ON THE UNIT GROUP AND THE 2-CLASS NUMBER OF $\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$

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Abstract. Let $p \equiv 1 \pmod 8$ and $q \equiv 7 \pmod 8$ be two prime numbers. The purpose of this paper is to compute the unit group of the fields $\mathbb{L} = \mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$ and give their 2-class numbers.

1. Introduction

Let k be a number field of degree n (i.e., $[k:\mathbb{Q}]=n$). Denote by E_k the unit group of k that is the group of the invertible elements of the ring $\mathcal{O}_k$ of algebraic integers of the number field k. By the well known Dirichlet's unit theorem, if $n = r + 2s$, where r is the number of real embeddings and s the number of conjugate pairs of complex embeddings of k, then there exist $r = r+s-1$ units of $\mathcal{O}_k$ that generate E_k (modulo the roots of unity), and these r units are called the *fundamental system of units of* k. Therefore, it is well known that

$$E_k \simeq \mu(k) \times \mathbb{Z}^{r+s-1},$$

where $\mu(k)$ is the group of roots of unity contained in k.

One major problem in algebraic number theory (more precisely in theory of units of number fields which is related to almost all areas of algebraic number theory) is the computation of the fundamental system of units. For quadratic fields, the problem is easily solved. An early study of unit groups of multiquadratic fields was established by Varmon [10]. For quartic bicyclic fields, Kubota [13] gave a method for finding a fundamental system of units. Wada [14] generalized Kubota's method, creating an algorithm for computing fundamental units in any given multiquadratic field. However, in general, it is not easy to compute the unit group of a number field especially for number fields of degree more than 4. Actually, in literature there are only few examples of computation of the unit group of a given number field k of degree 8 (see [6, 7, 8]). In the present work, we focus on the computation of the unit group of the real triquadratic fields of the form $\mathbb{L} = \mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$, where $p \equiv 1 \pmod 8$ and $q \equiv 7 \pmod 8$, and furthermore, we give the class number of these fields.

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Notice that the motivation behind the computations of the unit group of these fields is the fact that $\mathbb{L}$ is the first layer of the cyclotomic $\mathbb{Z}_2$-extension of the biquadratic field $\mathbb{Q}(\sqrt{p},\sqrt{q})$ (cf. [6, 7]). Furthermore, computing the unit group of the fields $\mathbb{L}$ is also first step to find the unit group of all fields of the form $\mathbb{L}(\sqrt{-\ell})$, where $\ell \geq 1$ is a positive square-free integer (cf. [2]). We note that the unit group of these fields are useful for the study of the Hilbert 2-class field tower of the subfields $\mathbb{L}(\sqrt{-\ell})$ (see for example [8]). We note also that this paper is a continuation of the the paper [5] and for further works in the same direction we refer the reader to [4, 10, 12].

Let ε_ℓ (resp. $h_2(\ell)$) denote the fundamental unit of (resp. the 2-class number of) a real quadratic field $\mathbb{Q}(\sqrt{\ell})$, where $\ell > 1$ is a positive square-free integer. Let $h_2(k)$ denote the 2-class number of a number fields k.

2. Preliminaries

Let us start this section by recalling the method given in [14], that describes a fundamental system of units of a real multiquadratic field K_0. Let σ_1 and σ_2 be two distinct elements of order 2 of the Galois group of $K_0/\mathbb{Q}$. Let K_1, K_2 and K_3 be the three subextensions of K_0 invariant by σ_1, σ_2 and $\sigma_3 = \sigma_1\sigma_3$, respectively. Let ε denote a unit of K_0. Then

$$\varepsilon^2 = \varepsilon\varepsilon^{\sigma_1}\varepsilon\varepsilon^{\sigma_2}(\varepsilon^{\sigma_1}\varepsilon^{\sigma_2})^{-1},$$

and we have, $\varepsilon\varepsilon^{\sigma_1} \in E_{K_1}$, $\varepsilon\varepsilon^{\sigma_2} \in E_{K_2}$ and $\varepsilon^{\sigma_1}\varepsilon^{\sigma_2} \in E_{K_3}$. It follows that the unit group of K_0 is generated by the elements of E_{K_1}, E_{K_2} and E_{K_3}, and the square roots of elements of $E_{K_1}E_{K_2}E_{K_3}$ which are perfect squares in K_0.

This method is very useful for computing a fundamental system of units of a real biquadratic number field, however, in the case of real triquadratic number field the problem of the determination of the unit group becomes very difficult and demands some specific computations and eliminations, as what we will see in the next section. We shall consider the field $\mathbb{L} = \mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$, where p and q are two distinct prime numbers. Thus, we have the following diagram:

$$\begin{tikzcd}
 & \mathbb{L}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q}) & \\
k_1=\mathbb{Q}(\sqrt{2},\sqrt{p}) \arrow[ur] & k_2=\mathbb{Q}(\sqrt{2},\sqrt{q}) \arrow[u] & k_3=\mathbb{Q}(\sqrt{2},\sqrt{pq}) \arrow[ul] \\
 & \mathbb{Q}(\sqrt{2}) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

FIGURE 1. Intermediate fields of $\mathbb{L}/\mathbb{Q}(\sqrt{2})$

Let τ_1, τ_2 and τ_3 be the elements of $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ defined by

$$\begin{array}{lll}\tau_1(\sqrt{2})=-\sqrt{2}, & \tau_1(\sqrt{p})=\sqrt{p}, & \tau_1(\sqrt{q})=\sqrt{q},\\ \tau_2(\sqrt{2})=\sqrt{2}, & \tau_2(\sqrt{p})=-\sqrt{p}, & \tau_2(\sqrt{q})=\sqrt{q},\\ \tau_3(\sqrt{2})=\sqrt{2}, & \tau_3(\sqrt{p})=\sqrt{p}, & \tau_3(\sqrt{q})=-\sqrt{q}.\end{array}$$

Note that $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})=\langle\tau_1,\tau_2,\tau_3\rangle$ and the subfields k_1, k_2 and k_3 are fixed by $\langle\tau_3\rangle$, $\langle\tau_2\rangle$ and $\langle\tau_2\tau_3\rangle$ respectively. Therefore, a fundamental system of units of $\mathbb{L}$ consists of seven units chosen from those of k_1, k_2 and k_3, and from the square roots of the elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$. With these notations, we have:

Lemma 2.1 ([5], Lemma 2.1). *Let $p\equiv 1 \pmod 8$ be a prime number. Put $\varepsilon_{2p}=\beta+\alpha\sqrt{2p}$ with $\beta,\alpha\in\mathbb{Z}$. If $N(\varepsilon_{2p})=1$, then $\sqrt{\varepsilon_{2p}}=\frac{1}{\sqrt{2}}(\alpha_1+\alpha_2\sqrt{2p})$, for some integers α_1,α_2 such that $\alpha=\alpha_1\alpha_2$. It follows that:*

σ	$1+\tau_2$	$1+\tau_1\tau_2$	$1+\tau_1\tau_3$	$1+\tau_2\tau_3$	$1+\tau_1$
$\sqrt{\varepsilon_{2p}}^{\sigma}$	$(-1)^u$	$-\varepsilon_{2p}$	$(-1)^{u+1}$	$(-1)^u$	$(-1)^{u+1}$

(1)

for some u in $\{0,1\}$ such that $\frac{1}{2}(\alpha_1^2-2p\alpha_2^2)=(-1)^u$.

Lemma 2.2 ([1], Lemma 5). *Let $d>1$ be a square-free integer and $\varepsilon_d=x+y\sqrt{d}$, where x, y are integers or semi-integers. If $N(\varepsilon_d)=1$, then $2(x+1)$, $2(x-1)$, $2d(x+1)$ and $2d(x-1)$ are not squares in $\mathbb{Q}$.*

Lemma 2.3 ([3], Theorem 6). *Let $p\equiv 1 \pmod 4$ be a prime number. We have*
1) *If $N(\varepsilon_{2p})=-1$, then $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ is a fundamental system of units of $k_1=\mathbb{Q}(\sqrt{2},\sqrt{p})$.*
2) *If $N(\varepsilon_{2p})=1$, then $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_{2p}}\}$ is a fundamental system of units of $k_1=\mathbb{Q}(\sqrt{2},\sqrt{p})$.*

Now we recall the following useful lemmas:

Lemma 2.4 ([11]). *Let K be a multiquadratic number field of degree 2^n, $n\in\mathbb{N}$, and k_i the $s=2^n-1$ quadratic subfields of K. Then*

$$h(K)=\frac{1}{2^v}(E_K:\prod_{i=1}^{s}E_{k_i})\prod_{i=1}^{s}h(k_i),$$

with

$$v=\begin{cases} n(2^{n-1}-1); & \text{if } K \text{ is real},\\ (n-1)(2^{n-2}-1)+2^{n-1}-1 & \text{if } K \text{ is imaginary.}\end{cases}$$

Lemma 2.5. *Let $q\equiv 3 \pmod 4$ and $p\equiv 1 \pmod 4$ be two distinct primes. Then*
1) *By* [9, Corollary 18.4], *we have $h_2(p)=h_2(q)=h_2(2q)=h_2(2)=h_2(-2)=h_2(-q)=h_2(-1)=1$.*
2) *If $\left(\dfrac{p}{q}\right)=-1$, then $h_2(pq)=h_2(2pq)=h_2(-pq)=2$, else $h_2(pq)$, $h_2(2pq)$ and $h_2(-pq)$ are divisible by 4 (cf.* [9, Corollaries 19.6 and 19.7]).

3) *If* $q\equiv 3 \pmod 8$, *then* $h_2(-2q)=2$ (*cf.* [9, Corollary 19.6]).

3. Unit groups computation

We close this section with the following lemmas that are very useful in what follows.

3.1. The case: $p\equiv 1 \pmod 8$, $q\equiv 7 \pmod 8$ and $\left(\frac{p}{q}\right)=-1$.

The following lemmas are very useful in what follows to prove our first main theorem.

Lemma 3.1. *Let* $p\equiv 1 \pmod 8$ *and* $q\equiv 7 \pmod 8$ *be two primes such that* $\left(\frac{p}{q}\right)=-1$.

