On some classes of Saphar type operators II

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Abstract. The purpose of this paper is to explore additional properties of left Drazin invertible, essentially left Drazin invertible, right Drazin invertible, and essentially right Drazin invertible operators on Banach spaces, building upon the groundwork laid in [8] and [19]. Specifically, we propose alternative definitions for these operators and investigate their behavior in powers. Furthermore, we employ a specific operator decomposition to characterize these operators, providing a deeper understanding of their structure and properties. The operators we study are distinguished from other operators bearing the same name in existing literature, see [1], [9]. By employing more refined definitions, we uncover a broader range of properties for these operators, setting them apart from their counterparts in the literature.

Mathematics Subject Classification (2010). 15A09, 47A05.

Keywords. Banach space; Saphar operators; left and right invertible operators; left and right Fredholm operators; decomposition of operators; powers of operators.

1. Introduction

Let X be an infinite-dimensional complex Banach space and let $\mathcal{B}(X)$ be the algebra of all linear bounded operators on X. For $T \in \mathcal{B}(X)$, let $\mathcal{N}(T)$ denotes its kernel and $\mathcal{R}(T)$ denotes its range, while $a(T)$ (resp., $d(T)$, $a_e(T)$, $d_e(T)$) denotes the ascent (resp., the descent, the essential ascent, the essential descent). If M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$, then we write $T = T_M \oplus T_N$ and say that T is completely reduced by the pair (M, N), denoting this by $(M, N) \in Red(T)$.

Recall now that an operator $T \in \mathcal{B}(X)$ is called Drazin invertible if there exist a nonnegative integer k and a (unique) bounded operator S such that $ST = TS$, $STS = S$, $ST^{k+1} = T^k$. The smallest k for which the definition is valid is called the index of T.

It is well known that Drazin invertible operators are a generalization of invertible operators and that they can be characterized in different ways. For instance, we have an equivalence between the following statements:

- T is Drazin invertible.
- $a(T) = d(T) < \infty$.
- There exist a pair $(M, N) \in Red(T)$ such that T_M is invertible and T_N is nilpotent.

The last characterization is called the Kato decomposition of Drazin invertible operators. In addition to that, Drazin invertible operators can be also characterized in the following way:

Theorem 1.1. [12] *Let $T \in \mathcal{B}(X)$. Then, the following two statements are equivalent:*

(i) *T is Drazin invertible.*
(ii) *There exist operators $S, R \in \mathcal{B}(X)$ such that:*
   (a) *$T = S + R$ and $SR = RS = 0$.*
   (b) *S is Drazin invertible with index 0 or 1.*
   (c) *R is nilpotent.*

Set

$$\begin{aligned} R_1 &= \{T \in \mathcal{B}(X) : a(T) < \infty,\ R(T^n) \text{ is closed for every n} \geq \text{a(T)}\}, \\ R_2 &= \{T \in \mathcal{B}(X) : d(T) < \infty,\ R(T^n) \text{ is closed for every n} \geq \text{d(T)}\}, \\ R_3 &= \{T \in \mathcal{B}(X) : a_e(T) < \infty,\ R(T^n) \text{ is closed for every n} \geq \text{a}_\text{e}\text{(T)}\}, \\ R_4 &= \{T \in \mathcal{B}(X) : d_e(T) < \infty,\ R(T^n) \text{ is closed for every n} \geq \text{d}_\text{e}\text{(T)}\}. \end{aligned}$$

It is well known that a bounded linear operator T acting on a Hilbert space T belongs to R_1 (resp., R_2, R_3, R_4) if and only if it can be decomposed into a direct sum of a nilpotent operator and a left invertible, (resp., right invertible, left Fredholm, right Fredholm) operator [4, Theorem 3.12]. In [19] it was proved that this is not valid in the general Banach space setting. Namely, the class of bounded linear operators on a Banach space which are characterized by the property that they can be decomposed into a direct sum of a nilpotent operator and a left invertible (resp., right invertible, left Fredholm, right Fredholm) operator can be a proper subset of the class R_1 (resp., R_2, R_3, R_4). For a bounded linear operators T acting on a Banach space which belongs to the class R_1 (resp., R_2, R_3, R_4) to be represented as a direct sum of a nilpotent operator and a left invertible (resp., right invertible, left Fredholm, right Fredholm) operator it is necessary and sufficient that T is of Saphar type, i.e. that the subspaces $\mathcal{N}(T) \cap \mathcal{R}(T^{\text{dis}(T)})$ and $\mathcal{R}(T) + \mathcal{N}(T^{\text{dis}(T)})$ are topologically complemented, where $\text{dis}(T)$ is the degree of stable iteration of T (see [19, Theorems 4.13, 4.15, Corolarries 4.23. 4.24], [8, Theorems 3.2,3.7]). Even though operators from the class R_1 (resp., R_2, R_3, R_4) were previously known in other literature as left Drazin invertible (resp., right Drazin inverible, essentially left Drazin invertible, essentially right Drazin invertible) operators, we refer to them as upper Drazin invertible (resp. lower Drazin invertible, essentially upper Drazin invertible, essentially lower Drazin invertible). We observed that the name left Drazin invertible (resp., right Drazin inverible, essentially left Drazin invertible, essentially right Drazin invertible) operators is more appropriate for operators that can be represented as a direct sum of a nilpotent operator and a left invertible (resp., right invertible, left Fredholm, right Fredholm) operator [8], [19].

The notion of Drazin invertibility has been relaxed in [8] by introducing the class of left Drazin invertible operators and the class of right Drazin invertible operators.

Definition 1.2. An operator $T \in \mathcal{B}(X)$ is called left Drazin invertible if $p := a(T) < \infty$ and $\mathcal{R}(T)+\mathcal{N}(T^p)$ is topologically complemented, while $T \in \mathcal{B}(X)$ is called right Drazin invertible if $q := d(T) < \infty$ and $\mathcal{N}(T) \cap \mathcal{R}(T^q)$ is topologically complemented.

Note that an operator $T \in \mathcal{B}(X)$ is Drazin invertible if and only if it is left and right Drazin invertible. Also, note that the class of left Drazin invertible operators subsumes the class of left invertible operators and that the class of right Drazin invertible operators subsumes that of right invertible operators. It turns out that left and right Drazin invertible operators possess rich properties, the most important of which is, similar to the case of Drazin invertible operators, the property of Kato decomposition.

Theorem 1.3. [8] *Let $T \in \mathcal{B}(X)$. Then T is left (right) Drazin invertible if and only if there exist a pair $(M, N) \in Red(T)$ such that T_M is left (right) invertible and T_N is nilpotent.*

More general classes of left and right Drazin invertible operators, which are the classes of essentially left Drazin invertible and essentially right Drazin invertible operators, have been introduced in [19] as follows:

Definition 1.4. An operator $T \in \mathcal{B}(X)$ is said to be essentially left Drazin invertible if $a_e(T) < \infty$ and the subspace $\mathcal{R}(T) + \mathcal{N}(T^{\text{dis}(T)})$ is topologically complemented, while $T \in \mathcal{B}(X)$ is said to be essentially right Drazin invertible if $d_e(T) < \infty$ and $\mathcal{N}(T) \cap \mathcal{R}(T^{\text{dis}(T)})$ is topologically complemented, where $\text{dis}(T)$ is the degree of stable iteration of T.

Similar to left (resp. right) Drazin invertible operators, essentially left (resp. right) Drazin invertible operators can be characterized by the property of Kato decomposition, namely, they can be decomposed into a direct sum of a left (resp. right) Fredholm operator and a nilpotent one [19].

