QUILLEN (CO)HOMOLOGY OF DIVIDED POWER ALGEBRAS OVER AN OPERAD

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Abstract. Barr–Beck cohomology, put into the framework of model categories by Quillen, provides a cohomology theory for any algebraic structure, for example André–Quillen cohomology of commutative rings. Quillen cohomology has been studied notably for divided power algebras and restricted Lie algebras, both of which are instances of divided power algebras over an operad $\mathcal{P}$: the commutative and Lie operad respectively. In this paper, we investigate the Quillen cohomology of divided power algebras over an operad $\mathcal{P}$, identifying Beck modules, derivations, and Kähler differentials in that setup. We also compare the cohomology of divided power algebras over $\mathcal{P}$ with that of $\mathcal{P}$-algebras, and work out some examples.

Contents

1. Introduction

Barr–Beck cohomology is a cohomology theory for general algebraic structures, based on simplicial resolutions [BB69]. It recovers (up to a shift in degree) group cohomology, Chevalley–Eilenberg cohomology of Lie algebras, and Hochschild cohomology of algebras over a field; see for instance [Dus75, Bar96]. One important example is André–Quillen cohomology of commutative rings [And67, And74], put into the framework of model categories by Quillen [Qui67, Qui70], which

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has various applications in algebra and algebraic geometry. Quillen cohomology later found applications in topology, often in the guise of inputs to spectral sequences [Mil84, Goe90] or obstructions in realization and classification problems [GH00, GH04, BDG04, Fra11, BB11, BJT12].

Divided power algebras

Divided power structures are algebraic structures which are fairly ubiquitous when working over a field of positive characteristic. N. Jacobson introduced the concept of restricted Lie algebra to study modular Lie theory [Jac62]. A restricted Lie algebra is a Lie algebra equipped with an additional operation called the p-map, which satisfies specific relations. The archetypal example of a restricted Lie algebra is an associative algebra in prime characteristic equipped with the Lie bracket given by the commutator, and with the p-map given by the p-th power. Restricted Lie algebras appear notably in field theory [Jac37], linear algebraic groups [Bor91], and the cohomology of the Steenrod algebra [May66].

G. Hochschild, and later B. Pareigis, defined cohomology groups of restricted Lie algebras [Hoc54, Par68]. In [Dok04], the first author developed the Quillen cohomology theory for restricted Lie algebras. In contrast to the cases of associative algebras and Lie algebras, this cohomology does not coincide with Hochschild cohomology for restricted Lie algebras. In [Dok15], 2-fold extensions of restricted Lie algebras are classified using Duskin's and Glenn's torsor cohomology ([Dus75, Gle82]) and the work of Cegarra and Aznar [CA86].

The notion of algebra with divided powers was introduced by H. Cartan in [Car56] to study certain bar constructions of commutative algebras, and further developed by N. Roby [Rob65]. A divided power algebra is a commutative algebra endowed with additional operations γ_n which satisfy specific relations. An algebra A over $\mathbb{Q}$ is naturally equipped with a structure of divided powers algebra such that $\gamma_n(x) = \frac{x^n}{n!}$ for all $x \in A$. Without any restriction on the ground ring, H. Cartan proves in [Car56] that the homotopy of a simplicial commutative algebra comes naturally equipped with the structure of a divided power algebra. The notion of divided power algebra is essential in the theory of crystalline cohomology for schemes introduced by A. Grothendieck [Gro68] and developed by P. Berthelot [Ber74]. The first author studied Quillen cohomology of divided power algebras [Dok09, Dok23].

Operads

Operads are an algebraic device which represent types of algebras, such as associative algebras, commutative algebras, and Lie algebras. The theory of operads has proven to be a powerful tool in studying different categories of algebras in a unified way. For instance, in the seminal book [LV12], J.-L. Loday and B. Vallette make a detailed account of many known homological and homotopical theories for algebras over an operad. This includes a notion of Quillen (co)homology for algebras over an operad, studied in [Liv98, Fre98, Mil11, GH00]. In the quadratic case, this is studied through a notion of Koszul duality due to V. Ginzburg and M. Kapranov [GK94] (see also [GJ94, Fre04]). It also includes a notion of deformation theory due to M. Gerstenhaber [Ger64] (see [Bal97, KS00, Kel05] for the operadic account), which led the way to the solution of the Deligne conjecture by D. Tamarkin [Tam98].

The notion of divided power algebras over an operad was introduced by B. Fresse in [Fre00]. While usual algebras over an operad $\mathcal{P}$ are algebras over a monad $S(\mathcal{P})$ built from $\mathcal{P}$ using coinvariant operations, divided power algebras over $\mathcal{P}$ are algebras over a monad $\Gamma(\mathcal{P})$ which is built using *invariant* operations. This recovers the classical notion of divided power algebras of Cartan, and Fresse shows that it also encompasses the notion of restricted Lie algebra of Jacobson. Furthermore, Fresse generalises Cartan's result by showing that the homotopy of a simplicial algebra over $\mathcal{P}$ is always equipped with a structure of divided power algebra over $\mathcal{P}$. Divided power algebras were further studied by the third author, who obtained a convenient characterisation for these objects using monomial operations [Iko20].

The aim of this article is to study the Quillen cohomology of divided power algebras over any operad $\mathcal{P}$, using the techniques due to Quillen, Barr and Beck. This generalises work of the first author [Dok04, Dok09, Dok23].

Outline and main results

Sections 2 and 3 are background sections. In Section 2, we recall the basic constructions leading to Quillen homology and cohomology. In Section 3, we recall a characterisation for divided power algebras obtained by the third author in [Iko20] which we will use throughout the article. As a preliminary useful result, we obtain the following:

Proposition (see Proposition 3.4)**.** *The structure of a $\Gamma(\mathcal{P})$-algebra is entirely determined by monomial operations whose monomial degrees are powers of the characteristic of the base field.*

The rest of the article is divided into three parts.

I. General theory. In Sections 4 to 8, we develop the general notion of Quillen cohomology for $\Gamma(\mathcal{P})$-algebras.

II. Examples. In Sections 9 and 10, we apply these constructions to the examples of classical divided power algebras, and of restricted Lie algebras, and recover known results.

III. Comparisons. In Sections 11 to 15, we investigate different comparison maps induced by adjunctions, which allow us to compare our cohomology for $\Gamma(\mathcal{P})$-algebras to more usual cohomology theories in certain categories of algebras.

In Section 4, we introduce a new notion of module over a $\Gamma(\mathcal{P})$-algebra and of abelian $\Gamma(\mathcal{P})$-algebra, which corresponds to a module over the trivial (or terminal) $\Gamma(\mathcal{P})$-algebra. Section 5 is devoted to the proof of the following result:

Theorem (see Theorem 5.2)**.** *The data of a Beck module over the $\Gamma(\mathcal{P})$-algebra A is equivalent to the data of an A-module as in Definition 4.5.*

One side of the equivalence is obtained by building a new notion of semidirect product for A-modules.

In Section 6, we build a ring $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ which represents the operations defining the notion of an A-module. We obtain the following result:

Theorem (See Theorem 6.11)**.** *The category of A-modules is equivalent to the category of left modules over $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$.*

In Section 7, we identify Beck derivations in the category of A-modules, and we build an A-module $\Omega_{\Gamma(\mathcal{P})}(A)$ which represents these derivations.

Section 8 concludes our general theory by identifying the left adjoint functor to the functor which sends an A module M to the $\Gamma(\mathcal{P})$-algebra over A obtained by the semidirect product $A \ltimes M$ of M by A. More precisely, we obtain:

Theorem (See Theorem 8.2)**.** *The following two functors form an adjoint pair:*

$$\Gamma(\mathcal{P})\text{-Alg}/A \underset{A\ltimes -}{\overset{\mathbb{U}_{\Gamma(\mathcal{P})}(A)\otimes_{\mathbb{U}_{\Gamma(\mathcal{P})}(-)}\Omega_{\Gamma(\mathcal{P})}(-)}{\rightleftarrows}} A\text{-Mod}.$$

This induces an analogue of Quillen's cotangent complex, and by deriving the simplicial extension of this left adjoint functor, we obtain the desired Quillen homology of A.

In Sections 9 and 10, we reconcile our general theory with the study of classical divided power algebras and of restricted Lie algebras by the first author in [Dok04, Dok09, Dok15, Dok23].

In the remaining sections of the article, we investigate the different comparison maps induced by certain adjunctions involving categories of divided power algebras. We focus on two types of such adjunctions. First, for any operad $\mathcal{P}$, there is an adjunction between the category of $\mathcal{P}$-algebras and the category of $\Gamma(\mathcal{P})$-algebras. Second, any morphism of operads $f\colon \mathcal{P} \to \mathcal{Q}$ induces an adjunction between the categories of $\Gamma(\mathcal{P})$- and $\Gamma(\mathcal{Q})$-algebras, similar to the extension/restriction of scalars adjunction for modules over a ring. In Section 11, we describe the comparison maps induced by both types of adjunctions in a general setting.

In Section 12, we study the first type of adjunction for the operad Com of commutative algebras, which produces comparison maps between the usual Quillen cohomology of a commutative algebra, and the Quillen cohomology of a certain associated divided power algebra. Similarly, in Section 13, we produce a comparison map between the usual Quillen cohomology of a Lie algebra, and the Quillen cohomology of a certain associated restricted Lie algebra.

In Section 14, we study an adjunction between associative algebras and restricted Lie algebras which is induced by an injection of the operad of Lie algebras into the operad of associative algebras. We obtain the following:

Theorem (See Theorem 14.2). *Let L be a restricted Lie algebra and M a $u(L)$-bimodule. Then there is an isomorphism*

$$\mathrm{HQ}^*_{\mathrm{As}}(u(L), M) \cong \mathrm{HQ}^*_{\mathrm{RLie}}(L, {}_{u(L)}M).$$

Here, $u(L)$ denotes an analogue of the universal enveloping algebra for restricted Lie algebras, $\mathrm{HQ}^*_{\mathrm{As}}$ is the usual Quillen cohomology for associative algebras, and $\mathrm{HQ}^*_{\mathrm{RLie}}$ is the Quillen cohomology for restricted Lie algebras. This result is analogous to a classical theorem of Cartan and Eilenberg, which gives an isomorphism between Chevalley–Eilenberg cohomology of a Lie algebra L and Hochschild cohomology of its universal enveloping algebra $U(L)$ [CE56, §XIII, Theorem 5.1].