1) *Let* x *and* y *be two integers such that* $\varepsilon_{2pq}=x+y\sqrt{2pq}$. *Then we have*
   i. $(x+1)$ *is a square in* $\mathbb{N}$,
   ii. $\sqrt{2\varepsilon_{2pq}}=y_1+y_2\sqrt{2pq}$ *and* $2=y_1^2-2pqy_2^2$, *for some integers* y_1 *and* y_2.
2) *Let* v *and* w *be two integers such that* $\varepsilon_{pq}=v+w\sqrt{pq}$. *Then we have*
   i. $(v+1)$ *is a square in* $\mathbb{N}$,
   ii. $\sqrt{2\varepsilon_{pq}}=w_1+w_2\sqrt{pq}$ *and* $2=w_1^2-pqw_2^2$, *for some integers* w_1 *and* w_2.

Proof. As it is known that $N(\varepsilon_{2pq})=1$, then, by the unique factorization in $\mathbb{Z}$ and Lemma 2.2 there exist some integers y_1 and y_2 ($y=y_1y_2$) such that

$$(1):\begin{cases} x\pm 1=y_1^2\\ x\mp 1=2pqy_2^2,\end{cases}\quad (2):\begin{cases} x\pm 1=py_1^2\\ x\mp 1=2qy_2^2,\end{cases}\quad \text{or}\quad (3):\begin{cases} x\pm 1=2py_1^2\\ x\mp 1=qy_2^2,\end{cases}$$

$*$ System (2) can not occur since it implies $-1=\left(\frac{2qy_1^2}{p}\right)=\left(\frac{x\mp1}{p}\right)=\left(\frac{x\pm1\mp2}{p}\right)=\left(\frac{\mp2}{p}\right)=\left(\frac{2}{p}\right)=1$, which is absurd.

$*$ We similarly show that System (3) can not occur.

$*$ Assume that $\begin{cases} x-1=y_1^2\\ x+1=2pqy_2^2\end{cases}$. So $1=\left(\frac{y_1^2}{q}\right)=\left(\frac{x-1}{q}\right)=\left(\frac{x+1-2}{q}\right)=\left(\frac{-2}{q}\right)=-1$, which is a contradiction.

Therefore $\begin{cases} x+1=y_1^2\\ x-1=2pqy_2^2\end{cases}$ which gives the first item. The proof of the second item is analogous. □

Lemma 3.2. *Let* $q\equiv 7 \pmod 8$ *be a prime number.*

1) *Let* c *and* d *be two integers such that* $\varepsilon_{2q}=c+d\sqrt{2q}$. *Then we have*
   i. $c+1$ *is a square in* $\mathbb{N}$,
   ii. $\sqrt{2\varepsilon_{2q}}=d_1+d_2\sqrt{2q}$ *and* $2=d_1^2-2qd_2^2$, *for some integers* d_1 *and* d_2.
2) *Let* α *and* β *be two integers such that* $\varepsilon_q=\alpha+\beta\sqrt{q}$. *Then we have*
   i. $\alpha+1$ *is a square in* $\mathbb{N}$,
   ii. $\sqrt{2\varepsilon_q}=\beta_1+\beta_2\sqrt{q}$ *and* $2=\beta_1^2-q\beta_2^2$, *for some integers* β_1 *and* β_2.

Furthermore, for any prime number $p\equiv 1 \pmod 4$ *we have:*

ε	ε_2	ε_p	$\sqrt{\varepsilon_q}$	$\sqrt{\varepsilon_{2q}}$
$\varepsilon^{1+\tau_1}$	-1	ε_p^2	$-\varepsilon_q$	-1
$\varepsilon^{1+\tau_2}$	ε_2^2	-1	ε_q	ε_{2q}
$\varepsilon^{1+\tau_3}$	ε_2^2	ε_p^2	1	1
$\varepsilon^{1+\tau_1\tau_2}$	-1	-1	$-\varepsilon_q$	-1
$\varepsilon^{1+\tau_1\tau_3}$	-1	ε_p^2	-1	$-\varepsilon_{2q}$
$\varepsilon^{1+\tau_2\tau_3}$	ε_2^2	-1	1	1

(2)

Proof. For the two items see [8, Lemma 4.1]. The computations in the table follows from the definitions of τ_i and the two items. □

Theorem 3.3. *Let* $p\equiv 1 \pmod 8$ *and* $q\equiv 7 \pmod 8$ *be two primes such that* $\left(\frac{p}{q}\right)=-1$. *Put* $\mathbb{L}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. *Then*

1) *If* $N(\varepsilon_{2p})=-1$, *we have*
   - *The unit group of* $\mathbb{L}$ *is :*
$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}},\sqrt{\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}}\rangle.$$
   - *The* 2*-class group of* $\mathbb{L}$ *is cyclic of order* $\frac{1}{2}h_2(2p)$.
2) *If* $N(\varepsilon_{2p})=1$, *we have*
   - *The unit group of* $\mathbb{L}$ *is :*
$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2p}}},\sqrt{\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}}\rangle,$$
   where $a\in\{0,1\}$ *such that* $a\equiv u+1 \pmod 2$ *and* u *is defined in Lemma 2.1.*
   - *The* 2*-class group of* $\mathbb{L}$ *is cyclic of order* $h_2(2p)$.

Proof. 1) Assume that $N(\varepsilon_{2p})=-1$. By Lemma 2.3, $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ is a fundamental system of units of k_1. Using Lemmas 3.2 and 3.1, we check that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,
$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$
Thus we shall determine elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$. Let ξ is an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. We can assume that
$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$
where a,b,c,d,e,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2}=1+\tau_2$. We have $\sqrt{\varepsilon_{pq}}^{1+\tau_2}=1$, $\sqrt{\varepsilon_{2pq}}^{1+\tau_2}=1$ and $\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^{1+\tau_2}=(-1)^v\varepsilon_2$, for some $v\in\{0,1\}$. Thus, by (2) we have:

$$\begin{array}{rcl} N_{\mathbb{L}/k_2}(\xi^2) &=& \varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot 1\cdot(-1)^{gv}\varepsilon_2^g\\ &=& \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gv}\varepsilon_2^g.\end{array}$$

Thus $b+gv\equiv 0 \pmod 2$ and $g=0$. Therefore, $b=0$ and

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. We have $\sqrt{\varepsilon_{pq}}^{1+\tau_1\tau_2}=-1$ and $\sqrt{\varepsilon_{2pq}}^{1+\tau_1\tau_2}=-\varepsilon_{2pq}$. Thus, by (2) we have:

$$\begin{array}{rcl} N_{\mathbb{L}/k_5}(\xi^2) &=& (-1)^a\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot(-1)^e\cdot(-1)^f\cdot\varepsilon_{2pq}^f\\ &=& (-1)^{a+c+d+e+f}\varepsilon_q^c\cdot\varepsilon_{2pq}^f.\end{array}$$

Thus $a+c+d+e+f=0 \pmod 2$ and $f=c$. Thus, $a+d+e=0 \pmod 2$. Therefore,

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. We have $\sqrt{\varepsilon_{pq}}^{1+\tau_1\tau_3}=-1$ and $\sqrt{\varepsilon_{2pq}}^{1+\tau_1\tau_3}=-\varepsilon_{2pq}$. Thus, by (2) we have:

$$\begin{array}{rcl} N_{\mathbb{L}/k_5}(\xi^2) &=& (-1)^a\cdot(-1)^c\cdot(-1)^d\cdot\varepsilon_{2q}^d\cdot(-1)^e\cdot(-1)^c\cdot\varepsilon_{2pq}^c\\ &=& (-1)^{a+d+e}\varepsilon_{2q}^d\varepsilon_{2pq}^c.\end{array}$$

Thus $a+d+e=0 \pmod 2$ and $d=c$. Therefore

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^c\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_3}=1+\tau_2\tau_3$, with $k_3=\mathbb{Q}(\sqrt{2},\sqrt{pq})$. We have $\sqrt{\varepsilon_{pq}}^{1+\tau_2\tau_3}=\varepsilon_{pq}$ and $\sqrt{\varepsilon_{2pq}}^{1+\tau_2\tau_3}=\varepsilon_{2pq}$. Thus, by (2) we have:

$$\begin{array}{rcl} N_{\mathbb{L}/k_3}(\xi^2) &=& \varepsilon_2^{2a}\cdot 1\cdot 1\cdot\varepsilon_{pq}^e\cdot\varepsilon_{2pq}^c\\ &=& \varepsilon_2^{2a}\varepsilon_{pq}^c\varepsilon_{2pq}^c.\end{array}$$

We have nothing to deduce from this. Therefore, we apply another norm.

➠ Let us apply the norm $N_{\mathbb{L}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. We have $\sqrt{\varepsilon_{pq}}^{1+\tau_1}=-\varepsilon_{pq}$ and $\sqrt{\varepsilon_{2pq}}^{1+\tau_1}=-1$. Thus, by (2) we have:

$$\begin{array}{rcl} N_{\mathbb{L}/k_5}(\xi^2) &=& (-1)^a\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^c\cdot(-1)^e\cdot\varepsilon_{pq}^e\cdot(-1)^c\cdot\\ &=& (-1)^{a+c+e}\varepsilon_q^c\varepsilon_{pq}^e.\end{array}$$

Thus $a+c+e=0 \pmod 2$ and $c=e$. Hence, $a=0$ and

$$\xi^2=\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^c\sqrt{\varepsilon_{pq}}^c\sqrt{\varepsilon_{2pq}}^c.$$

Let us show that the square root of $\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}$ is an element of $\mathbb{L}$. Note that one can easily check that the 2-class group of $k_5=\mathbb{Q}(\sqrt{2p},\sqrt{q})$ is cyclic and by Lemmas 2.4 and 2.5, we have $h_2(k_5)=\frac{1}{4}q(k_5)h_2(2p)h_2(q)h_2(2pq)=\frac{1}{2}q(k_5)h_2(2p)$. Using Lemmas 3.1 and 3.2 (and the algorithm given in page 2), we easily deduce that that $q(k_5) = 2$. Thus $h_2(k_5) = h_2(2p)$. Since $\mathbb{L}/k_5$ is an unramified quadratic extension, then

$$h_2(\mathbb{L}) = \frac{1}{2} \cdot h_2(k_5) = \frac{1}{2} \cdot h_2(2p). \tag{3}$$

Assume by absurd that $\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}$ is not a square in $\mathbb{L}$. Then $q(\mathbb{L}) = 2^5$. By Lemma 2.4, we have:

$$\begin{aligned} h_2(\mathbb{L}) &= \frac{1}{2^9} q(\mathbb{L}) h_2(2) h_2(p) h_2(q) h_2(2p) h_2(2q) h_2(pq) h_2(2pq) \qquad (4) \\ &= \frac{1}{2^9} \cdot 2^5 \cdot 1 \cdot 1 \cdot 1 \cdot h_2(2p) \cdot 1 \cdot 2 \cdot 2 = \frac{1}{4} \cdot h_2(2p). \end{aligned}$$

Which is a contradiction with (3). Therefore $c = 1$ and $\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}$ is a square in $\mathbb{L}$. So the first item.

