

A remark on the distortion of twisted sums

Jesús Suárez

ABSTRACT. Odell and Schlumprecht solved the distortion problem by proving that the classic sequence spaces ℓ_p for $1 < p < \infty$ admit an inevitable biorthogonal system. In particular, these spaces are arbitrarily distortable. Later, Maurey extended the result to asymptotic ℓ_p -spaces while Tomczak-Jaegermann did likewise for the Schatten classes. We observe that the Kalton-Peck spaces Z_p for $1 < p < \infty$ admit an inevitable biorthogonal system. Therefore, we may add this classic family to the list of arbitrarily distortable spaces.

1. Introduction

As it is well known, it was James who introduced us to the topic of the distortion. He proved that the spaces ℓ_1 and c_0 are not distortable [6]. Recall that given $\lambda > 1$, a norm $\|\cdot\|$ on an infinite-dimensional Banach space X is said to be λ -distortable if there is an equivalent norm $|||\cdot|||$ on X such that for every infinite-dimensional subspace Y of X

$$\sup\{|||x|||/|||y||| : x, y \in Y, \|x\| = \|y\| = 1\} \geq \lambda.$$

A given norm is said to be distortable if it is λ -distortable for some $\lambda > 1$ and arbitrarily distortable if it is λ -distortable for all $\lambda > 1$. We invite the reader to the survey of Odell and Schlumprecht [14] for a comprehensive study on distortion and its development. We just focus on some points to introduce our results, starting with Schlumprecht's construction in [15] of the first example of a Banach space which is arbitrarily distortable and that we denote by $\mathcal{S}$. This follows nowadays from the fact that $\mathcal{S}$ admits an inevitable biorthogonal system. Here, we are replacing the term asymptotic by inevitable as suggested by Maurey in [11]. Recall that a Banach space X is said to admit an inevitable biorthogonal system with constant $0 < \delta < 1/2$ if there are two sequences $\{A_j\}_{j=1}^\infty$ and $\{A_j^*\}_{j=1}^\infty$ of subsets of the unit sphere $S(X)$ of X and of the unit ball $B(X^*)$ of X^* , respectively, such that

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- (1) $d(H, A_j) = 0$ for every infinite dimensional subspace H of X and every $j \in \mathbb{N}$.
- (2) Given $j \in \mathbb{N}$ and $x \in A_j$, there is $x^* \in A_j^*$ so that $\langle x^*, x \rangle \geq 1 - \delta$.
- (3) If $j \neq k$, then $\langle x^*, x \rangle \leq \delta$ whenever $x \in A_j$ and $x^* \in A_k^*$.

The condition (1) above is usually termed as the set A_j is inevitable for $j \in \mathbb{N}$. The way to construct such a system in $\mathcal{S}$ deals with ℓ_1 -averages and rapidly increasing sequences, RIS sequences in short, which are tools introduced by Gowers and Maurey in the solution to the unconditional basic sequence problem in [5]. Once we have such a system for a fixed δ , the equivalent norm

$$\varepsilon \|x\| + \sup\{|\langle x^*, x \rangle| : x^* \in A_1^*\},$$

is a $(1 + \varepsilon - \delta)/(\varepsilon + \delta)$ -distortion, see [2, Proposition 13.28.] for the details.

A natural question remained opened for a long time: Whether or not the classical sequence spaces ℓ_p for $1 < p < \infty$ are also arbitrarily distortable. It was a surprise when Odell and Schlumprecht in [13] were able to transfer these inevitable sets from $\mathcal{S}$ to ℓ_p with $1 < p < \infty$ solving the “the distortion problem”. Later, Maurey extended the result to asymptotic ℓ_p -spaces with an unconditional basis [12]. These are spaces such that any n -blocks of the basis starting far enough are uniformly equivalent to the unit vector basis of ℓ_p . Modelled over Maurey’s proof, it was shown by Tomczak-Jaegermann in [19] that the Schatten classes C_p also admit such system of inevitable sets for $1 < p < \infty$.

A third class of spaces which fits with the previous idea is that of the twisted sums of ℓ_p . These are Banach spaces Z so that they contain a copy of ℓ_p for which the natural quotient map Z/ℓ_p is again isomorphic to ℓ_p . The main examples of this class are the so-called Kalton-Peck spaces Z_p for $1 < p < \infty$ which were introduced in the seminal paper [10]. The main purpose of this note is to extend the inevitable biorthogonal system from ℓ_p to a large class of twisted sums of ℓ_p that includes the Kalton-Peck spaces. So we add new examples to the works of Maurey [12] and Tomczak-Jaegermann [19] on families of arbitrarily distortable Banach spaces.

