RANK DISTRIBUTION IN CUBIC TWIST FAMILIES OF ELLIPTIC CURVES

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ABSTRACT. Let a be an integer which is not of the form n^2 or $-3n^2$ for $n \in \mathbb{Z}$. Let E_a be the elliptic curve with rational 3-isogeny defined by $E_a : y^2 = x^3 + a$, and $K := \mathbb{Q}(\mu_3)$. Assume that the 3-Selmer group of E_a over K vanishes. It is shown that there is an explicit infinite set of cubefree integers m such that the 3-Selmer groups over K of $E_{m^2 a}$ and $E_{m^4 a}$ both vanish. In particular, the ranks of these cubic twists are seen to be 0 over K. Our results are proven by studying stability properties of 3-Selmer groups in cyclic cubic extensions of K, via local and global Galois cohomological techniques.

1. INTRODUCTION

1.1. **Motivation and historical background.** Given an elliptic curve E over $\mathbb{Q}$, and a subgroup H of $\operatorname{Aut}_{\bar{\mathbb{Q}}}(E)$, it is natural to consider the twist family of elliptic curves E_ζ indexed by cohomology classes in

$$\zeta \in H^1(\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), H).$$

When $H = \{\pm 1\}$, the associated twists are called quadratic twists and are in bijection with $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$. Goldfeld's conjecture predicts that the average rank of quadratic twists E_m over $\mathbb{Q}$ is $1/2$. In fact, the conjecture predicts that when m is ordered according to its absolute value, exactly half of the quadratic twists have rank 0 and the other half have rank 1. The rank distribution in quadratic twist families is studied for instance in [BM92, HB94, KMR13, Smi17, KL19, BKOS19]. Set $K := \mathbb{Q}(\mu_3)$ and let $\mathcal{O}_K$ be its ring of integers. We consider certain families of elliptic curves with rational 3-isogeny that have complex multiplication by $\mathcal{O}_K$. For a sixth power free integer $a \neq 0$, set E_a to be the elliptic curve

$$E_a : y^2 = x^3 + a.$$

Then for a cubefree integer m, we have the cubic twists $E_{m^2 a}$ and $E_{m^4 a}$ of E_a. Let $\operatorname{Sel}_3(E_a/K)$ (resp. $\operatorname{Ш}(E_a/K)$) denote the 3-Selmer group (resp. Tate–Shafarevich group) of E_a over K. The Selmer group sits in a short exact sequence

$$0 \to E_a(K)/3E_a(K) \to \operatorname{Sel}_3(E_a/K) \to \operatorname{Ш}(E_a/K)[3] \to 0.$$

It is thus easy to see that $\operatorname{Sel}_3(E_a/K) = 0$ if and only if

$$E_a(K)[3] = 0, \operatorname{rank}(E_a(K)) = 0 \text{ and } \operatorname{Ш}(E_a/K)[3] = 0.$$

The main focus of this article is to study the following natural question.

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Question 1.1. *Fix a sixth power free integer a such that $\mathrm{Sel}_3(E_a/K) = 0$. What can one say about the family of cubefree integers m for which $\mathrm{Sel}_3(E_{m^2a}/K) = 0$ and $\mathrm{Sel}_3(E_{m^4a}/K) = 0$?*

We construct explicit infinite families of numbers m for which the above Selmer groups vanish.

1.2. **Main results.** We state the main results of this article. It is shown that there is an explicit positive density of primes $\mathcal{Q}_a$ defined by local congruence conditions, such that if m is divisible only by primes $\ell \in \mathcal{Q}_a$, then, $\mathrm{Sel}_3(E_{m^2a}/K) = 0$ and $\mathrm{Sel}_3(E_{m^4a}/K) = 0$. In particular,

$$\mathrm{rank}\, E_{m^2a}(K), \mathrm{rank}\, E_{m^4a}(K) = 0 \text{ and } \mathrussianText{Ш}(E_{m^2a}/K)[3], Ш(E_{m^4a}/K)[3] = 0.$$

When we specialize to the families E_{ℓ^2a} and E_{ℓ^4a}, where ℓ is a prime number, we find that our result holds for an explicit positive density of primes ℓ.

For a prime number ℓ, set $\mathbb{F}_\ell := \mathbb{Z}/\ell\mathbb{Z}$. If E_a has good reduction at ℓ, we set $\widetilde{E}_a(\mathbb{F}_\ell)$ to denote the group of $\mathbb{F}_\ell$-points on the reduced curve of E_a at ℓ. More generally, if $m = \ell_1 \ldots \ell_k$, for a fixed value of k and ℓ_i prime, then there is a positive density of such numbers for which our result holds.

Theorem A. *Let a be an integer and E_a be the elliptic curve defined by $y^2 = x^3 + a$. Assume that the following conditions are satisfied*

(1) a is not of the form n^2 or $-3n^2$ for $n \in \mathbb{Z}$,
(2) $\mathrm{Sel}_3(E_a/K) = 0$.

Set $\mathcal{Q}_a$ to denote the set of primes $\ell \nmid a$ such that the following conditions are satisfied

(1) $\ell \equiv 1 \pmod{18}$,
(2) ℓ is a cube modulo q for each prime q that divides a.
(3) $\widetilde{E}_a(\mathbb{F}_\ell)[3] = 0$.

Then, the set $\mathcal{Q}_a$ is infinite with natural density given by

$$\mathfrak{d}(\mathcal{Q}_a) = \frac{1}{8 \times 3^{s+1}},$$

where s is the number of prime divisors q of a such that $q \equiv 1 \pmod{3}$. Moreover, if m is any cubefree integer divisible only by primes in $\mathcal{Q}_a$, then,

$$\mathrm{Sel}_3(E_{m^2a}/K) = 0 \text{ and } \mathrm{Sel}_3(E_{m^4a}/K) = 0.$$

For integers $a \in [-20, 20]$, it is shown that $\mathrm{Sel}_3(E_a/K) = 0$ for

$$a = -17, -16, -14, -10, -9, -8, -6, -5, -1, 6, 7, 8, 13, 14, 20,$$

and thus Theorem A applies to these values of a. The values of $\mathfrak{d}(\mathcal{Q}_a)$ are given in Table 1. One of the main features of our result is that the set of primes $\mathcal{Q}_a$ can be made explicit. For instance, when $a = -1$, we have that

$$\mathcal{Q}_{-1} \cap [1, 1000] = \{19, 127, 163, 199, 271, 307, 379, 487, 523, 631, 739, 811, 883, 919, 991\}.$$

For further details on numerical computations, we refer to sections 4.3 and 4.4.

1.3. **Methodology.** The study of Selmer ranks in twist families is related to the study the rank growth and stability of elliptic curves defined over a number field in a $\mathbb{Z}/p\mathbb{Z}$-extension for an odd prime p. Let K be a number field containing the p-th roots of unity and E be an elliptic curve over K such that the p-Selmer group $\mathrm{Sel}_p(E/K) = 0$. Let L/K be a cyclic p-extension. Then, it is shown that under certain conditions on E and L, the Selmer group $\mathrm{Sel}_p(E/L) = 0$ as well. This strategy has been adapted from the technique of Mazur–Rubin [MR07, MRS07], Brau [Bra14] and Česnavičius [Č17]. In keeping with these works, we study stability questions in prime cyclic extensions for any prime number $p \geq 3$, cf. Theorem 3.12 and Theorem 3.15 for further details.

1.4. **Related work.** In recent years, there has been an increasing interest in rank distribution questions for families of elliptic curves from the perspective of *arithmetic statistics*. These techniques involve the parameterization of Selmer groups of elliptic curves and abelian varieties by integral forms in many variables. The distribution properties of such integral forms can be studied via techniques from analytic number theory and the geometry of numbers, and has gained momentum in works of Bhargava–Shankar [BS15a, BS15b, BS13], Bhargava–Elkies–Shnidman [BES20], Alpöge–Bhargava–Shnidman [ABS22], Shnidman–Weiss [SW21, SW23] etc. Our techniques on the other hand rely of local and global arguments in Galois cohomology (in keeping with those used by Mazur–Rubin, Brau, Česnavičius), and one of the consequences is that we obtain very explicit descriptions for the twist families for which the Selmer group group vanishes. It is conceivable that our results should generalize to the context of abelian varieties with complex multiplication. For instance, Jacobians of twist families of bicyclic trigonal curves $y^3 = f(x^2)$, cf. [SW21] for details.

1.5. **Organization.** Including the introduction, the article consists of four sections. In section 2, we set basic notation and recall definitions of Selmer groups and their properties. In section 3, we study the stability of p-Selmer groups in p-cyclic extensions of number fields, leveraging the technique of Mazur–Rubin, Brau and Česnavičius. We prove our main results for cubic twist families in section 4. These results are backed up by explicit numerical computations in subsections 4.3 and 4.4.

2. Notation and preliminaries

In this section, we introduce certain preliminary notions and set up relevant notation. Given an abelian group M, set $\#M$ to denote its size. Set $M[n]$ to denote its n-torsion subgroup. Given a prime number p, set $M[p^\infty] := \bigcup_k M[p^k]$. The p-adic Tate module associated to M is the inverse limit

$$\mathrm{T}_p(M) := \varprojlim_n M[p^n]$$

with respect to natural multiplication by p maps $M[p^{k+1}] \to M[p^k]$.