2) Assume that $N(\varepsilon_{2p}) = 1$. Then by Lemma 2.3, $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_{2p}}\}$ is a fundamental system of units of k_1 and from Lemmas 3.1 and 3.2 we deduce that $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. So we have:

$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_{2p}} \rangle.$$

Put

$$\xi^2 = \varepsilon_2^a \varepsilon_p^b \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}}^e \sqrt{\varepsilon_{2pq}}^f \sqrt{\varepsilon_{2p}}^g,$$

where a, b, c, d, e, f and g are in $\{0, 1\}$. We shall proceed as in the first item. Assume that $\xi \in \mathbb{L}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2} = 1 + \tau_2$. We have

$$\begin{aligned} N_{\mathbb{L}/k_2}(\xi^2) &= \varepsilon_2^{2a} \cdot (-1)^b \cdot \varepsilon_q^c \cdot \varepsilon_{2q}^d \cdot 1 \cdot 1 \cdot (-1)^{gu} \\ &= \varepsilon_2^{2a} \varepsilon_q^c \varepsilon_{2q}^d \cdot (-1)^{b+gu}. \end{aligned}$$

Thus, $b + gu \equiv 0 \pmod 2$.

➠ Let us apply the norm map $N_{\mathbb{L}/k_5} = 1 + \tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$. We have

$$\begin{aligned} N_{\mathbb{L}/k_5}(\xi^2) &= (-1)^a \cdot (-1)^b \cdot (-1)^c \cdot \varepsilon_q^c \cdot (-1)^d \cdot (-1)^e \cdot (-1)^f \cdot \varepsilon_{2pq}^f \cdot (-1)^g \cdot \varepsilon_{2p}^g \\ &= \varepsilon_{2p}^g (-1)^{a+b+c+d+e+f+g} \cdot \varepsilon_q^c \varepsilon_{2pq}^f \varepsilon_{2p}^g. \end{aligned}$$

Thus, $a + b + c + d + e + f + g \equiv 0 \pmod 2$ and $c + f + g \equiv 0 \pmod 2$. Therefore, $a + b + d + e \equiv 0 \pmod 2$.

➠ Let us apply the norm map $N_{\mathbb{L}/k_6} = 1 + \tau_1\tau_3$, with $k_6 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. We have

$$\begin{aligned} N_{\mathbb{L}/k_6}(\xi^2) &= (-1)^a \cdot \varepsilon_p^{2b} \cdot (-1)^c \cdot (-1)^d \cdot \varepsilon_{2q}^d \cdot (-1)^e \cdot (-1)^f \cdot \varepsilon_{2pq}^f \cdot (-1)^{gu+g} \\ &= \varepsilon_p^{2b} \cdot (-1)^{a+c+d+e+f+ug+g} \cdot \varepsilon_{2q}^d \varepsilon_{2pq}^f. \end{aligned}$$

Thus, $a+c+d+e+f+ug+g\equiv 0 \pmod 2$ and $d=f$. Then, $a+c+e+ug+g\equiv 0 \pmod 2$ and

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^d\sqrt{\varepsilon_{2p}}^g,$$

➠ By applying the norm map $N_{\mathbb{L}/k_3}=1+\tau_2\tau_3$, with $k_3=\mathbb{Q}(\sqrt{2},\sqrt{pq})$, we deduce nothing new.

➠ Let us apply the norm $N_{\mathbb{L}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. We have

$$\begin{aligned} N_{\mathbb{L}/k_5}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2b}\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot(-1)^e\cdot\varepsilon_{pq}^e\cdot(-1)^d\cdot(-1)^{gu+g}\\ &= \varepsilon_p^{2b}(-1)^{a+c+e+gu+g}\varepsilon_q^c\varepsilon_{pq}^e. \end{aligned}$$

Thus, $a+c+e+gu+g\equiv 0 \pmod 2$ and $c=e$. Then, $a+gu+g\equiv 0 \pmod 2$ and

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^e\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^d\sqrt{\varepsilon_{2p}}^g,$$

with

$$b+gu\equiv 0 \pmod 2 \tag{5}$$
$$e+d+g\equiv 0 \pmod 2 \tag{6}$$
$$a+b+d+e\equiv 0 \pmod 2 \tag{7}$$
$$a+ug+g\equiv 0 \pmod 2 \tag{8}$$

On the other hand, as in the proof of the first item, we show that $h_2(\mathbb{L})=h_2(2p)$ and that $q(\mathbb{L})=2^7$. So if $g=0$, then (5) and (8) $a=b=0$ and so by (7), $d=e$. Thus, $\xi^2=\sqrt{\varepsilon_q}^e\sqrt{\varepsilon_{2q}}^e\sqrt{\varepsilon_{pq}}^e\sqrt{\varepsilon_{2pq}}^e$, with $e=0$ or 1. In the two cases we have $q(\mathbb{L})\neq 2^7$. Therefore, $g=1$ and so by (6) $e\neq d$. By (5) and (8), $b=u\neq a$ and $a\equiv u+1 \pmod 2$. Hence, necessarily the two equations following equations have solution in $\mathbb{L}$: $\xi^2=\varepsilon_2^a\varepsilon_p^u\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ and $\xi^2=\varepsilon_2^a\varepsilon_p^u\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2p}}$, where $a\equiv u+1 \pmod 2$ and u is defined in Page 3. Since, $q(\mathbb{L})=2^7$, these two equations are necessarily solvable in $\mathbb{L}$, since . Which completes the proof.

□

3.2. The case: $p\equiv 1 \pmod 8$, $q\equiv 7 \pmod 8$ and $\left(\frac{p}{q}\right)=1$.

Lemma 3.4. *Let $p\equiv 1 \pmod 8$ and $q\equiv 7 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right)=1$.*

1) *Let x and y be two integers such that $\varepsilon_{2pq}=x+y\sqrt{2pq}$. Then*
   i. *$(x+1)$, $p(x+1)$ or $2p(x+1)$ is a square in $\mathbb{N}$,*
   ii. *Furthermore, we have*
      a) *If $(x+1)$, then $\sqrt{2\varepsilon_{2pq}}=y_1+y_2\sqrt{2pq}$ and $2=y_1^2-2pqy_2^2$.*
      b) *If $p(x+1)$, then $\sqrt{2\varepsilon_{2pq}}=y_1\sqrt{p}+y_2\sqrt{2q}$ and $2=py_1^2-2qy_2^2$.*
      c) *If $2p(x+1)$, then $\sqrt{2\varepsilon_{2pq}}=y_1\sqrt{2p}+y_2\sqrt{q}$ and $2=2py_1^2-qy_2^2$.*

Where y_1 and y_2 are two integers such that $y=y_1y_2$.

2) *Let v and w be two integers such that $\varepsilon_{pq}=v+w\sqrt{pq}$. Then we have*
   i. $(v+1)$, $p(v+1)$ *or* $2p(v+1)$ *is a square in* $\mathbb{N}$,
   ii. *Furthermore, we have*
      a) *If* $(v+1)$, *then* $\sqrt{2\varepsilon_{pq}}=w_1+w_2\sqrt{pq}$ *and* $2=w_1^2-pqw_2^2$.
      b) *If* $p(v+1)$, *then* $\sqrt{2\varepsilon_{pq}}=w_1\sqrt{p}+w_2\sqrt{q}$ *and* $2=pw_1^2-qw_2^2$.
      c) *If* $2p(v+1)$, *then* $\sqrt{\varepsilon_{pq}}=w_1\sqrt{p}+w_2\sqrt{q}$ *and* $1=pw_1^2-qw_2^2$.

   Where w_1 and w_2 are two integers such that $w=w_1w_2$ in a) and b), and $w=2w_1w_2$ in c).

Proof. We proceed as in the proof of 3.1. □

ε	Conditions	$\varepsilon^{1+\tau_2}$	$\varepsilon^{1+\tau_1\tau_2}$	$\varepsilon^{1+\tau_1\tau_3}$	$\varepsilon^{1+\tau_2\tau_3}$	$\varepsilon^{1+\tau_1}$
$\sqrt{\varepsilon_{2pq}}$	$(x+1)$ is a square in $\mathbb{N}$	1	$-\varepsilon_{2pq}$	$-\varepsilon_{2pq}$	ε_{2pq}	-1
	$p(x+1)$ is a square in $\mathbb{N}$	-1	ε_{2pq}	$-\varepsilon_{2pq}$	$-\varepsilon_{2pq}$	-1
	$2p(x+1)$ is a square in $\mathbb{N}$	-1	$-\varepsilon_{2pq}$	ε_{2pq}	$-\varepsilon_{2pq}$	1
$\sqrt{\varepsilon_{pq}}$	$(v+1)$ is a square in $\mathbb{N}$	1	-1	-1	ε_{pq}	$-\varepsilon_{pq}$
	$p(v+1)$ is a square in $\mathbb{N}$	-1	1	-1	$-\varepsilon_{pq}$	$-\varepsilon_{pq}$
	$2p(v+1)$ is a square in $\mathbb{N}$	-1	-1	1	$-\varepsilon_{pq}$	ε_{pq}

TABLE 1. Norms maps on units

Let $p\equiv 1 \pmod 8$ and $q\equiv 7 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right)=1$. Then, by Lemmas 2.4 and 2.4, we have:

$$\frac{1}{2^9}q(\mathbb{L})\cdot h_2(2p)\cdot h_2(pq)\cdot h_2(2pq). \tag{9}$$

The above lemma shows that we have nine cases as we distinguished in the following theorems.