2. Main

We shall use the terminology commonly used in Banach space theory as it appears in the book of Albiac and Kalton [1] where all unexplained notation may be found.

Let us briefly recall the formal construction of a twisted sum of ℓ_p . First, an homogeneous map $\Omega : \ell_p \longrightarrow \omega$ is called a centralizer if there is a constant $C > 0$ so that:

$$\|\Omega(ax) - a\Omega(x)\|_{\ell_p} \leq C\|a\|_{\infty}\|x\|_{\ell_p}, \quad a \in \ell_{\infty}, x \in \ell_p,$$

where ω denotes the vector space of all complex scalar sequences. By homogeneous we just mean that Ω must satisfy $\Omega(\lambda x) = \lambda\Omega(x)$ for every $\lambda \in \mathbb{C}$ and $x \in \ell_p$. Once

we have a centralizer Ω , we pick the pairs $(x, y) \in \omega \times \omega$ for which the quasi-norm

$$\|x - \Omega(y)\|_{\ell_p} + \|y\|_{\ell_p}$$

is finite. The quasi-norm above is equivalent to a norm for $1 < p < \infty$ by a result of Kalton [8, Theorem 2.6]. This set of couples, say Z , is a twisted sum of ℓ_p . Indeed, it is easy to see that $j(x) = (x, 0)$ is an embedding of ℓ_p into Z for which the quotient map is identified as $q(x, y) = y$ from Z onto ℓ_p . The most important centralizer is given by the so-called Kalton-Peck map

$$(1) \quad \Omega_p(x) = \sum_{j=1}^{\infty} x_j \cdot \log \frac{|x_j|}{\|x\|_p}$$

with the convention $0 \cdot \log 0 = 0$. The Kalton-Peck space Z_p is thus defined as the pairs $(x, y) \in \omega \times \omega$ for which the quasi-norm

$$\|x - \Omega_p(y)\|_{\ell_p} + \|y\|_{\ell_p}$$

is finite. The details are to be found in the survey of Kalton and Montgomery-Smith [9, Pages 1155–61]. It will be useful for us to recall the duality between a twisted sum of ℓ_p , say Z , and its dual Z^* . The natural pairing for $(x, y) \in Z$ and $(a, b) \in Z^*$ is given by the rule

$$\langle (x, y), (a, b) \rangle = \langle x, b \rangle + \langle y, a \rangle.$$

The proof of this fact for the Kalton-Peck map is [10, Theorem 5.1]. For an arbitrary centralizer and $p = 2$, a proof can be found in [17, Lemma 1] but the proof easily extends to $1 < p < \infty$. The class of twisted sums we work with is isolated by the following rule.

THE PRINCIPLE OF SMALL PERTURBATIONS. *A twisted sum Z of ℓ_p for $1 < p < \infty$ satisfies P.S.P. if for every sequence $(w_j)_{j=1}^{\infty}$ in Z which is equivalent to the unit vector basis of ℓ_p there is a subsequence $(w_{k_j})_{j=1}^{\infty}$ of $(w_j)_{j=1}^{\infty}$ and blocks $(y_j)_{j=1}^{\infty}$ in ℓ_p such that*

- (1) $\|(y_j, 0)\| = 1$ for every $j \in \mathbb{N}$.
- (2) $\|w_{k_j} - (y_j, 0)\| \leq 2^{-j}$ for every $j \in \mathbb{N}$.

The Principle of Small Perturbations, the P.S.P. in short, above is satisfied by the Kalton-Peck spaces as we will show in Example 1 in the next section. In the meantime, let us justify the importance of the P.S.P. for the distortion.

THEOREM 1. *Let Z be a twisted sum of ℓ_p with $1 < p < \infty$. If Z satisfies the Principle of Small Perturbations, then Z admits an inevitable biorthogonal system for every $0 < \delta < 1/2$. In particular, Z is arbitrarily distortable.*