Fix a number field K and denote by Σ_K the set of all nonarchimedian primes v of K. Given a prime v, we shall let ℓ be the rational prime such that $v|\ell$. Set $\mathbb{F}_\ell$ to denote the finite field with ℓ elements. For $v \in \Sigma_K$, set K_v to denote the completion of K at v. Take $\mathcal{O}_{K_v}$ to be the valuation ring of K_v and π_v a choice of uniformizer of $\mathcal{O}_{K_v}$. Let k_v be the residue field $\mathcal{O}_{K_v}/\pi_v$. Denote by $i_v : K \hookrightarrow K_v$ the natural inclusion of K into K_v. Let $\bar{K}$ (resp. $\bar{K}_v$) be an algebraic closure of K (resp. K_v) and G_K (resp. G_{K_v}) denote the absolute Galois group of K (resp. K_v). Let I_v denote the inertia subgroup of G_{K_v}. At each prime $v \in \Sigma_K$, choose an embedding $\iota_v : \bar{K} \hookrightarrow \bar{K}_v$ that extends i_v. Note that ι_v induces an inclusion $\iota_v^* : \mathrm{G}_{K_v} \hookrightarrow \mathrm{G}_K$. Let M be a G_K-module and $i \in \mathbb{Z}_{\geq 0}$. Take $H^i(K, M)$ (resp. $H^i(K_v, M)$) to denote the cohomology group $H^i(\mathrm{G}_K, M)$ (resp. $H^i(\mathrm{G}_{K_v}, M)$). The restriction map induced by ι_v^* is denoted

$$\mathrm{res}_v : H^i(K, M) \to H^i(K_v, M).$$

Let E be an elliptic curve over K and n be a positive integer. Given a prime $v \in \Sigma_K$ at which E has good reduction, set $\widetilde{E}$ to denote the reduction of E at v. Take $\widetilde{E}(k_v)$ to denote the group of k_v-points on $\widetilde{E}$. Set $E[n]$ to denote the n-torsion subgroup if $E(\bar{K})$. For $\mathcal{F} \in \{K, K_v\}$, we have the exact sequence of $\mathrm{G}_\mathcal{F}$-modules

$$0 \to E[n] \to E \xrightarrow{\times n} E \to 0,$$

from which we arrive at the following Kummer sequence

$$(2.1) \qquad 0 \to E(\mathcal{F})/nE(\mathcal{F}) \xrightarrow{\delta_{E,\mathcal{F}}} H^1(\mathcal{F}, E[n]) \to H^1(\mathcal{F}, E)[n] \to 0.$$

Definition 2.1. *Let n be an odd natural number. With respect to notation above, the n-Selmer group of E over K is defined as follows*

$$\mathrm{Sel}_n(E/K) := \ker \left\{ H^1(K, E[n]) \xrightarrow{\Phi} \prod_{v \in \Sigma_K} H^1(K_v, E)[n] \right\}.$$

Here the map Φ is the product of maps Φ_v, given by the composite

$$H^1(K, E[n]) \xrightarrow{\mathrm{res}_v} H^1(K_v, E[n]) \to H^1(K_v, E)[n].$$

Let Σ be a finite set of primes that contains the primes of K that lie above the primes that divide n and the primes at which E has bad reduction. Let K_Σ be the maximal algebraic extension of K in which the primes $\ell \notin \Sigma$ are unramified. Set $\mathrm{G}_{K,\Sigma}$ to denote the Galois group $\mathrm{Gal}(K_\Sigma/K)$. Given a $\mathrm{G}_{K,\Sigma}$-module M, set $H^i(K_\Sigma/K, M)$ to denote $H^i(\mathrm{G}_{K,\Sigma}, M)$. The Selmer group $\mathrm{Sel}_n(E/K)$ can also be described by finitely many local conditions

$$\mathrm{Sel}_n(E/K) = \ker \left\{ H^1(K_\Sigma/K, E[n]) \xrightarrow{\Phi_\Sigma} \bigoplus_{v \in \Sigma} H^1(K_v, E)[n] \right\},$$

where the map $\Phi_\Sigma := \bigoplus_{v \in \Sigma} \Phi_v$, cf. [CR00, Chapter 1]. Given a prime p, the natural map $E[p^k] \hookrightarrow E[p^{k+1}]$ induces a map at the level of p-primary Selmer groups

$$\mathrm{Sel}_{p^n}(E/K) \to \mathrm{Sel}_{p^{n+1}}(E/K).$$

The p-primary Selmer group is then defined to be the direct limit

$$\mathrm{Sel}_{p^\infty}(E/K) := \varinjlim_n \mathrm{Sel}_{p^n}(E/K).$$

Let $\mathrm{Ш}(E/K)$ denote the Tate–Shafarevich group of E over K. The Selmer group fits into a short exact sequence

$$(2.2) \qquad 0 \to E(K)/nE(K) \xrightarrow{\delta_{E,n}} \mathrm{Sel}_n(E/K) \to \mathrm{Ш}(E/K)[n] \to 0.$$

Given a prime p, passing to the direct limit for integers n of the form p^k, we have the following short exact sequence

$$(2.3) \qquad 0 \to E(K) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(E/K) \to \mathrm{Ш}(E/K)[p^\infty] \to 0.$$

As is well known, the Tate–Shafarevich group $\mathrm{Ш}(E/K)$ is conjectured to be finite. If $\mathrm{Ш}(E/K)[p^\infty]$ is known to be finite, then from (2.3),

$$\mathrm{corank}_{\mathbb{Z}_p}\left(\mathrm{Sel}_{p^\infty}(E/K)\right) = \mathrm{rank}\, E(K).$$

For each integer $k \geq 1$, multiplication by p induces a map of Galois modules $E[p^{n+1}] \xrightarrow{\times p} E[p^n]$. The *compact Selmer group* is defined to be the inverse limit

$$\mathfrak{S}_p(E/K) := \varprojlim_n \mathrm{Sel}_{p^n}(E/K)$$

with respect to these maps. From (2.2), we obtain the following exact sequence

$$(2.4) \qquad 0 \to E(K) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathfrak{S}_p(E/K) \to \mathrm{T}_p\left(\mathrm{Ш}(E/K)\right) \to 0.$$

If the p-primary Tate–Shafarevich group $\mathrm{Ш}(E/K)[p^\infty]$ is finite, then its Tate–module vanishes. Thus, if $\mathrm{Ш}(E/K)[p^\infty]$ is finite, then,

$$\mathrm{rank}_{\mathbb{Z}_p}\left(\mathfrak{S}_p(E/K)\right) = \mathrm{rank}\, E(K).$$

It follows from [CR00, Lemma 1.8] that there is a short exact sequence

$$(2.5) \qquad 0 \to E(K)[p^\infty] \to \mathfrak{S}_p(E/K) \to \mathrm{T}_p\left(\mathrm{Sel}_{p^\infty}(E/K)\right) \to 0,$$

and thus,

$$(2.6) \qquad \mathrm{corank}_{\mathbb{Z}_p}\left(\mathrm{Sel}_{p^\infty}(E/K)\right) = \mathrm{rank}_{\mathbb{Z}_p}\left(\mathfrak{S}_p(E/K)\right).$$

Note if $\mathrm{Sel}_p(E/K) = 0$, then, $\mathrm{Ш}(E/K)[3^\infty] = 0$, $\mathrm{rank}\, E(K) = 0$ and $E(K)[3^\infty] = 0$. Thus it follows from (2.5) that $\mathfrak{S}_p(E/K) = 0$. It then follows from the Cassels–Poitou–Tate exact sequence [CR00, Ch. 1 (3)] that the map Φ_Σ is surjective, from which we obtain a short exact sequence

$$(2.7) \qquad 0 \to \mathrm{Sel}_n(E/K) \to H^1(K_\Sigma/K, E[n]) \xrightarrow{\Phi_\Sigma} \bigoplus_{v \in \Sigma} H^1(K_v, E)[n] \to 0.$$

3. Rank stability in prime cyclic extensions

Let E be an elliptic curve over a number field K and p be an odd prime number. Denote by Σ_p the set of primes v of K that lie above p. In this section, we assume that $\mathrm{Sel}_p(E/K) = 0$. Note that this is equivalent to requiring that $E(K)$ has rank 0, and both $E(K)$ and $\mathrm{III}(E/K)$ have no nontrivial p-torsion.

3.1. **Local contributions to the growth of Selmer groups.** We take L/K to be a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$ isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Let σ be a generator of G. Set S to be the set of primes $v \in \Sigma_K$ such that either one of the following conditions are satisfied

- $v \in \Sigma_p$,
- E has bad reduction at v,
- v is ramified in L.

Denote by $S(L)$ the primes $w \in \Sigma_L$ such that $w|v$ for some prime $v \in S$. Then there are exact sequences

$$0 \to \mathrm{Sel}_p(E/K) \to H^1(K_S/K, E[p]) \xrightarrow{\Phi_K} \bigoplus_{v \in S} H^1(K_v, E)[p] \to 0,$$

$$0 \to \mathrm{Sel}_p(E/L)^G \to H^1(K_S/L, E[p])^G \xrightarrow{\Phi_L} \left(\bigoplus_{w \in S(L)} H^1(L_w, E)[p] \right)^G.$$

From the Cassels-Poitou-Tate sequence (cf. (2.7) and the discussion preceding it), the map Φ_K is surjective. For $v \in S$, consider the restriction map

$$\gamma_v : H^1(K_v, E)[p] \to \left(\bigoplus_{w|v} H^1(L_w, E)[p] \right)^G,$$

where w runs through all primes of L that lie above v. Let

$$\gamma : \bigoplus_{v \in S} H^1(K_v, E)[p] \to \left(\bigoplus_{w \in S(L)} H^1(L_w, E)[p] \right)^G$$

denote the direct sum of maps $\bigoplus_{v \in S} \gamma_v$.