Theorem 3.5. *Let $p\equiv 1 \pmod 8$ and $q\equiv 7 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right)=1$. Put $\mathbb{L}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $x+1$ and $v+1$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 3.4.*

1) *If* $N(\varepsilon_{2p})=-1$, *we have*
   - *The unit group of* $\mathbb{L}$ *is :*
$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}},\sqrt{\sqrt{\varepsilon_q}^a\sqrt{\varepsilon_{2q}}^a\sqrt{\varepsilon_{pq}}^a\sqrt{\varepsilon_{2pq}}^{1+b}}\rangle$$
   where $a,b\in\{0,1\}$ *such that* $a\neq b$ *and* $a=1$ *if and only if* $\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}$ *is a square in* $\mathbb{L}$.
   - *The* 2*-class number of* $\mathbb{L}$ *equals* $\frac{1}{2^{4-a}}h_2(2p)h_2(pq)h_2(2pq)$.
2) *If* $N(\varepsilon_{2p})=1$ *and let* $a\in\{0,1\}$ *such that* $a\equiv 1+u \pmod 2$. *we have*
   - *The unit group of* $\mathbb{L}$ *is :*
$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_2^{ar'}\varepsilon_p^{ar'}\sqrt{\varepsilon_q}^{r'}\sqrt{\varepsilon_{pq}}^{r'}\sqrt{\varepsilon_{2p}}^{1+s'}},\sqrt{\varepsilon_2^{ar}\varepsilon_p^{ar}\sqrt{\varepsilon_{2q}}^{r}\sqrt{\varepsilon_{2pq}}^{1+s}\sqrt{\varepsilon_{2p}}^{r}}\rangle$$
   where $r,r',s,\ s'\in\{0,1\}$ *such that* $r\neq s$ *(resp.* $r'\neq s'$*) and* $r=1$ *(resp.* $r'=1$*) if and only if* $\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ *(resp.* $\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2p}}$*) is a square in* $\mathbb{L}$.
   - *The* 2*-class number of* $\mathbb{L}$ *equals* $\frac{1}{2^{4-r-r'}}h_2(2p)h_2(pq)h_2(2pq)$.

Proof. The same computations as in the proof of Theorem 3.3 give the result. □

Theorem 3.6. *Let* $p\equiv 1 \pmod 8$ *and* $q\equiv 7 \pmod 8$ *be two primes such that* $\left(\frac{p}{q}\right)=1$. *Put* $\mathbb{L}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. *Assume furthermore that* $x+1$ *and* $p(v+1)$ *are squares in* $\mathbb{N}$, *where* x *and* v *are defined in Lemma 3.4. We have*

1) *If* $N(\varepsilon_{2p})=-1$, *then*
   - *The unit group of* $\mathbb{L}$ *is :*
$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$
   - *The* 2*-class number of* $\mathbb{L}$ *equals* $\frac{1}{2^4}h_2(2p)h_2(pq)h_2(2pq)$.
2) *If* $N(\varepsilon_{2p})=1$ *and let* $a\in\{0,1\}$ *such that* $a\equiv 1+u \pmod 2$. *We have*
   - *The unit group of* $\mathbb{L}$ *is :*
$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{u\alpha}\sqrt{\varepsilon_{2q}}^{\alpha}\sqrt{\varepsilon_{2pq}}^{\alpha}\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$
   where $\alpha,\gamma\in\{0,1\}$ *such that* $\alpha\neq\gamma$ *and* $\alpha=1$ *if and only if* $\varepsilon_2^a\varepsilon_p^u\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ *is a square in* $\mathbb{L}$.
   - *The* 2*-class number of* $\mathbb{L}$ *equals* $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.

Proof. 1) Assume that $N(\varepsilon_{2p})=-1$. By Lemma 2.3, $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ is a fundamental system of units of k_1. Using Lemmas 3.2 and 3.4, we check that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,
$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\varepsilon_{pq},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$
Thus we shall determine elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$. Notice that by Lemma 3.4, ε_{pq} is a square in $\mathbb{L}$. Let ξ is an element of $\mathbb{L}$ which is the

square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. We can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2}=1+\tau_2$. We have $\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^{1+\tau_2}=(-1)^v\varepsilon_2$, for some $v\in\{0,1\}$. By means of (2) and Table 1, we get:

$$\begin{aligned}N_{\mathbb{L}/k_2}(\xi^2)&=\varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^{gv}\varepsilon_2^g\\&=\varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gv}\varepsilon_2^g.\end{aligned}$$

Thus $b+gv\equiv 0 \pmod 2$ and $g=0$. Therefore, $b=0$ and

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_5}(\xi^2)&=(-1)^a\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot(-1)^f\cdot\varepsilon_{2pq}^f\\&=(-1)^{a+c+d+f}\varepsilon_q^c\cdot\varepsilon_{2pq}^f.\end{aligned}$$

Thus $a+c+d+f=0 \pmod 2$ and $f=c$. Thus, $a=d$. Therefore,

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^a\sqrt{\varepsilon_{2pq}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_6}(\xi^2)&=(-1)^a\cdot(-1)^c\cdot(-1)^a\cdot\varepsilon_{2q}^a\cdot(-1)^c\cdot\varepsilon_{2pq}^c\\&=\varepsilon_{2q}^a\varepsilon_{2pq}^c.\end{aligned}$$

Thus $a=c$. Therefore

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^a\sqrt{\varepsilon_{2q}}^a\sqrt{\varepsilon_{2pq}}^a.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_4}(\xi^2)&=(-1)^a\cdot(-1)^a\cdot\varepsilon_q^a\cdot(-1)^a\cdot(-1)^a\\&=\varepsilon_q^a.\end{aligned}$$

Thus $a=0$. It follows that the only element of $E_{k_1}E_{k_2}E_{k_3}$ that is a square in $\mathbb{L}$ is ε_{pq}. So the first item.

2) Assume that $N(\varepsilon_{2p})=1$. By Lemma 2.3, $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_{2p}}\}$ is a fundamental system of units of k_1. Using Lemmas 3.2 and 3.4, we check that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\varepsilon_{pq},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_{2p}}\rangle.$$

Thus we shall determine elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$. Note that ε_{pq} is a square in $\mathbb{L}$. Let ξ is an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. Assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a,b,c,d,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2}=1+\tau_2$. By means of (1), (2) and Table 1, we get:

$$\begin{aligned} N_{\mathbb{L}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^{gu}\\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gu}. \end{aligned}$$

Thus $b+gu\equiv 0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{L}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. By (1), (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^b\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot(-1)^f\cdot\varepsilon_{2pq}^f\cdot(-1)^g\cdot\varepsilon_{2p}^g\\ &= (-1)^{a+b+c+d+f+g}\varepsilon_q^c\cdot\varepsilon_{2pq}^f\cdot\varepsilon_{2p}^g. \end{aligned}$$

Thus $a+b+c+d+f+g=0 \pmod 2$ and $c+f+g=0 \pmod 2$. So $a+b+d=0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{L}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By (1), (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_6}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2b}\cdot(-1)^c\cdot(-1)^d\cdot\varepsilon_{2q}^d\cdot(-1)^f\cdot\varepsilon_{2pq}^f\cdot(-1)^{gu+g}\\ &= \varepsilon_p^{2b}(-1)^{a+c+d+f+gu+g}\varepsilon_{2q}^d\cdot\varepsilon_{2pq}^f \end{aligned}$$

Thus $a+c+d+f+gu+g=0 \pmod 2$ and $d=f$. So $a+c+gu+g=0 \pmod 2$. Therefore,

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{2pq}}^d\sqrt{\varepsilon_{2p}}^g,$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. By (1), (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_4}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2b}\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot(-1)^d\cdot(-1)^{gu+g}\\ &= \varepsilon_p^{2b}(-1)^{a+c+gu+g}\varepsilon_q^c \end{aligned}$$

Thus $c=0$ and so $a+gu+g=0 \pmod 2$. Since $c+f+g=0 \pmod 2$, this implies that $g=f=d$. As $b+gu=0=b+du \pmod 2$, then we have:

$$\xi^2=\varepsilon_2^a\varepsilon_p^{du}\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{2pq}}^d\sqrt{\varepsilon_{2p}}^d,$$

where $a+du+d=0 \pmod 2$. So we have the second item.