In this paper, we aim at developing additional properties of left Drazin invertible, essentially left Drazin invertible, right Drazin invertible and essentially right Drazin invertible operators on Banach spaces. The paper is organized as follows. Section 2 contains some definitions and preliminary results. In section 3, we prove that $T \in \mathcal{B}(X)$ is essentially left (resp. essentially right) Drazin invertible if and only if $a_e(T) < \infty$ (resp., $d_e(T) < \infty$) and $\mathcal{R}(T^n)$ and $\mathcal{N}(T^n)$ are topologically complemented for every $n \geq a_e(T)$

(resp., $n \geq d_e(T)$). In that way the difference between essentially upper (resp. essentially lower) Drazin invertible operators and essentially left (resp. essentially right) Drazin invertible operators is more noticeable. Also we show that $T \in \mathcal{B}(X)$ is left (resp. right, essentially left, essentially right) Drazin invertible if and only if T is upper (resp. lower, essentially upper, essentially lower) Drazin invertible and T can be decomposed into a direct sum of a Saphar operator and a meromorphic operator. Furthermore, if $d(T) < \infty$ (resp., $d_e(T) < \infty$) for $T \in \mathcal{B}(X)$, we prove that T is right Drazin invertible (resp. essentially right Drazin invertible) if and only if $\mathcal{R}(T^n)$ and $\mathcal{N}(T^{n+1})$ are topologically complemented in X for some $n \geq d(T)$ (resp. $n \geq d_e(T)$). In section 4 the behavior of powers of (essentially) left Drazin invertible and (essentially) right Drazin invertible operators are examined. Section 5 contains the corresponding results to Theorem 1.1 for left Drazin invertible (resp., right Drazin inverible, essentially left Drazin, essentially right Drazin invertible) operators. Finally, section 6 is devoted to investigating adjoints of operators of Saphar type, and as a consequence, those of (essentially) left Drazin invertible and (essentially) right Drazin invertible operators.

2. Definitions and preliminary results

Let X be an infinite-dimensional complex Banach space. We denote by X' the dual of X, and by T' the dual of $T \in \mathcal{B}(X)$. Here $\mathbb{N}$ ($\mathbb{N}_0$) denotes the set of all positive (non-negative) integers, and $\mathbb{C}$ denotes the set of all complex numbers. Recall the definitions of the ascent $a(T)$ and the descent $d(T)$ of T: we say that T has finite ascent if the chain $\mathcal{N}(T^0) \subseteq \mathcal{N}(T) \subseteq \mathcal{N}(T^2)$... becomes constant after a finite number of steps. The smallest $n \in \mathbb{N}_0$ such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ is defined to be $a(T)$. The descent is defined similarly for the chain $\mathcal{R}(T^0) \supseteq \mathcal{R}(T) \supseteq \mathcal{R}(T^2)$.... We define the infimum of the empty set to be ∞. For $T \in \mathcal{B}(X)$ and $n \in \mathbb{N}_0$ we set $\alpha_n(T) = \dim(\mathcal{N}(T) \cap \mathcal{R}(T^n))$ and $\beta_n(T) = \operatorname{codim}(\mathcal{R}(T) + \mathcal{N}(T^n))$. The smallest integer n such that $\alpha_n(T) < \infty$ is defined to be the essential ascent $a_e(T)$ of T, while the smallest integer n such that $\beta_n(T) < \infty$ is defined to be the essential descent $d_e(T)$ of T. If there is $d \in \mathbb{N}_0$ for which the sequence $(\mathcal{N}(T) \cap \mathcal{R}(T^n))$ is constant for $n \geq d$, then T is said to have uniform descent for $n \geq d$. For $T \in \mathcal{B}(X)$ the degree of stable iteration $\operatorname{dis}(T)$ is defined by:

$$\operatorname{dis}(T) = \inf\{n \in \mathbb{N}_0 : m \geq n, m \in \mathbb{N} \Longrightarrow \mathcal{N}(T) \cap \mathcal{R}(T^n) = \mathcal{N}(T) \cap \mathcal{R}(T^m)\}.$$

An operator $T \in \mathcal{B}(X)$ is said to be *quasi-Fredholm of degree d* if there exists a $d \in \mathbb{N}_0$ such that $\operatorname{dis}(T) = d$ and $\mathcal{R}(T^n)$ is closed for each $n \geq d$. An operator $T \in \mathcal{B}(X)$ is *Kato* if $\mathcal{R}(T)$ is closed and $\mathcal{N}(T) \subset \mathcal{R}(T^n)$ for every $n \in \mathbb{N}_0$. We recall that $T \in \mathcal{B}(X)$ is Kato if and only if T is quasi-Fredholm of degree 0.

An operator $T \in \mathcal{B}(X)$ is called upper (resp. lower) semi-Fredholm, or $T \in \Phi_+(X)$ (resp. $T \in \Phi_-(X)$) if $\alpha(T) := \dim\mathcal{N}(T) < \infty$ and $\mathcal{R}(T)$ is closed (resp. $\beta(T) := \operatorname{codim}\mathcal{R}(T) < \infty$), and it is said to be relatively regular if $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are topologically complemented subspaces of X.

An operator $T \in \mathcal{B}(X)$ is called left (resp. right) Fredholm, or $T \in \Phi_l(X)$ (resp. $T \in \Phi_r(X)$) if it is relatively regular and upper (resp. lower) semi-Fredholm.

If $T \in \mathcal{B}(X)$ is a relatively regular Kato operator, then we say that T is a *Saphar operator*. The degree of a nilpotent operator T is the smallest $d \in \mathbb{N}_0$ such that $T^d = 0$. Let $d \in \mathbb{N}_0$. An operator $T \in \mathcal{B}(X)$ is said to be *of Saphar type of degree* d if there exists a pair $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is nilpotent of degree d. In that case, since $\text{dis}(T_N) = a(T_N) = d(T_N) = d$ and $\text{dis}(T_M) = 0$, it follows that $\text{dis}(T) = \max\{\text{dis}(T_M), \text{dis}(T_N)\} = d$. We recall that $T \in \mathcal{B}(X)$ is of Saphar type of degree d if and only if it is a quasi-Fredholm operator with $\text{dis}(T) = d$ and the subspaces $\mathcal{R}(T) + \mathcal{N}(T^d)$ and $\mathcal{N}(T) \cap \mathcal{R}(T^d)$ are topologically complemented [19, Theorem 4.2], [7, Theorem 4.1]. An operator $T \in \mathcal{B}(X)$ is said to admit a *generalized Saphar decomposition* if there exists a pair $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is quasinilpotent [6].

An operator $T \in \mathcal{B}(X)$ is meromorphic if its non-zero spectral points are poles of its resolvent. We shall say $T \in \mathcal{B}(X)$ admits a *generalized Saphar-meromorphic decomposition* if there exists a pair $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is meromorphic.

Let F, G and H be subspaces of X. We will copiously use the following modular law:

$$\text{if } F \subseteq H, \text{ then } (F + G) \cap H = F + G \cap H.$$

We shall recall the following algebraic isomorphisms as they are crucial for our work.

Lemma 2.1. [10] *For all $k, i \in \mathbb{N}$, one has:*

(i) $\dfrac{\mathcal{N}(T^{i+k})}{\mathcal{N}(T^i)} \simeq \mathcal{N}(T^k) \cap \mathcal{R}(T^i)$.

(ii) $\dfrac{\mathcal{R}(T^i)}{\mathcal{R}(T^{i+k})} \simeq \dfrac{X}{\mathcal{R}(T^k) + \mathcal{N}(T^i)}$.

Lemma 2.2. *For all $n \in \mathbb{N}$, one has:*

(i) $a(T) \leq n a(T^n)$.

(ii) $d(T) \leq n d(T^n)$.

We shall now recall Lemma 2.3 from [8].

Lemma 2.3. *Let M and N be closed in X such that $X = M \oplus N$. Let F be a subspace of M and G be a subspace of N. Then, $F \oplus G$ is topologically complemented in X if and only if F is topologically complemented in M and G is topologically complemented in N.*

Lemma 2.4. *Let M, N be subspaces of X such that M is topologically complemented and N is finite-dimensional. Then $M + N$ is a topologically complemented subspace of X.*

Proof. It follows from [17, Lemma 2.6]. □

Lemma 2.5. [19] *Let M be a topologically complemented subspace of X and let N be a closed subspace of X such that* $\operatorname{codim} N < \infty$. *Then $M \cap N$ is topologically complemented in X.*

Lemma 2.6. [19] *Let M be complemented subspace of X and let M_1 be a closed subspace of X such that $M \subset M_1$. Then M is complemented in M_1.*

Lemma 2.7. [19] *For $T \in \mathcal{B}(X)$ let there exists a pair $(M, N) \in Red(T)$. Then T is Saphar if and only if T_M and T_N are Saphar.*

Proposition 2.8. *Let $T \in \mathcal{B}(X)$. The following statements are equivalent:*

(i) *T is essentially left Drazin invertible.*

(ii) *$a_e(T) < \infty$ and $\mathcal{N}(T) \cap \mathcal{R}(T^n)$ and $\mathcal{R}(T) + \mathcal{N}(T^n)$ are topologically complemented for every $n \geq a_e(T)$.*

Proof. (i)$\Longrightarrow$(ii): Let T be essentially left Drazin invertible. Then from the equivalence (i)$\Longleftrightarrow$(iii) in [19, Theorem 4.11] it follows that $\mathcal{R}(T)+\mathcal{N}(T^n)$ is topologically complemented for every $n \geq a_e(T)$. Since $\dim(\mathcal{N}(T)\cap\mathcal{R}(T^n)) = \alpha_n(T) < \infty$ for every $n \geq a_e(T)$, we conclude that $\mathcal{N}(T) \cap \mathcal{R}(T^n)$ is topologically complemented in X for every $n \geq a_e(T)$.

