = 1$  and  $g$  does not contain  $x$ . Suppose  $k_1 \neq 0$ . Then, since  $x^k \in \pi_1(Y_2)$ , for some  $k_2$  (in particular,  $k_2 = k_1 k$ ),  $g^{k_2} \in \pi_1(Y_2)$ . Therefore  $g^{k_2} \in \pi_1(Y)$  (indeed, making a step we do

not glue together different vertices). But then  $x^{k_1 k_2} \in \pi_1(Y)$ , contradiction with the assumption that we are in Cases 1-3 and that  $k_1 \neq 0$ . Therefore  $k_1 = 0$  and  $g \in \pi_1(Y)$ . Relation  $[x^k, g] = 1$  is an HNN-extension relation. We can apply it, decrease  $t$  and use induction.

Therefore, if  $x^k$  appears in a relation of  $\pi_1(Y_2)$ , then this relation is a consequence of relations  $[x^k, g] = 1$  for some  $g \in \pi_1(Y)$  such that  $[g, x] = 1$ . This proves the claim. Since the canonical completion is constructed by a sequence of such steps, the proposition is proved.  $\square$

### 3 Representations

**Definition 2.** [8] Let  $G$  be a finitely generated group and  $H$  a finitely generated subgroup of  $G$ . For a complex affine algebraic group  $\mathbf{G}$  and any representation  $\rho_0 \in \text{Hom}(G, \mathbf{G})$ , we have the closed affine subvariety

$$R_{\rho_0, H}(G, \mathbf{G}) = \{\rho \in \text{Hom}(G, \mathbf{G}) : \rho_0(h) = \rho(h) \text{ for all } h \in H\}$$

The representation  $\rho_0$  is said to *strongly distinguish*  $H$  in  $G$  if there exist representations  $\rho, \rho' \in R_{\rho_0, H}(G, \mathbf{G})$  such that  $\rho(g) \neq \rho'(g)$  for all  $g \in G - H$ .

**Lemma 7.** [8, Lemma 3.1] *Let  $G$  be a finitely generated group,  $\mathbf{G}$  a complex algebraic group, and  $H$  a finitely generated subgroup of  $G$ . If  $H$  is strongly distinguished by a representation  $\rho \in \text{Hom}(G, \mathbf{G})$ , then there exists a representation  $\varrho : G \rightarrow \mathbf{G} \times \mathbf{G}$  such that  $\varrho(G) \cap \overline{\varrho(H)} = \overline{\varrho(H)}$ , where  $\overline{\varrho(H)}$  is the Zariski closure of  $\varrho(H)$  in  $\mathbf{G} \times \mathbf{G}$ .*

**Proposition 8.** *Let  $L$  be a RAAG and  $H$  a word quasiconvex finitely generated subgroup. There exist a finite-index subgroup  $K \leq L$  and a faithful representation  $\rho_\omega : K \rightarrow \mathbf{G}$  that strongly distinguishes  $H$  in  $K$ .*

*Proof.* Let  $K$  be a finite index subgroup from Proposition 6. Let  $\rho$  be a faithful representation of  $K$  in  $\mathbf{G}$ . Since  $K$  is obtained from  $H$  as in Proposition 6, we can write

$$H = K_0 < \dots K_i < \dots K_n = K,$$

where  $K_i = \langle K_{i-1}, t_i \mid [H_{i-1}, t_i] = 1 \rangle$ . Then  $H$  is strongly distinguished in  $K$  by the representation  $\rho$  because we can take  $\rho$  and  $\rho'$  to be the same on  $H$  and  $\rho'(t_i) = \rho(t_i)^k$ , for  $k > 1$ ,  $i = 1, \dots, n$ .  $\square$

Let us prove Theorem 1. The proof of [8, Theorem 1.1] shows that it is sufficient to have a representation of  $K$  that strongly distinguishes  $H$ . Indeed, like in [8, Corollary 3.4], we can construct a representation  $\Phi : K \rightarrow GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$  such that  $\Phi(g) \in \text{Diag}(GL(2, \mathbb{C}))$  if and only if  $g \in H$ . Setting  $d_H = [L : K]$ , we have the induced representation

$$\text{Ind}_K^G(\Phi) : L \rightarrow GL(2d_H, \mathbb{C}) \times GL(2d_H, \mathbb{C}).$$