□

Theorem 3.7. *Let $p\equiv 1 \pmod 8$ and $q\equiv 7 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right)=1$. Put $\mathbb{L}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $x+1$ and $2p(v+1)$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 3.4.*

1) *Assume that $N(\varepsilon_{2p})=-1$. We have*
   - *The unit group of $\mathbb{L}$ is :*
   $$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$
   - *The 2-class number of $\mathbb{L}$ equals $\frac{1}{2^4}h_2(2p)h_2(pq)h_2(2pq)$.*
2) *Assume that $N(\varepsilon_{2p})=1$ and let $a\in\{0,1\}$ such that $a\equiv 1+u \pmod 2$. We have*
   - *The unit group of $\mathbb{L}$ is :*
   $$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{u\alpha}\sqrt{\varepsilon_{2q}}^{\alpha}\sqrt{\varepsilon_{2pq}}^{\alpha}\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$
   where $\alpha,\gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ is a square in $\mathbb{L}$.
   - *The 2-class number of $\mathbb{L}$ equals $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.*

Proof. 1) Assume that $N(\varepsilon_{2p})=-1$. By Lemma 2.3, $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ is a fundamental system of units of k_1. Using Lemmas 3.2 and 3.4, we check that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,
$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\varepsilon_{pq},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$
Thus we shall determine elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$. Notice that by Lemma 3.4, ε_{pq} is a square in $\mathbb{L}$. Let ξ is an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. We can assume that
$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$
where a,b,c,d,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2}=1+\tau_2$. We have $\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^{1+\tau_2}=(-1)^v\varepsilon_2$, for some $v\in\{0,1\}$. By means of (2) and Table 1, we get:
$$\begin{aligned}N_{\mathbb{L}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^{gv}\varepsilon_2^g\\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gv}\varepsilon_2^g.\end{aligned}$$
Thus $b+gv\equiv 0 \pmod 2$ and $g=0$. Therefore, $b=0$ and
$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. By (2) and Table 1, we have:
$$\begin{aligned}N_{\mathbb{L}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot(-1)^f\cdot\varepsilon_{2pq}^f\\ &= (-1)^{a+c+d+f}\varepsilon_q^c\cdot\varepsilon_{2pq}^f.\end{aligned}$$

Thus $a+c+d+f=0 \pmod 2$ and $f=c$. Thus, $a=d$. Therefore,

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^a\sqrt{\varepsilon_{2pq}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_6}(\xi^2) &= (-1)^a\cdot(-1)^c\cdot(-1)^a\cdot\varepsilon_{2q}^a\cdot(-1)^c\cdot\varepsilon_{2pq}^c\\ &= \varepsilon_{2q}^a\varepsilon_{2pq}^c. \end{aligned}$$

Thus $a=c$. Therefore

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^a\sqrt{\varepsilon_{2q}}^a\sqrt{\varepsilon_{2pq}}^a.$$

2) To check that we have the same computations as in the previous theorem

□

Theorem 3.8. *Let $p\equiv 1 \pmod 8$ and $q\equiv 7 \pmod 8$ be two primes such that $\left(\dfrac{p}{q}\right)=1$. Put $\mathbb{L}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $p(x+1)$ and $v+1$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 3.4.*

1) *Assume that $N(\varepsilon_{2p})=-1$. We have*
- *The unit group of $\mathbb{L}$ is :*

$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

- *The 2-class number of $\mathbb{L}$ equals $\frac{1}{2^4}h_2(2p)h_2(pq)h_2(2pq)$.*

2) *Assume that $N(\varepsilon_{2p})=1$ and let $a\in\{0,1\}$ such that $a\equiv 1+u \pmod 2$. We have*
- *The unit group of $\mathbb{L}$ is :*

$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{u\alpha}\sqrt{\varepsilon_q}^{\alpha}\sqrt{\varepsilon_{pq}}^{\alpha}\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$

where $\alpha,\gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\varepsilon_2^a\varepsilon_p^u\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2p}}$ is a square in $\mathbb{L}$.
- *The 2-class number of $\mathbb{L}$ equals $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.*

Proof. 1) Assume that $N(\varepsilon_{2p})=-1$. By Lemma 2.3, $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ is a fundamental system of units of k_1. Using Lemmas 3.2 and 3.4, we check that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{2pq},\sqrt{\varepsilon_{pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\varepsilon_{2pq},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Thus we shall determine elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$. Notice that by Lemma 3.4, ε_{2pq} is a square in $\mathbb{L}$. Let ξ is an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. We can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2} = 1+\tau_2$. We have $\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^{1+\tau_2} = (-1)^v\varepsilon_2$, for some $v \in \{0, 1\}$. By means of (2) and Table 1, we get:

$$\begin{aligned} N_{\mathbb{L}/k_2}(\xi^2) &= \varepsilon_2^{2a} \cdot (-1)^b \cdot \varepsilon_q^c \cdot \varepsilon_{2q}^d \cdot 1 \cdot (-1)^{gv}\varepsilon_2^g \\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d \cdot (-1)^{b+gv}\varepsilon_2^g. \end{aligned}$$

Thus $b + gv = 0 \pmod 2$ and $g = 0$. Therefore, $b = 0$ and

$$\xi^2 = \varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_5} = 1 + \tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_5}(\xi^2) &= (-1)^a \cdot (-1)^c \cdot \varepsilon_q^c \cdot (-1)^d \cdot (-1)^f \\ &= (-1)^{a+c+d+f}\varepsilon_q^c. \end{aligned}$$

Thus $a + c + d + f = 0 \pmod 2$ and $c = 0$. Thus, $a + d + f = 0 \pmod 2$. Therefore,

$$\xi^2 = \varepsilon_2^a\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6} = 1 + \tau_1\tau_3$, with $k_6 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_6}(\xi^2) &= (-1)^a \cdot (-1)^d \cdot \varepsilon_{2q}^d \cdot (-1)^f \\ &= (-1)^{a+d+f}\varepsilon_{2q}^d. \end{aligned}$$

Thus $d = 0$ and $a = f$. Therefore,

$$\xi^2 = \varepsilon_2^a\sqrt{\varepsilon_{pq}}^a.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4} = 1 + \tau_1$, with $k_4 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. By (2) and Table 1, we have:

$$N_{\mathbb{L}/k_4}(\xi^2) = (-1)^a \cdot (-1)^a \cdot \varepsilon_{pq}^a = \varepsilon_{pq}^a.$$

Thus $a = 0$. It follows that the only element of $E_{k_1}E_{k_2}E_{k_3}$ that is a square in $\mathbb{L}$ is ε_{2pq} and so we have the first item.

2) Assume that $N(\varepsilon_{2p}) = 1$. So we have:

$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \varepsilon_{2pq}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2p}}\rangle.$$

To determine the elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$, let us consider ξ an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. As ε_{2pq} is a square in $\mathbb{L}$, we can assume that

$$\xi^2 = \varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a, b, c, d, f and g are in $\{0, 1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2} = 1+\tau_2$. By (1), (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^{gu}\\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gu}. \end{aligned}$$

Thus $b+gu=0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{L}/k_5} = 1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^b\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot(-1)^f\cdot(-1)^g\cdot\varepsilon_{2p}^g\\ &= (-1)^{a+b+c+d+f+g}\varepsilon_q^c\cdot\varepsilon_{2p}^g. \end{aligned}$$

Thus $c=g$ and so $a+b+d+f=0 \pmod 2$. Thus, $a+d+f=0 \pmod 2$. Therefore,

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^f\sqrt{\varepsilon_{2p}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6} = 1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_6}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2b}\cdot(-1)^c\cdot(-1)^d\cdot\varepsilon_{2q}^d\cdot(-1)^f\cdot(-1)^{cu+c}\\ &= \varepsilon_p^{2b}(-1)^{a+d+f+cu}\varepsilon_{2q}^d. \end{aligned}$$

Thus $d=0$ and so $a+f+cu=0 \pmod 2$. Therefore,

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{pq}}^f\sqrt{\varepsilon_{2p}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4} = 1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_4}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2b}\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^f\cdot\varepsilon_{pq}^f\cdot(-1)^{cu+c},\\ &= \varepsilon_p^{2b}(-1)^{a+f+cu}\varepsilon_q^c\varepsilon_{pq}^f. \end{aligned}$$

Thus $c=f$. Since $b+gu=0=b+cu \pmod 2$, we have

$$\xi^2=\varepsilon_2^a\varepsilon_p^{cu}\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{pq}}^c\sqrt{\varepsilon_{2p}}^c,$$

with $a+c+cu=0 \pmod 2$.

□

Theorem 3.9. *Let $p\equiv 1 \pmod 8$ and $q\equiv 7 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right)=1$. Put $\mathbb{L}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $p(x+1)$ and $p(v+1)$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 3.4.*

1) *Assume that $N(\varepsilon_{2p})=-1$. We have*
- *The unit group of $\mathbb{L}$ is :*

$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

- *The 2-class number of $\mathbb{L}$ equals $\frac{1}{2^4}h_2(2p)h_2(pq)h_2(2pq)$.*

2) *Assume that* $N(\varepsilon_{2p})=1$ *and let* $a\in\{0,1\}$ *such that* $a\equiv 1+u \pmod 2$. *We have*
- *The unit group of* $\mathbb{L}$ *is :*

$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{u\alpha}\sqrt{\varepsilon_q}^{\alpha}\sqrt{\varepsilon_{2q}}^{\alpha}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^{\alpha}\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$

where $\alpha,\gamma\in\{0,1\}$ *such that* $\alpha\neq\gamma$ *and* $\alpha=1$ *if and only if* $\varepsilon_2^a\varepsilon_p^u\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ *is a square in* $\mathbb{L}$.
- *The* 2*-class number of* $\mathbb{L}$ *equals* $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.

Proof. 1) Assume that $N(\varepsilon_{2p})=-1$. By Lemma 2.3, $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ is a fundamental system of units of k_1. Using Lemmas 3.2 and 3.4, we check that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Thus we shall determine elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$. Notice that by Lemma 3.4, ε_{pq} is a square in $\mathbb{L}$. Let ξ is an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. We can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2}=1+\tau_2$. We have $\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^{1+\tau_2}=(-1)^v\varepsilon_2$, for some $v\in\{0,1\}$. By means of (2) and Table 1, we get:

$$\begin{aligned}N_{\mathbb{L}/k_2}(\xi^2)&=\varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^{gv}\varepsilon_2^g\\&=\varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gv}\varepsilon_2^g.\end{aligned}$$

Thus $b+gv=0 \pmod 2$ and $g=0$. Therefore, $b=0$ and

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_5}(\xi^2)&=(-1)^a\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot\varepsilon_{2pq}^f\\&=\varepsilon_{2pq}^f(-1)^{a+c+d}\varepsilon_q^c.\end{aligned}$$

Thus $c=0$ and so $a=d$. Therefore,

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_{2q}}^a\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_6}(\xi^2)&=(-1)^a\cdot(-1)^a\cdot\varepsilon_{2q}^a\cdot\varepsilon_{2pq}^f\\&=\varepsilon_{2q}^a\varepsilon_{2pq}^f.\end{aligned}$$

Thus $a=f$. Therefore,

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_{2q}}^a\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^a.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. By (2) and Table 1, we have:

$$N_{\mathbb{L}/k_4}(\xi^2) \;=\; (-1)^a\cdot(-1)^a\cdot\varepsilon_{pq}^a=\varepsilon_{pq}^a.$$

Thus $a=0$. It follows that the only element of $E_{k_1}E_{k_2}E_{k_3}$ that is a square in $\mathbb{L}$ is ε_{pq} and so we have the first item.