(ii)$\Longrightarrow$(i): It follows from the equivalence (i)$\Longleftrightarrow$(iii) in [19, Theorem 4.11]. □

Proposition 2.9. *Let $T \in \mathcal{B}(X)$. The following statements are equivalent:*

(i) *T is essentially right Drazin invertible.*

(ii) *$d_e(T) < \infty$ and $\mathcal{N}(T) \cap \mathcal{R}(T^n)$ and $\mathcal{R}(T) + \mathcal{N}(T^n)$ are topologically complemented for every $n \geq d_e(T)$.*

Proof. (i)$\Longrightarrow$(ii): Suppose that T is essentially right Drazin invertible. Then from the equivalence (i)$\Longleftrightarrow$(iii) in [19, Theorem 4.12] it follows that $\mathcal{N}(T) \cap \mathcal{R}(T^n)$ is topologically complemented for every $n \geq d_e(T)$. According to [19, Proposition 4.10] we have that $\mathcal{R}(T^n)$ is closed for every $n \geq d_e(T)$. Hence $T^{-n}(\mathcal{R}(T^{n+1})) = \mathcal{R}(T) + \mathcal{N}(T^n)$ is closed for every $n \geq d_e(T)$. As $\operatorname{codim}(\mathcal{R}(T) + \mathcal{N}(T^n)) = \beta_n(T) < \infty$ for every $n \geq d_e(T)$, we conclude that $\mathcal{R}(T) + \mathcal{N}(T^n)$ is topologically complemented in X for every $n \geq d_e(T)$.

(ii)$\Longrightarrow$(i): It follows from the equivalence (i)$\Longleftrightarrow$(iii) in [19, Theorem 4.12]. □

3. Equivalent definitions of (essentially) left Drazin invertible and (essentially) right Drazin invertible operators

The following theorem provides necessary and sufficient conditions for an upper (resp. lower, essentially upper, essentially lower) Drazin invertible operator to be left (resp. lower, essentially upper, essentially lower) Drazin invertible.

Theorem 3.1. *Let $T \in \mathcal{B}(X)$ is an upper (resp. lower, essentially upper, essentially lower) Drazin invertible operator. Then the following conditions are equivalent:*

(i) *T is left (resp. right, essentially left, essentially right) Drazin invertible.*
(ii) *T is of Saphar type.*
(iii) *T admits a generalized Saphar decomposition.*
(iv) *T admits a generalized Saphar-meromorphic decomposition.*

Proof. (i)$\Longrightarrow$(ii) Suppose that T is left Drazin invertible. Then according to [19, Corollary 4.23] it follows that T is of Saphar type.

(ii)$\Longrightarrow$(i) Let T be of Saphar type. Since T is upper Drazin invertible, it follows that $a(T) < \infty$. Now from [19, Corollary 4.23] we conclude that T is left Drazin invertible.

(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv) It is clear.

(iv)$\Longrightarrow$(ii) Suppose that T admits a generalized Saphar-meromorphic decomposition. Then there exists a pair $(M, N) \in Red(T)$ such that T_M is Saphar and T_N is meromorphic. It follows that $T_N - \lambda I_N$ is Drazin invertible for every $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and hence $T_N - \lambda I_N$ is upper Drazin invertible for every $\lambda \neq 0$. Since T is upper Drazin invertible, we have that T_N is upper Drazin invertible. Therefore, $\{\lambda \in \mathbb{C} : T_N - \lambda I_N \text{ is not upper Drazin invertible}\} = \emptyset$, which according to [9, Corollary 3.15] implies that the Drazin spectrum of T_N is empty. Thus T_N is Drazin invertible, and hence there exists a pair $(N_1, N_2) \in Red(T_N)$ such that T_{N_1} is invertible and T_{N_2} is nilpotent. Applying Lemma 2.7 we obtain that $T_M \oplus T_{N_1}$ is Saphar. Consequently, T is of Saphar type.

The rest of the assertions can be proved similarly. $\square$

Therefore, an operator $T \in \mathcal{B}(X)$ is left (resp. right, essentially left, essentially right) Drazin invertible if and only if T is upper (resp. lower, essentially upper, essentially lower) Drazin invertible and admits a generalized Saphar-meromorphic decomposition.

There exists a bounded linear operator T acting on a Banach space that is bounded below (resp., surjective) with not complemented range (resp., null-space) [5, Example 1.1]. This operator is upper (resp., lower) Drazin invertible, and hence essentially upper (resp., essentially lower) Drazin invertible. Also, this operator is of Kato type, but it is not of Saphar type (see [19, p. 170]), and hence according to Theorem 3.1 it does not admit a generalized Saphar decomposition nor a generalized Saphar-meromorphic decomposition. This example shows that for an upper (resp. lower, essentially upper, essentially lower) Drazin invertible operator to be left (resp. right, essentially left, essentially right) Drazin invertible it is not enough to be of Kato type. Also this example shows that the condition that T admits a generalized Saphar-meromorphic decomposition in the previous assertion can not be omitted.

By using Theorem 1.3, we have given in [8] an equivalent definition of left Drazin invertible operators [8, Theorem 3.3]. We will show that the converse implication in [8, Theorem 3.3] is merely a consequence of the following lemma.

Lemma 3.2. [7] *Let $T \in \mathcal{B}(X)$ be such that $\mathcal{N}(T)$ is topologically complemented. If M is topologically complemented and $M \subseteq \mathcal{R}(T)$, then $T^{-1}(M)$ is topologically complemented.*

Theorem 3.3. *Let $T \in \mathcal{B}(X)$. The following statements are equivalent:*

(i) *T is left Drazin invertible.*

(ii) *$a(T) < \infty$ and $\mathcal{N}(T^n)$ and $\mathcal{R}(T^n)$ are topologically complemented for every $n \geq a(T)$.*

(iii) *$a(T) < \infty$ and there exists $n \geq a(T)$ such that $\mathcal{N}(T^n)$ and $\mathcal{R}(T^{n+1})$ are topologically complemented.*

Proof. (i)$\Longrightarrow$(ii) Suppose that T is left Drazin invertible. By Theorem 1.3 there exist closed T-invariant subspaces M and N such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is left invertible and T_N is nilpotent of degree $a(T)$. For every $n \geq a(T)$ we have that $T^n = T_M^n \oplus 0_N$, and so $\mathcal{N}(T^n) = \mathcal{N}(T_M^n) \oplus N = N$ which is topologically complemented and $\mathcal{R}(T^n) = \mathcal{R}(T_M^n)$. Since T_M^n is left invertible it follows that $\mathcal{R}(T_M^n)$ is topologically complemented in M. Applying Lemma 2.3 we conclude that $\mathcal{R}(T^n)$ are topologically complemented in X.

(ii)$\Longrightarrow$(iii): It is obvious.

(iii)$\Longrightarrow$(i): Assume that $a(T) < \infty$, and $\mathcal{N}(T^n)$ and $\mathcal{R}(T^{n+1})$ are topologically complemented for some $n \geq a(T)$. Since $\mathcal{R}(T) + \mathcal{N}(T^n) = T^{-n}(\mathcal{R}(T^{n+1}))$ and $\mathcal{R}(T^{n+1}) \subseteq \mathcal{R}(T^n)$, from Lemma 3.2 it follows that $\mathcal{R}(T) + \mathcal{N}(T^{a(T)}) = \mathcal{R}(T) + \mathcal{N}(T^n)$ is topologically complemented. Therefore, T is left Drazin invertible. □

Similar to the case of left Drazin invertible operators, we may come up with an alternative definition of right Drazin invertible operators. This was not done in [8] since at the time, the following lemma was not discovered yet.