There is a natural restriction map

$$\alpha : \mathrm{Sel}_p(E/K) \to \mathrm{Sel}_p(E/L)^G$$

which fits into the fundamental diagram

(3.1)

$$0 = \mathrm{Sel}_p(E/K) \longrightarrow H^1(K_S/K, E[p]) \xrightarrow{\Phi_K} \bigoplus_{v \in S} H^1(K_v, E)[p] \longrightarrow 0$$

$$\downarrow \alpha \qquad\qquad \downarrow \beta \qquad\qquad \downarrow \gamma$$

$$0 \longrightarrow \mathrm{Sel}_p(E/L)^G \longrightarrow H^1(L_S/L, E[p])^G \longrightarrow \left(\bigoplus_{w \in S(L)} H^1(L_w, E)[p]\right)^G,$$

From the snake lemma one obtains an exact sequence

(3.2) $\qquad 0 = \ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \to \mathrm{cok}(\alpha) \to \mathrm{cok}(\beta) \to \mathrm{cok}(\gamma)$

The following basic result will be useful in many of our arguments.

Lemma 3.1. *Let V be a finite dimensional $\mathbb{F}_p$-vector space and G be a finite p-group which acts on V. Then, the following implication holds*

$$V^G = 0 \Rightarrow V = 0.$$

Proof. We have that

$$\#V = \sum_i \#\mathcal{O}_i,$$

where the sum ranges over G-orbits of V. Suppose that $V^G = 0$, then $\{0\}$ is a G-orbit of V which contains only 1-element. Every other G-orbit has p^{n_i} elements with $n_i > 0$. This implies that $\#V \equiv 1 \mod p$, hence, $V = 0$. $\qquad\square$

Proposition 3.2. *Let K be a number field and E be an elliptic curve over K such that $\mathrm{Sel}_p(E/K) = 0$. Let L/K be a cyclic p-extension with Galois group $G := \mathrm{Gal}(L/K)$. Assume moreover that for all $v \in S$, the map γ_v is injective. Then, we have that $\mathrm{Sel}_p(E/L) = 0$. In particular, the rank of $E(L)$ is 0.*

Proof. Since it is assumed that $\mathrm{Sel}_p(E/K) = 0$, we find that $E(K)[p] = 0$. Note that $E(K)[p] = E(L)[p]^G$. We deduce from Lemma 3.1 that $E(L)[p] = 0$. The restriction map β in (3.1) sits in an inflation-restriction sequence

$$0 \to H^1(L/K, E(L)[p]) \to H^1(K_S/K, E[p]) \xrightarrow{\beta} H^1(K_S/L, E[p])^G \to H^2(L/K, E(L)[p]).$$

The groups $H^i(L/K, E(L)[p]) = 0$ and hence, β is an isomorphism. The exact sequence (3.2) then implies that

$$\mathrm{cok}(\alpha) \simeq \ker(\gamma) \simeq \bigoplus_{v \in S} \ker \gamma_v.$$

Since it is assumed that γ_v is injective for all $v \in S$, we deduce that $\mathrm{cok}(\alpha) = 0$. Since $\mathrm{Sel}_p(E/K) = 0$, it follows that $\mathrm{cok}(\alpha) = \mathrm{Sel}_p(E/L)^G = 0$. It then follows from Lemma 3.1 that $\mathrm{Sel}_p(E/L) = 0$, and this concludes the proof of the result. $\quad\square$

We analyze conditions for γ_v to be injective for $v \in S$. Let $w|v$ be a chosen prime of L and consider the restriction map

$$\gamma'_v : H^1(K_v, E)[p] \to H^1(L_w, E)[p]^{\mathrm{Gal}(L_w/K_v)}.$$

Since G permutes the primes w of L that lie above v transitively, it follows that

(3.3) $\ker \gamma_v = \ker \gamma'_v.$

Lemma 3.3. *Let $v \in S$ and $w|v$ be a prime of L. Then, we have that*

$$\ker \gamma_v \simeq H^1(L_w/K_v, E(L_w))[p].$$

In particular, if v splits in L, then, γ_v is injective.

Proof. From the inflation-restriction sequence, we find that

$$\ker \gamma'_v \simeq H^1(L_w/K_v, E(L_w))[p].$$

Since $\ker \gamma_v \simeq \ker \gamma'_v$, the result follows. $\square$

Since γ_v is injective when v splits in L, we consider two cases

(1) $v \nmid p$ and v is inert or ramified in L,
(2) $v|p$ and v is inert or ramified in L.

We note that in the above cases, there is a unique prime $w|v$ and hence, $\gamma'_v = \gamma_v$.

Lemma 3.4. *For $v \in S \setminus \Sigma_p$ assume that either of the following conditions is satisfied*

(1) v is a prime of bad reduction and is split in L,

(2) v is a prime of good reduction, v is ramified in L and $\widetilde{E}(k_v)[p] = 0$.

Then, γ_v is injective.

We remark that a prime $v \in S \setminus \Sigma_p$ at which E has good reduction is necessarily ramified in L.

Proof. The result follows from [PR24, Proposition 2.8]. $\square$

Throughout the rest of this section, we shall study the cases when $v|p$. This analysis is based on unpublished work of Brau [Bra14], results of Mazur–Rubin [MR07] and Mazur–Rubin–Silverberg [MRS07]. For a prime v of a number field F and a G_{F_v}-module V, set

$$H^1_{\mathrm{nr}}(F_v, V) := \mathrm{image}\left\{\inf : H^1(\mathrm{G}_{F_v}/I_v, V^{I_v}) \longrightarrow H^1(K_v, V)\right\}.$$

Definition 3.5. *Let p be an odd prime and F be a number field. For a continuous p-primary G_F-module V, a Selmer structure $\mathcal{F}$ on V is a collection of cohomology classes*

$$H^1_{\mathcal{F}}(F_v, V) \subseteq H^1(F_v, V)$$

for $v \in \Sigma_F$ such that for all but finitely many primes v,

$$H^1_{\mathcal{F}}(F_v, V) = H^1_{\mathrm{nr}}(F_v, V).$$

Associated to $\mathcal{F}$ is the Selmer group

$$H^1_{\mathcal{F}}(F, V) := \ker \left\{ H^1(F, V) \longrightarrow \prod_{v \in \Sigma_F} \left(\frac{H^1(F_v, V)}{H^1_{\mathcal{F}}(F_v, V)} \right) \right\}.$$

Given two Selmer structures $\mathcal{F}$ and $\mathcal{G}$, set

$$H^1_{\mathcal{F}+\mathcal{G}}(F_v, V) := H^1_{\mathcal{F}}(F_v, V) + H^1_{\mathcal{G}}(F_v, V),$$
$$H^1_{\mathcal{F}\cap\mathcal{G}}(F_v, V) := H^1_{\mathcal{F}}(F_v, V) \cap H^1_{\mathcal{G}}(F_v, V).$$

We would like to draw attention to two Selmer conditions $\mathcal{F}$ and $\mathcal{A}$ associated to the G_K-module $E[p]$. For each prime $v \in \Sigma_K$, set

$$H^1_{\mathcal{F}}(K_v, E[p]) := \text{image} \left\{ \frac{E(K_v)}{pE(K_v)} \xrightarrow{\delta_{E,K_v}} H^1(K_v, E[p]) \right\},$$

where δ_{E,K_v} is the local Kummer map (2.1). Observe that $\text{Sel}_p(E/K) = H^1_{\mathcal{F}}(K, E[p])$. Let L/K be a G-extension where $G \simeq \mathbb{Z}/p\mathbb{Z}$. Set $\text{Res}^L_K E$ to denote the Weil–restriction of scalars of E from L to K. For every K-algebra X, we have that

$$(\text{Res}^L_K E)(X) = E(L \otimes_K X).$$

Let A denote the kernel of the norm map $\text{Res}^L_K E \xrightarrow{N_{L/K}} E$. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\mu_p)$ and $\mathfrak{p}$ be the ideal of $\mathcal{O}$ that lies above p. The natural inclusion $\mathbb{Z}[G] \hookrightarrow \text{End}_K(\text{Res}^L_K E)$ induces a natural map $\mathbb{Z}[G] \to \text{End}_K(A)$, which factors as

$$\mathbb{Z}[G] \twoheadrightarrow \mathcal{O} \hookrightarrow \text{End}_K(A).$$

There is a canonical G_K-isomorphism $A[\mathfrak{p}] \xrightarrow{\sim} E[p]$, cf. [MR07, Proposition 4.1]. Let π be a generator of $\mathfrak{p}$ and consider the composed map

$$\lambda_{A,v} : \frac{A(K_v)}{\pi A(K_v)} \hookrightarrow H^1(K_v, A[\mathfrak{p}]) \xrightarrow{\sim} H^1(K_v, E[p]).$$

Let $H^1_{\mathcal{A}}(K_v, E[p])$ be the image of this composed map. This defines $\mathcal{A}$, which is referred to as the *twisted Selmer condition* with respect to L/K.