PROOF. Let $\{A_j, A_j^*\}_{j=1}^{\infty}$ be an inevitable biorthogonal system of ℓ_p with constant $\delta > 0$. We define sets in Z for every $j \in \mathbb{N}$ as

$$\hat{A}_j := \{(x, 0) : x \in A_j\}.$$

We turn to Z^* so let Ω be a centralizer for Z^* and define the following sets in Z^* for every $j \in \mathbb{N}$

$$\widehat{A}_j^* := \{(\Omega(x), x) : x \in A_j^*\}.$$

Observe that the quasinorm of an element $(\Omega(x), x)$ with $x \in A_j^*$ is given by

$$\|\Omega(x) - \Omega(x)\| + \|x\| = \|x\| < \infty,$$

so the sets $\widehat{A}_j^*$ are well defined. The proof consists in showing that the sets $\{\widehat{A}_j\}_{j=1}^\infty$ and $\{\widehat{A}_j^*\}_{j=1}^\infty$ defined above form an inevitable biorthogonal system for Z . We only need to check that the set $\widehat{A}_j$ is inevitable for each $j \in \mathbb{N}$. Indeed, the remaining two claims of the definition are trivial using the natural pairing duality

$$\langle (x, y), (a, b) \rangle = \langle x, b \rangle + \langle y, a \rangle$$

and the fact that $\{A_j, A_j^*\}_{j=1}^\infty$ is biorthogonal. Thus, pick $j_0 \in \mathbb{N}$ and let W be any infinite dimensional subspace of Z . Our aim is to show that $d(W, \widehat{A}_{j_0}) = 0$. Thus, let us fix $\varepsilon > 0$. Let us recall first that every infinite-dimensional subspace W of a twisted sum of ℓ_p contains an isomorphic copy of ℓ_p : First, every infinite dimensional subspace of ℓ_p contains an isomorphic copy of ℓ_p , see [1, Proposition 2.2.1]. Now, if $W \cap \text{Ker } q$ is infinite-dimensional, where $q(a, b) = b$ denotes the natural quotient map onto ℓ_p , this is clear by the first argument. Otherwise, q is an isomorphism on a finite codimensional subspace of W and we are done again. This argument is just taken for the case $p = 2$ from the proof of [2, Theorem 16.16., Second paragraph]. By the P.S.P., we may find a subsequence $(w_{k_n})_{n=1}^\infty$ of $(w_n)_{n=1}^\infty$ and normalized blocks $(u_n)_{n=1}^\infty$ in ℓ_p so that

$$\|w_{k_n} - (u_n, 0)\| \leq \frac{\varepsilon}{2^n}, \quad n \in \mathbb{N}.$$

Define U as the closed linear span of $(u_j)_{j=1}^\infty$ in ℓ_p , usually termed as $U = [u_n]_{n=1}^\infty$. Since A_{j_0} is inevitable in ℓ_p , given our $\varepsilon > 0$ and U , there must be a vector $u \in U$ with $\|u\|_{\ell_p} \leq 1$ and $a \in A_{j_0}$ so that

$$\|u - a\|_{\ell_p} \leq \varepsilon.$$

We now consider Z , so write $u = \sum_{n=1}^\infty \lambda_n u_n$ and thus let $w = \sum_{n=1}^\infty \lambda_n w_{k_n} \in W$. Observe that w is a well defined element of W . Indeed, $\|u\|_{\ell_p} \leq 1$ and the normalized blocks $(u_n)_{n=1}^\infty$ are equivalent to the canonical ℓ_p basis. This means that

$$\|u\|_{\ell_p}^p = \sum_{n=1}^\infty |\lambda_n|^p \|u_n\|_{\ell_p}^p = \sum_{n=1}^\infty |\lambda_n|^p \leq 1.$$

Since $(w_{k_n})_{n=1}^\infty$ is again equivalent to the canonical ℓ_p -basis, a similar computation as before proves our claim. Then, we have

$$\begin{aligned}
\|w - (a, 0)\|_Z &\leq \|w - (u, 0)\|_Z + \|(u, 0) - (a, 0)\|_Z \\
&= \|w - (u, 0)\|_Z + \|u - a\|_{\ell_p} \\
&\leq \|w - (u, 0)\|_Z + \varepsilon \\
&\leq \sum_{n=1}^{\infty} |\lambda_n| \|w_{k_n} - (u_n, 0)\| + \varepsilon \\
&\leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} + \varepsilon \\
&= 2\varepsilon.
\end{aligned}$$

Since obviously $(a, 0) \in \widehat{A}_{j_0}$, the inequality above gives that $d(W, \widehat{A}_{j_0}) \leq 2\varepsilon$. But $\varepsilon > 0$ is arbitrary, so $d(W, \widehat{A}_{j_0}) = 0$ and we are done. Therefore, Z admits an inevitable biorthogonal system for every $0 < \delta < 1/2$. Now, to conclude that Z is arbitrarily distortable we just need to apply [2, Proposition 13.28]. $\square$

3. Examples

Let us show that a large class of twisted Hilbert spaces satisfy the Principle for Small Perturbations and thus we may apply Theorem 1.