Recall, that when  $\Phi$  is represented by the action on the vector space  $V$  and  $L = \cup_{i=0}^t g_i K$ , then the induced representation acts on the disjoint union  $\sqcup_{i=0}^t g_i V$  as follows

$$g \Sigma g_i v_i = \Sigma g_{j(i)} \Phi(k_i) v_i,$$

where  $g g_i = g_{j(i)} k_i$ , for  $k_i \in K$ . Taking  $\rho = \text{Ind}_K^G(\Phi)$ , it follows from the construction of  $\rho$  and definition of induction that  $\rho(g) \in \overline{(\rho(H))}$  if and only if  $g \in H$ . If we set  $\rho = \rho_H$ , then Theorem 1 is proved.

Since a special group is a subgroup of a RAAG and we can extend a representation from a finite index subgroup to the whole group, Theorem 1 implies Theorem 2.

All definitions of quasiconvexity coincide in hyperbolic groups and Theorem 3 follows from Theorem 2.

## 4 Proof of Corollary 4

Given a complex algebraic group  $\mathbf{G} < GL(n, \mathbb{C})$ , there exist polynomials  $P_1, \dots, P_r \in \mathbb{C}[X_{i,j}]$  such that

$$\mathbf{G} = \mathbf{G}(\mathbb{C}) = V(P_1, \dots, P_r) = \left\{ X \in \mathbb{C}^{n^2} \mid P_k(X) = 0, k = 1, \dots, r \right\}$$

We refer to the polynomials  $P_1, \dots, P_r$  as *defining polynomials* for  $\mathbf{G}$ . We will say that  $\mathbf{G}$  is  $K$ -defined for a subfield  $K \subset \mathbb{C}$  if there exists defining polynomials  $P_1, \dots, P_r \in K[X_{i,j}]$  for  $\mathbf{G}$ . For a complex affine algebraic subgroup  $\mathbf{H} < \mathbf{G} < GL(n, \mathbb{C})$ , we will pick the defining polynomials for  $\mathbf{H}$  to contain a defining set for  $\mathbf{G}$  as a subset. Specifically, we have polynomials  $P_1, \dots, P_{r_{\mathbf{G}}}, P_{r_{\mathbf{G}}+1}, \dots, P_{r_{\mathbf{H}}}$  such that

$$\mathbf{G} = V(P_1, \dots, P_{r_{\mathbf{G}}}) \text{ and } \mathbf{H} = V(P_1, \dots, P_{r_{\mathbf{H}}}) \quad (1)$$

If  $\mathbf{G}$  is defined over a number field  $K$  with associated ring of integers  $\mathcal{O}_K$ , we can find polynomials  $P_1, \dots, P_r \in \mathcal{O}_K[X_{i,j}]$  as a defining set by clearing denominators. For instance, in the case when  $K = \mathbb{Q}$  and  $\mathcal{O}_K = \mathbb{Z}$ , these are multivariable integer polynomials.

For a fixed finite set  $X = \{x_1, \dots, x_t\}$  with associated free group  $F(X)$  and any group  $G$ , the set of homomorphisms from  $F(X)$  to  $G$ , denoted by  $\text{Hom}(F(X), G)$ , can be identified with  $G^t = G_1 \times \dots \times G_t$ . For any point  $(g_1, \dots, g_t) \in G^t$ , we have an associated homomorphism  $\varphi_{(g_1, \dots, g_t)} : F(X) \rightarrow G$  given by  $\varphi_{(g_1, \dots, g_t)}(x_i) = g_i$ . For any word  $w \in F(X)$ , we have a function  $\text{Eval}_w : \text{Hom}(F(X), G) \rightarrow G$  defined by  $\text{Eval}_w(\varphi_{(g_1, \dots, g_t)})(w) = w(g_1, \dots, g_t)$ . For a finitely presented group  $\Gamma$ , we fix a finite presentation  $\langle \gamma_1, \dots, \gamma_t \mid r_1, \dots, r_{t'} \rangle$ , where  $X = \{\gamma_1, \dots, \gamma_t\}$  generates  $\Gamma$  as a monoid and  $\{r_1, \dots, r_{t'}\}$  is a finite set of relations. If  $\mathbf{G}$  is a complex affine algebraic subgroup of  $GL_n(n, \mathbb{C})$ , the set  $\text{Hom}(\Gamma, \mathbf{G})$  of homomorphisms  $\rho : \Gamma \rightarrow \mathbf{G}$  can be identified with an affine subvariety of  $G^t$ . Specifically,