Finally, in Section 15, we come back to the case were the base field is of characteristic 0, and we study the case of a good triple of operads $(\mathcal{C}, \mathcal{A}, \mathcal{P})$. In this case, the notions of $\mathcal{P}$-algebras and $\Gamma(\mathcal{P})$-algebras coincide. Good triples of operads generalise the mutual behaviours of the operads of commutative, of associative, and of Lie algebras. In particular, we see the operad $\mathcal{P}$ as a suboperad $\mathcal{A}$ of a certain type. In this setting, we can generalise the result of the previous section, and we obtain:

Theorem (See Theorem 15.2). *Let $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ be a good triple of operads. Let P be an $\mathcal{P}$-algebra and M a Beck $U(P)$-module. Then we have the following isomorphism*

$$\mathrm{HQ}^*_{\mathcal{A}\text{-Alg}}(U(P), M) \simeq \mathrm{HQ}^*_{\mathcal{P}\text{-Alg}}(P, {}_P M).$$

Here again, $U(P)$ denotes a certain notion of universal enveloping $\mathcal{A}$-algebra, $\mathrm{HQ}^*_{\mathcal{A}\text{-Alg}}$ is the Quillen cohomology for $\mathcal{A}$-algebras, and $\mathrm{HQ}^*_{\mathcal{P}\text{-Alg}}$ is the Quillen cohomology for $\mathcal{P}$-algebras. If we consider the good triple (Com, As, Lie), then Theorem 15.2 recovers the Cartan–Eilenberg result in characteristic zero.

While this last section focusses on the characteristic zero setting, there is evidence that the notion of good triple of operads can be generalised in positive characteristic, and we expect that our result still holds under the correct assumptions. Formulating these assumptions is—to our knowledge—an open problem for future consideration.

Conventions and notations

All the operads considered in this article will be algebraic, symmetric operads. We assume the reader has a good familiarity with this notion of operad. For more details, we refer to [LV12].

For this whole article, we fix a base field $\mathbb{F}$ and an operad $\mathcal{P}$ in $\mathbb{F}$-vector spaces that is reduced, i.e., satisfying $\mathcal{P}(0) = 0$. A $\Gamma(\mathcal{P})$-algebra will be generically denoted A.

Notation 1.1. We will denote:

- $\mathrm{Vect}_{\mathbb{F}}$ the category of $\mathbb{F}$-vector spaces.
- $\mathcal{P}$-Alg the category of $\mathcal{P}$-algebras in $\mathrm{Vect}_{\mathbb{F}}$, with forgetful functor $U^{\mathcal{P}}_{\mathbb{F}} \colon \mathcal{P}\text{-Alg} \to \mathrm{Vect}_{\mathbb{F}}$ and its left adjoint $F^{\mathcal{P}}_{\mathbb{F}} \colon \mathrm{Vect}_{\mathbb{F}} \to \mathcal{P}\text{-Alg}$.

- $\Gamma(\mathcal{P})$-Alg the category of $\Gamma(\mathcal{P})$-algebras in $\mathrm{Vect}_{\mathbb{F}}$, with forgetful functor $U_{\mathcal{P}}^{\Gamma(\mathcal{P})} \colon \Gamma(\mathcal{P})\text{-Alg} \to \mathcal{P}\text{-Alg}$ and its left adjoint $F_{\mathcal{P}}^{\Gamma(\mathcal{P})} \colon \mathcal{P}\text{-Alg} \to \Gamma(\mathcal{P})\text{-Alg}$.

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2. Preliminaries on Quillen cohomology

Throughout this section, let $\mathcal{C}$ be an algebraic category. More details about the kinds of categories we may consider are given in [Fra15, §2]. For the purposes of this paper, the categories of $\mathcal{P}$-algebras and $\Gamma(\mathcal{P})$-algebras are examples of algebraic categories, in fact, the one-sorted kind: sets equipped with certain operations satisfying certain equations. Let us recall some terminology, also found in [Bar96] and [Fra15, §1.3].

Definition 2.1. For an object X of $\mathcal{C}$, a **Beck module** over X is an abelian group object in the slice category $\mathcal{C}/X$. The category of Beck modules over X is denoted $(\mathcal{C}/X)_{\mathrm{ab}}$.

Definition 2.2. The **abelianization** over X is the functor $Ab_X \colon \mathcal{C}/X \to (\mathcal{C}/X)_{\mathrm{ab}}$ which is left adjoint to the forgetful functor $U_X \colon (\mathcal{C}/X)_{\mathrm{ab}} \to \mathcal{C}/X$.

Definition 2.3. For a map $f \colon X \to Y$ in $\mathcal{C}$, the pullback functor $f^* \colon \mathcal{C}/Y \to \mathcal{C}/X$ preserves limits and thus induces a functor $f^* \colon (\mathcal{C}/Y)_{\mathrm{ab}} \to (\mathcal{C}/X)_{\mathrm{ab}}$, also called **pullback**. The **pushforward** along f is the left adjoint $f_! \colon (\mathcal{C}/X)_{\mathrm{ab}} \to (\mathcal{C}/Y)_{\mathrm{ab}}$ of f^*.

Lemma 2.4. *Given a map $f \colon X \to Y$ in $\mathcal{C}$ viewed as an object of the slice category $\mathcal{C}/Y$, its abelianization is given by*

$$Ab_Y(X \xrightarrow{f} Y) \cong f_!(Ab_X X)$$

where $Ab_X X$ is shorthand for $Ab_X(X \xrightarrow{\mathrm{id}} X)$.

Definition 2.5. Given a Beck module $\mathrm{pr} \colon E \twoheadrightarrow X$ over X, a (Beck) **derivation** from X to E is a section of pr. The set of derivations is denoted

$$\mathrm{Der}(X, E) := \mathrm{Hom}_{\mathcal{C}/X}(X \xrightarrow{\mathrm{id}} X, E \xrightarrow{\mathrm{pr}} X) \cong \mathrm{Hom}_{(\mathcal{C}/X)_{\mathrm{ab}}}(Ab_X X, E \xrightarrow{\mathrm{pr}} X).$$

Note that $\mathrm{Der}(X, E)$ is canonically an abelian group.

The module of **Kähler differentials** of X is $\Omega_{\mathcal{C}}(X) := Ab_X X$, which represents derivations.

Definition 2.6. The **cotangent complex** $\mathbf{L}_X$ of X is the derived abelianization of X, i.e., the simplicial module over X given by $\mathbf{L}_X := Ab_X(C_\bullet \to X)$, where $C_\bullet \to X$ is a cofibrant replacement of X in $s\mathcal{C}$.

Definition 2.7. The n^{th} **Quillen homology** module of X is the n^{th} (simplicially) derived functor of abelianization, given by $\mathrm{HQ}_n(X) := \pi_n(\mathbf{L}_X)$. Note that $\mathrm{HQ}_n(X)$ is a Beck module over X.

Definition 2.8. The n^{th} **Quillen cohomology** group of X with coefficients in a module M is the n^{th} (simplicially) derived functor of derivations, given by $\mathrm{HQ}^n(X; M) := \pi^n \mathrm{Hom}(\mathbf{L}_X, M)$. Note that $\mathrm{HQ}^n(X; M)$ is an abelian group.

Example 2.9. Assuming $\mathcal{C}$ is a one-sorted algebraic category, consider the underlying set functor $U = U_{\mathrm{Set}}^{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Set}$ and its left adjoint $F = F_{\mathrm{Set}}^{\mathcal{C}} \colon \mathrm{Set} \to \mathcal{C}$. Iterating the free-of-forget comonad FU yields the standard augmented simplicial object

$$C_\bullet := (FU)^{\bullet+1}(X) \to X,$$

which is a cofibrant replacement of X in $s\mathcal{C}$. This was the original approach used by Barr and Beck [BB69], and by M. André in his work on André–Quillen cohomology [And67, And74].

3. Recollections on divided power algebras

In this section, we review the general notion of a divided power algebra over an operad $\mathcal{P}$, which are also called $\Gamma(\mathcal{P})$-algebras, due to Fresse [Fre00]. Throughout this article, we use the characterisation of divided power algebras in terms of monomial operations due to the third author [Iko20]. We rely heavily on the notation introduced in [Iko20] for those operations, and operations on partitions of integers.

Recall that an operad $\mathcal{P}$ is a sequence $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ such that for all n, $\mathcal{P}(n)$ is an $\mathbb{F}$-linear representation of the symmetric group $\mathfrak{S}_n$ on n letters. An operad is called reduced when $\mathcal{P}(0)=0$, and Fresse showed the following:

Proposition 3.1 ([Fre00, §1.1.18]). *For a reduced operad $\mathcal{P}$, the endofunctor $\Gamma(\mathcal{P})$ in vector spaces defined on objects by:*

$$\Gamma(\mathcal{P},V)=\bigoplus_{n>0}(\mathcal{P}(n)\otimes V^{\otimes n})^{\mathfrak{S}_n}$$

is equipped with a monad structure.

Here, $(\mathcal{P}(n)\otimes V^{\otimes n})^{\mathfrak{S}_n}$ stands for the module of invariant under the diagonal action of $\mathfrak{S}_n$, where the action of $\mathfrak{S}_n$ on $V^{\otimes n}$ is the usual permutation of factors.

Definition 3.2. A **divided power $\mathcal{P}$-algebra**, or **$\Gamma(\mathcal{P})$-algebra**, is an algebra over the resulting monad $\Gamma(\mathcal{P})$.

Theorem 3.3 ([Iko20]). *A divided power algebra over a reduced operad $\mathcal{P}$ is a vector space A endowed with a family of operations $\beta_{x,\underline{r}}\colon A^{\times s}\to A$, given for all $\underline{r}=(r_1,\dots,r_s)$ such that $r_1+\cdots+r_s=n$ and $x\in\mathcal{P}(n)^{\mathfrak{S}_{\underline{r}}}$, and which satisfy the relations:*

- *($\beta1$) $\beta_{x,\underline{r}}((a_i)_i)=\beta_{\rho^*\cdot x,\underline{r}^\rho}((a_{\rho^{-1}(i)})_i)$ for all $\rho\in\mathfrak{S}_s$, where ρ^* denotes the block permutation with blocks of size (r_i) associated to ρ.*
- *($\beta2$) $\beta_{x,(0,r_1,r_2,\dots,r_s)}(a_0,a_1,\dots,a_s)=\beta_{x,(r_1,r_2,\dots,r_s)}(a_1,\dots,a_s)$.*
- *($\beta3$) $\beta_{x,\underline{r}}(\lambda a_1,a_2,\dots,a_s)=\lambda^{r_1}\beta_{x,\underline{r}}(a_1,\dots,a_s)\quad\forall\lambda\in\mathbb{F}$.*
- *($\beta4$) If $r_1+\cdots+r_s=n$ and $q_1+\cdots+q_{s'}=s$, then*
$$\beta_{x,\underline{r}}(\underbrace{a_1,\dots,a_1}_{q_1},\underbrace{a_2,\dots,a_2}_{q_2},\dots,\underbrace{a_s,\dots,a_s}_{q_{s'}})=\beta_{\left(\sum_{\sigma\in\mathfrak{S}_{\underline{q}\triangleright\underline{r}}/\mathfrak{S}_{\underline{r}}}\sigma\cdot x\right),\ \underline{q}\triangleright\underline{r}}(a_1,a_2,\dots,a_s).$$
- *($\beta5$) $\beta_{x,\underline{r}}(a_0+a_1,\dots,a_s)=\sum_{l+m=r_1}\beta_{x,\underline{r}\circ_1(l,m)}(a_0,a_1,\dots,a_s)$.*
- *($\beta6$) $\beta_{\lambda x+y,\underline{r}}=\lambda\beta_{x,\underline{r}}+\beta_{y,\underline{r}}$, for all $x,y\in\mathcal{P}(n)^{\mathfrak{S}_{\underline{r}}}$,*
- *($\beta7$) $\beta_{1_{\mathcal{P}},(1)}(a)=a\quad\forall a\in A$.*
- *($\beta8$) Let $r_1+\cdots+r_s=n$, $x\in\mathcal{P}(n)^{\mathfrak{S}_{\underline{r}}}$ and for all $i\in[s]$, let $q_{i,1}+\cdots+q_{i,k_i}=m_i$, $x_i\in\mathcal{P}(m_i)^{\mathfrak{S}_{\underline{q}_i}}$ and $(b_{ij})_{1\le j\le k_i}\in A^{\times k_i}$. Denote by $b=(b_{ij})_{i\in[s],j\in[k_i]}$ and for all $i\in[s]$, $b_i=(b_{i,j})_{j\in[k_i]}$. Then:*
$$\beta_{x,\underline{r}}(\beta_{x_1,\underline{q}_1}(b_1),\dots,\beta_{x_s,\underline{q}_s}(b_s))=\beta_{\sum_\tau\tau\cdot x\left(x_1^{\times r_1},\dots,x_s^{\times r_s}\right),\underline{r}\diamond(\underline{q}_i)_{i\in[s]}}(b),$$
where $\underline{r}\diamond(\underline{q}_i)_{i\in[s]}$ is defined in [Iko20], where $\beta_{\cdot,\underline{r}\diamond(\underline{q}_i)_{i\in[s]}}$ is defined in [Iko20] and where τ ranges over $\mathfrak{S}_{\underline{r}\diamond(q_i)_{i\in[s]}}/(\prod_{i=1}^s\mathfrak{S}_{r_i}\wr\mathfrak{S}_{\underline{q}_i})$ in the sum.