EXAMPLE 1. *The Kalton-Peck spaces Z_p for $1 < p < \infty$ admit an inevitable biorthogonal system for every $0 < \delta < 1/2$. In particular, they are arbitrarily distortable.*

PROOF. We only need to prove that Z_p satisfies the P.S.P. and then apply Theorem 1. The idea that Z_p satisfies the P.S.P. traces back to Kalton and Peck's paper [10]. The exact argument appears explicitly in the proof of [2, Theorem 16.16.] of Benyamini and Lindenstrauss for Z_2 . We reproduce it for the sake of completeness. First, we would like to recall an important fact that is: the sequence

$$(2) \quad \{(e_1, 0), (0, e_1), (e_2, 0), \dots\}$$

is (in this order) a basis of Z_p , [10, Theorem 4.10]. So, let $(w_j)_{j=1}^\infty$ be a sequence in Z_p which is equivalent to the ℓ_p basis. Since $w_j \rightarrow 0$ weakly, we may apply The Bessaga-Pełczyński Selection Principle [10, Proposition 1.3.10] and pass to a subsequence (we do not relabel) so that there exists blocks (x_j, u_j) of the basis (2) with

$$\|w_j - (x_j, u_j)\|_{Z_p} \leq 2^{-j}, \quad j \in \mathbb{N}.$$

We claim that $\|u_j\|_p \rightarrow 0$. Let us observe that we always have the following inequality that will be crucial

$$\begin{aligned} \left\| \left(\sum_{j=1}^n x_j, \sum_{j=1}^n u_j \right) \right\| &\geq \left\| \sum_{j=1}^n x_j - \Omega_p \left(\sum_{j=1}^n u_j \right) \right\| \\ &\geq \left\| \Omega_p \left(\sum_{j=1}^n u_j \right) - \sum_{j=1}^n \Omega_p(u_j) \right\| - \left\| \sum_{j=1}^n (x_j - \Omega_p(u_j)) \right\|, \end{aligned}$$

where Ω_p denotes the Kalton-Peck map described in (1). The inequality above turns out to be impossible if $\inf_{j \in \mathbb{N}} \|u_j\| > 0$. Indeed,

$$\left\| \sum_{j=1}^n (x_j - \Omega_p(u_j)) \right\|_p^p \leq \sum_{j=1}^n \|x_j - \Omega_p(u_j)\|_p^p \leq n,$$

while

$$(3) \quad \left\| \left(\sum_{j=1}^n x_j, \sum_{j=1}^n u_j \right) \right\| \sim n^{\frac{1}{p}}$$

because the blocks (x_j, u_j) are a small perturbation of (w_j) . If we assume that there is some $\eta > 0$ so that $\|u_j\| \geq \eta > 0$ for all $j \in \mathbb{N}$, then the blocks u_j are semi-normalized and thus

$$\left\| \Omega_p \left(\sum_{j=1}^n u_j \right) - \sum_{j=1}^n \Omega_p(u_j) \right\| \geq \rho n^{\frac{1}{p}} \log n,$$

by the definition of Ω_p for some $\rho > 0$. If we insert these estimates in the crucial inequality, we find on the right side the lower bound

$$\rho n^{\frac{1}{p}} \log n - n^{\frac{1}{p}},$$

which is impossible because the left side is of the order $n^{\frac{1}{p}}$. Thus, it must be $\|u_j\|_p \rightarrow 0$ as $j \rightarrow \infty$ and the claim is proved. So let us assume, passing to a subsequence if necessary, that $\|u_j\|_p \leq 2^{-j}$ and hence

$$\begin{aligned} \|w_j - (x_j - \Omega_p(u_j), 0)\| &\leq \|w_j - (x_j, u_j)\| + \|(\Omega_p(u_j), u_j)\| \\ &\leq 2^{-j} + \|u_j\|_p \\ &\leq 2^{-j+1}. \end{aligned}$$

Thus, pick $y_j := x_j - \Omega_p(u_j)$. Replacing w_j by $\lambda_j w_j$, where the sequence λ_j satisfies $\inf_j \lambda_j > 0$ and $\sup_j \lambda_j < \infty$, we may assume that $\|y_j\|_p = 1$. Therefore, Z_p satisfies the P.S.P. and we are ready to apply Theorem 1. $\square$