Lemma 3.4. [7] *Let $T \in \mathcal{B}(X)$ be such that $\mathcal{R}(T)$ is topologically complemented. If M is topologically complemented and $\mathcal{N}(T) \subseteq M$, then $T(M)$ is topologically complemented.*

Theorem 3.5. *Let $T \in \mathcal{B}(X)$. The following statements are equivalent:*

(i) *T is right Drazin invertible.*

(ii) *$d(T) < \infty$ and $\mathcal{N}(T^n)$ and $\mathcal{R}(T^n)$ are topologically complemented for every $n \geq d(T)$.*

(iii) *$d(T) < \infty$ and there exists $n \geq d(T)$ such that $\mathcal{N}(T^{n+1})$ and $\mathcal{R}(T^n)$ are topologically complemented.*

Proof. (i)$\Longrightarrow$(ii): If T is right Drazin invertible, then by Theorem 1.3, there exist closed T-invariant subspaces M and N such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is right invertible and $T_N^q = 0$ where $q := d(T)$. It follows that $T^n = T_M^n \oplus 0$ for every $n \geq q$. Hence, $\mathcal{R}(T^n) = M$ which is topologically complemented and $\mathcal{N}(T^n) = \mathcal{N}(T_M^n) \oplus N$. Since T_M^n is right invertible, $\mathcal{N}(T_M^n)$ is topologically complemented in M, so from Lemma 2.3 it follows that $\mathcal{N}(T^n)$ is topologically complemented in X.

(ii)$\Longrightarrow$(iii): It is clear.

(iii)$\Longrightarrow$(i): Assume that $d(T) < \infty$ and that $\mathcal{N}(T^{n+1})$ and $\mathcal{R}(T^n)$ are topologically complemented for some $n \geq d(T)$. Notice that $T^n(\mathcal{N}(T^{n+1})) = \mathcal{N}(T) \cap \mathcal{R}(T^n) = \mathcal{N}(T) \cap \mathcal{R}(T^{d(T)})$. Since $\mathcal{N}(T^n) \subseteq \mathcal{N}(T^{n+1})$, it follows from Lemma 3.4 that $\mathcal{N}(T) \cap \mathcal{R}(T^{d(T)})$ is topologically complemented. □

Our upcoming work heavily relies on the subsequent two lemmas.

Lemma 3.6. *Let $T \in \mathcal{B}(X)$ and M be a paracomplete subspace (i.e. range of a bounded operator) of X such that $M \cap \mathcal{N}(T)$ is topologically complemented. If $T(M)$ is topologically complemented, then so is M.*

Proof. Let N and G be closed subspaces of X such that $T(M) \oplus N = X$ and $(M \cap \mathcal{N}(T)) \oplus G = X$. We claim that M is topologically complemented in X by $G \cap T^{-1}(N)$. Indeed, first we have:

$$\begin{aligned} M + G \cap T^{-1}(N) &= M + M \cap \mathcal{N}(T) + G \cap T^{-1}(N) \\ &= M + (M \cap \mathcal{N}(T) + G) \cap T^{-1}(N) \\ &= M + X \cap T^{-1}(N) \\ &= M + T^{-1}(N). \end{aligned}$$

On the other hand, since $T(M) \subset \mathcal{R}(T)$, one has:

$$T^{-1}(T(M) + N) = T^{-1}(T(M)) + T^{-1}(N).$$

But

$$T^{-1}(T(M)) = M + \mathcal{N}(T).$$

Thus,

$$\begin{aligned} X &= T^{-1}(X) \\ &= T^{-1}(T(M) + N) \\ &= M + \mathcal{N}(T) + T^{-1}(N) \\ &= M + T^{-1}(N). \end{aligned}$$

Consequently,

$$M + G \cap T^{-1}(N) = X.$$

Now, if $x \in M \cap (G \cap T^{-1}(N))$, then $T(x) \in T(M) \cap N = \{0\}$ so that $x \in \mathcal{N}(T)$. Therefore, since $M \cap G \cap \mathcal{N}(T) = \{0\}$, it follows that $x = 0$, and whence $X = M \oplus (G \cap T^{-1}(N))$. Now to finish the proof, since $G \cap T^{-1}(N)$ is closed, we only need to prove that M is closed. Towards this, we will apply some results from Chapter II in [13]. Since M and $\mathcal{N}(T)$ are paracomplete and since $T^{-1}(T(M)) = M + \mathcal{N}(T)$ and $M \cap \mathcal{N}(T)$ are closed, it follows from [13, Proposition 2.1.1] that M is closed too. □

Lemma 3.7. *Let $T \in \mathcal{B}(X)$ and M be paracomplete such that $M + \mathcal{R}(T)$ is topologically complemented. If $T^{-1}(M)$ is topologically complemented, then so is M.*

Proof. Let N and H be closed such that $T^{-1}(M)\oplus N = X$ and $(M+\mathcal{R}(T))\oplus H = X$. We claim that M is topologically complemented in X by $T(N)\oplus H$. Indeed, first, one has

$$\begin{aligned} T(T^{-1}(M)+N) &= TT^{-1}(M)+T(N) \\ &= M\cap\mathcal{R}(T)+T(N). \end{aligned}$$

Hence

$$M\cap\mathcal{R}(T)+T(N)=\mathcal{R}(T)$$

Since $M+\mathcal{R}(T)+H=X$, it follows that

$$\begin{aligned} M+(T(N)\oplus H) &= M+M\cap\mathcal{R}(T)+T(N)+H \\ &= M+\mathcal{R}(T)+H \\ &= X. \end{aligned}$$

On the other hand, if $x\in M\cap(T(N)\oplus H)$, then $x=Tn+h$ for some $n\in N$ and $h\in H$. Hence, $h=x-Tn$ so that $h\in(M+\mathcal{R}(T))\cap H=\{0\}$. Therefore, $n\in T^{-1}(M)\cap N=\{0\}$ which implies that $x=0$. Summing up, $M\oplus(T(N)\oplus H)=X$. We shall lastly prove that M and $T(N)\oplus H$ are closed. But this is a consequence of [13, Proposition 2.1.1] since M and $T(N)\oplus H$ are paracomplete (CH. II [13]) and $M+(T(N)\oplus H)$ and $M\cap(T(N)\oplus H)$ are closed. □

Similar to left and right Drazin invertible operators, essentially left and right Drazin invertible operators can be equivalently defined in the following manner.

Theorem 3.8. *Let $T\in\mathcal{B}(X)$. The following statements are equivalent:*

(i) *T is essentially left Drazin invertible.*
(ii) *$a_e(T)<\infty$ and $\mathcal{N}(T^n)$ and $\mathcal{R}(T^n)$ are topologically complemented for every $n\geq a_e(T)$.*
(iii) *$a_e(T)<\infty$ and there exists $n\geq a_e(T)$ such that $\mathcal{N}(T^n)$ and $\mathcal{R}(T^{n+1})$ are topologically complemented.*

Proof. (i)$\Longrightarrow$(ii): Suppose that T is essentially left Drazin invertible. Then $a_e(T)\leq \mathrm{dis}(T)<\infty$. From the proof of the implication (iv)$\Longrightarrow$(v) in [19, Theorem 4.13] it follows that there exist closed T-invariant subspaces M and N such that $X=M\oplus N$ and $T=T_M\oplus T_N$ where T_M is left Fredholm and $T_N^d=0$ for $d:=\mathrm{dis}(T)$. For every $n\geq d$ we have that $T^n=T_M^n\oplus 0_N$, and so $\mathcal{R}(T^n)=\mathcal{R}(T_M^n)$ and $\mathcal{N}(T^n)=\mathcal{N}(T_M^n)\oplus N$. Since T_M^n is left Fredholm it follows that $\mathcal{R}(T_M^n)$ and $\mathcal{N}(T_M^n)$ are topologically complemented in M. Applying Lemma 2.3, we conclude that $\mathcal{N}(T^n)$ and $\mathcal{R}(T^n)$ are topologically complemented in X for every $n\geq d$.