$$\text{Hom}(\Gamma, \mathbf{G}) = \left\{ (g_1, \dots, g_t) \in \mathbf{G}^t \mid r_j(g_1, \dots, g_t) = I_n \text{ for all } j \right\} \quad (2)$$

If  $\Gamma$  is finitely generated,  $\text{Hom}(\Gamma, \mathbf{G})$  is an affine algebraic variety by the Hilbert Basis Theorem.

The set  $\text{Hom}(\Gamma, \mathbf{G})$  also has a topology induced by the analytic topology on  $G^t$ . There is a Zariski open subset of  $\text{Hom}(\Gamma, \mathbf{G})$  that is smooth in this topology called the smooth locus, and the functions  $\text{Eval}_w : \text{Hom}(\Gamma, \mathbf{G}) \rightarrow \mathbf{G}$  are analytic on the smooth locus. For any subset  $S \in G$  and representation  $\rho \in \text{Hom}(\Gamma, \mathbf{G})$ ,  $\overline{\rho(S)}$  will denote the Zariski closure of  $\rho(S)$  in  $\mathbf{G}$ .

**Lemma 9.** (*[8, Lemma 5.1]*) *Let  $\mathbf{G} \leq GL(n, \mathbb{C})$  be a  $\overline{\mathbb{Q}}$ -algebraic group,  $L \leq \mathbf{G}$  be a finitely generated subgroup, and  $\mathbf{A} \leq \mathbf{G}$  be a  $\overline{\mathbb{Q}}$ -algebraic subgroup. Then,  $H = L \cap \mathbf{A}$  is closed in the profinite topology.*