2) Assume that $N(\varepsilon_{2p})=1$. So we have:

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_{2p}}\rangle.$$

To determine the elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$, let us consider ξ an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. As ε_{pq} is a square in $\mathbb{L}$, we can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a,b,c,d,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2}=1+\tau_2$. By (1), (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^{gu}\\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gu}.\end{aligned}$$

Thus $b+gu=0 \pmod 2$. So $b=gu$.

➠ Let us apply the norm $N_{\mathbb{L}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^b\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot\varepsilon_{2pq}^f\cdot(-1)^g\cdot\varepsilon_{2p}^g\\ &= \varepsilon_{2pq}^f(-1)^{a+b+c+d+g}\varepsilon_q^c\varepsilon_{2p}^g.\end{aligned}$$

Thus $c=g$ and so $a+b+d=0 \pmod 2$. Therefore,

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_6}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2b}\cdot(-1)^c\cdot(-1)^d\cdot\varepsilon_{2q}^d\cdot\varepsilon_{2pq}^f\cdot(-1)^{cu+c}\\ &= \varepsilon_p^{2b}(-1)^{a+d+cu}\varepsilon_{2q}^d\varepsilon_{2pq}^f.\end{aligned}$$

Thus $a+d+cu=0 \pmod 2$ and $d=f$. Therefore,

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^d\sqrt{\varepsilon_{2p}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_4}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2b}\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot\varepsilon_{pq}^d\cdot(-1)^{cu+c}\\ &= \varepsilon_p^{2b}(-1)^{a+d+cu}\varepsilon_q^c\varepsilon_{pq}^d. \end{aligned}$$

Thus $d=c$ and so $a+c+cu=0 \pmod 2$. It follows that

$$\xi^2=\varepsilon_2^a\varepsilon_p^{cu}\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^c\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^c\sqrt{\varepsilon_{2p}}^c,$$

where $a+cu+c=0 \pmod 2$.

□

Theorem 3.10. *Let $p\equiv 1 \pmod 8$ and $q\equiv 7 \pmod 8$ be two primes such that $\left(\dfrac{p}{q}\right)=1$. Put $\mathbb{L}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $p(x+1)$ and $2p(v+1)$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 3.4.*

1) *Assume that $N(\varepsilon_{2p})=-1$. We have*
   - *The unit group of $\mathbb{L}$ is :*

$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}},\sqrt{\sqrt{\varepsilon_{2q}}^{1+\gamma}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^{\alpha}}\rangle,$$

   where $\alpha,\ \gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}$ is a square in $\mathbb{L}$.
   - *The 2-class number of $\mathbb{L}$ equals $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.*

2) *Assume that $N(\varepsilon_{2p})=1$. We have*
   - *The unit group of $\mathbb{L}$ is :*

$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_{2p}},\sqrt{\sqrt{\varepsilon_{2q}}^{1+\gamma}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^{\alpha}}\rangle$$

   where $\alpha,\ \gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}$ is a square in $\mathbb{L}$.
   - *The 2-class number of $\mathbb{L}$ equals $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.*

Proof. 1) Assume that $N(\varepsilon_{2p})=-1$. By Lemma 2.3, $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ is a fundamental system of units of k_1. Using Lemmas 3.2 and 3.4, we check that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Thus we shall determine elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$. Notice that by Lemma 3.4, ε_{pq} is a square in $\mathbb{L}$. Let ξ is an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. We can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2}=1+\tau_2$. We have $\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^{1+\tau_2}=(-1)^v\varepsilon_2$, for some $v\in\{0,1\}$. By means of (2) and Table 1, we get:

$$\begin{array}{lll}N_{\mathbb{L}/k_2}(\xi^2) & = & \varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^{gv}\varepsilon_2^g\\ & = & \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gv}\varepsilon_2^g.\end{array}$$

Thus $b+gv=0 \pmod 2$ and $g=0$. Therefore, $b=0$ and

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{array}{lll}N_{\mathbb{L}/k_5}(\xi^2) & = & (-1)^a\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot(-1)^f\cdot\varepsilon_{2pq}^f\\ & = & \varepsilon_{2pq}^f(-1)^{a+c+d+f}\varepsilon_q^c.\end{array}$$

Thus $c=0$ and so $a+d+f=0 \pmod 2$. Therefore,

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{array}{lll}N_{\mathbb{L}/k_6}(\xi^2) & = & (-1)^a\cdot(-1)^d\cdot\varepsilon_{2q}^d\cdot(-1)^f\cdot\varepsilon_{2pq}^f\\ & = & (-1)^{a+d+f}\varepsilon_{2q}^d\varepsilon_{2pq}^f.\end{array}$$

Thus $d=f$ and $a=0$. Therefore,

$$\xi^2=\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. By (2) and Table 1, we have:

$$\begin{array}{lll}N_{\mathbb{L}/k_4}(\xi^2) & = & (-1)^d\cdot(-1)^f\cdot\varepsilon_{pq}^f\\ & = & \varepsilon_{pq}^f(-1)^{d+f}.\end{array}$$

Thus $d=f$. Therefore,

$$\xi^2=\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^d.$$

2) Assume that $N(\varepsilon_{2p})=1$. So we have:

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_{2p}}\rangle.$$

To determine the elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$, let us consider ξ an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. As ε_{pq} is a square in $\mathbb{L}$, we can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a,b,c,d,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2} = 1 + \tau_2$. By (1), (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_2}(\xi^2) &= \varepsilon_2^{2a} \cdot (-1)^b \cdot \varepsilon_q^c \cdot \varepsilon_{2q}^d \cdot 1 \cdot (-1)^{gu} \\ &= \varepsilon_2^{2a} \varepsilon_q^c \varepsilon_{2q}^d \cdot (-1)^{b+gu}. \end{aligned}$$

Thus $b + gu = 0 \pmod 2$. So $b = gu$.

➠ Let us apply the norm $N_{\mathbb{L}/k_5} = 1 + \tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_5}(\xi^2) &= (-1)^a \cdot (-1)^b \cdot (-1)^c \cdot \varepsilon_q^c \cdot (-1)^d \cdot (-1)^f \cdot \varepsilon_{2pq}^f \cdot (-1)^g \cdot \varepsilon_{2p}^g \\ &= \varepsilon_{2pq}^f (-1)^{a+b+c+d+f+g} \varepsilon_q^c \varepsilon_{2p}^g. \end{aligned}$$

Thus $c = g$ and so $a + b + d + f = 0 \pmod 2$. Therefore,

$$\xi^2 = \varepsilon_2^a \varepsilon_p^b \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f \sqrt{\varepsilon_{2p}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6} = 1 + \tau_1\tau_3$, with $k_6 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_6}(\xi^2) &= (-1)^a \cdot \varepsilon_p^{2b} \cdot (-1)^c \cdot (-1)^d \cdot \varepsilon_{2q}^d \cdot (-1)^f \cdot \varepsilon_{2pq}^f \cdot (-1)^{cu+c} \\ &= \varepsilon_p^{2b} (-1)^{a+d+f+cu} \varepsilon_{2q}^d \varepsilon_{2pq}^f. \end{aligned}$$

Thus $d = f$ and so $a + cu = 0 \pmod 2$. As $a + b + d + f = 0 \pmod 2$, then $a = b = cu$. Therefore,

$$\xi^2 = \varepsilon_2^{cu} \varepsilon_p^{cu} \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^d \sqrt{\varepsilon_{2p}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4} = 1 + \tau_1$, with $k_4 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_4}(\xi^2) &= (-1)^{cu} \cdot \varepsilon_p^{2cu} \cdot (-1)^c \cdot \varepsilon_q^c \cdot (-1)^d \cdot (-1)^d \cdot \varepsilon_{pq}^d \cdot (-1)^{cu+c} \\ &= \varepsilon_p^{2cu} \varepsilon_{pq}^d \cdot \varepsilon_q^c. \end{aligned}$$

Thus $c = 0$. It follows that

$$\xi^2 = \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^d,$$

□

Theorem 3.11. *Let $p \equiv 1 \pmod 8$ and $q \equiv 7 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right) = 1$. Put $\mathbb{L} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$. Assume furthermore that $2p(x+1)$ and $v+1$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 3.4.*

1) *Assume that $N(\varepsilon_{2p}) = -1$. We have*
- *The unit group of $\mathbb{L}$ is :*

$$E_{\mathbb{L}} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}} \rangle,$$

- *The 2-class number of $\mathbb{L}$ equals $\frac{1}{2^4} h_2(2p) h_2(pq) h_2(2pq)$.*

2) *Assume that* $N(\varepsilon_{2p})=1$ *and let* $a\in\{0,1\}$ *such that* $a\equiv 1+u \pmod 2$. *We have*

- *The unit group of* $\mathbb{L}$ *is :*

$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{u\alpha}\sqrt{\varepsilon_q}^{\alpha}\sqrt{\varepsilon_{pq}}^{\alpha}\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$

where $\alpha,\gamma\in\{0,1\}$ *such that* $\alpha\neq\gamma$ *and* $\alpha=1$ *if and only if* $\varepsilon_2^a\varepsilon_p^u\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2p}}$ *is a square in* $\mathbb{L}$.

- *The* 2*-class number of* $\mathbb{L}$ *equals* $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.