Lemma 3.6. *For every prime $v \in S$, we have that*

$$\ker \gamma_v \simeq \frac{H^1_{\mathcal{F}+\mathcal{A}}(K_v, E[p])}{H^1_{\mathcal{A}}(K_v, E[p])} \simeq \frac{H^1_{\mathcal{F}}(K_v, E[p])}{H^1_{\mathcal{F}\cap\mathcal{A}}(K_v, E[p])}.$$

Proof. The result above is [Bra14, Lemma 5.1], and follows from Proposition 4.4 of *loc. cit.* $\qquad\square$

Lemma 3.7. *Let $v \in \Sigma_p$ be a prime at which E has good ordinary reduction, and assume that v is ramified in L. Then if $\widetilde{E}(k_v)[p] = 0$, the map γ_v is injective.*

Proof. Lemma 3.6 asserts that

$$(3.4) \qquad \ker \gamma_v \simeq \frac{H^1_{\mathcal{F}}(K_v, E[p])}{H^1_{\mathcal{F} \cap \mathcal{A}}(K_v, E[p])}.$$

It follows from [MR07, Proposition 5.2] that

$$(3.5) \qquad \frac{H^1_{\mathcal{F}}(K_v, E[p])}{H^1_{\mathcal{F} \cap \mathcal{A}}(K_v, E[p])} \simeq \frac{E(K_v)}{\left(E(K_v) \cap N_{L_w/K_v} E(L_w) \right)}.$$

Denote by $E_1(K_v)$ (resp. $E_1(L_w)$) denote the kernel of the reduction map $E(K_v) \to \widetilde{E}(k_v)$ (resp. $E(L_w) \to \widetilde{E}(k_w)$). Since $\widetilde{E}(k_v)[p] = 0$ and L/K is a p-extension, it follows that $\widetilde{E}(k_w)[p] = 0$ (cf. Lemma 3.1). Therefore, we find that

$$E(K_v)/pE(K_v) \simeq E_1(K_v)/pE_1(K_v),$$
$$E(L_w)/pE(L_w) \simeq E_1(L_w)/pE_1(L_w).$$

Hence it is easy to see that

$$\frac{E(K_v)}{\left(E(K_v) \cap N_{L_w/K_v} E(L_w) \right)} \simeq \frac{E_1(K_v)}{\left(E_1(K_v) \cap N_{L_w/K_v} E_1(L_w) \right)}.$$

From [MR07, Proposition B.2] it follows that

$$\frac{E_1(K_v)}{\left(E_1(K_v) \cap N_{L_w/K_v} E_1(L_w) \right)} = 0.$$

Combining this with (3.4) and (3.5), we conclude that γ_v is injective. $\square$

Lemma 3.8. *Let $v \in \Sigma_p$ be a prime at which E has good reduction and assume that v is inert in L. Then, γ_v is injective.*

Proof. From the proof of Lemma 3.7, we find that

$$\ker \gamma_v \simeq \frac{E_1(K_v)}{\left(E_1(K_v) \cap N_{L_w/K_v} E_1(L_w) \right)}.$$

The norm map $N_{L_w/K_v} : E_1(L_w) \to E_1(K_v)$ is surjective when L_w/K_v is unramified (cf. [Haz74, Proposition 3.1]). Therefore, γ_v is injective and this proves the result. $\square$

Next, we deal with the case when $v \in \Sigma_p$ is a prime at which E has good supersingular reduction. Let $\mathcal{K}$ be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_{\mathcal{K}}$, maximal ideal $\mathfrak{m}_{\mathcal{K}}$ and residue field $k = \mathcal{O}_{\mathcal{K}}/\mathfrak{m}_{\mathcal{K}}$. Let $\mathcal{F}(X,Y) \in \mathcal{O}[\![X,Y]\!]$ be a formal group law and $\mathcal{F}(\mathfrak{m}_{\mathcal{K}})$ be the induced abelian group. The group operation is defined as follows

$$x \oplus_{\mathcal{F}} y := \mathcal{F}(x,y)$$

for $x, y \in \mathfrak{m}_{\mathcal{K}}$. Let $\mathcal{L}/\mathcal{K}$ be a finite Galois extension with Galois group $\mathcal{G} = \mathrm{Gal}(\mathcal{L}/\mathcal{K})$. Writing $\mathcal{G} = \{\sigma_1, \ldots, \sigma_n\}$, consider the induced norm map

$$N_{\mathcal{L}/\mathcal{K}} : \mathcal{F}(\mathfrak{m}_{\mathcal{L}}) \to \mathcal{F}(\mathfrak{m}_{\mathcal{K}}),$$

defined by

$$N_{\mathcal{L}/\mathcal{K}}(x) := \sigma_1(x) \oplus_{\mathcal{F}} \sigma_2(x) \oplus_{\mathcal{F}} \cdots \oplus_{\mathcal{F}} \sigma_n(x).$$

Assume that $\mathcal{G} \simeq \mathbb{Z}/p\mathbb{Z}$ and σ be a generator of $\mathcal{G}$. Let $v_{\mathcal{L}}$ (resp. $v_{\mathcal{K}}$) denote the normalized valuation of $\mathcal{O}_{\mathcal{L}}$ (resp. $\mathcal{O}_{\mathcal{K}}$). Assume that $\mathcal{L}/\mathcal{K}$ is a ramified extension. Set $\pi_{\mathcal{L}}$ (resp. $\pi_{\mathcal{K}}$) to denote a uniformizer of $\mathcal{O}_{\mathcal{L}}$ (resp. $\mathcal{O}_{\mathcal{K}}$). Setting

(3.6) $$t := v_{\mathcal{L}}\left(\sigma(\pi_{\mathcal{L}}) - \pi_{\mathcal{L}}\right) - 1,$$

we note that $t \geq 1$ since $\mathcal{L}/\mathcal{K}$ must be a wildly ramified extension. On the other hand, suppose $\mathcal{L}/\mathcal{K}$ is unramified, then, $t := -1$.

Suppose now that L/K is a $\mathbb{Z}/p\mathbb{Z}$-extension of number fields, $v \in \Sigma_K$ and $w|v$ is a prime of L. Let $t_v := t$ be the invariant defined above (3.6) for $\mathcal{L} := L_w$ and $\mathcal{K} := K_v$.

Proposition 3.9. *Let E be an elliptic curve with good supersingular reduction at a prime $v \in \Sigma_p$. Suppose that v is ramified in L. Then, γ_v is injective if and only if $t_v = 1$.*

Proof. The result is [Bra14, Proposition 5.10]; we give a brief sketch. Let w be the prime of L that lies above v. We identify $\text{Gal}(L_w/K_v)$ with G. Since E has good supersingular reduction at v, we find that

$$a_v(E) := \text{Norm}_{K/\mathbb{Q}}(v) + 1 - \#\widetilde{E}(k_v)$$

is divisible by p. This implies that $\widetilde{E}(k_v)[p] = 0$, in other words,

$$\left(\widetilde{E}(k_w)[p]\right)^G = 0.$$

It follows from Lemma 3.1 that $\widetilde{E}(k_w)[p] = 0$. Let E_1 denote the formal group of E. Consider the short exact sequence of G-modules

$$0 \to E_1(L_w) \to E(L_w) \to \widetilde{E}(k_w) \to 0$$

and the associated long exact sequence in cohomology

$$\widetilde{E}(k_v) \to H^1(G, E_1(L_w)) \to H^1(G, E(L_w)) \to H^1(G, \widetilde{E}(k_w)).$$

Since $p \nmid \widetilde{E}(k_w)$, it follows from the restriction-corestriction sequence that

$$H^1(G, \widetilde{E}(k_w)) = 0.$$

On the other hand, $H^1(G, E_1(L_w))$ is a p-primary group, and $p \nmid \#\widetilde{E}(k_v)$. Hence, there is an isomorphism

$$H^1(G, E(L_w)) \simeq H^1(G, E_1(L_w)).$$

The proof of [Bra14, Proposition 3.6] shows that the Herbrand-quotient of $E_1(L_w)$ is 1, and hence,

$$H^1(G, E_1(L_w)) \simeq H^2(G, E_1(L_w)) \simeq E_1(K_v)/N_{L_w/K_v}\left(E_1(L_w)\right).$$

Then [Bra14, Proposition 3.5] shows that

$$E_1(K_v) = N_{L_w/K_v}\big(E_1(L_w)\big) \Leftrightarrow t_v = 1.$$

$\square$

Lemma 3.10. *Let E be an elliptic curve and v be a prime of K. Assume that there is a finite Galois extension K'/K with $p \nmid [K' : K]$ such that $\gamma_{v'}$ is injective for some prime $v'|v$ of K'. Then, γ_v itself is injective.*

Proof. Assume without loss of generality that v is inert or ramified in L. Let w' be the prime of L' that lies above v'. Setting $L' := K' \cdot L$, let w be a prime of L that lies below w', as depicted

Since K'/K is a prime to p-extension, $\mathrm{Gal}(L'_{w'}/K'_{v'})$ is naturally isomorphic to $\mathrm{Gal}(L_w/K_v)$. By hypothesis, the kernel of $\gamma_{v'}$ is trivial, hence,

$$H^1(\mathrm{Gal}(L'_{w'}/K'_{v'}), E)[p] = 0.$$

Consider the inflation-restriction sequence for the field extensions $L'_{w'}/K'_{v'}/K_v$

$$0 \to H^1(K'_{v'}/K_v, E)[p] \to H^1(L'_{w'}/K_v, E)[p] \to H^1(L'_{w'}/K'_{v'}, E)[p].$$

Since K'/K is a prime to p-extension, it follows that

$$H^1(K'_{v'}/K_v, E)[p] = 0,$$

and we deduce that

$$H^1(\mathrm{Gal}(L'_{w'}/K_v), E)[p] = 0.$$

Once again, from the inflation-restriction sequence for the fields $L'_{w'}/L_w/K_v$, we deduce that

$$H^1(\mathrm{Gal}(L_w/K_v), E)[p] = 0$$

and hence, γ_v is injective. $\square$

Lemma 3.11. *Let E be an elliptic curve and $v|p$ be a prime of K at which E has potentially good reduction. Assume that there is a finite Galois extension K'/K such that $p \nmid [K' : K]$ and for a prime $v'|v$ of K', either of the following conditions hold*

(1) E has good ordinary reduction at v' and $\widetilde{E}(k_{v'})[p] = 0$, or,
(2) E has good supersingular reduction at v' and $t_{v'} \leq 1$.

