ON NONCOMMUTATIVE HÖLDER INEQUALITY OF SUKOCHEV AND ZANIN FOR WEAK SCHATTEN CLASS

YI C. HUANG AND SIJIE LUO

Abstract. Sukochev and Zanin resolved an open problem due to B. Simon concerning optimal constants in Hölder inequality for the weak Schatten classes of compact operators. In this note we observe that these constants, by introducing the modified weak Schatten quasi-norms, can be renormalised so that the original Simon's conjecture (with optimal constant 1) does hold. We also provide an unexpectedly simple proof for the modified Hölder inequality and its sharpness.

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space. Denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded operators on $\mathcal{H}$ equipped with uniform operator norm $\|\cdot\|_\infty$ and by $\mathcal{K}(\mathcal{H})$ the ideal of all compact operators on $\mathcal{H}$. For $T \in \mathcal{K}(\mathcal{H})$, let $\mu(T) = \{\mu(k,T)\}_{k=0}^{\infty}$ be the sequence of eigenvalues of $|T| = \sqrt{T^*T}$ arranged in non-increasing order with multiplicities. It is convenient to replace the sequence $\mu(T)$ by the function

$$\mu(t,T) = \sum_{k=0}^{\infty} \mu(k,T)\mathbf{1}_{[k,k+1)}(t), \quad t \geq 0,$$

and one has the following crucial property

$$\mu(t_1 + t_2, T_1T_2) \leq \mu(t_1, T_1)\mu(t_2, T_2), \tag{1.1}$$

where $T_1, T_2 \in \mathcal{K}(\mathcal{H})$ and $t_1, t_2 \geq 0$.

Definition 1.1. Define for $0 < p < \infty$ the weak Schatten class $\mathcal{L}_{p,\infty}$ by those T with

$$\|T\|_{p,\infty} := \sup_{t>0} t^{\frac{1}{p}}\mu(t,T) < \infty.$$

For further materials about the classes $\mathcal{L}_{p,\infty}$ (in particular $\mathcal{L}_{1,\infty}$) and their applications, see Connes [Con94], Lord-Sukochev-Zanin [LSZ13] and Ponge [Pon23].

B. Simon asked (see [Sim79, p. 32] and [Sim05, p. 21]) the following

Conjecture 1.2. *Let $p, q, r > 0$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Is it true that*

$$\|TS\|_{r,\infty} \leq \|T\|_{p,\infty}\|S\|_{q,\infty}, \quad T \in \mathcal{L}_{p,\infty}, S \in \mathcal{L}_{q,\infty}?$$

This was answered in the negative by Sukochev and Zanin in [SZ21]. They proved

Date: March 28, 2024.

2010 *Mathematics Subject Classification.* Primary 46L52. Secondary 46E30.

Key words and phrases. Weak Schatten classes, Hölder inequalities, diagonal operators.

Research of the authors is partially supported by the National NSF grants of China (nos. 11801274 and 12201646) and the Natural Science Foundation of Hunan (no. 2023JJ40696). YCH would like to thank Shangquan Bu for kind support and Raphaël Ponge for helpful communications.

Theorem 1.3 (Sukochev-Zanin)**.** *Let* $p, q, r > 0$ *be such that* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$*. Then*

$$\|TS\|_{r,\infty} \le \frac{(p+q)^{\frac{1}{p}+\frac{1}{q}}}{q^{\frac{1}{p}} p^{\frac{1}{q}}} \|T\|_{p,\infty} \|S\|_{q,\infty}, \quad T \in \mathcal{L}_{p,\infty}, S \in \mathcal{L}_{q,\infty},$$

and the constant is optimal.