If $a_e(T)=d$, then we have that $\mathcal{N}(T^n)$ and $\mathcal{R}(T^n)$ are topologically complemented for every $n\geq a_e(T)$. Suppose that $a_e(T)<d$. Then $a_e(T)\leq d-1$ and hence according to Proposition 2.8 the subspace $\mathcal{N}(T)\cap\mathcal{R}(T^{d-1})$ is topologically complemented. Since $\mathcal{R}(T^{d-1})$ is paracomplete and $T(\mathcal{R}(T^{d-1}))=\mathcal{R}(T^d)$ is topologically complemented, from Lemma 3.6, it follows that

$\mathcal{R}(T^{d-1})$ is topologically complemented. By repeating this method we can conclude that $\mathcal{R}(T^n)$ is complemented for every $n \geq a_e(T)$. On the other side, by Proposition 2.8, we have that $\mathcal{R}(T)+\mathcal{N}(T^{d-1})$ is topologically complemented. As $\mathcal{N}(T^{d-1})$ is paracomplete and since $T^{-1}(\mathcal{N}(T^{d-1})) = \mathcal{N}(T^d)$ is topologically complemented, Lemma 3.7 ensures that $\mathcal{N}(T^{d-1})$ is topologically complemented. By repeating this method we can conclude that $\mathcal{N}(T^n)$ is complemented for every $n \geq a_e(T)$.

(ii)$\Longrightarrow$(iii): It is obvious.

(iii)$\Longrightarrow$(i): It follows from [19, Theorem 4.13]. □

Theorem 3.9. *Let $T \in \mathcal{B}(X)$. The following statements are equivalent:*

(i) *T is essentially right Drazin invertible.*

(ii) *$d_e(T) < \infty$ and $\mathcal{N}(T^n)$ and $\mathcal{R}(T^n)$ are topologically complemented for every $n \geq d_e(T)$.*

(iii) *$d_e(T) < \infty$ and there exists $n \geq d_e(T)$ such that $\mathcal{N}(T^{n+1})$ and $\mathcal{R}(T^n)$ are topologically complemented.*

Proof. (i)$\Longrightarrow$(ii): From [19, Theorem 4.15] and Lemma 2.3, similarly to the proof of the implication (i)$\Longrightarrow$(ii) in Theorem 3.8 we conclude that $\mathcal{N}(T^n)$ and $\mathcal{R}(T^n)$ are topologically complemented for every $n \geq \mathrm{dis}(T)$. Further by using Poposition 2.9, Lemma 3.6, Lemma 3.7 and the same method as in the proof of Theorem 3.8, we can conclude that $\mathcal{N}(T^n)$ and $\mathcal{R}(T^n)$ are topologically complemented for every $n \geq d_e(T)$.

(ii)$\Longrightarrow$(iii): It is obvious.

(iii)$\Longrightarrow$(i): Suppose that $d_e(T) < \infty$ and let $\mathcal{N}(T^{n+1})$ and $\mathcal{R}(T^n)$ be topologically complemented for some $n \geq d_e(T)$. As in the proof of Theorem 3.5, applying Lemma 3.4 we obtain that $\mathcal{N}(T) \cap \mathcal{R}(T^n)$ is topologically complemented. Using the equivalence (i)$\Longleftrightarrow$(iv) in [19, Theorem 4.12] we conclude that T is essentially right Drazin invertible. □

4. Powers of (essentially) left Drazin invertible and (essentially) right Drazin invertible operators

In this section, we will be interested in investigating the behaviour of powers of left (resp. right) and essentially left (resp. essentially right) Drazin invertible operators. Towards this, we shall begin with the following proposition.

Proposition 4.1. *Let $T \in \mathcal{B}(X)$.*

(i) *If T is (essentially) left Drazin invertible, then T^n is (essentially) left Drazin invertible for all $n \in \mathbb{N}$.*

(ii) *If T is (essentially) right Drazin invertible, then T^n is (essentially) right Drazin invertible for all $n \in \mathbb{N}$.*

Proof. (i) If T is (essentially) left Drazin invertible, then by Theorem 1.3 ([19, Theorem 4.13]), there exist closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is left invertible (left Fredholm) and T_N is nilpotent. For any $n \in \mathbb{N}$, we have $T^n = T_M^n \oplus T_N^n$. But

T^n_M is left invertible (left Fredholm) and T^n_N is nilpotent. Thus, T^n is left Drazin invertible, again by Theorem 1.3 ([19, Theorem 4.13]).

(ii) This can be proved in a way similar to part (i). $\square$

What if T^m is a left or an essentially left Drazin invertible operator for some $m \in \mathbb{N}$, what do we expect for T? We answer this question in the subsequent theorems.

Theorem 4.2. *Let $T \in \mathcal{B}(X)$. If T^m is left Drazin invertible for some $m \in \mathbb{N}$, then T is left Drazin invertible.*

Proof. Assume that T^m is left Drazin invertible for some $m \in \mathbb{N}$. Let $d := a(T^m)$. From Lemma 2.2 (i) it follows that $p := a(T) \leq md$. According to the equivalence (i)$\Longleftrightarrow$(iii) in Theorem 3.3, to prove that T is left Drazin invertible it suffices to prove that $\mathcal{N}(T^{md})$ and $\mathcal{R}(T^{md+1})$ are topologically complemented.

Since T^m is left Drazin invertible, from the equivalence (i)$\Longleftrightarrow$(ii) in Theorem 3.3 it follows that $\mathcal{N}((T^m)^d) = \mathcal{N}(T^{md})$ and $\mathcal{R}((T^m)^{d+1}) = \mathcal{R}(T^{md+m})$ are topologically complemented. Notice that $T^{m-1}(\mathcal{R}(T^{md+1})) = \mathcal{R}(T^{md+m})$. Also, since $a(T) \leq md$, we know from Lemma 2.1 (i) that $\mathcal{N}(T^{m-1}) \cap \mathcal{R}(T^{md+1}) = \{0\}$. On the other hand, $\mathcal{R}(T^{md+1})$ is paracomplete. So an application of Lemma 3.6 gives that $\mathcal{R}(T^{md+1})$ is topologically complemented and the proof is completed. $\square$

Theorem 4.3. *Let $T \in \mathcal{B}(X)$. If T^m is essentially left Drazin invertible for some $m \in \mathbb{N}$, then T is essentially left Drazin invertible.*

Proof. Let T^m be essentially left Drazin invertible for some $m \in \mathbb{N}$. Then $d := a_e(T^m) < \infty$. From the inequality $\alpha_{mn}(T) \leq \alpha_n(T^m)$, $n \in \mathbb{N}$ [16, p. 137], we have that $a_e(T) \leq m a_e(T^m) = md$. According to the equivalence (i)$\Longleftrightarrow$(iii) in Theorem 3.8, to prove that T is essentially left Drazin invertible it suffices to prove that $\mathcal{N}(T^{md})$ and $\mathcal{R}(T^{md+1})$ are topologically complemented.

As T^m is essentially left Drazin invertible, from the equivalence (i)$\Longleftrightarrow$(ii) in Theorem 3.8 it follows that $\mathcal{N}((T^m)^d) = \mathcal{N}(T^{md})$ and $\mathcal{R}((T^m)^{d+1}) = \mathcal{R}(T^{md+m})$ are topologically complemented. We have that

$$\dim(\mathcal{N}(T^m) \cap \mathcal{R}(T^{md})) = \alpha_d(T^m) < \infty,$$

and hence $\dim T(\mathcal{N}(T^m) \cap \mathcal{R}(T^{md})) < \infty$. Since $T(\mathcal{N}(T^m) \cap \mathcal{R}(T^{md})) = \mathcal{N}(T^{m-1}) \cap \mathcal{R}(T^{md+1})$, we obtain that $\mathcal{N}(T^{m-1}) \cap \mathcal{R}(T^{md+1})$ is topologically complemented. As $\mathcal{R}(T^{md+1})$ is paracomplete and $T^{m-1}(\mathcal{R}(T^{md+1})) = \mathcal{R}(T^{md+m})$ is topologically complemented, from Lemma 3.6 it follows that $\mathcal{R}(T^{md+1})$ is topologically complemented. $\square$

By combining Proposition 4.1 (i) and Theorems 4.2 and 4.3, the following result arises.

Theorem 4.4. *For $T \in \mathcal{B}(X)$, the following statements are equivalent:*

(i) *T is (essentially) left Drazin invertible.*
(ii) *T^n is (essentially) left Drazin invertible for all $n \in \mathbb{N}$.*

(iii) *T^n is (essentially) left Drazin invertible for some $n \in \mathbb{N}$.*

Now, we switch the focus to right and essentially right Drazin invertible operators.