We can slightly refine this characterisation if we specify the characteristic of the base field:

Proposition 3.4. *Let A be a $\Gamma(\mathcal{P})$-algebra over a field $\mathbb{F}$ of characteristic p. Then, the operations $\beta_{x,\underline{r}}$ are generated by the family of operations $\beta_{x,\underline{r}}$ such that all the r_is are powers of p.*

Proof. Let $n \in \mathbb{N}$, $r_1 + \cdots + r_s = n$, and $x \in \mathcal{P}(n)^{\mathfrak{S}_{\underline{r}}}$. For each $i \in [s]$, denote $r_i = \sum_{j=0}^{k_{r_i}} r_{i,j} p^j$ the mod p expansion of r_i. For all $i \in [s]$ and $j \in \{0, \dots, k_i\}$, denote by $R_{ij} = r_{i,j} p^j$. We get a partition $\underline{R} = (R_{1,0}, \dots, R_{1,k_{r_1}}, R_{2,0}, \dots, R_{s,k_s})$ of $[n]$ into $k_{r_1} + \cdots + k_{r_s} + s$ integers. Note that since $\underline{R}$ is finer than $\underline{r}$, x is fixed by the action of $\mathfrak{S}_{\underline{R}}$. Since $x \in \mathcal{P}(n)^{\underline{r}}$, we obtain

$$\sum_{\sigma \in \mathfrak{S}_{\underline{r}}/\mathfrak{S}_{\underline{R}}} x = \left(\prod_{i=1}^{s} \binom{r_i}{r_{i,0}p^0, \dots, r_{i,k_{r_i}} p^{k_{r_i}}} \right) x.$$

Using Lucas's Theorem, the product $\prod_{i=1}^{k} \binom{r_i}{r_{i,0}p^0, \dots, r_{i,k_{r_i}} p^{r_{k_i}}}$ is equal to 1 modulo p. Using relation ($\beta 4$), we then get:

$$\beta_{x,\underline{r}}(a_1, \dots, a_s) = \beta_{x,\underline{R}}(\underbrace{a_1, \dots, a_1}_{k_{r_1}+1}, \dots, \underbrace{a_s, \dots, a_s}_{k_{r_s}}).$$

Now, consider the partition

$$\underline{Q} := (\underbrace{p^0, \dots, p^0}_{r_{1,0}}, \dots, \underbrace{p^{k_{r_1}}, \dots, p^{k_{r_1}}}_{r_{1,k_{r_1}}}, \dots, \underbrace{p^{k_{r_s}}, \dots, p^{k_{r_s}}}_{r_{s,k_{r_s}}}).$$

Since $\underline{Q}$ is finer than $\underline{R}$, x is stable under the action of $\mathfrak{S}_{\underline{Q}}$. Using Lucas's Theorem again, we get:

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_{\underline{R}}/\mathfrak{S}_{\underline{Q}}} \sigma x &= \left(\prod_{i=1}^{s} \prod_{j=1}^{k_{r_i}} \binom{r_{i,j} p^j}{p^j, \dots, p_j} \right) x \\ &= \left(\prod_{i=1}^{s} \prod_{j=1}^{k_{r_i}} r_{i,j}! \right) x. \end{aligned}$$

Since $r_{i,j} < p$ for all i, j, $r_{i,j}!$ is invertible modulo p. So, we can write:

$$x = \left(\prod_{i=1}^{s} \prod_{j=1}^{k_{r_i}} (r_{i,j}!)^{-1} \right) \sum_{\sigma \in \mathfrak{S}_{\underline{R}}/\mathfrak{S}_{\underline{Q}}} \sigma x,$$

and so, using relation ($\beta 8$),

$$\beta_{x,\underline{R}}(\underbrace{a_1, \dots, a_1}_{k_{r_1}+1}, \dots, \underbrace{a_s, \dots, a_s}_{k_{r_s}}) = \left(\prod_{i=1}^{s} \prod_{j=1}^{k_{r_i}} (r_{i,j}!)^{-1} \right) \beta_{x,\underline{Q}}(\underbrace{a_1, \dots, a_1}_{r_{1,0}+\cdots+r_{1,k_{r_1}}}, \dots, \underbrace{a_s, \dots, a_s}_{r_{s,0}+\cdots+r_{s,k_{r_s}}}).$$

Here, the integers in $\underline{Q}$ are either 0s or powers of p. Using relation ($\beta 2$) allows us to remove the 0s, and we obtain the result. $\square$

4. Abelian $\Gamma(\mathcal{P})$-algebras, A-modules

In order to obtain an analogue of the André–Quillen cohomology of operadic algebras [LV12, §12.3] on divided power algebras, we will study the notion of a module over a divided power algebra, which is obtained by looking at certain abelian (or square-zero) extensions. In the setting of usual (non-unital) $\mathbb{F}$-algebras, since any vector space can be equipped with trivial algebra structure, the category of abelian (square-zero) $\mathbb{F}$-algebras coincide with the category of $\mathbb{F}$-vector spaces. An A-module then becomes a vector space equipped with an A-action. However, we will see that, in the category of divided power algebras, abelian $\Gamma(\mathcal{P})$-algebras correspond to vector spaces equipped with additional internal operations, and therefore, requires a separate definition. This section contains our candidates for the definitions of abelian $\Gamma(\mathcal{P})$-algebras and of A-modules, and we will show in the next section that these are indeed the right definitions (see Theorem 5.2, Corollary 5.4).

Definition 4.1. An **abelian** $\Gamma(\mathcal{P})$-algebra is a $\Gamma(\mathcal{P})$-algebra M such that all the operations $\beta_{x,\underline{r}}$ are trivial as soon as $\underline{r}$ contains two non-zero integers. Abelian $\Gamma(\mathcal{P})$-algebras with $\Gamma(\mathcal{P})$-algebra morphisms form a category $(\Gamma(\mathcal{P})\text{-Alg})_{\mathrm{Ab}}$.

Proposition 4.2. *Equivalently, an abelian $\Gamma(\mathcal{P})$-algebra is a vector space M equipped, for all $n \in \mathbb{N}$ and $x \in \mathcal{P}(n)^{\mathfrak{S}_n}$, with an operation (a set map) $\beta_x \colon M \to M$ satisfying:*

(Abβ3) $\beta_x(\lambda m) = \lambda^n \beta_x(m)$,

(Abβ4) Suppose there is $\underline{r} \in \mathrm{Comp}_s(n)$, with $s > 1$ and $\underline{r}$ contains at least two non-zero integers, and $y \in \mathcal{P}(n)$ such that $x = \sum_{\sigma \in \mathfrak{S}_n/\mathfrak{S}_{\underline{r}}} y$. Then $\beta_x = 0$,

(Abβ5) $\beta_x(m_1 + m_2) = \beta_x(m_1) + \beta_x(m_2)$,

(Abβ6) $\beta_{\lambda x + y}(m) = \lambda \beta_x(m_1) + \beta_y(m_2)$,

(Abβ7) $\beta_{1_{\mathcal{P}}}(m) = m$,

(Abβ8) $\beta_x(\beta_y(m)) = \beta_{\sum_\tau \tau\cdot x(y,\ldots,y)}(m)$*, where $y \in \mathcal{P}(l)$ and τ ranges over $\mathfrak{S}_{nl}/\mathfrak{S}_n \wr \mathfrak{S}_l$.*

Proof. Let M be a $\Gamma(\mathcal{P})$ algebra such that all the operations $\beta_{x,\underline{r}}$ are trivial as soon as $\underline{r}$ contains two non-zero integers. For all $x \in \mathcal{P}(n)^{\mathfrak{S}_n}$, denote by $\beta_x := \beta_{x,(n)}$. We will prove that relations (Abβ3) to (Abβ8) are satisfied:

(Abβ3) is a direct consequence of (β3),

(Abβ4) is deduced from (β4) in the following way: if there is $\underline{r} \in \mathrm{Comp}_s(n)$ with $s > 1$ and $\underline{r}$ contains at least two non-zero integers, and $y \in \mathcal{P}(n)$ such that $x = \sum_{\sigma \in \mathfrak{S}_n/\mathfrak{S}_{\underline{r}}} y$, then

$$\beta_x(m) = \beta_{x,(n)}(m) = \beta_{y,\underline{r}}(m, \ldots, m) = 0.$$

(Abβ5) is deduced from (β5) and (β2) in the following way:

$$\begin{aligned}\beta_x(m + m') = \beta_{x,(n)}(m + m') &= \sum_{i+j=n} \beta_{x,(i,j)}(m, m') = \beta_{x,(n,0)}(m, m') + \beta_{x,(0,n)}(m, m') \\ &= \beta_x(m) + \beta_x(m').\end{aligned}$$

(Abβ6) is a direct consequence of (β6),

(Abβ7) is a direct consequence of (β7),

(Abβ8) is a direct consequence of (β8).

Let now M be a vector space equipped, for all $x \in \mathcal{P}(n)^{\mathfrak{S}_n}$, with an operation $\beta_x \colon M \to M$, satisfying the relations (Abβ3) to (Abβ8). For all $x \in \mathcal{P}(n)^{\mathfrak{S}_n}$ and $\underline{r} \in \mathrm{Comp}_s(n)$, define an s-ary operation $\beta_{x,\underline{r}}$, such that:

$$\beta_{x,\underline{r}}(m_1, \ldots, m_s) = \begin{cases} \beta_x(m_i) & \text{if } r_i = n, \\ 0 & \text{if } \underline{r} \text{ contains at least two non-zero integers.} \end{cases}$$

We want to show that $\beta_{x,\underline{r}}$ satisfy the relations (β1) to (β8). Relations (β1) and (β2), are a direct consequence of the definition. Relations (β3), (β5), (β6), (β7) are direct consequences respectively of relations (Abβ3), (Abβ5), (Abβ6), and (Abβ7).