Proof. 1) Assume that $N(\varepsilon_{2p})=-1$. By Lemma 2.3, $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ is a fundamental system of units of k_1. Using Lemmas 3.2 and 3.4, we check that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{2pq},\sqrt{\varepsilon_{pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\varepsilon_{2pq},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Thus we shall determine elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$. Notice that by Lemma 3.4, ε_{2pq} is a square in $\mathbb{L}$. Let ξ is an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. We can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2}=1+\tau_2$. We have $\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^{1+\tau_2}=(-1)^v\varepsilon_2$, for some $v\in\{0,1\}$. By means of (2) and Table 1, we get:

$$\begin{aligned}N_{\mathbb{L}/k_2}(\xi^2)&=\varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^{gv}\varepsilon_2^g\\&=\varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gv}\varepsilon_2^g.\end{aligned}$$

Thus $b+gv=0 \pmod 2$ and $g=0$. So $b=0$ and

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_5}(\xi^2)&=(-1)^a\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot(-1)^f\\&=(-1)^{a+c+d+f}\varepsilon_q^c.\end{aligned}$$

Thus $c=0$ and so $a+d+f=0 \pmod 2$. Therefore,

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_6}(\xi^2)&=(-1)^a\cdot(-1)^d\cdot\varepsilon_{2q}^d\cdot(-1)^f\\&=(-1)^{a+d+f}\varepsilon_{2q}^d.\end{aligned}$$

Thus $d = 0$ and $a = f$. Therefore,

$$\xi^2 = \varepsilon_2^a \sqrt{\varepsilon_{pq}}^a.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4} = 1 + \tau_1$, with $k_4 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. By (2) and Table 1, we have:

$$N_{\mathbb{L}/k_4}(\xi^2) = (-1)^a \cdot (-1)^a \cdot \varepsilon_{pq}^a = \varepsilon_{pq}^a.$$

Thus $a = 0$. So we have the first item.

2) Assume that $N(\varepsilon_{2p}) = 1$. So we have:

$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \varepsilon_{2pq}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2p}} \rangle.$$

To determine the elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$, let us consider ξ an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. As ε_{2pq} is a square in $\mathbb{L}$, we can assume that

$$\xi^2 = \varepsilon_2^a \varepsilon_p^b \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}}^f \sqrt{\varepsilon_{2p}}^g,$$

where a, b, c, d, f and g are in $\{0, 1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2} = 1 + \tau_2$. By (1), (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_2}(\xi^2) &= \varepsilon_2^{2a} \cdot (-1)^b \cdot \varepsilon_q^c \cdot \varepsilon_{2q}^d \cdot 1 \cdot (-1)^{gu} \\ &= \varepsilon_2^{2a} \varepsilon_q^c \varepsilon_{2q}^d \cdot (-1)^{b+gu}. \end{aligned}$$

Thus $b + gu = 0 \pmod 2$. So $b = gu$.

➠ Let us apply the norm $N_{\mathbb{L}/k_5} = 1 + \tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_5}(\xi^2) &= (-1)^a \cdot (-1)^b \cdot (-1)^c \cdot \varepsilon_q^c \cdot (-1)^d \cdot (-1)^f \cdot (-1)^g \cdot \varepsilon_{2p}^g \\ &= (-1)^{a+b+c+d+f+g} \varepsilon_q^c \varepsilon_{2p}^g. \end{aligned}$$

Thus $c = g$ and so $a + b + d + f = 0 \pmod 2$. Therefore,

$$\xi^2 = \varepsilon_2^a \varepsilon_p^b \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}}^f \sqrt{\varepsilon_{2p}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6} = 1 + \tau_1\tau_3$, with $k_6 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_6}(\xi^2) &= (-1)^a \cdot \varepsilon_p^{2b} \cdot (-1)^c \cdot (-1)^d \cdot \varepsilon_{2q}^d \cdot (-1)^f \cdot (-1)^{cu+c} \\ &= \varepsilon_p^{2b} (-1)^{a+d+f+cu} \varepsilon_{2q}^d. \end{aligned}$$

Thus $d = 0$ and so $a + f + cu = 0 \pmod 2$. As $a + b + d + f = 0 \pmod 2$, then $a + b + f = 0 \pmod 2$. Therefore,

$$\xi^2 = \varepsilon_2^a \varepsilon_p^b \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{pq}}^f \sqrt{\varepsilon_{2p}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. By (2) and Table 1, we have:

$$\begin{array}{rcl} N_{\mathbb{L}/k_4}(\xi^2) & = & (-1)^a\cdot\varepsilon_p^{2b}\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^f\cdot\varepsilon_{pq}^f\cdot(-1)^{cu+c}\\ & = & \varepsilon_p^{2b}(-1)^{a+f+cu}\varepsilon_q^c\varepsilon_{pq}^f. \end{array}$$

Thus $c=f$ and $a+c+cu=0 \pmod 2$. Therefore,

$$\xi^2=\varepsilon_2^a\varepsilon_p^{cu}\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{pq}}^c\sqrt{\varepsilon_{2p}}^c.$$

□

Theorem 3.12. *Let $p\equiv 1 \pmod 8$ and $q\equiv 7 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right)=1$. Put $\mathbb{L}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $2p(x+1)$ and $p(v+1)$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 3.4.*

1) *Assume that $N(\varepsilon_{2p})=-1$. We have*
- *The unit group of $\mathbb{L}$ is :*

$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}},\sqrt{\sqrt{\varepsilon_q}^{1+\gamma}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^{\alpha}}\rangle,$$

where $\alpha,\gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}$ is a square in $\mathbb{L}$.
- *The 2-class number of $\mathbb{L}$ equals $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.*

2) *Assume that $N(\varepsilon_{2p})=1$. We have*
- *The unit group of $\mathbb{L}$ is :*

$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_{2p}},\sqrt{\sqrt{\varepsilon_q}^{1+\gamma}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^{\alpha}}\rangle$$

where $\alpha,\gamma\in\{0,1\}$ such that $\alpha\neq\gamma$ and $\alpha=1$ if and only if $\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}$ is a square in $\mathbb{L}$.
- *The 2-class number of $\mathbb{L}$ equals $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.*

Proof. 1) Assume that $N(\varepsilon_{2p})=-1$. By Lemma 2.3, $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ is a fundamental system of units of k_1. Using Lemmas 3.2 and 3.4, we check that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Thus we shall determine elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$. Notice that by Lemma 3.4, ε_{pq} is a square in $\mathbb{L}$. Let ξ is an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. We can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2} = 1+\tau_2$. We have $\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^{1+\tau_2} = (-1)^v\varepsilon_2$, for some $v \in \{0,1\}$. By means of (2) and Table 1, we get:

$$\begin{aligned} N_{\mathbb{L}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^{gv}\varepsilon_2^g \\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gv}\varepsilon_2^g. \end{aligned}$$

Thus $b+gv = 0 \pmod 2$ and $g = 0$. Therefore, $b = 0$ and

$$\xi^2 = \varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_5} = 1+\tau_1\tau_2$, with $k_5 = \mathbb{Q}(\sqrt{q}, \sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot(-1)^f\cdot\varepsilon_{2pq}^f \\ &= (-1)^{a+c+d+f}\varepsilon_q^c\varepsilon_{2pq}^f. \end{aligned}$$

Thus $c = f$ and $a+c+d+f = 0 \pmod 2$. Therefore, $a = d$ and

$$\xi^2 = \varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^a\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6} = 1+\tau_1\tau_3$, with $k_6 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_6}(\xi^2) &= (-1)^a\cdot(-1)^c\cdot(-1)^a\cdot\varepsilon_{2q}^a\cdot(-1)^c\cdot\varepsilon_{2pq}^c \\ &= \varepsilon_{2pq}^c\cdot\varepsilon_{2q}^a. \end{aligned}$$

Thus $a = 0$. Therefore,

$$\xi^2 = \sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4} = 1+\tau_1$, with $k_4 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. By (2) and Table 1, we have:

$$N_{\mathbb{L}/k_4}(\xi^2) = (-1)^c\cdot\varepsilon_q^c\cdot(-1)^c\cdot\varepsilon_{pq}^c = \varepsilon_q^c\varepsilon_{pq}^c.$$

We deduce nothing.

2) Assume that $N(\varepsilon_{2p}) = 1$. So we have:

$$E_{k_1}E_{k_2}E_{k_3} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \varepsilon_{pq}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}, \sqrt{\varepsilon_{2p}}\rangle.$$

To determine the elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$, let us consider ξ an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. As ε_{pq} is a square in $\mathbb{L}$, we can assume that

$$\xi^2 = \varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a, b, c, d, f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2} = 1+\tau_2$. By (1), (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^{gu} \\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gu}. \end{aligned}$$

Thus $b+gu=0 \pmod 2$. So $b=gu$.

➠ Let us apply the norm $N_{\mathbb{L}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^b\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot(-1)^f\cdot\varepsilon_{2pq}^f\cdot(-1)^g\cdot\varepsilon_{2p}^g\\ &= (-1)^{a+b+c+d+f+g}\varepsilon_q^c\varepsilon_{2pq}^f\varepsilon_{2p}^g. \end{aligned}$$

Thus $a+b+c+d+f+g=0 \pmod 2$ and $c+f+g=0 \pmod 2$. Therefore, $a+b+d=0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{L}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_6}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2b}\cdot(-1)^c\cdot(-1)^d\cdot\varepsilon_{2q}^d\cdot(-1)^f\cdot\varepsilon_{2pq}^f\cdot(-1)^{gu+g}\\ &= \varepsilon_p^{2b}\varepsilon_{2pq}^f(-1)^{a+c+d+f+gu+g}\varepsilon_{2q}^d. \end{aligned}$$

Thus $d=0$ and so $a+c+f+gu+g=0 \pmod 2$. As $a+b+d=0 \pmod 2$, then $a=b$. Therefore,

$$\xi^2=\varepsilon_2^a\varepsilon_p^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. By (2) and Table 1, we have:

$$\begin{aligned} N_{\mathbb{L}/k_4}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2a}\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^f\cdot\varepsilon_{pq}^f\cdot(-1)^{gu+g}\\ &= \varepsilon_p^{2b}(-1)^{a+c+f+gu+g}\varepsilon_q^c\varepsilon_{pq}^f. \end{aligned}$$

Thus $a+c+f+gu+g=0 \pmod 2$ and $c=f$. So $a+gu+g=0 \pmod 2$. As $c+f+g=0 \pmod 2$, then $g=0$ and so $a=0$. Therefore,

$$\xi^2=\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^c.$$

So the second item.