Theorem 4.5. *Let $T \in \mathcal{B}(X)$. If T^m is right Drazin invertible for some $m \in \mathbb{N}$, then T is right Drazin invertible.*

Proof. Assume that T^m is right Drazin invertible for some $m \in \mathbb{N}$. Let $d := d(T^m)$. From Lemma 2.2 (ii) it follows that $d(T) \leq md$. According to the equivalence (i)$\Longleftrightarrow$(iii) in Theorem 3.5, in order to prove that T is right Drazin invertible, it suffices to prove that $\mathcal{N}(T^{md+1})$ and $\mathcal{R}(T^{md})$ are topologically complemented.

Since T^m is right Drazin invertible, from the equivalence (i)$\Longleftrightarrow$(ii) in Theorem 3.5 it follows that $\mathcal{N}((T^m)^{d+1}) = \mathcal{N}(T^{md+m})$ and $\mathcal{R}((T^m)^d) = \mathcal{R}(T^{md})$ are topologically complemented. Notice that $T^{-(m-1)}(\mathcal{N}(T^{md+1})) = \mathcal{N}(T^{md+m})$. Also, since $d(T) \leq md$, we know from Lemma 2.1 (ii) that $\mathcal{R}(T^{m-1}) + \mathcal{N}(T^{md+1}) = X$. As the subspace $\mathcal{N}(T^{md+1})$ is paracomplete, an application of Lemma 3.7 gives that $\mathcal{N}(T^{md+1})$ is topologically complemented and the proof is completed. □

Theorem 4.6. *Let $T \in \mathcal{B}(X)$. If T^m is essentially right Drazin invertible for some $m \in \mathbb{N}$, then T is essentially right Drazin invertible.*

Proof. Let T^m be essentially right Drazin invertible for some $m \in \mathbb{N}$. Then $d := d_e(T^m) < \infty$. From [16, Lemma 3] it follows that $d_e(T) \leq md_e(T^m) = md$. According to the equivalence (i)$\Longleftrightarrow$(iii) in Theorem 3.9, to prove that T is essentially right Drazin invertible it suffices to prove that $\mathcal{N}(T^{md+1})$ and $\mathcal{R}(T^{md})$ are topologically complemented.

As T^m is essentially right Drazin invertible, according to the equivalence (i)$\Longleftrightarrow$(ii) in Theorem 3.9 we obtain that $\mathcal{N}((T^m)^{d+1}) = \mathcal{N}(T^{md+d})$ and $\mathcal{R}((T^m)^d) = \mathcal{R}(T^{md})$ are topologically complemented. Since

$$\operatorname{codim}(\mathcal{R}(T^m) + \mathcal{N}(T^{md})) = \beta_d(T^m) < \infty,$$

we have that $\operatorname{codim} T^{-1}(\mathcal{R}(T^m) + \mathcal{N}(T^{md})) < \infty$. From [16, Lemma 12] it follows that $\mathcal{R}(T^m) + \mathcal{N}(T^{md})$ is closed and hence $T^{-1}(\mathcal{R}(T^m) + \mathcal{N}(T^{md}))$ is closed. Since $T^{-1}(\mathcal{R}(T^m) + \mathcal{N}(T^{md})) = \mathcal{R}(T^{m-1}) + \mathcal{N}(T^{md+1})$, we get that $\mathcal{R}(T^{m-1}) + \mathcal{N}(T^{md+1})$ is topologically complemented. As $\mathcal{N}(T^{md+1})$ is paracomplete and $T^{-(m-1)}(\mathcal{N}(T^{md+1})) = \mathcal{N}(T^{md+m})$ is topologically complemented, Lemma 3.7 ensures that $\mathcal{N}(T^{md+1})$ is topologically complemented and the proof is completed. □

As a consequence of Proposition 4.1 (ii) and Theorems 4.5 and 4.6, one can draw the conclusion that:

Theorem 4.7. *For $T \in \mathcal{B}(X)$, the following statements are equivalent:*

(i) *T is (essentially) right Drazin invertible.*
(ii) *T^n is (essentially) right Drazin invertible for all $n \in \mathbb{N}$.*
(iii) *T^n is (essentially) right Drazin invertible for some $n \in \mathbb{N}$.*

5. Another decomposition of (essentially) left Drazin invertible and (essentially) right Drazin invertible operators

The purpose in this section is to prove a corresponding result to Theorem 1.1 for (essentially) left Drazin invertible and (essentially) right Drazin invertible operators. Towards this, we begin by proving such a result for operators of Saphar type. This result complements [7, Theorem 5.2].

Theorem 5.1. *Let $T \in \mathcal{B}(X)$. The following statements are equivalent:*

(i) *T is of Saphar type.*

(ii) *There exist operators $S, R \in \mathcal{B}(X)$ such that $T = S + R$, $SR = RS = 0$, S is of Saphar type,* $\mathrm{dis}(S) \leq 1$ *and R is nilpotent.*

(iii) *There exist operators $S, R \in \mathcal{B}(X)$ such that $T = S + R$, $SR = RS = 0$, S is of Saphar type and R is nilpotent.*

Proof. (i)$\Longrightarrow$(ii): It is proved in [7, Theorem 5.2].

(ii)$\Longrightarrow$(iii) is trivial.

(iii)$\Longrightarrow$(i): Suppose that there exist operators $S, R \in \mathcal{B}(X)$ such that $T = S + R$ and $SR = RS = 0$, S is of Saphar type and R is nilpotent of some degree d. As $RS = 0$, we get that

$$\mathcal{N}(T) \cap \mathcal{R}(S^n) = \mathcal{N}(S) \cap \mathcal{R}(S^n) \text{ for all } n \in \mathbb{N}.$$

Since $SR = RS = 0$, it follows that $T^n = S^n$ for all $n \geqslant d$, and so for all $n \geqslant d$, it holds:

$$\begin{aligned} \mathcal{N}(T) \cap \mathcal{R}(T^n) &= \mathcal{N}(T) \cap \mathcal{R}(S^n) \\ &= \mathcal{N}(S) \cap \mathcal{R}(S^n). \end{aligned}$$

Let $r = \max\{d, \mathrm{dis}(S)\}$. Then for all $n \geq r$, we have that

$$\mathcal{N}(T) \cap \mathcal{R}(T^n) = \mathcal{N}(S) \cap \mathcal{R}(S^{\mathrm{dis}(S)}).$$

Therefore, T has uniform descent for $n \geq r$ and $\mathcal{N}(T) \cap \mathcal{R}(T^r)$ is topologically complemented.

We shall prove that $\mathcal{R}(T) + \mathcal{N}(T^n) = \mathcal{R}(S) + \mathcal{N}(S^n)$ for all $n \geq d$. Let $y \in \mathcal{R}(T) + \mathcal{N}(T^n)$. Then $y = T(x) + z$ where $z \in \mathcal{N}(T^n)$. As $T^n = S^n$ we have that $y = (S + R)(x) + z = S(x) + R(x) + z$ which belongs to $\mathcal{R}(S) + \mathcal{N}(S^n)$. Thus $\mathcal{R}(T) + \mathcal{N}(T^n) \subseteq \mathcal{R}(S) + \mathcal{N}(S^n)$. Now, the converse inclusion can be demonstrated in the same way. Consequently, $\mathcal{R}(T) + \mathcal{N}(T^r) = \mathcal{R}(S) + \mathcal{N}(S^{\mathrm{dis}(S)})$ and hence $\mathcal{R}(T) + \mathcal{N}(T^r)$ is topologically complemented. Applying [19, Theorem 4.2] we conclude that T is of Saphar type. □

Theorem 5.2. *Let $T \in \mathcal{B}(X)$. The following statements are equivalent:*

(i) *T is left Drazin invertible.*

(ii) *There exist operators $S, R \in \mathcal{B}(X)$ such that:*

  (a) *$T = S + R$ and $SR = RS = 0$,*

  (b) *S is left Drazin invertible with ascent* 0 *or* 1*,*

  (c) *R is nilpotent.*

(iii) *There exist operators $S, R \in \mathcal{B}(X)$ such that:*

(a) $T = S + R$ *and* $SR = RS = 0$,
(b) S *is left Drazin invertible,*
(c) R *is nilpotent.*

Proof. $(i) \Rightarrow (ii)$ Suppose that T is left Drazin invertible. Then $a(T) < \infty$. From [19, Corollary 4.23] it follows that T is of Saphar type. According to Theorem 5.1 there exist operators $S, R \in \mathcal{B}(X)$ such that $T = S + R$, $SR = RS = 0$, S is of Saphar type, $\mathrm{dis}(S) \leq 1$ and R is nilpotent. Since $TR = RT$, from [11, Theorem 2.2] it follows that $a(S) < \infty$. As $\mathrm{dis}(S) = a(S)$ we conclude that $a(S) \leq 1$. Using [19, Corollary 4.23] we get that S is left Drazin invertible.