Then, γ_v is injective.

Proof. The result follows from Lemma 3.7, Proposition 3.9 and Lemma 3.10. □

3.2. **Rank stability in prime cyclic extensions.** In this subsection, we study applications of our local computations to rank stability questions in $\mathbb{Z}/p\mathbb{Z}$-extensions of K.

Theorem 3.12. *Let K be a number field and E be an elliptic curve over K. Let p be an odd prime number and assume that $\mathrm{Sel}_p(E/K) = 0$. Let L/K be a degree p Galois extension and S be the set of primes v of K such that either one of the following hold*

- *$v|p$,*
- *E has bad reduction at v,*
- *v is ramified in L.*

Write $S = \Sigma_1 \cup \Sigma_2 \cup \Sigma_p$, where Σ_1 is the set of primes of K at which E has bad reduction and $\Sigma_2 := S \setminus (\Sigma_1 \cup \Sigma_p)$. Assume that the following conditions are satisfied.

- *(1) Each prime $v \in \Sigma_1$ splits completely in L.*
- *(2) For each prime $v \in \Sigma_2$, we have that $\widetilde{E}(k_v)[p] = 0$.*
- *(3) For each prime $v \in \Sigma_p$ which is not completely split in L, there is a prime to p Galois extension K'/K and a prime $v'|v$ of K' such that either*
  - *E has good ordinary reduction at v' and $\widetilde{E}(k_{v'})[p] = 0$, or,*
  - *E has good supersingular reduction at v' and $t_{v'} \leq 1$.*

Then, the p-Selmer group $\mathrm{Sel}_p(E/L) = 0$. In particular, $\mathrm{rank}\, E(L) = 0$ and $\text{Ш}(E/L)[p] = 0$.

Proof. The Proposition 3.2 asserts that if γ_v is injective for all primes $v \in S$, then, $\mathrm{Sel}_p(E/L) = 0$. For each prime $v \in \Sigma_1$, it follows from the Lemma 3.3 that the map γ_v is injective. For each prime $v \in \Sigma_2$ such that $v \nmid p$, it follows from Lemma 3.4 that γ_v is injective. On the other hand, for $v|p$, it follows from Lemma 3.11 that γ_v is injective. □

Assuming that K contains μ_p, we shall consider Kummer extensions $L = K(\alpha^{\frac{1}{p}})$ for $\alpha \in K^*/(K^*)^p$. The following is a well known criterion for the ramification and splitting behavior of primes in such Kummer extensions. Given a prime $v \in \Sigma_K$, let $v(\alpha)$ denote $v_{K_v}(\alpha)$ by abuse of notation.

Proposition 3.13. *Let L/K be as above and $v \in \Sigma_K$ and $i_v : K \hookrightarrow K_v$ denote the natural embedding. Then the following assertions hold.*

- *(1) Assume that $v \nmid p$. Then, v is unramified in L if and only if $v(\alpha) \equiv 0 \pmod{p}$.*
- *(2) Assume that $v|p$. Then, v is unramified in L if and only if the congruence*

$$\frac{\alpha}{x_v^p} \equiv 1 \pmod{\pi_v^{pe_v}}$$

has a solution for $x_v \in K_v^\times$, where e_v is the ramification index of v in $K/\mathbb{Q}(\mu_p)$.
(3) The prime v splits in L if and only if $i_v(\alpha) \in (K_v^\times)^p$.

Proof. The above result is [Gra03, Theorem 6.3]. $\qquad\qquad\qquad\qquad\square$

Let us consider the special case where $K := \mathbb{Q}(\mu_p)$.

Lemma 3.14. *Suppose that $K = \mathbb{Q}(\mu_p)$ and let v be the unique prime of K that lies above p. Let m be a rational integer such that $m \notin (K^\times)^p$. Setting $L := K(m^{\frac{1}{p}})$, we make the following assumptions.*
(1) *Suppose that $m^{p-1} \equiv 1 \mod p^2$. Then, v splits in L.*
(2) *Suppose that $p \nmid m$, then, $t_v \leq 1$.*

Proof. The condition that $m^{p-1} \equiv 1 \mod p^2$ implies that $m \in (\mathbb{Q}_p^\times)^p$. In particular, this implies that $m \in (K_v^\times)^p$ and thus v splits in L by Proposition 3.13. This proves part (1). For Part (2), assume without loss of generality that $p||(m^{p-1}-1)$. The result then follows from [Viv04, Theorem 5.6]. $\qquad\qquad\square$

Theorem 3.15. *Let p be an odd prime number and $K = \mathbb{Q}(\mu_p)$. Assume that $\mathrm{Sel}_p(E/K) = 0$ and m be a rational integer such that $m \notin (K^\times)^p$. Let Q_1 be the set of primes $v \nmid p$ at which E has good reduction and $\widetilde{E}(k_v)[p] = 0$. Let Q_2 be the set of primes $v \nmid p$ at which E has bad reduction. Then, assume that the following conditions are satisfied.*
(1) *The only primes $v \nmid p$ for which $v(m) \not\equiv 0 \mod p$ are contained in Q_1.*
(2) *For each prime $v \in Q_2$, we have that $i_v(m) \in (K_v^\times)^p$.*
(3) *For each prime $v \mid p$ at which E has good ordinary reduction and $\widetilde{E}(k_v)[p] \neq 0$, we have that $i_v(m) \in (K_v^\times)^p$.*
(4) *For each prime $v \mid p$ at which E has supersingular reduction, we have that $v(m) \equiv 0 \mod p$.*
(5) *For each prime $v \mid p$ of bad reduction for E, we have that $i_v(m) \in (K_v^\times)^p$.*

Then, we have that $\mathrm{Sel}_p(E/K(m^{1/p})) = 0$.

Proof. Setting $L := K(m^{1/p})$ it suffices to show that the conditions of Theorem 3.12 are satisfied. This is a direct consequence of Proposition 3.13 and Lemma 3.14. $\qquad\qquad\qquad\qquad\square$

4. ELLIPTIC CURVES WITH RATIONAL 3-ISOGENY : STABILITY RESULTS

We illustrate our results from the previous section to the case when $p = 3$. In this setting, we may consider families of elliptic curves $E_{/\mathbb{Q}}$ that have a rational 3-isogeny. More specifically, we consider the family of elliptic curves given by $E := E_a : y^2 = x^3 + a$, where a is a cubefree integer. Let $\varphi_a : E \to E'$ be a rational 3-isogeny, where $E' : y^2 = x^3 - 27a$. Set $\zeta := \frac{-1+\sqrt{-3}}{2}$ and $\varpi := 1 - \zeta$. We have

that $\varpi = -\sqrt{-3}\zeta$, and hence, $\varpi^6 = -27$. Thus, over $K := \mathbb{Q}(\mu_3)$, the curves E' and E are isomorphic via $\theta_\varpi : E' \xrightarrow{\sim} E$, defined by

$$\theta_\varpi(x, y) := \left(\frac{x}{\varpi^2}, \frac{y}{\varpi^3} \right).$$

Hence, E is 3-isogenous to itself over K. Let $\varphi := \theta_\varpi \circ \varphi_a : E \to E$ be the isogeny which is explicitly given as follows

$$\varphi(x, y) = \left(\frac{x^3 + 4a}{\varpi^2 x^2}, \frac{y(x^3 - 8a)}{\varpi^3 x^3} \right).$$

The kernel of φ is the order 3 subgroup $\{O, (0, \sqrt{a}), (0, -\sqrt{a})\} \subset E(\bar{K})$, cf. [Top93, p. 3]. Denote by $E(K)[\varphi]$ the kernel of the map $\varphi : E(K) \to E(K)$. Since φ is self-dual, we find that $E(K)[\varphi^2] = E(K)[3]$, and thus, $E(K)[\varphi]$ is contained in $E(K)[3]$.

Lemma 4.1. *With respect to notation above, the following are equivalent*

  (1) $a \notin (K^\times)^2$,
  (2) $E(K)[\varphi] = 0$,
  (3) $E(K)[3] = 0$.

Proof. Note that the point $(0, \sqrt{a})$ of $E(\bar{K})$ has order 3 and is the generator of $E[\varphi]$. Thus, a is not a square in $K^\times$ if and only if $(0, \sqrt{a})$ is not contained in $E(K)$. This shows that

$$a \notin (K^\times)^2 \Leftrightarrow E(K)[\varphi] = 0,$$

i.e., conditions (1) and (2) are equivalent.

There is a natural short exact sequence

$$0 \to E[\varphi] \to E[3] \xrightarrow{\varphi} E[\varphi] \to 0.$$

Since φ is defined over K, taking Galois invariants gives the exact sequence

$$0 \to E(K)[\varphi] \to E(K)[3] \xrightarrow{\varphi} E(K)[\varphi].$$

Thus, one has $E(K)[\varphi] = 0$ if and only if $E(K)[3] = 0$. This proves that conditions (2) and (3) are equivalent, completing the proof. $\qquad\square$

4.1. **Results of Mazur and Rubin on the parity of ranks.** Note that E is isogenous to its quadratic twist by -3 over $\mathbb{Q}$ and hence $\operatorname{rank} E(K) = 2 \operatorname{rank} E(\mathbb{Q})$. In particular, the rank of $E(K)$ is even. It is however still natural to study the parity of the rank of $E(K)$ as an $\mathcal{O}_K$-module. We briefly discuss parity results of Mazur and Rubin [MR07] for cubic twist families.