To prove relation (β4), note that the left term of the equality,

$$\beta_{x,\underline{r}}(\underbrace{a_1, \ldots, a_1}_{q_1}, \underbrace{a_2, \ldots, a_2}_{q_2}, \ldots, \underbrace{a_s, \ldots, a_s}_{q_{s'}}),$$

is equal to 0 except if $r_i = n$ and $q_j = m$ for certain i and j, in which case, it is equal to $\beta_x(a_i)$. On the other hand, the right term of the equality,

$$\beta_{\left(\sum_{\sigma \in \mathfrak{S}_{\underline{q} \triangleright \underline{r}}/\mathfrak{S}_{\underline{r}}} \sigma\cdot x\right),\ \underline{q} \triangleright \underline{r}}(a_1, a_2, \ldots, a_s),$$

is 0 unless $\underline{q} \triangleright \underline{r}$ contains a single non-zero integers. But this happens if and only if both $\underline{r}$ and $\underline{q}$ contain a single non-zero integer, namely, if there is an i and a j such that $r_i = n$ and $q_j = m$, and if so, $\mathfrak{S}_{\underline{q} \triangleright \underline{r}} / \mathfrak{S}_{\underline{r}}$ is trivial, so this term is also equal to $\beta_x(a_i)$.

To prove relation ($\beta 8$), note that the left term in the equality,

$$\beta_{x,\underline{r}}(\beta_{x_1,\underline{q}_1}(b_1), \dots, \beta_{x_s,\underline{q}_s}(b_s)),$$

Is 0 unless $r_i = n$, $q_{i,j} = m_i$, and in this case, this term is equal to $\beta_x(\beta_{x_i}(b_{i,j}))$. On the other hand, the right term in the equality,

$$\beta_{\sum_\tau \tau \cdot x\left(x_1^{\times r_1}, \dots, x_s^{\times r_s}\right)), \underline{r} \diamond (\underline{q}_i)_{i \in [s]}}(b),$$

Is equal to 0 unless $\underline{r} \diamond (\underline{q}_i)_{i \in [s]}$ has a single non-zero integer. But again, this happens if and only if there exists an i such that $r_i = n$, a j such that $q_{i,j} = m_i$, and in this case, $\underline{r} \diamond (\underline{q}_i)_{i \in [s]} = (nm_i)$,$\mathfrak{S}_{\underline{r} \diamond (q_i)_{i \in [s]}} / (\prod_{i=1}^s \mathfrak{S}_{r_i} \wr \mathfrak{S}_{\underline{q}_i}) = \mathfrak{S}_{nm_i} / \mathfrak{S}_n \wr \mathfrak{S}_{m_i}$, and so, this term is equal to

$$\beta_{\sum_\tau \tau x(x_i, \dots, x_i)}(b_{i,j}),$$

where τ ranges over $\mathfrak{S}_{nm_i} / \mathfrak{S}_n \wr \mathfrak{S}_{m_i}$ in the sum.

We have left to prove that

$$\beta_x(\beta_{x_i}(b_{i,j})) = \beta_{\sum_\tau \tau x(x_i, \dots, x_i)}(b_{i,j}),$$

where τ ranges over $\mathfrak{S}_{nm_i} / \mathfrak{S}_n \wr \mathfrak{S}_{m_i}$ in the sum, but that is exactly relation (Abβ8). □

Proposition 4.3. *Let M be an abelian $\Gamma(\mathcal{P})$-algebra over a field $\mathbb{F}$ of characteristic p. Then $\beta_x(m) = 0$ for all $x \in \mathcal{P}(n)^{\mathfrak{S}_n}$, for all $n \neq p^i$, $i \in \mathbb{N}$.*

Proof. This is similar to the proof of 3.4. Denote by $n_0 p^0 + \cdots + n_k p^k$ the mod p expansion of n. Denote by $\underline{Q} = (\underbrace{p^0, \dots, p^0}_{n_0}, \dots, \underbrace{p^k, \dots, p^k}_{n_k})$. Denote by $y = (\prod_{j=0}^k n_j!)^{-1}$. Using Lucas's Theorem, one has:

$$\sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{\underline{Q}}} \sigma \cdot y = \left(\prod_{j=0}^k \binom{n_j p^j}{p^j, \dots, p^j} \right) y = \left(\prod_{j=0}^k n_j! \right) y = x.$$

Either n is a power of p, or $\underline{Q}$ has at least two non-zero integers, which, using relation (Abβ4), implies that $\beta_x = 0$. □

Corollary 4.4. *Let M be an abelian $\Gamma(\mathcal{P})$-algebra over a field $\mathbb{F}$ of characteristic p. Then, the operations β_x are generated by the family of operations β_x such that $x \in \mathcal{P}(p^i)^{\mathfrak{S}_{p^i}}$.*

Definition 4.5. A **module** over the $\Gamma(\mathcal{P})$-algebra A is an abelian $\Gamma(\mathcal{P})$-algebra M equipped with a divided power action of A on M represented by operations $\beta_{x,\underline{r}}\colon A^{\times s-1} \times M \to M$ for all $x \in \mathcal{P}(n)$ and $\underline{r} \in \mathrm{Comp}_s(n)$ such that $r_s \neq 0$, satisfying:

(βAM6) $\beta_{\lambda x + y, \underline{r}} = \lambda \beta_{x,\underline{r}} + \beta_{y,\underline{r}}$, for all $x, y \in \mathcal{P}(n)^{\mathfrak{S}_{\underline{r}}}$,

(βAM7) If $s = 1$, then $\beta_{x,(n)}(m) = \beta_x(m)$.

(βAM8) Let $r_1 + \cdots + r_s = n$, $x \in \mathcal{P}(n)^{\mathfrak{S}_{\underline{r}}}$ and for all $i \in [s]$, let $q_{i,1} + \cdots + q_{i,u_i} = k_i$, $x_i \in \mathcal{P}(k_i)^{\mathfrak{S}_{\underline{q}_i}}$. Let $(b_{ij})_{1 \leq j \leq u_i} \in A^{\times u_i}$ for $i \in [s-1]$ and $b_{s,1}, \dots, b_{s,u_s - 1} \in A$. For all $i \in [s-1]$, denote by $b_i = (b_{i,j})_{j \in [u_i]}$. Then:

$$\beta_{x,\underline{r}}(\beta_{x_1,\underline{q}_1}(b_1), \dots, \beta_{x_{s-1},\underline{q}_{s-1}}(b_{s-1}), \beta_{x_s,\underline{q}_s}(b_{s,1}, \dots, b_{s,u_s-1}, m)) = \beta_{\sum_\tau \tau \cdot x(x_1^{r_1}, \dots, x_s^{\times r_s}), \underline{r} \diamond (\underline{q}_1, \dots, \underline{q}_s)}(b_{1,1}, \dots, b_{1,u_1}, \dots, b_{s,u_s-1}, m),$$

where $\underline{r} \diamond (\underline{q}_i)_{i \in [s]}$ is defined in [Iko20] and where τ ranges over $\mathfrak{S}_{\underline{r} \diamond \underline{q}} / \prod_{i=1}^s \mathfrak{S}_{r_i} \wr \mathfrak{S}_{\underline{q}_i})$ in the sum.

For $s>1$, the two sets of relations:

(βA1) $\beta_{x,\underline{r}}((a_i)_i,m)=\beta_{\rho^*\cdot x,\underline{r}^\rho}((a_{\rho^{-1}(i)})_i,m)$ for all $\rho\in\mathfrak{S}_{s-1}$, where ρ^* denotes the block permutation with blocks of size (r_i) associated to ρ.

(βA2) $\beta_{x,(0,r_1,r_2,\dots,r_s)}(a_0,a_1,\dots,a_{s-1},m)=\beta_{x,(r_1,r_2,\dots,r_s)}(a_1,\dots,a_{s-1},m)$.

(βA3) $\beta_{x,\underline{r}}(\lambda a_1,a_2,\dots,a_{s-1},m)=\lambda^{r_1}\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},m)\quad\forall\lambda\in\mathbb{F}$.

(βA4) If $\underline{r}\in\mathrm{Comp}_s(n)$ and $\underline{q}\in\mathrm{Comp}_{s'}(s-1)$, then

$$\beta_{x,\underline{r}}(\underbrace{a_1,\dots,a_1}_{q_1},\underbrace{a_2,\dots,a_2}_{q_2},\dots,\underbrace{a_{s-1},\dots,a_{s-1}}_{q_{s'}},m)=\beta_{\left(\sum_{\sigma\in\mathfrak{S}_{\underline{q}'\triangleright\underline{r}}/\mathfrak{S}_{\underline{r}}}\sigma\cdot x\right),\ \underline{q}'\triangleright\underline{r}}(a_1,a_2,\dots,a_{s-1},m),$$

where $\underline{q}'=(\underline{q},0)$,

(βA5) $\beta_{x,\underline{r}}(a_0+a_1,\dots,a_{s-1},m)=\sum_{l+l'=r_1}\beta_{x,\underline{r}\circ_1(l,l')}(a_0,a_1,\dots,a_{s-1},m)$,

and,

(βM3) $\beta_{x,\underline{r}}(a_1,a_2,\dots,a_{s-1},\lambda m)=\lambda^{r_1}\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},m)\quad\forall\lambda\in\mathbb{F}$.

(βM4) Suppose there is $\underline{q}\in\mathrm{Comp}_{s'}(r_s)$ with $s'>0$, where $\underline{q}$ contains at least two non-zero integers, and there is a $y\in\mathcal{P}(n)^{\mathfrak{S}_1^{\times n-r_s}\times\mathfrak{S}_{\underline{q}}}$ such that $x=\sum_{\sigma\in\mathfrak{S}_{n-r_s}\times\mathfrak{S}_{r_s}/\mathfrak{S}_1^{\times n-r_s}\times\mathfrak{S}_{\underline{q}}}y$, then $\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},m)=0$,

(βM5) $\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},m_1+m_2)=\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},m_1)+\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},m_2)$,

And the two additional relations:

(βAM4) Let $s,s'>0$, $\underline{q}\in\mathrm{Comp}_{s'}(n)$, and $\underline{r}\in\mathrm{Comp}_s(\underline{q})$. Let $x\in\mathcal{P}(n)^{\mathfrak{S}_{\underline{q}\circ_i\underline{r}}}$. Then,

$$\sum_{k=1}^{q_i}\beta_{\sum_{\rho\in\mathfrak{S}_1^{\times(q_1+\dots+q_{i-1})}\times\overline{E}_{\underline{r},k}\times\mathfrak{S}_1^{\times(q_{i+1}+\dots+q_{s'})}}\rho\cdot\sigma_i^*x,\underline{q}\circ_i(q_i-k,k)}(a_1,\dots,a_{i-1},a_{i+1},\dots,a_{s'},a_i,m)=0,$$

where σ_i^* is the block permutation for blocks of size $q_1,q_2,\dots,q_{s'}$ associated to the transposition of i and s' in $\mathfrak{S}_{s'}$, and where:

$$\overline{E}_{\underline{r},k}=\left\{\rho'\in Sh(\underline{r}):\left|\{\rho'(r_1+\dots+r_j):j\in[s]\}\cap\{q_i-k+1,\dots,q_i\}\right|\geq 2\right\},$$

where $Sh(\underline{r})$ is the set of $(r_1,\dots,r_s)$-shuffles.