$(ii) \Rightarrow (iii)$ It is obvious.

$(iii) \Rightarrow (i)$ Assume now that there exist operators $S, R \in \mathcal{B}(X)$ such that $T = S + R$ and $SR = RS = 0$, S is left Drazin invertible and R is nilpotent. Then S is of Saphar type according to [19, Corollary 4.23]. Now from the equivalence (iii)$\Longleftrightarrow$(i) in Theorem 5.1 it follows that T is of Saphar type, while from [11, Theorem 2.2] it follows that $a(T) < \infty$. Using [19, Corollary 4.23] we obtain that T is left Drazin invertible. $\square$

Theorem 5.3. *Let* $T \in \mathcal{B}(X)$*. The following statements are equivalent:*

(i) T *is right Drazin invertible.*
(ii) *There exist operators* $S, R \in \mathcal{B}(X)$ *such that:*
  (a) $T = S + R$ *and* $SR = RS = 0$,
  (b) S *is right Drazin invertible with descent* 0 *or* 1,
  (c) R *is nilpotent.*
(iii) *There exist operators* $S, R \in \mathcal{B}(X)$ *such that:*
  (a) $T = S + R$ *and* $SR = RS = 0$,
  (b) S *is right Drazin invertible,*
  (c) R *is nilpotent.*

Proof. It may be proved by using [19, Corollary 4.24], Theorem 5.1 and [11, Theorem 2.2] in a similar way to Theorem 5.2. $\square$

The goal now is to prove a corresponding result to Theorem 5.1 for essentially left Drazin invertible and essentially right Drazin invertible operators.

Theorem 5.4. *Let* $T \in \mathcal{B}(X)$*. The following two statements are equivalent:*

(i) T *is essentially left Drazin invertible.*
(ii) *There exist operators* $S, R \in \mathcal{B}(X)$ *such that:*
  (a) $T = S + R$ *and* $SR = RS = 0$,
  (b) S *is essentially left Drazin invertible with essential ascent* 0 *or* 1,
  (c) R *is nilpotent.*
(iii) *There exist operators* $S, R \in \mathcal{B}(X)$ *such that:*
  (a) $T = S + R$ *and* $SR = RS = 0$,
  (b) S *is essentially left Drazin invertible,*
  (c) R *is nilpotent.*

Proof. $(i) \Rightarrow (ii)$ Suppose that T is essentially left Drazin invertible. From [19, Theorem 4.13] it follows that there exist closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is left Fredholm and $T_N^p = 0$ where $p = \text{dis}(T)$. Let $S = T_M \oplus 0_N$ and $R = 0_M \oplus T_N$. Then $S + R = T$, $SR = RS = 0$ and $R^p = 0$.

As $\mathcal{N}(S) = \mathcal{N}(T_M) \oplus N$ and $\mathcal{R}(S) = \mathcal{R}(T_M) \subset M$, we have that

$$\begin{aligned}\mathcal{N}(S) \cap \mathcal{R}(S) &= (\mathcal{N}(T_M) + N) \cap \mathcal{R}(T_M) \subset (\mathcal{N}(T_M) + N) \cap M \\ &= \mathcal{N}(T_M) + (N \cap M) = \mathcal{N}(T_M)\end{aligned}$$

so that $\dim(\mathcal{N}(S) \cap \mathcal{R}(S)) \le \dim(\mathcal{N}(T_M) < \infty$. Consequently, $a_e(S) \le 1$.

Further, we have:

$$\mathcal{R}(S) + \mathcal{N}(S) = \mathcal{R}(T_M) + \mathcal{N}(T_M) + N. \tag{5.1}$$

Since T_M s left Fredholm, we have that $\mathcal{R}(T_M)$ is topologically complemented in M and that $\dim \mathcal{N}(T_M) < \infty$. Lemma 2.4 ensures that $\mathcal{R}(T_M) + \mathcal{N}(T_M)$ is topologically complemented in M. Now from (5.1) and Lemma 2.3 it follows that $\mathcal{R}(S) + \mathcal{N}(S)$ is topologically complemented in X. Using the equivalence (i)$\Longleftrightarrow$(iv) in [19, Theorem 4.11] we conclude that S is essentially left Drazin invertible.

$(ii) \Rightarrow (iii)$ It is clear.

$(iii) \Rightarrow (i)$ Suppose that there exist operators $S, R \in \mathcal{B}(X)$ such that $T = S + R$ and $SR = RS = 0$, S is essentially left Drazin invertible and R is nilpotent. Then S is of Saphar type according to [19, Theorem 4.13]. Theorem 5.1 ensures that T is of Saphar type. From [3, Proposition 3.1] it follows that $a_e(T) < \infty$. Using the equivalence (i)$\Longleftrightarrow$(iv) in [19, Theorem 4.13] we get that T is essentially left Drazin invertible. □

Remark 5.5. The implication $(i) \Rightarrow (ii)$ in Theorem 5.4 can be also proved by using Theorem 5.1 and [3, Proposition 3.1] analogously to the proof of the implication $(i) \Rightarrow (ii)$ in Theorem 5.2. Namely, from the fact that T is essentially left Drazin invertible it follows that $a_e(T) < \infty$ and T is of Saphar type according to [19, Theorem 4.13]. Theorem 5.1 ensures that there exist operators $S, R \in \mathcal{B}(X)$ such that $T = S + R$, $SR = RS = 0$, S is of Saphar type, $\text{dis}(S) \le 1$ and R is nilpotent. Since $TR = RT$, from [3, Proposition 3.1] it follows that $a_e(S) < \infty$. As $a_e(S) \le \text{dis}(S)$ we conclude that $a_e(S) \le 1$. Using [19, Theorem 4.13] we get that S is essentially left Drazin invertible.

Theorem 5.6. *Let $T \in \mathcal{B}(X)$. The following statements are equivalent:*

(i) *T is essentially right Drazin invertible.*

(ii) *There exist operators $S, R \in \mathcal{B}(X)$ such that:*
   (a) *$T = S + R$ and $SR = RS = 0$,*
   (b) *S is essentially right Drazin invertible with essential descent 0 or 1,*
   (c) *R is nilpotent.*

(iii) *There exist operators $S, R \in \mathcal{B}(X)$ such that:*
   (a) *$T = S + R$ and $SR = RS = 0$,*
   (b) *S is essentially right Drazin invertible,*

(c) *R is nilpotent.*

Proof. $(i) \Rightarrow (ii)$ Suppose that T is essentially right Drazin invertible. From the proof of the implication (iv)$\Longrightarrow$(v) in [19, Theorem 4.15] it follows that there exist closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T = T_M \oplus T_N$ where T_M is right Fredholm and $T_N^p = 0$ where $p = \operatorname{dis}(T)$. Let define bounded operators S and R to be: $S = T_M \oplus 0_N$ and $R = 0_M \oplus T_N$. Then $S + R = T$, $SR = RS = 0$, $R^p = 0$ and

$$\mathcal{N}(S) + \mathcal{R}(S) \quad = \quad (\mathcal{N}(T_M) + \mathcal{R}(T_M)) + N. \tag{5.2}$$

Since T_M is right Fredholm it follows that $\dim(M/\mathcal{R}(T_M)) < \infty$, and so $\dim(M/(\mathcal{N}(T_M)+\mathcal{R}(T_M)) < \infty$. Now from (5.2) we obtain that $\operatorname{codim}(\mathcal{N}(S)+\mathcal{R}(S)) < \infty$, and so $d_e(S) \leq 1$. Since $\mathcal{R}(S) = \mathcal{R}(T_M)$ is closed in M, it follows that $\mathcal{R}(S)$ is closed in X.