□

Theorem 3.13. *Let $p\equiv 1 \pmod 8$ and $q\equiv 7 \pmod 8$ be two primes such that $\left(\frac{p}{q}\right)=1$. Put $\mathbb{L}=\mathbb{Q}(\sqrt{2},\sqrt{p},\sqrt{q})$. Assume furthermore that $2p(x+1)$ and $2p(v+1)$ are squares in $\mathbb{N}$, where x and v are defined in Lemma 3.4.*

1) *Assume that $N(\varepsilon_{2p})=-1$. We have*
   - *The unit group of $\mathbb{L}$ is :*
$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle,$$
   - *The 2-class number of $\mathbb{L}$ equals $\frac{1}{2^4}h_2(2p)h_2(pq)h_2(2pq)$.*
2) *Assume that $N(\varepsilon_{2p})=1$ and let $a\in\{0,1\}$ such that $a\equiv 1+u \pmod 2$. We have*

- *The unit group of* $\mathbb{L}$ *is :*

$$E_{\mathbb{L}}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\sqrt{\varepsilon_{pq}},\sqrt{\varepsilon_{2pq}},\sqrt{\varepsilon_2^{a\alpha}\varepsilon_p^{u\alpha}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^{\alpha}\sqrt{\varepsilon_{2p}}^{1+\gamma}}\rangle$$

where $\alpha,\gamma\in\{0,1\}$ *such that* $\alpha\neq\gamma$ *and* $\alpha=1$ *if and only if* $\varepsilon_2^a\varepsilon_p^u\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\sqrt{\varepsilon_{2p}}$ *is a square in* $\mathbb{L}$.

- *The 2-class number of* $\mathbb{L}$ *equals* $\frac{1}{2^{4-\alpha}}h_2(2p)h_2(pq)h_2(2pq)$.

Proof. 1) Assume that $N(\varepsilon_{2p})=-1$. By Lemma 2.3, $\{\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\}$ is a fundamental system of units of k_1. Using Lemmas 3.2 and 3.4, we check that $\{\varepsilon_2,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}}\}$ and $\{\varepsilon_2,\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ are respectively fundamental systems of units of k_2 and k_3. It follows that,

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}\rangle.$$

Thus we shall determine elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$. Notice that by Lemma 3.4, ε_{pq} is a square in $\mathbb{L}$. Let ξ is an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. We can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g,$$

where a,b,c,d,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2}=1+\tau_2$. We have $\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^{1+\tau_2}=(-1)^v\varepsilon_2$, for some $v\in\{0,1\}$. By means of (2) and Table 1, we get:

$$\begin{aligned}N_{\mathbb{L}/k_2}(\xi^2)&=\varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^{gv}\varepsilon_2^g\\&=\varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gv}\varepsilon_2^g.\end{aligned}$$

Thus $b+gv=0 \pmod 2$ and $g=0$. Therefore, $b=0$ and

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_5}(\xi^2)&=(-1)^a\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot\varepsilon_{2pq}^f\\&=(-1)^{a+c+d}\varepsilon_q^c\varepsilon_{2pq}^f.\end{aligned}$$

Thus $c=f$ and $a+c+d=0 \pmod 2$. Therefore,

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^c.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_6}(\xi^2)&=(-1)^a\cdot(-1)^c\cdot(-1)^d\cdot\varepsilon_{2q}^d\cdot\varepsilon_{2pq}^c\\&=\varepsilon_{2pq}^c(-1)^{a+c+d}\cdot\varepsilon_{2q}^d.\end{aligned}$$

Thus $d=0$ and $a+c+d=0 \pmod 2$. Therefore, $a=c$ and

$$\xi^2=\varepsilon_2^a\sqrt{\varepsilon_q}^a\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^a.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. By (2) and Table 1, we have:

$$N_{\mathbb{L}/k_4}(\xi^2) = (-1)^a\cdot(-1)^a\cdot\varepsilon_q^a\cdot\varepsilon_{pq}^a=\varepsilon_{pq}^a\cdot\varepsilon_q^a.$$

Thus, $a=0$. So we have the first item.

2) Assume that $N(\varepsilon_{2p})=1$. So we have:

$$E_{k_1}E_{k_2}E_{k_3}=\langle -1,\varepsilon_2,\varepsilon_p,\sqrt{\varepsilon_q},\sqrt{\varepsilon_{2q}},\varepsilon_{pq},\sqrt{\varepsilon_{pq}\varepsilon_{2pq}},\sqrt{\varepsilon_{2p}}\rangle.$$

To determine the elements of $E_{k_1}E_{k_2}E_{k_3}$ which are squares in $\mathbb{L}$, let us consider ξ an element of $\mathbb{L}$ which is the square root of an element of $E_{k_1}E_{k_2}E_{k_3}$. As ε_{pq} is a square in $\mathbb{L}$, we can assume that

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{2q}}^d\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g,$$

where a,b,c,d,f and g are in $\{0,1\}$.

➠ Let us start by applying the norm map $N_{\mathbb{L}/k_2}=1+\tau_2$. By (1), (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_2}(\xi^2) &= \varepsilon_2^{2a}\cdot(-1)^b\cdot\varepsilon_q^c\cdot\varepsilon_{2q}^d\cdot 1\cdot(-1)^{gu}\\ &= \varepsilon_2^{2a}\varepsilon_q^c\varepsilon_{2q}^d\cdot(-1)^{b+gu}.\end{aligned}$$

Thus $b+gu=0 \pmod 2$. So $b=gu$.

➠ Let us apply the norm $N_{\mathbb{L}/k_5}=1+\tau_1\tau_2$, with $k_5=\mathbb{Q}(\sqrt{q},\sqrt{2p})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_5}(\xi^2) &= (-1)^a\cdot(-1)^b\cdot(-1)^c\cdot\varepsilon_q^c\cdot(-1)^d\cdot\varepsilon_{2pq}^f\cdot(-1)^g\cdot\varepsilon_{2p}^g\\ &= (-1)^{a+b+c+d+g}\varepsilon_q^c\varepsilon_{2pq}^f\varepsilon_{2p}^g.\end{aligned}$$

Thus $a+b+c+d+g=0 \pmod 2$ and $c+f+g=0 \pmod 2$.

➠ Let us apply the norm $N_{\mathbb{L}/k_6}=1+\tau_1\tau_3$, with $k_6=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_6}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2b}\cdot(-1)^c\cdot(-1)^d\cdot\varepsilon_{2q}^d\cdot\varepsilon_{2pq}^f\cdot(-1)^{gu+g}\\ &= \varepsilon_p^{2b}\varepsilon_{2pq}^f(-1)^{a+c+d+gu+g}\varepsilon_{2q}^d.\end{aligned}$$

Thus $d=0$ and so $a+c+gu+g=0 \pmod 2$. Therefore,

$$\xi^2=\varepsilon_2^a\varepsilon_p^b\sqrt{\varepsilon_q}^c\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f\sqrt{\varepsilon_{2p}}^g.$$

➠ Let us apply the norm $N_{\mathbb{L}/k_4}=1+\tau_1$, with $k_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$. By (2) and Table 1, we have:

$$\begin{aligned}N_{\mathbb{L}/k_4}(\xi^2) &= (-1)^a\cdot\varepsilon_p^{2a}\cdot(-1)^c\cdot\varepsilon_q^c\cdot\varepsilon_{pq}^f\cdot(-1)^{gu+g}\\ &= \varepsilon_p^{2b}\varepsilon_{pq}^f(-1)^{a+c+gu+g}\varepsilon_q^c.\end{aligned}$$

Thus $c=0$ and so $a+gu+g=0 \pmod 2$. Since $c+f+g=0 \pmod 2$, we have $f=g$. Therefore,

$$\xi^2=\varepsilon_2^a\varepsilon_p^{gu}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^g\sqrt{\varepsilon_{2p}}^g,$$

with $a + gu + g = 0 \pmod 2$. So we have the second item.

□

References

[1] A. Azizi, Sur la capitulation des 2-classes d'idéaux de $\Bbbk = \mathbb{Q}(\sqrt{2pq}, i)$, où $p \equiv -q \equiv 1 \pmod 4$, Acta. Arith. 94 (2000), 383-399.

[2] A. Azizi, *Unités de certains corps de nombres imaginaires et abéliens sur $\mathbb{Q}$,* Ann. Sci. Math. Québec, 23 (1999), 15-21.

[3] A. Azizi et M. Talbi, Capitulation des 2-classes d'idéaux de certains corps biquadratiques cycliques, Acta Arith., 127 (2007), 231-248.

[4] M. Bulant, On the parity of the class number of the field $\mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{r})$, J. Number Theory 68 (1998) 72-86.

[5] M. M. Chems-Eddin, Arithmetic of some real triquadratic fields; Units and 2-class groups, arXiv:2108.04171

[6] M. M. Chems-Eddin, Unit groups of some multiquadratic number fields and 2-class groups, Period. Math. Hung. 84 (2022), 235–249.

[7] M. M. Chems-Eddin, A. Azizi and A. Zekhnini, Unit groups and Iwasawa lambda invariants of some multiquadratic number fields, Bol. Soc. Mat. Mex. 27, 24 (2021). https://doi.org/10.1007/s40590-021-00329-z

[8] M. M. Chems-Eddin, A. Zekhnini and A. Azizi, *Units and 2-class field towers of some multiquadratic number fields*, Turk J. Math., 44 (2020), 1466-1483.

[9] P. E. Conner and J. Hurrelbrink, Class number parity, Ser. Pure. Math. 8, World Scientific, 1988.

[10] J. Varmon, Über Abelsche Körper, deren alle Gruppeninvarianten aus einer Primzahl bestehen, und über Abelsche Körper als Kreiskörper. JFM 51.0123.05 Lund, Hakan Ohlssons Boktryckeri (1925).

[11] S. Kuroda, Über die Klassenzahlen algebraischer Zahlkörper, Nagoya Math. J., 1 (1950), 1–10.

[12] R. Kučera, On the parity of the class number of a biquadratic field, *J. Number Theory* **52** (1995), 43–52.

[13] T. Kubota, *Über den bizyklischen biquadratischen Zahlkörper.* Nagoya. Math. J. 10 (1956), 65-85 (in German). Zbl 0074.03001, MR0083009

[14] H. Wada, On the class number and the unit group of certain algebraic number fields. J. Fac. Sci. Univ. Tokyo, 13 (1966), 201–209.

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