(βAM9) Let $r_1+\dots+r_s=n$, $x\in\mathcal{P}(n)^{\mathfrak{S}_{\underline{r}}}$, let $q_1+\dots+q_u=k$, $y\in\mathcal{P}(k)^{\mathfrak{S}_{\underline{q}}}$. Let $(a_i)_{i\in[s+u]}\in A^{\times s+u}$ and $m\in M$. Denote by $z=x\left(1_{\mathcal{P}}^{\times n-r_s},y^{\times r_s}\right)\in\mathcal{P}(n+r_s(k-1))$ Then,

$$\sum_{t+t'=r_sq_u,t'>0}\beta_{\sum_\tau\tau z,\underline{r}\diamond_s(\underline{q})\circ_{s+u}(t,t')}(a_1,\dots,a_{s+u},m)=$$

$$\sum_{\lambda+\lambda'=r_s,\lambda'>0}\ \sum_{l+l'=q_u,l'>0}\beta_{\sum_{\sigma'}\sigma'z,R_{\lambda\lambda'}^{ll'}}(a_1,\dots,a_{s+u},m),$$

where, for all l,l',λ,λ', $R_{\lambda\lambda'}^{ll'}=((R_{ij})_{i\in[s],j\in[u_s]},R_{s,u+1})$ is the partition of $[n+r_s(k-1)]$ into $s+u+1$ parts such that $R_i=(\underline{r}\diamond_s(\underline{q}))_i$ for all $i\in[s+u-1]$, and such that:

$$R_{s+u}=\{n-r_s+k-q_u+\alpha k+\gamma,\alpha\in[\lambda],\gamma\in[q_u]\}\cup\{\lambda k+(k-q_u)+\alpha k+\gamma,\alpha\in[\lambda'],\gamma\in[l]\}.$$

$$R_{s+u+1}=\{n-r_s+\lambda k+(k-l')+\alpha k+\gamma,\alpha\in[\lambda'],\gamma\in[l']\}.$$

where τ ranges over $\mathfrak{S}_{\underline{r}\diamond_s\underline{q}}/\left(\prod_{i=1}^{s-1}\mathfrak{S}_{r_i}\right)\times\mathfrak{S}_{r_s}\wr\mathfrak{S}_{\underline{q}}$ in the sum, and where σ' ranges over $\mathfrak{S}_{R_{\lambda\lambda'}^{ll'}}/\left(\prod_{i=1}^{s-1}\mathfrak{S}_{r_i}\right)\times\mathfrak{S}_\lambda\wr\mathfrak{S}_{\underline{q}}\times\mathfrak{S}_{\lambda'}\wr\mathfrak{S}_{\underline{q}\circ_u(l,l')}$ in the sum.

Modules over a $\Gamma(\mathcal{P})$-algebra A form a category A-Mod.

Notation 4.6. We will allow the notation:

$$\beta_{x,\underline{r}}(a_1,\dots,a_{i-1},m,a_{i+1},\dots,a_s) := \beta_{\sigma_i^*,\underline{r}^{\sigma_i}}(a_1,\dots,a_{i-1},a_{i+1},\dots,a_s,m),$$

where $\sigma_i \in \mathfrak{S}_s$ sends i to the s-th spot and keeps all other indices in order, that is:

$$\sigma_i(j) = \begin{cases} j, & \text{if } j < i, \\ s, & \text{if } j = i, \\ j-1, & \text{if } j > i, \end{cases}$$

and where $\sigma_i^* \in \mathfrak{S}_n$ is the block permutation with blocks of size $(r_i)_{i\in[s]}$ associated to σ_i.

Once again, we can slightly refine this characterisation if we specify the characteristic of the base field:

Proposition 4.7. *Let M be an A-module over a field $\mathbb{F}$ of characteristic p. Then the operations $\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},m)$ are generated by the operations $\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},m)$ where all the r_is are powers of p, and $\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},m) = 0$ when r_s is not a power of p.*

Proof. The proof, very similar to that of Propositions 3.4 and 4.3, is omitted. □

5. Beck modules over A

This section is devoted to the notion of a Beck module over a $\Gamma(\mathcal{P})$-algebra A. The general notion of Beck module is equivalent to the notion of abelian extensions over an object (see [Bec67], [Qui70, §2–4], [Bar96, §6], or [Fra10, §A]). The main result of this section, Theorem 5.2, is that the notion of Beck module over a $\Gamma(\mathcal{P})$-algebra A corresponds to the notion of A-module from Definition 4.5. As a particular case of this result, we obtain Corollary 5.4, which shows that the notion of abelian $\Gamma(\mathcal{P})$-algebra from Definition 4.1 corresponds to the notion of an abelian object in the category of $\Gamma(\mathcal{P})$-algebras.

Remark 5.1. If $B \xrightarrow{pr} A$ and $C \xrightarrow{pr'} A$ are $\Gamma(\mathcal{P})$-algebras over A, we can endow $B \times_A C$ of a structure of $\Gamma(\mathcal{P})$-algebra over A. The map $B \times_A C \to A$ is given by $pr \times_A pr'$, and the evaluation map $\mathcal{P}\,\tilde{\circ}\,(B\times_A C) \to B\times_A C$ is given by universal property of the pullback, using the two maps $\mathcal{P}\,\tilde{\circ}\,(B\times_A C) \to \mathcal{P}\,\tilde{\circ}\,B \to B$ and $\mathcal{P}\,\tilde{\circ}\,(B\times_A C) \to \mathcal{P}\,\tilde{\circ}\,C \to C$. This means that for s elements $(b_1,c_1),\dots,(b_s,c_s,) \in B\times C$ such that $pr(b_i) = pr'(c_i)$,

$$\beta_{x,\underline{r}}\left((b_1,c_1),\dots,(b_s,c_s)\right) = \left(\beta_{x,\underline{r}}(b_1,\dots,b_s), \beta_{x,\underline{r}}(c_1,\dots,c_s)\right).$$

Theorem 5.2. *The data of a Beck module over the $\Gamma(\mathcal{P})$-algebra A is equivalent to the data of an A-module as in Definition 4.5.*

Proof. We will define a pair of functors:

$$\ker\colon (\Gamma(\mathcal{P})\text{-Alg}/A)_{\mathrm{ab}} \to A\text{-Mod},$$

and

$$A \ltimes -\colon A\text{-Mod} \to (\Gamma(\mathcal{P})\text{-Alg}/A)_{\mathrm{ab}},$$

and show that these are inverse to each other.
Let us prove that the data of a Beck module over A yields an A-module. Let $B \xrightarrow{pr} A$ be a $\Gamma(\mathcal{P})$-algebra over A. Denote by $M := \ker(p)$. The data of a Beck module structure on B is equivalent to the data of a map $A \xrightarrow{z} B$, and of a multiplication map $B\times_A B \xrightarrow{\mu} B$ over A, satisfying the following conditions:

(1) z is a map of $\Gamma(\mathcal{P})$-algebras over A,

(2) μ is a map of $\Gamma(\mathcal{P})$-algebras over A,

(3) μ is commutative,
(4) z is a unit map for the multiplication μ, that is,

$$\begin{tikzcd} B \times_A B \arrow[drr, "\mu"'] & A \times_A B \arrow[l, "z \times_A B"'] = B = B \times_A A \arrow[d, "B"] \arrow[r, "B \times_A z"] & B \times B \arrow[dl, "\mu"] \\ & B & \end{tikzcd}$$

Condition (1) implies that z splits pr, so we have a split exact sequence of vector spaces:

$$\mathbf{0} \longrightarrow M \longrightarrow B \underset{z}{\overset{pr}{\rightleftarrows}} A \longrightarrow \mathbf{0} ,$$

which provides us with a linear isomorphism $B \cong A \oplus M$.

The fact that pr is a map of $\Gamma(\mathcal{P})$-algebras implies that

$$pr(\beta_{x,\underline{r}}((a_1, m_1), \ldots, (a_s, m_s))) = \beta_{x,\underline{r}}(pr(a_1, m_1), \ldots, pr(a_s, m_s)) = \beta_{x,\underline{r}}(a_1, \ldots, a_s),$$

so there exists a set map $\nabla_{x,\underline{r}} \colon A^{\times s} \times M^{\times s} \to M$ such that:

$$\beta_{x,\underline{r}}((a_1, m_1), \ldots, (a_s, m_s)) = \left(\beta_{x,\underline{r}}(a_1, \ldots, a_s), \nabla_{x,\underline{r}}(a_1, \ldots, a_s, m_1, \ldots, m_s)\right).$$

Note that, as a vector space, $B \times_A B \cong A \oplus M \oplus M$. One can rewrite condition (4) as the equality $\mu(a, m, 0) = \mu(a, 0, m) = (a, m)$. Since μ is linear,

$$\mu(a, m, m') = \mu\left((a, m, 0) + (0, 0, m')\right) = \mu(a, m, 0) + \mu(0, 0, m') = (a, m) + (0, m') = (a, m + m'),$$

from which the commutativity of μ is in fact necessary.

Inspecting the $\Gamma(\mathcal{P})$-algebra structure on $B \times_A B$ (see previous section), one has:

$$\begin{aligned}\beta_{x,\underline{r}}\left((a_1, m_1, m'_1), \ldots, (a_s, m_s, m'_s)\right) = (\beta_{x,\underline{r}}(a_1, \ldots, a_s), \nabla_{x,\underline{r}}(a_1, \ldots, a_s, m_1, \ldots, m_s),& \\ \nabla_{x,\underline{r}}(a_1 \ldots, a_s, m'_1, \ldots, m'_s)).&\end{aligned}$$

On the one hand, this means that:

$$\begin{aligned}\mu\left(\beta_{x,\underline{r}}\left((a_1, m_1, m'_1), \ldots, (a_s, m_s, m'_s)\right)\right) = \mu\Bigg(\beta_{x,\underline{r}}(a_1, \ldots, a_s), \nabla_{x,\underline{r}}(a_1, \ldots, a_s, m_1, \ldots, m_s),& \\ \nabla_{x,\underline{r}}(a_1 \ldots, a_s, m'_1, \ldots, m'_s)\Bigg),&\end{aligned}$$

Which is equal to:

$$\Bigg(\beta_{x,\underline{r}}(a_1, \ldots, a_s), \nabla_{x,\underline{r}}(a_1, \ldots, a_s, m_1, \ldots, m_s) + \nabla_{x,\underline{r}}(a_1 \ldots, a_s, m'_1, \ldots, m'_s)\Bigg).$$