Let m be a cubefree integer. Consider the cubic twists $E_{(m)}$ and $E_{(m^2)}$ of E, defined as follows

$$E_{(m)} := E_{m^2 a} : y^2 = x^3 + m^2 a;$$
$$E_{(m^2)} := E_{m^4 a} : y^2 = x^3 + m^4 a.$$

In the next subsection, it is proven that for many values of m, the groups $E_{(m)}(K)$ and $E_{(m^2)}(K)$ both have rank 0. Setting $L := K(\sqrt[3]{m})$, we take $\chi = \chi_m :$ $\mathrm{Gal}(L/K) \to \mu_3$ to be the cubic character of order 3 defined as follows

$$\chi(\sigma) := \frac{\sigma(\sqrt[3]{m})}{\sqrt[3]{m}},$$

for $\sigma \in \mathrm{Gal}(L/K)$. Since $\mathrm{Aut}_{\bar{K}}(E) \simeq \mathcal{O}_K^\times = \mu_6$, we find that the action of G_K on $\mathrm{Aut}_{\bar{K}}(E)$ is trivial. We view μ_3 as a subgroup of $\mathrm{Aut}_{\bar{K}}(E)$ and thus view χ as a cohomology class as follows

$$\chi \in \mathrm{Hom}\left(\mathrm{G}_K, \mathrm{Aut}_{\bar{K}}(E)\right) = H^1\left(\mathrm{G}_K, \mathrm{Aut}_{\bar{K}}(E)\right).$$

The elliptic curve $E_{(m)}$ (resp. $E_{(m^2)}$) is the twist E_χ (resp. E_{χ^2}). In other words, there is an isomorphism $\phi : E_\chi \xrightarrow{\sim} E$ over $\bar{K}$ such that $\phi^\sigma = \chi(\sigma)\phi$, cf. [Sil09, section X.2]. For $P \in E_\chi(\bar{K})$, we have that

(4.1) $\sigma(\phi(P)) = \chi(\sigma)\phi(\sigma(P)).$

Given a G_K-module M, take

$$M^\chi := \{m \in M \mid \sigma(m) = \chi(\sigma)m \text{ for all } \sigma \in \mathrm{G}_K\}.$$

In view of (4.1), we have the identification $E_\chi(K) = E(L)^\chi$, and therefore,

$$E(L) \simeq E(K) \oplus E(L)^\chi \oplus E(L)^{\chi^2} \simeq E(K) \oplus E_\chi(K) \oplus E_{\chi^2}(K).$$

From (2.4), one obtains the short exact sequence

$$0 \to E_\chi(K) \otimes \mathbb{Z}_3 \to \mathfrak{S}_3(E/L)^\chi \to \mathrm{T}_3\left(\mathrusseШ(E/L)[3^\infty]\right)^\chi \to 0,$$

and thus,

$$\mathrm{rank}\left(\mathfrak{S}_3(E/L)^\chi\right) = \mathrm{rank}\, E_\chi(K)$$

provided $Ш(E/L)[3^\infty]$ is finite.

Let S be the set of primes of K consisting of the prime above 3, the primes at which E has bad reduction and the primes that ramify in L. For each prime $v \in S$, there is a locally defined invariant $\delta_v \in \mathbb{Z}/2\mathbb{Z}$, cf. [MR07, Definition 4.5]. Let $\mathbb{Z}_3[\chi]$ be the extension of $\mathbb{Z}_3$ generated by the values of χ. The following result follows from Theorem A of *loc. cit.*

Theorem 4.2 (Mazur–Rubin). *With respect to notation above, we have that*

$$\mathrm{rank}_{\mathbb{Z}_3}\left(\mathfrak{S}_3(E/K)\right) - \mathrm{rank}_{\mathbb{Z}_3[\chi]}\left(\mathfrak{S}_3(E/L)^\chi\right) \equiv \sum_{v \in S} \delta_v \pmod{2}.$$

Note that the value of $\mathrm{rank}_{\mathbb{Z}_3}\left(\mathfrak{S}_3(E/K)\right)$ is expected to be even, and thus the above result predicts that

(4.2) $\mathrm{rank}\, E_{(m)}(K), \mathrm{rank}\, E_{(m^2)}(K) \equiv 2\left(\sum_{v \in S} \delta_v\right) \pmod{4}.$

Since rank $E_{(m)}(K) = 2 \operatorname{rank} E_{(m)}(\mathbb{Q})$, the above relation implies that

$$\operatorname{rank} E_{(m)}(\mathbb{Q}), \operatorname{rank} E_{(m^2)}(\mathbb{Q}) \equiv \left(\sum_{v \in S} \delta_v \right) \pmod{2}.$$

4.2. **Main results.** In this section, we prove the main results of this article. As in the previous subsection, let m be a cubefree integer. Assume now that m is coprime to 3. Set $L_m := K(\sqrt[3]{m})$, note that L_m/K is a cyclic cubic extension. Notice that the substitution

$$(x, y) \mapsto (m^{2/3}x, my)$$

gives us that E and $E_{(m)}$ (resp. $E_{(m)}$ and $E_{(m^2)}$) are isomorphic over $\mathbb{Q}(\sqrt[3]{m})$. We find there are isomorphisms of the Selmer groups

$$\operatorname{Sel}_3(E/L_m) \simeq \operatorname{Sel}_3(E_{(m)}/L_m) \simeq \operatorname{Sel}_3(E_{(m^2)}/L_m).$$

Thus in particular, if $\operatorname{Sel}_3(E/L_m) = 0$, then rank $E_{(m)}(K) = 0$ and rank $E_{(m^2)}(K) = 0$. Let $\mathfrak{p} := \varpi \mathcal{O}_K$ be the unique prime of K that lies above (3). Note that E, $E_{(m)}$ and $E_{(m^2)}$ all have additive reduction at $\mathfrak{p}$.

From here on in, denote by ℓ a prime divisor of m, and q a prime divisor of a.

Definition 4.3. *Let $\mathcal{M}_a$ be the set of rational primes ℓ such that the following conditions are satisfied*

- $\ell \nmid a$,
- $\ell \equiv 1 \pmod{6}$,
- $\widetilde{E}(\mathbb{F}_\ell)[3] = 0$.

Theorem 4.4. *With respect to notation above, assume that*

(1) $m \equiv 1 \pmod 9$,
(2) $\operatorname{Sel}_3(E/K) = 0$,
(3) m is only divisible by primes of $\mathcal{M}_a$,
(4) m is a cube modulo q for each prime $q|a$.

Then, $\operatorname{Sel}_3(E/L_m) = 0$, in particular, rank $E_{(m)}(K) = 0$ and rank $E_{(m^2)}(K) = 0$.

Proof. We show that the conditions of Theorem 3.12 are indeed satisfied for $L = L_m$.

(1) First, we consider the condition for the primes of Σ_1. Recall that these are primes $v \nmid 3$ at which E has bad reduction. We must show that such v must be completely split in L. By part (3) of Proposition 3.13, we have that v splits in L if and only if $m \in (K_v^\times)^3$. Let q be the rational prime such that $v|q$. Since v is a prime of bad reduction, it follows that $q|2a$. Consider two cases. First, suppose that $q|a$. Note that $q \nmid m$ and $q \neq 3$. Thus, $m \in (\mathbb{Z}_q^\times)^3$ if and only if m is a cube modulo q. Since m is a cube modulo q by assumption, we conclude that $m \in (\mathbb{Z}_q^\times)^3$ and hence, $m \in (K_v^\times)^3$. Thus v is completely split in L_m for all primes $v \in \Sigma_1$. Now consider the other

case, namely, $q = 2$. In this case, since m is odd, $m \in \mathbb{Z}_2^\times$. However, it is easy to see that $\mathbb{Z}_2^\times = (\mathbb{Z}_2^\times)^3$, and thus, in particular $m \in (K_v^\times)^3$ and thus, v splits in L.

(2) Recall that Σ_2 is the set of primes $v \nmid 3$ of K at which E has good reduction which are ramified in L. The condition (2) of Theorem 3.12 requires that $\widetilde{E}(k_v)[3] = 0$ for all primes $v \in \Sigma_2$. This follows from the condition that m is only divisible by primes of $\mathcal{M}_a$. In greater detail, let $v \in \Sigma_2$ and ℓ be the rational prime such that $v | \ell$. Since v is ramified in L, it follows that $\ell | m$. Thus, $\ell \in \mathcal{M}_a$, and by definition, $\widetilde{E}(\mathbb{F}_\ell)[3] = 0$. We are to show that $\widetilde{E}(k_v)[3] = 0$. Since $\ell \in \mathcal{M}_a$, it follows by definition that $\ell \equiv 1 \mod 3$, thus ℓ splits in K. Thus, $k_v = \mathbb{F}_\ell$, and therefore, $\widetilde{E}(k_v)[3] = 0$.

(3) Since $m \equiv 1 \mod 9$, it follows that $m \in (\mathbb{Z}_3^\times)^3$. Thus in particular, it follows that $m \in (K_v^\times)^3$ and thus v splits in L for the prime v of K that lies above 3. Therefore, the condition (3) of Theorem 3.12 is satisfied.