*Proof.* Given  $g \in L - H$ , we need a homomorphism  $\varphi : L \rightarrow Q$  such that  $|Q| < \infty$  and  $\varphi(g) \notin \varphi(H)$ . We first select polynomials  $P_1, \dots, P_{r_{\mathbf{G}}}, \dots, P_{r_{\mathbf{A}}} \in \mathbb{C}[X_{i,j}]$  satisfying (1). Since  $\mathbf{G}$  and  $\mathbf{A}$  are  $\overline{\mathbb{Q}}$ -defined, we can select  $P_j \in \mathcal{O}_{K_0}[X_{i,j}]$  for some number field  $K_0/\mathbb{Q}$ . We fix a finite set  $\{l_1, \dots, l_{r_L}\}$  that generates  $L$  as a monoid. In order to distinguish between elements of  $L$  as an abstract group and the explicit elements in  $\mathbf{G}$ , we set  $l = M_l \in \mathbf{G}$  for each  $l \in L$ . In particular, we have a representation given by  $\rho_0 : L \rightarrow \mathbf{G}$  given by  $\rho_0(l_t) = M_{l_t}$ . We set  $K_L$  to be the field generated over  $K_0$  by the set of matrix entries  $\left\{ (M_{l_t})_{i,j} \right\}_{t,i,j}$ . It is straightforward to see that  $K_L$  is independent of the choice of the generating set for  $L$ . Since  $L$  is finitely generated, the field  $K_L$  has finite transcendence degree over  $\mathbb{Q}$  and so  $K_L$  is isomorphic to a field of the form  $K(T)$  where  $K/\mathbb{Q}$  is a number field and  $T = \{T_1, \dots, T_d\}$  is a transcendental basis (See [8]). For each,  $M_{l_t}$ , we have  $(M_{l_t})_{i,j} = F_{i,j,t}(T) \in K_L$ . In particular, we can view the  $(i,j)$ -entry of the matrix  $M_{l_t}$  as a rational function in  $d$  variables with coefficients in some number field  $K$ . Taking the ring  $R_L$  generated over  $\mathcal{O}_{K_0}$  by the set  $\left\{ (M_{l_t})_{i,j} \right\}_{t,i,j}$ ,  $R_L$  is obtained from  $\mathcal{O}_{K_0}[T_1, \dots, T_d]$  by inverting a finite number of integers and polynomials. Any ring homomorphism  $R_L \rightarrow R$  induces a group homomorphism  $GL(n, R_L) \rightarrow GL(n, R)$ , and since  $L \leq GL(n, R_L)$ , we obtain  $L \rightarrow GL(n, R)$ . If  $g \in L - H$  then there exists  $r_{\mathbf{G}} < j_g \leq r_{\mathbf{A}}$  such that  $Q_g = P_{j_g} \left( (M_{l_1})_{1,1}, \dots, (M_{l_{n,n}})_{n,n} \right) \neq 0$ . Using Lemma 2.1 in [3], we have a ring homomorphism  $\psi_R : R_L \rightarrow R$  with  $|R| < \infty$  such that  $\psi_R(Q_g) \neq 0$ . Setting,  $\rho_R : GL(n, R_L) \rightarrow GL(n, R)$  we assert that  $\rho_R(g) \notin \rho_R(H)$ . To see this, set  $\overline{M}_\eta = \rho_R(\eta)$  for each  $\eta \in L$ , and note that  $\psi_R(P_{j_g}((M_\eta)_{1,1}, \dots, (M_\eta)_{n,n})) = P_{j_g}((\overline{M}_\eta)_{1,1}, \dots, (\overline{M}_\eta)_{n,n})$ . For each  $h \in H$ , we know that  $P_{j_i}((M_h)_{i,j}) = 0$  and so  $P_{j_i}((\overline{M}_\eta)_{1,1}, \dots, (\overline{M}_\eta)_{n,n}) = 0$ . However, by selection of  $\psi_R$ , we know that  $\psi_R(Q_g) \neq 0$  and so  $\rho_R(g) \notin \rho_R(H)$ .  $\square$

Theorems 1, 2, 3 and Lemma 9 imply Corollary 4.

*Proof.* Since  $H \leq L$  is word quasiconvex, by Theorems 1, 2, 3 there is a faithful representation

$$\rho_H : L \rightarrow GL(n, \mathbb{C})$$

such that  $\overline{\rho_H(H)} \cap \rho_H(L) = \rho_H(H)$ . We can construct the representation in Theorem 1 so that  $\mathbf{G} = \overline{\rho_H(L)}$  and  $\mathbf{A} = \overline{\rho_H(H)}$  are both  $\mathbb{Q}$ -defined. So, by Lemma 9, we can separate  $H$  in  $L$ . Next, we quantify the separability of  $H$  in  $L$ . Toward that end, we need to bound the order of the ring  $R$  in the proof of Lemma 9 in terms of the word length of the element  $g$ . Lemma 2.1 from [3] bounds the size of  $R$  in terms of the coefficient size and degree of the polynomial  $Q_g$ . It follows from a discussion on pp 412-413 of [3] that the coefficients and degree can be bounded in terms of the word length of  $g$ , and that the coefficients and degrees of the polynomials  $P_j$ . Because the  $P_j$  are independent of the word  $g$ , there exists a constant  $N_0$  such that  $|R| \leq \|g\|^{N_0}$ . By construction, the group  $Q$  we seek is a subgroup of  $GL(n, R)$ . Thus,  $|Q| \leq |R|^{n^2} \leq \|g\|^{N_0 n^2}$ . Taking  $N = N_0 n^2$  completes the proof.  $\square$

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