On the other hand, since μ is a $\Gamma(\mathcal{P})$-algebra map, one has:

$$\begin{aligned}\mu(\beta_{x,\underline{r}}\left((a_1, m_1, m'_1), \ldots, (a_s, m_s, m'_s)\right)) &= \beta_{x,\underline{r}}\left(\mu(a_1, m_1, m'_1), \ldots, \mu(a_s, m_s, m'_s)\right) \\ &= \beta_{x,\underline{r}}\left((a_1, m_1 + m'_1), \ldots, (a_s, m_s + m'_s)\right) \\ &= \left(\beta_{x,\underline{r}}, \nabla_{x,\underline{r}}(a_1, \ldots, a_s, m_1 + m'_1, \ldots, m_s + m'_s)\right),\end{aligned}$$

And hence,

$$\nabla_{x,\underline{r}}(a_1, \ldots, a_s, m_1 + m'_1, \ldots, m_s + m'_s) = \nabla_{x,\underline{r}}(a_1, \ldots, a_s, m_1, \ldots, m_s) + \nabla_{x,\underline{r}}(a_1 \ldots, a_s, m'_1, \ldots, m'_s).$$

This implies that

$$\nabla_{x,\underline{r}}(a_1, \ldots, a_s, m_1, \ldots, m_s) = \sum_{i=0}^{s} \nabla_{x,\underline{r}}(a_1, \ldots, a_s, 0, \ldots, 0, m_i, 0, \ldots, 0), \tag{E1}$$

and that

$$\nabla_{x,\underline{r}}(a_1,\dots,a_s,0,\dots,0)=0, \tag{E2}$$

which is also a consequence of condition (1). From relation ($\beta 3$), we also obtain:

$$\beta_{x,\underline{r}}((a_1,m_1),\dots,(a_{i-1},m_{i-1}),0,(a_{i+1},m_{i+1}),\dots,(a_s,m_s))=0,$$

for all i such that $r_i>0$, which also implies:

$$\nabla_{x,\underline{r}}(a_1,\dots,a_{i-1},0,a_{i+1},\dots,a_s,m_1,\dots,m_{i-1},0,m_{i+1},\dots,m_s)=0 \tag{E3}$$

Using the decomposition $(a,m)=(a,0)+(0,m)$ and the relation ($\beta 5$), one then gets:

$$\beta_{x,\underline{r}}\left((a_1,m_1),\dots,(a_s,m_s)\right)=\sum_{(l_i+l_i'=r_i)_i}\beta_{x,(l_1,l_1',l_2,\dots,l_s,l_s')}\left((a_1,0),(0,m_1),(a_2,0)\dots,(a_s,0),(0,m_s)\right).$$

Using relations ($\beta 3$) and ($\beta 2$), we then get:

$$\begin{aligned}\beta_{x,\underline{r}}\left((a_1,m_1),\dots,(a_s,m_s)\right)&=(\beta_{x,\underline{r}}(a_1,\dots,a_s),0)\\&+\sum_{(l_i+l_i'=r_i,l_i'>0)_i}\Big(0,\nabla_{x,(l_1,l_1',l_2,\dots,l_s,l_s')}(a_1,0,a_2,0,\dots,a_s,0,0,m_1,0,m_2,\dots,0,m_s)\Big),\end{aligned}$$

which, using (E1) is equal to:

$$\begin{aligned}&\big(\beta_{x,\underline{r}}(a_1,\dots,a_s),0\big)\\&\quad+\sum_{(l_i+l_i'=r_i,l_i'>0)_i}\sum_{j=1}^{s}\Big(0,\nabla_{x,(l_1,l_1',l_2,\dots,l_s,l_s')}(a_1,0,a_2,0,\dots,a_s,0,0,0,\dots,0,\underbrace{m_j}_{(2s+2j)\text{th}},0,\dots,0)\Big).\end{aligned}$$

Using (E3), this is equal to:

$$\begin{aligned}&\big(\beta_{x,\underline{r}}(a_1,\dots,a_s),0\big)\\&\quad+\sum_{j=1}^{s}\sum_{l+l'=r_j,l'>0}(0,\nabla_{x,\underline{r}\circ_j(l,l')}(a_1,\dots,a_j,0,a_{j+1},\dots,a_s,0,\dots,0,\underbrace{m_j}_{(s+j+2)\text{th}},0,\dots,0)).\end{aligned}$$

For all j, denote by $\sigma_j\in\mathfrak{S}_s$ the permutation that moves j to the last spot and keeps all the other numbers in order, and denote by $\sigma_j^*\in\mathfrak{S}_n$ the block permutation of the blocks of size $(r_i)_i$ associated to σ_j. Using ($\beta 1$), one can see that

$$\begin{aligned}&\nabla_{x,\underline{r}\circ_j(l,l')}(a_1,\dots,a_j,0,a_{j+1},\dots,a_s,0,\dots,0,\underbrace{m_j}_{(s+j+2)\text{th}},0,\dots,0)=\\&\qquad\nabla_{\sigma_j^*x,\underline{r}^{\sigma_j}\circ_s(l,l')}(a_1,\dots,a_{j-1},a_{j+1},\dots,a_s,a_j,0,\dots,0,\underbrace{m_j}_{(2s+2)\text{th}}).\end{aligned}$$

For all $y\in\mathcal{P}(k)$, $b_1,\dots,b_{s-1}\in A$, $m\in M$ and $q_1+\dots+q_s=k$ such that $q_s>0$, we set:

$$\beta_{y,\underline{q}}(b_1,\dots,b_{s-1},m):=\nabla_{y,\underline{q}}(b_1,\dots,b_{s-1},0,\dots,0,\underbrace{m}_{(2q)\text{th}}). \tag{$*$}$$

From what precedes, we finally get the explicit $\Gamma(\mathcal{P})$-algebra structure on $A \oplus M$ by:

$$\beta_{x,\underline{r}}\left((a_1,m_1),\ldots,(a_s,m_s)\right) =$$
$$(\beta_{x,\underline{r}}(a_1,\ldots,a_s),0) + \sum_{j=1}^{s} \sum_{l+l'=r_j, l'>0} \Big(0, \beta_{\sigma_j^* x, (\underline{r}^{\sigma_j})\circ_s(l,l')}(a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_s,a_j,m_j)\Big). \qquad (**)$$

We claim that M is an abelian $\Gamma(\mathcal{P})$-algebra as defined in 4.1 and that the assignment $(*)$ endows M with an A-module structure as defined in 4.5.

M is an abelian algebra: Suppose $\underline{r}$ has at least two non-zero integers. For the sake of clarity we will assume that $s=2$. Then we want to prove that $\beta_{x,(r_1,r_2)}(m_1,m_2)=0$. In $B = A\oplus M$, this can be rewritten $\beta_{x,(r_1,r_2)}\left((0,m_1),(0,m_2)\right)$, and we know that:

$$\begin{aligned}\beta_{x,(r_1,r_2)}\left((0,m_1),(0,m_2)\right) &= (0,\nabla_{x,\underline{r}}(0,0,m_1,m_2))\\ &= (0,\nabla_{x,\underline{r}}(0,0,m_1,0)+\nabla_{x,\underline{r}}(0,0,0,m_2)) = 0,\end{aligned}$$

using (E1) and (E3).

M is an A-module: Using the assignment $(*)$, note that

$$\beta_{y,\underline{q}}\left((b_1,0),\ldots,(b_{s-1},0),(0,m)\right) = \Big(0,\beta_{y,\underline{q}}(b_1,\ldots,b_{s-1},m)\Big).$$

Then, the relations we have to verify for this assignment to equip M with an A-module structure all are implied by the relations (β1) to (β8) in B.

Only relations (βAM4) and (βAM9) are not completely straightforward. Suppose all other relations are proven.

Let us show that relation (βAM4) is satisfied. Using relation (β1), it suffices to show (βAM4) for $i=s'$ Let $s,s'>0$, $\underline{q}\in \mathrm{Comp}_{s'}(n)$, and $\underline{r}\in\mathrm{Comp}_s(q_{s'})$. Let $x\in\mathcal{P}(n)^{\mathfrak{S}_{\underline{q}\circ_{s'}\underline{r}}}$. We want to show:

$$\sum_{k=1}^{q_{s'}} \beta_{\sum_{\rho\in\mathfrak{S}_1^{\times(n-q_{s'})}\times\overline{E}_{\underline{r},k}} \rho x, \underline{q}\circ_{s'}(q_{s'}-k,k)}(a_1,\ldots,a_{s'},m) = 0.$$

On one hand, using the assignment $(**)$, we have:

$$\begin{aligned}\beta_{x,\underline{q}\circ_{s'}\underline{r}}((a_1,0),\ldots,(a_{s'-1},0),\underbrace{(a_{s'},m),\ldots,(a_{s'},m)}_{s}) &= \left(\beta_{x,\underline{q}\circ_{s'}\underline{r}}(a_1,\ldots,a_{s'-1},\underbrace{a_{s'},\ldots,a_{s'}}_{s}),0\right) + \\ &\sum_{j=1}^{s}\sum_{l+l'=r_j,l'>0}\left(0,\beta_{\tau_j^* x,\underline{q}\circ_{s'}(\underline{r}^{\tau_j})\circ_s(l,l')}(a_1,\ldots,a_{s'-1},\underbrace{a_{s'},\ldots,a_{s'}}_{s},m)\right),\end{aligned}$$

where $\tau_j^*\in\mathfrak{S}_1^{\times(n-q_{s'})}\times\mathfrak{S}_{q_{s'}}$ denotes the block permutation for blocks of size $(r_1,\ldots,r_s)$ associated with the transposition of j and s in $\mathfrak{S}_s$. Using relations (β4) on A, (βA4) on M, and re-indexing $k=l'$, $l=r_j-k$ we then get:

$$\begin{aligned}\beta_{x,\underline{q}\circ_{s'}\underline{r}}((a_1,0),\ldots,(a_{s'-1},0),\underbrace{(a_{s'},m),\ldots,(a_{s'},m)}_{s}) &= \left(\beta_{\sum_{\rho\in\mathfrak{S}_{\underline{q}}/\mathfrak{S}_{\underline{q}\circ_{s'}\underline{r}}}\sigma x,\underline{q}\circ_{s'}\underline{r}}(a_1,\ldots,a_{s'-1},a_{s'}),0\right)\\ &+\sum_{j=1}^{s}\sum_{k=1}^{r_j}\left(0,\beta_{\sum_{\mathfrak{S}_{(\underline{q}\circ_{s'}\underline{r})\circ_{s'+j-1}(r_j-k,k))}}\rho\tau_j^* x,\underline{q}\circ_{s'}(q_{s'}-k,k)}(a_1,\ldots,a_{s'},m)\right). \qquad (5.\mathrm{eq}.1)\end{aligned}$$

On the other hand, using relation (β4) in B, we get:

$$\beta_{x,\underline{q}\circ_{s'}\underline{r}}((a_1,0),\ldots,(a_{s'-1},0),\underbrace{(a_{s'},m),\ldots,(a_{s'},m)}_{s}) = \beta_{\sum_{\sigma\in\mathfrak{S}_{\underline{q}}/\mathfrak{S}_{\underline{q}\circ_{s'}\underline{r}}}\sigma x,\underline{q}}((a_1,0),\ldots,(a_{s'-1},0),(a_{s'},m)).$$