The result thus follows from Theorem 3.12. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Definition 4.5. *Let $\mathcal{Q}_a$ be the set of primes ℓ such that*

(1) $\ell \nmid a$,
(2) $\ell \equiv 1 \pmod{18}$,
(3) ℓ is a cube modulo q for each prime q that divides a.
(4) $\widetilde{E}(\mathbb{F}_\ell)[3] = 0$.

Corollary 4.6. *Let $K = \mathbb{Q}(\mu_3)$ and assume that $\mathrm{Sel}_3(E/K) = 0$. Let m be an integer that is divisible only by primes of $\mathcal{Q}_a$. Then, we have that $\mathrm{rank}\, E_{(m)}(K) = 0$ and $\mathrm{rank}\, E_{(m^2)}(K) = 0$.*

Proof. It is clear that the conditions of Theorem 4.4 are satisfied for integers m that are divisible exclusively by primes in $\mathcal{Q}_a$. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In light of Corollary 4.6, it is natural to investigate the density of the set of primes $\mathcal{Q}_a$. Let $\rho_{E,3} : \mathrm{G}_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F}_3)$ denote the Galois representation on $E[3]$. Set $\mathbb{Q}(E[3])$ to denote the Galois extension of $\mathbb{Q}$ which is *cut out by* $E[3]$. In greater detail, $\mathbb{Q}(E[3])$ is defined as $\bar{\mathbb{Q}}^{\ker \rho_{E,3}}$. It is clear that

$$\mathrm{Gal}(\mathbb{Q}(E[3])/\mathbb{Q}) \simeq \frac{\mathrm{G}_\mathbb{Q}}{\ker \rho_{E,3}} \simeq \mathrm{image}\, \rho_{E,3}.$$

Given $\sigma \in \mathrm{Gal}(\mathbb{Q}(E[3])/\mathbb{Q})$, set $\rho(\sigma) := \rho_{E,3}(\sigma)$. Note that E admits a rational 3-isogeny $\varphi : E \to E'$, and therefore, there is a $\mathrm{G}_\mathbb{Q}$-stable subgroup $C \subset E[3]$ that has order 3. This means that the representation $\rho_{E,3}$ is reducible. Choose an $\mathbb{F}_3$-basis of $E[3]$ such that the image of $\rho_{E,3}$ is contained B, the group of upper triangular matrices $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{F}_3)$. In other words, let v_1 be a generator of C and let $v_2 \in E[3]$ be another vector not contained in C. Then, with respect to

the basis $\{v_1, v_2\}$, one obtains an upper triangular representation. Thus, there are characters $\varphi, \psi : G_\mathbb{Q} \to \mathbb{F}_3^\times = \{\pm 1\}$ such that

$$\rho_{E,3} = \begin{pmatrix} \varphi & * \\ 0 & \psi \end{pmatrix}.$$

As a Galois module $C \simeq \mathbb{F}_3(\varphi)$. Let ω denote the mod-3 cyclotomic character and note that $\det \rho_{E,3} = \varphi\psi = \omega$. Write $B = T \ltimes U$, where T consists of diagonal matrices $\begin{pmatrix} * & \\ & * \end{pmatrix}$ and U consists of unipotent matrices $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

Lemma 4.7. *With respect to notation above, the following conditions are equivalent*

(1) $E(K)[3] \neq 0$.
(2) We have that $\{\varphi, \psi\} = \{1, \omega\}$,
(3) the image of $\rho_{E,3}$ does not contain $-\operatorname{Id} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proof. First, we prove (1) is equivalent to (2). Since φ and ψ take values in $\{\pm 1\}$, we have that $\varphi^2 = \psi^2 = 1$. Since $\varphi\psi = \omega$, we find that $\psi = \varphi^{-1}\omega = \varphi\omega$. Since $\omega_{|G_K} = 1$, we find that $\varphi_{|G_K} = \psi_{|G_K}$. The condition that $E(K)[3] \neq 0$ is equivalent to the condition that $\varphi_{|G_K} = \psi_{|G_K} = 1$. Note that any character $\epsilon : G_\mathbb{Q} \to \mathbb{F}_3^\times$ such that $\epsilon_{|G_K} = 1$ is either 1 or ω. Thus, the condition (1) implies that $\varphi, \psi \in \{1, \omega\}$. Since $\varphi = \psi\omega$, we find that $\{\varphi, \psi\} = \{1, \omega\}$. Conversely, if $\{\varphi, \psi\} = \{1, \omega\}$, then, $\varphi_{|G_K} = \psi_{|G_K} = 1$, and thus, $E(K)[3] \neq 0$. This implies that (2) implies (1). Thus, (1) and (2) are equivalent.

Suppose that $\{\varphi, \psi\} = \{1, \omega\}$, then, $-\operatorname{Id} \notin \operatorname{image} \rho_{E,3}$ since either φ or ψ is 1. Conversely, suppose that $\{\varphi, \psi\} \neq \{1, \omega\}$, then, $\varphi \neq 1$. Thus, there exists $\sigma \in G_K$ such that $\varphi(\sigma) = -1$. Since $\varphi(\sigma)\psi(\sigma) = \omega(\sigma) = 1$, we find that $\psi(\sigma) = -1$ and hence $\rho_{E,3}(\sigma)^3 = -\operatorname{Id}$. This proves that (2) and (3) are equivalent. $\square$

Let q be a prime dividing a. Suppose that $q \equiv 1 \mod 3$, then set $\mathbb{L}_q$ to denote the unique $\mathbb{Z}/3\mathbb{Z}$ extension of $\mathbb{Q}$ contained in $\mathbb{Q}(\mu_q)$. On the other hand, if $q \not\equiv 1 \mod 3$, set $\mathbb{L}_q := \mathbb{Q}$. Denote by

$$\mathbb{L} := \left(\prod_{q|a} \mathbb{L}_q \right) \cdot \mathbb{Q}(\mu_{18})$$

the compositum of all the fields $\mathbb{L}_q$ as q ranges over the primes dividing a, and the field $\mathbb{Q}(\mu_{18})$.

Lemma 4.8. *Let $\ell \nmid 6a$ be a prime. Then, the conditions (1)–(3) of Definition 4.5 are satisfied by ℓ if and only if ℓ splits completely in $\mathbb{L}$.*

Proof. A prime ℓ splits in $\mathbb{Q}(\mu_{18})$ if and only if $\ell \equiv 1 \mod 18$. Note that ℓ splits in $\mathbb{L}_q$ if and only if ℓ is a cube modulo q. This proves the result. $\square$

Note that the field $\mathbb{Q}(\mu_{18})$ is unramified at all primes other than 2 and 3. On the other hand, $\mathbb{L}_q$ is totally ramified at q, and unramified at all other primes. Recall that $\mathbb{L}_q := \mathbb{Q}$ unless $q \equiv 1 \mod 3$. This means that the fields $\{\mathbb{L}_q\}$ and $\mathbb{Q}(\mu_{18})$ are all mutually disjoint. Therefore, we find that

$$[\mathbb{L} : \mathbb{Q}] = \prod_{q|a}[\mathbb{L}_q : \mathbb{Q}] \times [\mathbb{Q}(\mu_{18}) : \mathbb{Q}] = 3^s \times 6 = 2 \times 3^{s+1},$$

where s is the number of primes $q \equiv 1 \mod 3$ which divide a. Set $F := \mathbb{Q}(E[3])$, $G := \mathrm{Gal}(F/\mathbb{Q})$ and $A \subset G$ be the subset of $\sigma \in G$ such that

$$\det \rho_{E,3}(\sigma) = 1 \text{ and } \mathrm{trace}\, \rho_{E,3}(\sigma) \neq 2.$$

Lemma 4.9. *With respect to notation above, let $\ell \nmid 6a$ be a prime number. Then ℓ is unramified in $\mathbb{Q}(E[3])$ and the following are equivalent*

(1) $\ell \equiv 1 \mod 3$ and $\widetilde{E}(\mathbb{F}_\ell)[3] = 0$,
(2) $\mathrm{Frob}_\ell \in A$.

Proof. Since $\ell \nmid 6a$, it is a prime of good reduction and it follows from the Neron–Ogg–Shafarevich theorem that ℓ is unramified in $\mathbb{Q}(E[3])$. We then note that

$$\det \rho_{E,\ell}(\mathrm{Frob}_\ell) = \ell \text{ and } \mathrm{trace}\, \rho_{E,3}(\mathrm{Frob}_\ell) = \ell + 1 - \#\widetilde{E}(\mathbb{F}_\ell).$$

The result thus follows. $\qquad\square$

Lemma 4.10. *With respect to the notation above, we have that $F \cap \mathbb{L} = K$.*

Proof. Set $H := \mathrm{Gal}(F/K) \subset G$ and $H' := \mathrm{Gal}(F/F \cap \mathbb{L}) \subseteq H$. We show that $H' = H$. Note that $G \simeq \rho_{E,3}(G)$ and $H \simeq \rho_{E,3}(H)$, and hence we may view these as subgroups of B. Since $\omega = \det \rho_{E,3}$, we find that G contains a matrix with determinant -1. Note that $B = \langle -\mathrm{Id}\rangle \times B_0$, where B_0 consists of matrices of the form $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$. Note that B_0 contains U as its unique normal subgroup of order 3, and contains 3 subgroups C_1, C_2 and C_3 that are of order 2. The possibilities for G are B, B_0, C_i and $\langle -\mathrm{Id}\rangle \times C_i$ as $i = 1, 2, 3$. Note that since $-\mathrm{Id} \in G$, the groups U and $\langle -\mathrm{Id}\rangle \times U$ are excluded from our list. All abelian quotients of these groups are 2-groups. Hence, $G/H' = \mathrm{Gal}(F \cap \mathbb{L}/\mathbb{Q})$ is an abelian 2-group. Since the abelian 2-quotient of $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ is $\mathrm{Gal}(K/\mathbb{Q})$, it follows that $F \cap \mathbb{L} = K$. $\quad\square$

Proposition 4.11. *Assume that a is not a square in K and is prime to 3. Then, with respect to notation above, the set of primes $\mathcal{Q}_a$ has positive density given by*

$$\mathfrak{d}(\mathcal{Q}_a) = \frac{1}{8 \times 3^{s+1}},$$

where s is the number of prime divisors q of a such that $q \equiv 1 \pmod 3$.