We have that

$$\mathcal{R}(S) \cap \mathcal{N}(S) = \mathcal{R}(T_M) \cap (\mathcal{N}(T_M) + N) = \mathcal{R}(T_M) \cap \mathcal{N}(T_M). \tag{5.3}$$

Since $\mathcal{N}(T_M)$ is topologically complemented in M and $\dim(M/\mathcal{R}(T_M)) < \infty$, from Lemma 2.5 it follows that $\mathcal{R}(T_M) \cap \mathcal{N}(T_M)$ is topologically complemented in M. From (5.3) according to Lemma 2.3 we conclude that $\mathcal{R}(S) \cap \mathcal{N}(S)$ is topologically complemented in X. Using the equivalence (i)$\Longleftrightarrow$(iv) in [19, Theorem 4.12] we obtain that S is essentially right Drazin invertible.

$(ii) \Rightarrow (iii)$ It is clear.

$(iii) \Rightarrow (i)$ It can be proved by using [19, Theorem 4.15], Theorem 5.1 and [2, Theorem 3.1] similarly to the proof of the implication $(iii) \Rightarrow (i)$ in Theorem 5.4. □

Remark 5.7. The implication $(i) \Rightarrow (ii)$ in Theorem 5.6 can be also proved by using [19, Theorem 4.15], Theorem 5.1 and [2, Theorem 3.1], analogously to the proof for essentially left Drazin invertible operators presented in Remark 5.5.

6. Adjoint of (essentially) left Drazin invertible and (essentially) right Drazin invertible operators

In this section, we study the adjoint of (essentially) left Drazin invertible and (essentially) right Drazin invertible operators. To this end, we begin by investigating the adjoint of an operator of Saphar type.

Proposition 6.1. *Let $T \in \mathcal{B}(X)$ be of Saphar type of degree d. Then T' is of Saphar type of degree d.*

Proof. From [7, Theorem 4.4] it follows that there exists a projection $P \in \mathcal{B}(X)$, such that $TP = PT$, $T + P$ is Saphar and TP is nilpotent. Then $P' \in \mathcal{B}(X')$ is a projection, $(TP)' = T'P' = P'T'$ is nilpotent. Since $T + P$ is Kato, from [15, Corollary 12.4] it follows that $(T+P)' = T'+P'$ is Kato, and since $T+P$ is relatively regular, we have that $T'+P'$ is relatively regular. Thus

$T' + P'$ is Saphar. From [14, Lemma 4] it follows that $\text{dis}(T') = \text{dis}(T) = d$. Again using [7, Theorem 4.4] we conclude that T' is of Saphar type of degree d. □

For $M \subset X$ the annihilator of M is defined by

$$M^{\perp} = \{f \in X' : f(x) = 0 \text{ for all } x \in M\}.$$

Corollary 6.2. *Let $T \in \mathcal{B}(X)$. Then the following implications hold:*

(i) *T is left Drazin invertible $\Longrightarrow$ T' is right Drazin invertible.*
(ii) *T is right Drazin invertible $\Longrightarrow$ T' is left Drazin invertible.*
(iii) *T is essentially left Drazin invertible $\Longrightarrow$ T' is essentially right Drazin invertible.*
(iv) *T is essentially right Drazin invertible $\Longrightarrow$ T' is essentially left Drazin invertible.*

Proof. (i) Let T be left Drazin invertible. From the equivalence (i)$\Longleftrightarrow$(ii) in [19, Corollary 4.23] it follows that T is of Saphar type and $a(T) < \infty$. Hence T is quasi-Fredholm with $\text{dis}(T) = a(T)$, and so $\mathcal{R}(T^n)$ is closed for every $n \geq a(T)$. Then, according to the proof of [18, Lemma 2.12] we have that $d(T') = a(T) < \infty$. From Proposition 6.1, it follows that T' is of Saphar type. Applying the equivalence (i)$\Longleftrightarrow$(ii) in [19, Corollary 4.23] we conclude that T' is right Drazin invertible.

(ii) This can be proved by using [19, Corollaries 4.23 and 4.24] and Proposition 6.1 in a similar way to part (i).

(iii) Suppose that T is essentially left Drazin invertible. Then from the equivalence (i)$\Longleftrightarrow$(iv) in [19, Theorem 4.13] it follows that T is of Saphar type and $a_e(T) < \infty$. Let $d = a_e(T)$. Then $\alpha_d(T) < \infty$ and $\mathcal{R}(T^n)$ is closed for every $n \geq d$. From Lemma 2.1 and [18, Lemma 2.2] it follows that $\alpha_d(T) = \dim \mathcal{N}(T^{d+1})/\mathcal{N}(T^d) = \dim \mathcal{N}(T^d)^{\perp}/\mathcal{N}(T^{d+1})^{\perp} = \dim \mathcal{R}(T'^d)/\mathcal{R}(T'^{d+1}) = \beta_d(T')$, and hence $d_e(T') \leq d$. Since T' is of Saphar type according to Proposition 6.1, from [19, Theorem 4.15] it follows that T' is essentially right Drazin invertible.

(iv) This can be proved in a similar way to part (iii). □

Author contributions. : All authors contributed to the study conception and design. Also, all authors read and approved the final manuscript.

Competing Interests. The authors declare that they have no financial interests.

Funding. : Snežana Č. Živković-Zlatanović is supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, grant no. 451-03-68/2022-14/200124.

Data availability statement. : Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Statements and Declarations. : Author have no conflict of interest to declare.

References

[1] P. Aiena, *Fredholm and local spectral theory II, with application to Weyl-type theorems*, Lecture Notes in Mathematics, 2235, Springer (2018).

[2] O. Bel Hadj Fredj, *Essential descent spectrum and commuting compact perturbations*, Extracta Math. 21 2006, 261-271.

[3] O. Bel Hadj Fredj, M. Burgos, and M. Oudghiri, *Ascent spectrum and essential ascent spectrum*, Studia Math. 187 2008, 59-73.

[4] M. Berkani, *Restriction of an operator to the range of its powers*, Studia Mathematica, 140 (2) (2000), 163-174.

[5] L. Burlando, *Continuity of spectrum and spectral radius in algebras of operators*, Annales Faculté des Sciences de Toulouse, Vol. IX (1), 1988.

[6] M. D. Dimitrijevi¸ S. Č. Živkovi-̧Zlatanovi¸ *Essentially left and right generalized Drazin invertible operators and generalized Saphar decomposition*, Filomat 37(28) (2023), 9511-9529.

[7] A. Ghorbel, *On Generalized Saphar Operators On Banach Spaces*, Banach J. Math. Anal. 17, 6 (2023).

[8] A. Ghorbel, M. Mnif, *On a generalization of left and right invertible operators on Banach spaces*, Operators and Matrices, 16(2) (2022) 401-413.

[9] Q. Jiang, H. Zhong and S. Zhang, *Components of topological uniform descent resolvent set and local spectral theory*, Linear Algebra Appl., 438 (2013), 1149-1158.

[10] M. A. Kaashoek, *Ascent, descent, nullity and defect, a note on a paper by A. E. Taylor*, Math. Ann. 172 (1967), 105-115.

[11] M.A. Kaashoek, D.C. Lay, *Ascent, descent, and commuting perturbations*, Trans. Amer. Math. Soc. 169(1972), 35-47.

[12] C. F. King, *A note on Drazin inverses*, Pacific J. Math. 70 (1977), 383-390.

[13] J. Ph. Labrousse, *Les opérateurs quasi-Fredholm: Une généralisation des opérateurs semi-Fredholm*, Rend. Circ. Mat. Palermo (2) 29 (1980) 161-258.

[14] V. Müller, *On the Kato decomposition of quasi-Fredholm operators and B -Fredholm operators*, Proc. Workshop Geometry in Functional Analysis, Erwin Schrodinger Institute, Wien, (2000).

[15] V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebras*, Birkhäuser (2007).

[16] M. Mbekhta, V. Müller, *On the axiomatic theory of spectrum II*, Studia Math. 119(2) (1996), 129-147.

[17] L. Svensson, *Sums of complemented subspaces in locally convex spaces*, Ark. Mat. 25 issue 1 (1987), 147-153.

[18] S. Č. Živković-Zlatanović, *An introduction into Fredholm theory and generalized Drazin-Riesz invertible operators*, Zbornik Radova, Matematički institut SANU, Beograd, 20(28), 2022.

[19] S. Č. Živković-Zlatanović, S.V. Djordjević, *On some classes of Saphar type operators*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie a Matemáticas 116:170 (2022).

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