Then, the assignment $**$ gives:

$$\beta_{x,\underline{q}\circ_{s'}\underline{r}}((a_1,0),\ldots,(a_{s'-1},0),\underbrace{(a_{s'},m),\ldots,(a_{s'},m)}_{s}) = (\beta_{\sum_{\sigma\in\mathfrak{S}_{\underline{q}}/\mathfrak{S}_{\underline{q}\circ_{s'}\underline{r}}}\sigma x,\underline{q}}(a_1,\ldots,a_{s'}),0)+ \sum_{k=1}^{q_j}(0,\beta_{\sum_{\sigma\in\mathfrak{S}_{\underline{q}}/\mathfrak{S}_{\underline{q}\circ_{s'}\underline{r}}}\sigma x,\underline{q}\circ_{s'}(q_{s'}-k,k)}(a_1,\ldots,a_{s'},m)). \quad (5.\text{eq}.2)$$

Subtracting equation 5.eq.2 from equation 5.eq.1, and projecting onto M, we get:

$$\left(\sum_{k=1}^{q_j}\beta_{\sum_{\sigma\in\mathfrak{S}_{\underline{q}}/\mathfrak{S}_{\underline{q}\circ_{s'}\underline{r}}}\sigma x,\underline{q}\circ_{s'}(q_{s'}-k,k)}(a_1,\ldots,a_{s'},m)\right) - \left(\sum_{j=1}^{s}\sum_{k=1}^{r_j}\beta_{\sum_{\mathfrak{S}_{(\underline{q}\circ_{s'}\underline{r})\circ_{s'+j-1}(r_j-k,k))}}\rho\tau_j^* x,\underline{q}\circ_{s'}(q_{s'}-k,k)}(a_1,\ldots,a_{s'},m)\right) = 0.$$

Observe that a set of representative for $\mathfrak{S}_{\underline{q}}/\mathfrak{S}_{\underline{q}\circ_{s'}\underline{r}}$ is given by $\mathfrak{S}_1^{\times n-q_{s'}}\times Sh(\underline{r})$, and, if $k\leq r_j$ a set of representative for $\mathfrak{S}_{\underline{q}\circ_{s'}(q_{s'}-k,k)}/\mathfrak{S}_{(\underline{q}\circ_{s'}\underline{r})\circ_{s'+j-1}(r_j-k,k))}$ is given by:

$$\mathfrak{S}_1^{\times n-q_{s'}}\times Sh(r_1,\ldots,r_{j-1},r_{j+1},\ldots,r_s,r_j-k)\times\mathfrak{S}_1^{\times k}.$$

Then, observe that, for all $j\in[s]$,

$$\{\rho\tau_j^*:\rho\in\mathfrak{S}_1^{\times n-q_{s'}}\times Sh(r_1,\ldots,r_{j-1},r_{j+1},\ldots,r_s,r_j-k)\times\mathfrak{S}_1^{\times k}\} = \mathfrak{S}_1^{\times n-q_{s'}}\times\{\rho'\in Sh(\underline{r}):\rho'(r_1+\cdots+r_j-k+1)=q_{s'}-k+1\},$$

since both sets are the sets of permutations of $\mathfrak{S}_n$ which shuffle the last blocks of size $r_1,\ldots,r_s$ and send $n-q_{s'}+r_1+\cdots+r_j-k+l$ on $n-k+l$ for all $l\in[k]$. Using relation (βAM6), we then get:

$$\sum_{k=1}^{q_{s'}}\beta_{\sum_{\rho\in\mathfrak{S}_1^{\times n-q_{s'}}\times\overline{F}_{\underline{r},k}}}\rho x,\underline{q}\circ_{s'}(q_{s'}-k,k)(a_1,\ldots,a_{s'},m)=0,$$

where $\overline{F}_{\underline{r},k}=Sh(\underline{r})\setminus\left(\coprod_{j\in[s]}\{\rho'\in Sh(\underline{r}):\rho'(r_1+\cdots+r_j-k+1)=q_{s'}-k+1\}\right)$. Note that, if $k>r_j$, then $\{\rho'\in Sh(\underline{r}):\rho'(r_1+\cdots+r_j-k+1)=q_{s'}-k+1\}=\emptyset$.

Remains to show that $\overline{F}_{\underline{r},k}=\overline{E}_{\underline{r},k}$. But, for $\rho'\in Sh(\underline{r})$, ρ' does not satisfy $\rho'(r_1+\cdots+r_j-k+1)=q_{s'}-k+1$ for any $j\in[s]$ if and only if $(\rho')^{-1}(\{q_{s'}-k+1,\ldots,q_{s'}\})$ intersects at least two of the blocks $(r_1,\ldots,r_s)$, and this happens if and only if it satisfies

$$\left|\{\rho'(r_1+\cdots+r_j):j\in[s]\}\cap\{q_i-k+1,\ldots,q_i\}\right|\geq 2.$$

Thus, we have shown that relation (βAM4) is satisfied.

Let us now show that relation (βAM9) is satisfied. Let $r_1+\cdots+r_s=n$, $x\in\mathcal{P}(n)^{\mathfrak{S}_{\underline{r}}}$, let $q_1+\cdots+q_u=k$, $y\in\mathcal{P}(k)^{\mathfrak{S}_{\underline{q}}}$. Let $(a_i)_{i\in[s+u]}\in A^{\times s+u}$ and $m\in M$. Denote by $z=x\left(1_{\mathcal{P}}^{\times n-r_s},y^{\times r_s}\right)\in$

$\mathcal{P}(n + r_s(k-1))$. On the one hand, using $(**)$, observe that:

$$\beta_{x,\underline{r}}\Big((a_1,0),\dots,(a_{s-1},0),\beta_{y,\underline{q}}((a_s,0),\dots,(a_{s+u-1},0),(a_{s+u},m))\Big)$$
$$= \beta_{x,\underline{r}}\left((a_1,0),\dots,(a_{s-1},0),\sum_{l+l'=q_s,l'>0}(\beta_{y,\underline{q}}(a_s,\dots,a_{s+u-1}),\beta_{y,\underline{q}\circ_u(l,l')}(a_s,\dots,a_{s+u},m))\right)$$
$$= \Big(\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},\beta_{y,\underline{q}}(a_s,\dots,a_{s+u})),0\Big) +$$
$$\sum_{\lambda,\lambda'}\left(0,\beta_{x,\underline{r}\circ_s(\lambda,\lambda')}\left(a_1,\dots,a_{s-1},\beta_{y,\underline{q}}(a_s,\dots,a_{s+u}),\sum_{l+l'=q_s,l'>0}\beta_{y,\underline{q}\circ_u(l,l')}(a_s,\dots,a_{s+u},m)\right)\right),$$

where (λ,λ') runs over the pairs of non-negative integers satisfying $\lambda+\lambda'=r_s, \lambda'>0$. Using relation $(\beta 8)$ on A gives us:

$$(\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},\beta_{y,\underline{q}}(a_s,\dots,a_{s+u})),0) = \Big(\beta_{\sum_\tau \tau z,\underline{r}\diamond_s(\underline{q})}(a_1,\dots,a_{s+u}),0\Big).$$

Using relation $(\beta$M5) and $(\beta$AM8) on M, we get

$$\sum_{\lambda,\lambda'}\left(0,\beta_{x,\underline{r}\circ_s(\lambda,\lambda')}\left(a_1,\dots,a_{s-1},\beta_{y,\underline{q}}(a_s,\dots,a_{s+u}),\sum_{l+l'=q_s,l'>0}\beta_{y,\underline{q}\circ_u(l,l')}(a_s,\dots,a_{s+u},m)\right)\right)$$
$$=\sum_{\lambda,\lambda'}\sum_{l+l'=q_s,l'>0}\Big(0,\beta_{\sum_\sigma \sigma z,(\underline{r}\circ_s(\lambda,\lambda')\diamond((1)^{\times s-1},\underline{q},\underline{q}\circ_u(l,l')))}(a_1,\dots,a_{s+u},a_s,\dots,a_{s+u},m)\Big),$$

where σ ranges over $\mathfrak{S}_{(\underline{r}\circ_s(\lambda,\lambda')\diamond_s(\underline{q}\circ_u(l,l')))}/\left(\prod_{i=1}^{s-1}\mathfrak{S}_{r_i}\right)\times\mathfrak{S}_\lambda\wr\mathfrak{S}_{\underline{q}}\times\mathfrak{S}_{\lambda'}\wr\mathfrak{S}_{\underline{q}\circ_u(l,l')}$ in the sum. Observe the repetition of input. Using relation $(\beta$A4), and noting that:

$$\sum_{\sigma''\in\mathfrak{S}_{R^{ll'}_{\lambda\lambda'}}/\mathfrak{S}_{(\underline{r}\circ_s(\lambda,\lambda')\diamond((1)^{\times s-1},\underline{q},\underline{q}\circ_u(l,l')))}}\sigma''\sum_{\sigma\in\mathfrak{S}_{(\underline{r}\circ_s(\lambda,\lambda')\diamond_s(\underline{q}\circ_u(l,l')))}/(\prod_{i=1}^{s-1}\mathfrak{S}_{r_i})\times\mathfrak{S}_{(\lambda,\lambda')}\wr\mathfrak{S}_{\underline{q}\circ_u(l,l')}}\sigma z$$
$$=\sum_{\sigma'\in\mathfrak{S}_{R^{ll'}_{\lambda\lambda'}}/(\prod_{i=1}^{s-1}\mathfrak{S}_{r_i})\times\mathfrak{S}_\lambda\wr\mathfrak{S}_{\underline{q}}\times\mathfrak{S}_{\lambda'}\wr\mathfrak{S}_{\underline{q}\circ_u(l,l')}}\sigma' z,$$

we get

$$\sum_{\lambda,\lambda'}\left(0,\beta_{x,\underline{r}\circ_s(\lambda,\lambda')}\left(a_1,\dots,a_{s-1},\beta_{y,\underline{q}}(a_s,\dots,a_{s+u}),\sum_{l+l'=q_s,l'>0}\beta_{y,\underline{q}\circ_u(l,l')}(a_s,\dots,a_{s+u},m)\right)\right)$$
$$=\sum_{\lambda+\lambda'=r_s,\lambda'>0}\;\sum_{l+l'=q_u,l'>0}\beta_{\sum_{\sigma'}\sigma' z,R^{ll'}_{\lambda\lambda'}}(a_1,\dots,a_{s+u},m).$$

Finally, we obtain the equality:

$$\beta_{x,\underline{r}}\Big((a_1,0),\dots,(a_{s-1},0),\beta_{y,\underline{q}}((a_s,0),\dots,(a_{s+u-1},0),(a_{s+u},m))\Big)=$$
$$\Big(\beta_{\sum_\tau \tau z,\underline{r}\diamond_s(\underline{q})}(a_1,\dots,a_{s+u}),0\Big)+\sum_{\lambda+\lambda'=r_s,\lambda'>0}\;\sum_{l+l'=q_u,l'>0}\beta_{\sum_{\sigma'}\sigma' z,R^{ll'}_{\lambda\lambda'}}(a_1,\dots,a_{s+u},m). \tag{5.eq.3}$$

On the other hand, using relation (β8),

$$\beta_{x,\underline{r}}\Big((a_1,0),\ldots,(a_{s-1},0),\beta_{y,\underline{q}}((a_s,0),\ldots,(a_{s+u-1},0),(a_{s+u},m))\Big)$$
$$= \beta_{\sum_\tau \tau z,\underline{r}\diamond_s(\underline{q})}((a_1,0),\ldots,(a_{s+u-1},0),(a_{s+u},m)),$$

where τ ranges over $\mathfrak{S}_{\underline{r}\diamond_s \underline{q}}/\left(\prod_{i=1}^{s-1}\mathfrak{S}_{r_i}\right)\times\mathfrak{S}_{r_s}\wr\mathfrak{S}_{\underline{q}}$ in the sum. Using $(**)$, we then get:

$$\beta_{x,\underline{r}}\Big((a_1,0),\ldots,(a_{s-1},0),\beta_{y,\underline{q}}((a_s,0),\ldots,(a_{s+u-1},0),(a_{s+u},m))\Big) =$$
$$\Big(\beta_{\sum_\tau \tau z,\underline{r}\diamond_s(\underline{q})}(a_1,\ldots,a_{s+u}),0\Big) + \sum_{t+t'=r_sq_u,t'>0}\Big(0,\beta_{\sum_\tau \tau z,\underline{r}\diamond_s(\underline{q})\circ_{s+u}(t,t')}(a_1,\ldots,a_{s+u},m)\Big). \quad (5.\text{eq}.4)$$

Comparing the equalities (5.eq.3) and (5.eq.4), and projecting in M, yields the relation (βAM9).