For other versions of noncommutative Hölder inequalities, see Guido-Isola [GI03], Dykema-Skripka [DS15] and Sukochev [Suk16]. The aim of this note is to point out the following amusing observation: the Sukochev-Zanin constants can be written as

$$\frac{(p+q)^{\frac{1}{p}+\frac{1}{q}}}{q^{\frac{1}{p}} p^{\frac{1}{q}}} = \frac{p^{\frac{1}{p}} q^{\frac{1}{q}}}{r^{\frac{1}{r}}}.$$

This motivates us to introduce for $T \in \mathcal{L}_{p,\infty}$ the following renormalised quantity

$$\|T\|'_{p,\infty} := \sup_{t>0} (pt)^{\frac{1}{p}} \mu(t, T) = \sup_{t>0} t^{\frac{1}{p}} \mu\left(\frac{t}{p}, T\right). \tag{1.2}$$

So the original conjecture of Simon does hold for the scale of quantities $\|\cdot\|'_{p,\infty}$.

Theorem 1.4 ($\simeq$ Sukochev-Zanin)**.** *Let* $p, q, r > 0$ *be such that* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$*. Then*

$$\|TS\|'_{r,\infty} \le \|T\|'_{p,\infty} \|S\|'_{q,\infty}, \quad T \in \mathcal{L}_{p,\infty}, S \in \mathcal{L}_{q,\infty},$$

and the constant is optimal.

Our proof for this result is unexpectedly simple and we believe that the renormalised quantities (1.2) would be useful in other quantitative studies of $\mathcal{L}_{p,\infty}$.

2. Proof of Theorem 1.4

Let $p, q, r > 0$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $T \in \mathcal{L}_{p,\infty}$ and $S \in \mathcal{L}_{q,\infty}$. Thus, by (1.1),

$$t^{\frac{1}{r}} \mu\left(\frac{t}{r}, TS\right) = t^{\frac{1}{p}+\frac{1}{q}} \mu\left(\frac{t}{p} + \frac{t}{q}, TS\right) \le t^{\frac{1}{p}} \mu\left(\frac{t}{p}, T\right) \, t^{\frac{1}{q}} \mu\left(\frac{t}{q}, S\right).$$

Taking the supremum for $t > 0$ completes the proof of the Hölder inequality.

For the sharpness of the Hölder inequality, it suffices to note that in above reasoning, the inequality is saturated by the particular choices where T (resp. S) is given by the rank n diagonal operator with diagonal values $\{k^{-\frac{1}{p}}\}_{1\le k\le n}$ (resp. $\{k^{-\frac{1}{q}}\}_{1\le k\le n}$).

Compliance with ethical standards

Conflict of interest The authors have no known competing financial interests or personal relationships that could have appeared to influence this reported work.

Availability of data and material Not applicable.

References

[Con94] Alain Connes. *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994.

[DS15] Ken Dykema and Anna Skripka. Hölder's inequality for roots of symmetric operator spaces. *Studia Math.*, 228(1):47–54, 2015.

[GI03] Daniele Guido and Tommaso Isola. Dimensions and singular traces for spectral triples, with applications to fractals. *J. Funct. Anal.*, 203(2):362–400, 2003.

[LSZ13] Steven Lord, Fedor Sukochev, and Dmitriy Zanin. *Singular traces*, volume 46 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, 2013. Theory and applications.

[Pon23] Raphaël Ponge. Connes' integration and Weyl's laws. *J. Noncommut. Geom.*, 17(2):719–767, 2023.

[Sim79] Barry Simon. *Trace ideals and their applications*, volume 35 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1979.

[Sim05] Barry Simon. *Trace ideals and their applications*, volume 120 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2005.

[Suk16] Fedor Sukochev. Hölder inequality for symmetric operator spaces and trace property of K-cycles. *Bull. Lond. Math. Soc.*, 48(4):637–647, 2016.

[SZ21] Fedor Sukochev and Dmitriy Zanin. Optimal constants in non-commutative Hölder inequality for quasi-norms. *Proc. Amer. Math. Soc.*, 149(9):3813–3817, 2021.

School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, People's Republic of China
E-mail: Yi.Huang.Analysis@gmail.com
Homepage: https://orcid.org/0000-0002-1297-7674

School of Mathematics and Statistics, Central South University, HNP-LAMA, Changsha 410075, People's Republic of China
E-mail: sijieluo@csu.edu.cn