Proof. Recall that $F := \mathbb{Q}(E[3])$, $G := \mathrm{Gal}(F/\mathbb{Q})$ and $H := \mathrm{Gal}(F/K)$. The representation

$$\rho = \rho_{E,3} : \mathrm{G}_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F}_3),$$

induces a natural isomorphism between G and the image of ρ. Let $\ell \nmid 6a$ be a prime, then, ℓ is unramified in F. Denote by $\mathrm{Frob}_\ell \in G$ a Frobenius element of ℓ. Recall that A is the subset of G consisting of σ such that

$$\mathrm{trace}\left(\rho(\sigma)\right) \neq 2 \text{ and } \det\left(\rho(\sigma)\right) = 1.$$

By Lemma 4.9, the following are equivalent

(1) $\ell \equiv 1 \mod 3$ and $\widetilde{E}(\mathbb{F}_\ell)[3] = 0$,
(2) $\mathrm{Frob}_\ell \in A$.

It is clear that A is contained in H. Since a is assumed to not be square in K, we find that $E(K)[3] = 0$. By Lemma 4.7, $-\mathrm{Id} \in A$. Given any matrix $h \in H$, we have that $\det(\rho(h)) = 1$ and $\mathrm{trace}(\rho(h))$ is either 1 or 2. Note that the trace of a matrix with determinant 1 cannot be 0. If $h \in H$, then since $-\mathrm{Id} \in H$, we have that $-h \in H$. Thus, we find that

$$(4.3) \qquad\qquad\qquad\qquad \#A = \frac{\#H}{2}.$$

We compute the density of $\mathcal{Q}_a$ by applying the Chebotarev density theorem to the field $F \cdot \mathbb{L}$. Note that $F \cap \mathbb{L} = K$ by Lemma 4.10. Thus, the Galois group $\mathrm{Gal}(F \cdot \mathbb{L}/K) \simeq H \times \mathrm{Gal}(\mathbb{L}/K)$. Let $\mathcal{S}$ be the subset $A \times \{1\}$ contained in $\mathrm{Gal}(F \cdot \mathbb{L}/K)$ and view $\mathcal{S}$ as a subset of $\mathrm{Gal}(F \cdot \mathbb{L}/\mathbb{Q})$. Note that the set $\mathcal{S}$ is stable with respect to the action of conjugation. Let ℓ be a prime which is unramified in $F \cdot \mathbb{L}$ and such that $\mathrm{Frob}_\ell \in \mathcal{S}$. Then, by Lemma 4.8, the prime ℓ splits completely in $\mathbb{L}$, and consequently, conditions (1)–(3) of Definition 4.5 are satisfied. Also $\mathrm{Frob}_\ell \in A$, it follows from Lemma 4.9 that (4) is also satisfied. Thus, by the Chebotarev density theorem and (4.3), we find that

$$\mathfrak{d}(\mathcal{Q}_a) = \frac{\#\mathcal{S}}{[F \cdot \mathbb{L} : \mathbb{Q}]} = \frac{\#\mathcal{S}}{2[F \cdot \mathbb{L} : K]} = \frac{\#A}{2\#H \times [\mathbb{L} : \mathbb{Q}]} = \frac{1}{4[\mathbb{L} : \mathbb{Q}]} = \frac{1}{8 \times 3^{s+1}}.$$

$\square$

We prove the main result of this article.

Proof of Theorem A. It is easy to see that $a \notin (K^\times)^2$. The result is a direct consequence of Corollary 4.6 and Proposition 4.11. $\square$

Corollary 4.12. *Let E and K be as in Theorem A. Then, there is a positive density of primes ℓ such that*

$$\mathrm{rank}\, E_{\ell^2 a}(K) = 0 \text{ and } \mathrm{rank}\, E_{\ell^4 a}(K) = 0.$$

Proof. The result is an immediate consequence of Theorem A, upon setting $m = \ell$ to be a prime number in $\mathcal{Q}_a$. $\square$

4.3. **An illustrative example.** Let us illustrate the result in Theorem A through an explicit example. Consider the curve $E_{-1} : y^2 = x^3 - 1$. We check that $\mathrm{Sel}_3(E_{-1}/K) = 0$. The quadratic twist of E by -3 is given by E_{27}. It suffices to show that

$$\mathrm{Sel}_3(E_{-1}/\mathbb{Q}) = 0 \text{ and } \mathrm{Sel}_3(E_{27}/\mathbb{Q}) = 0.$$

This is computed via the following Magma code

```
E:=EllipticCurve([0,0,0,0,-1]);
ThreeSelmerGroup(E);
Et := EllipticCurve([0,0,0,0,27]);
ThreeSelmerGroup(Et);

Ouput:
Abelian Group of order 1.
Abelian Group of order 1
```

Note that $\mathcal{Q}_{-1}$ (cf. Definition 4.5) consists of the primes $\ell \equiv 1 \mod 18$ such that $\widetilde{E}_{-1}(\mathbb{F}_\ell)[3] = 0$. Corollary 4.6 then asserts that if m is divisible only by primes of $\mathcal{Q}_{-1}$, then $E_{-m^2}(K)$ and $E_{-m^4}(K)$ both have rank 0. Proposition 4.11 asserts that the density of $\mathcal{Q}_{-1}$ is $\frac{1}{24}$. We arrive at the following concrete application of Theorem 4.4 and Proposition 4.11 for the elliptic curve $E := E_{-1}$.

Theorem 4.13. *Let $E_{/\mathbb{Q}}$ be the elliptic curve $y^2 = x^3 - 1$. Given a cube-free integer m, let $E_{(m)}$ (resp. $E_{(m^2)}$) denote the cubic twist of E defined by $y^2 = x^3 - m^2$ (resp. $y^2 = x^3 - m^4$). Let $\mathcal{Q} := \mathcal{Q}_{-1}$ be the set of primes $\ell \equiv 1 \mod 18$ such that $\widetilde{E}(\mathbb{F}_\ell)[3] = 0$. The set $\mathcal{Q}$ is infinite with density $1/24$. Furthermore, if m is divisible by only the primes in $\mathcal{Q}$, then,*

$$\mathrm{rank}\, E_{(m)}(K) = 0 \text{ and } \mathrm{rank}\, E_{(m^2)}(K) = 0.$$

Proof. The result follows from Theorem 4.4 and Proposition 4.11, see above for more details. $\square$

We remark that there are 15 primes in $\mathcal{Q}$ that are < 1000, given as follows

$$\mathcal{Q} \cap [1, 1000] = \{19, 127, 163, 199, 271, 307, 379, 487, 523, 631, 739, 811, 883, 919, 991\}.$$

The Theorem 4.13 implies in particular that for any of the primes $\ell \in \mathcal{Q} \cap [1, 1000]$ above, the ranks of $E_{-\ell^2}(K)$ and $E_{-\ell^4}(K)$ are both zero. This has been checked by the authors using Magma.

4.4. **Numerical data.** In the table 1 below (on the next page), we collect some numerical data for the values of $a \in [-20, 20]$ satisfying the conditions of Theorem 3.12, along with the density of $\mathcal{Q}_a$. The data shows that Theorem A applies to

$$a \in \{-17, -16, -14, -10, -9, -8, -6, -5, -1, 6, 7, 8, 13, 14, 20\}.$$

Thus, the result applies to 15 out of a total of 34 values considered for a. All computations were done on Magma.

a	$\mathrm{Sel}_3(E_a/\mathbb{Q})$	$\mathrm{Sel}_3(E_{-27a}/\mathbb{Q})$	$\mathrm{Sel}_3(E_a/K)$	s	$\mathfrak{d}(\mathcal{Q}_a)$
-20	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
-19	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
-18	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
-17	0	0	0	0	$1/24$
-16	0	0	0	0	$1/24$
-15	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
-14	0	0	0	1	$1/72$
-13	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
-11	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
-10	0	0	0	0	$1/24$
-9	0	0	0	0	$1/24$
-8	0	0	0	0	$1/24$
-7	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
-6	0	0	0	0	$1/24$
-5	0	0	0	0	$1/24$
-4	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
-2	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
-1	0	0	0	0	$1/24$
2	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
3	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
5	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
6	0	0	0	0	$1/24$
7	0	0	0	1	$1/72$
8	0	0	0	0	$1/24$
10	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
11	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
12	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
13	0	0	0	1	$1/72$
14	0	0	0	1	$1/72$
15	$(\mathbb{Z}/3\mathbb{Z})^2$	$(\mathbb{Z}/3\mathbb{Z})^2$	$(\mathbb{Z}/3\mathbb{Z})^4$	$-$	$-$
17	$(\mathbb{Z}/3\mathbb{Z})^2$	$(\mathbb{Z}/3\mathbb{Z})^2$	$(\mathbb{Z}/3\mathbb{Z})^4$	$-$	$-$
18	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
19	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$-$	$-$
20	0	0	0	0	$1/24$

TABLE 1. Numerical data for Theorem A

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