The rest of our examples deal with twisted sums of ℓ_2 which are usually called twisted Hilbert spaces. Kalton provided us in [7] with a procedure so that for a given space X with a shrinking unconditional basis, we may construct a twisted

Hilbert space that we denote $Z(X)$. In its simplest form we could describe the process as follows: First, the complex interpolation formula $(X, X^*)_{1/2} = \ell_2$ holds true, see [20] or [4]. This means that every element $x \in \ell_2$, say with $\|x\|_2 = 1$, may be represented by certain holomorphic function F ; here “represented” means that $F(1/2) = x$ and $\|F\| \leq 2$ (this last norm is taken on the so-called Calderón space where the norm of X is relevant). It is easy to check that the assignment

$$\Omega(x) := F'(1/2)$$

is a centralizer. Thus, we may select our pairs as before and end up with $Z(X)$. It is not surprising in view of the formula above that $Z(X)$ is usually termed as the derived space of X . It is also not hard to check that $Z(\ell_r) = Z_2$ for $r \neq 2$, and $Z(c_0) = Z_2$. The survey of Kalton and Montgomery-Smith [9] gives a short but accurate version of the process.

We would like to say that unlike the Kalton-Peck map, we have no an explicit description of such Ω in general, we only know the existence of such map by the result of Nigel quoted above. This fact complicates very much the computation of the norm of an element $\|(x, y)\|$ in a twisted Hilbert space Z with centralizer Ω . However, using recent results of [18], we have a large class of twisted Hilbert spaces for which it is ‘easy’ now to construct an inevitable biorthogonal system. This class contains the derived spaces of Maurey’s class of asymptotic ℓ_p -spaces [12]. Recall that a space X with an unconditional basis is an asymptotic ℓ_p -space if there is $C > 0$ such that for every $n \in \mathbb{N}$ and every n -normalized blocks of the basis with $n < u_1 < \dots < u_n$, we have that $[u_j]_{j=1}^n$ is C -equivalent to the natural basis of ℓ_p^n .

EXAMPLE 2. *Let X be a space with a shrinking unconditional basis $(e_j)_{j=1}^\infty$. Assume X satisfies one of the two following conditions:*

- (1) *X is an asymptotic ℓ_r -space with $1 \leq r \neq 2 < \infty$.*
- (2) *The only spreading model of a block basic sequence of $(e_j)_{j=1}^\infty$ is c_0 .*

Then, the twisted Hilbert space $Z(X)$ admits an inevitable biorthogonal system for every $0 < \delta < 1/2$. In particular, it is arbitrarily distortable.

PROOF. The idea of the proof is the following. Since X behaves asymptotically as, say ℓ_r , then $Z(X)$ behaves asymptotically as $Z(\ell_r) = Z_2$. This notion of limit allows to share information between the spaces which is given by arbitrary large but finite data segments. As it is witnessed by the proof of Example 1, this is enough to get the P.S.P. so if Z_2 has the P.S.P. then so has $Z(X)$. Formally speaking, for the first case, we apply [18, Theorem 4] and find that the only spreading model of $Z(X)$ is Z_2 . For the second, we use [18, Proposition 9] and also find that the only spreading model of $Z(X)$ is Z_2 . Anyway, we may apply now [18, Theorem 5] and deduce that either in case (1) or (2) the space $Z(X)$ satisfies the P.S.P., so we conclude using Theorem 1. $\square$

The result above does not deal with $r = 2$ but we may still give a nontrivial example of this; the Hilbert copy ℓ_2 is a trivial example of this phenomena since it is trivial to see that $Z(\ell_2) = \ell_2$. Let T denotes the Tsirelson space and T^2 the 2-convexification which is well known to be an asymptotic ℓ_2 -space with an unconditional basis (see [3]). In particular, by Maurey's result [12], the space T^2 admits an inevitable biorthogonal system. The derived space $Z(T^2)$, see [16] for more details on this space, also does.

EXAMPLE 3. *The twisted Hilbert space $Z(T^2)$ admits an inevitable biorthogonal system for every $0 < \delta < 1/2$. In particular, it is arbitrarily distortable.*

PROOF. We only need to prove that $Z(T^2)$ satisfies the P.S.P. and then apply Theorem 1. The proof that $Z(T^2)$ satisfies the P.S.P., which is indeed quite technical, is to be found at [18, Theorem 3]. $\square$

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE EXTREMADURA, AVENIDA DE ELVAS
S/N, 06011 BADAJOZ, SPAIN.

Email address: `jesus@unex.es`