Assigning, to each Beck module $B \xrightarrow{p} A$, the abelian $\Gamma(\mathcal{P})$-algebra $\ker(p)$ with the above A module structure provides a functor:

$$\ker\colon (\Gamma(\mathcal{P})\text{-Alg}/A)_{\text{ab}} \to A\text{-Mod}.$$

Let us now prove that the data of an A-module yields a Beck module over A. Let M be an A-module. Consider the vector space $B := A\oplus M$, equipped with:

(1) For all $\underline{r}\in \mathrm{Comp}_s(n)$, $x\in\mathcal{P}(n)^{\mathfrak{S}_{\underline{r}}}$, an operation $\beta_{x,\underline{r}}$ defined as in $(**)$,
(2) A map $pr\colon B\to A$ given by the projection orthogonally to M,
(3) A map $z\colon A\to B$ given by the natural injection,
(4) A map $\mu\colon B\times_A B\to A$ given by $\mu(a,m,m') = (a,m+m')$.

We claim that:

(A) The assignment (1) endows B with a $\Gamma(\mathcal{P})$-algebra structure,
(B) pr is a $\Gamma(\mathcal{P})$-algebra map,
(C) z and μ are maps of $\Gamma(\mathcal{P})$-algebras over A, and finally,
(D) z and μ endow B with the structure of a Beck module in A.

Assuming (A), conditions (B), (C) and (D) are obvious. To prove (A) we have to check that the operations defined in (1) satisfy the conditions (β1) to (β8). Using the notation 4.6, we can rewrite $(**)$ as:

$$\beta_{x,\underline{r}}((a_1,m_1),\ldots,(a_s,m_s))$$
$$= \big(\beta_{x,\underline{r}}(a_1,\ldots,a_s),0\big) + \sum_{j=1}^{s}\sum_{l+l'=r_j,l'>0}\Big(0,\beta_{x,\underline{r}\circ_j(l,l')}(a_1,\ldots,a_j,m_j,a_{j+1},\ldots,a_s)\Big).$$

Then, relation (β1) is directly deduced from (β1) on A and (βA1). Relation (β2) is deduced using (β2) on A and using the fact that there is no positive l,l' with $l+l'=0$ such that $l'>0$. Relation (β3) is a direct consequence of (β3) on A, (βA3) and (βM3) on M.

Let us prove relation (β4). For clarity we can assume $s'=1$. Then $q_1=s$, and we need to prove that:

$$\beta_{x,\underline{r}}((a,m),\ldots,(a,m)) = \beta_{\sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_{\underline{r}}}\sigma\cdot x,(n)}((a,m)).$$

Using $(**)$, one has:

$$\beta_{\sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_{\underline{r}}}\sigma\cdot x,(n)}((a,m)) = \Big(\beta_{\sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_{\underline{r}}}\sigma x,(n)}(a),0\Big) + \sum_{l+l'=n,l'>0}\Big(0,\beta_{\sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_{\underline{r}}}\sigma x,(l,l')}(a,m)\Big),$$

which, re-indexing $k=l'$, yields

$$\Big(\beta_{\sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_{\underline{r}}}\sigma x,(n)}(a),0\Big) + \sum_{k=1}^{n}\Big(0,\beta_{\sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_{\underline{r}}}\sigma x,(n-k,k)}(a,m)\Big)$$

Using relation (βAM4), this becomes:

$$\left(\beta_{\sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_{\underline{r}}}\sigma x,(n)}(a),0\right)+\sum_{k=1}^{n}\left(0,\beta_{\sum_\rho \rho x,(n-k,k)}(a,m)\right),$$

where ρ runs over the set of shuffles $Sh(\underline{r})$ satisfying:

$$\left|\{\rho'(r_1+\cdots+r_j) : j\in[s]\}\cap\{q_i-k+1,\ldots,q_i\}\right|=1.$$

From what precedes, this set is equal to $\coprod_{j\in[s]}\{\rho'\in Sh(\underline{r}) : \rho'(r_1+\cdots+r_j-k+1)=n-k+1\}$, which is equal to

$$\coprod_{j\in[s]}\{\rho\sigma_j^* : \rho\in Sh(r_1,\ldots,r_{j-1},r_{j+1},\ldots,r_s,r_j-k)\times\mathfrak{S}_1^{\times k}\}.$$

The set $Sh(r_1,\ldots,r_{j-1},r_{j+1},\ldots,r_s,r_j-k)\times\mathfrak{S}_1^{\times k}$ is empty if $k>r_j$, and otherwise, it is a set of representative for $\mathfrak{S}_{n-k}\times\mathfrak{S}_k/\mathfrak{S}_{\underline{r}^{\sigma_j}\circ_s(r_j-k,k)}$. So, we get:

$$\begin{aligned}\beta_{x,\underline{r}}\left((a,m),\ldots,(a,m)\right)=&\left(\beta_{\sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_{\underline{r}}}\sigma x,(n)}(a),0\right)+\\&\sum_{j=1}^{s}\sum_{k=1}^{n}\left(0,\beta_{\sum_{\rho\in\mathfrak{S}_{n-k}\times\mathfrak{S}_k/\mathfrak{S}_{\underline{r}^{\sigma_j}\circ_s(r_j-k,k)}}\rho\sigma_j^* x,(n-k,k)}(a,m)\right).\end{aligned}$$

Using relation (β4) on A and relation (βA4) on M, we get:

$$\beta_{x,\underline{r}}\left((a,m),\ldots,(a,m)\right)=\left(\beta_{x,\underline{r}}(a,\ldots,a),0\right)+\sum_{j=1}^{s}\sum_{k=1}^{n}\left(0,\beta_{\sigma_j^* x,\underline{r}^\sigma\circ_s(r_j-k,s)}(a,\ldots,a,m)\right),$$

and, using the assignment $*$, we finally obtain:

$$\beta_{x,\underline{r}}\left((a,m),\ldots,(a,m)\right)=\beta_{\sum_{\sigma\in\mathfrak{S}_n/\mathfrak{S}_{\underline{r}}}\sigma\cdot x,(n)}\left((a,m)\right).$$

Let us prove relation (β5). For clarity, we will assume $s=2$. On the one hand, we have:

$$\begin{aligned}\beta_{x,\underline{r}}\left((a,m)+(b,m'),(a_2,m_2)\right)&=\beta_{x,\underline{r}}\left((a+b,m+m'),(a_2,m_2)\right)\\&=\left(\beta_{x,\underline{r}}(a+b,a_2),0\right)+\sum_{l+l'=r_1,l'>0}\left(0,\beta_{x,\underline{r}\circ_1(l,l')}(a+b,m+m',a_2)\right)\\&\quad+\sum_{l+l'=r_2,l'>0}\left(0,\beta_{x,\underline{r}\circ_2(l,l')}(a+b,a_2,m_2)\right),\end{aligned}$$

which, using relations (β5) on A, (βM5) on M and (βA5) on M, is equal to:

$$\begin{aligned}&\sum_{k_1+k_2=r_1}\left(\beta_{x,\underline{r}\circ_1(k_1,k_2)}(a,b,a_2),0\right)\\&\quad+\sum_{l+l'=r_1,l'>0}\sum_{k_1+k_2=l}\left(0,\beta_{x,\underline{r}\circ_1(k_1,k_2,l')}(a,b,m,a_2)+\beta_{x,\underline{r}\circ_1(k_1,k_2,l')}(a,b,m',a_2)\right)\\&\quad\quad+\sum_{k_1+k_2=r_1}\sum_{l+l'=r_2,l'>0}\left(0,\beta_{x,(\underline{r}\circ_2(l,l'))\circ_1(k_1,k_2)}(a,b,a_2,m_2)\right).\end{aligned}$$

Note that the middle summand can be rewritten, changing the indices:

$$\sum_{k_1+k_2+k_3=r_1,k_3>0}\left(0,\beta_{x,\underline{r}\circ_1(k_1,k_2,k_3)}(a,b,m,a_2)+\beta_{x,\underline{r}\circ_1(k_1,k_2,k_3)}(a,b,m',a_2)\right).$$

On the other hand, note that the assignment $(**)$ gives:

$$\begin{aligned}\sum_{k_1+k_2=r_1}\beta_{x,\underline{r}\circ_1(k_1,k_2)}\big((a,m),(b,m'),(a_2,m_2)\big) &= \sum_{k_1+k_2=r_1}(\beta_{x,\underline{r}\circ_1(k_1,k_2)}(a,b,a_2),0)\\ &+\sum_{k_1+k_2=r_1}\sum_{l+l'=k_1,l'>0}\beta_{x,\underline{r}\circ_1(l,l',k_2)}(a,m,b,a_2),0)\\ &+\sum_{k_1+k_2=r_1}\sum_{l+l'=k_2,l'>0}\beta_{x,\underline{r}\circ_1(k_1,l,l')}(a,b,m',a_2),0)\\ &+\sum_{k_1+k_2=r_1}\sum_{l+l'=r_2,l'>0}\big(0,\beta_{x,(\underline{r}\circ_2(l,l'))\circ_1(k_1,k_2)}(a,b,a_2,m_2)\big)\,.\end{aligned}$$

The sum of the second and third summand can be rewritten:

$$\sum_{k_1+k_2+k_3=r_1,k_3>0}\big(0,\beta_{x,\underline{r}\circ_1(k_1,k_2,k_3)}(a,b,m,a_2)+\beta_{x,\underline{r}\circ_1(k_1,k_2,k_3)}(a,b,m',a_2)\big)\,.$$

This proves that relation $(\beta 5)$ holds.

Relation $(\beta 6)$ is a consequence of relation $(\beta 6)$ on A and $(\beta\mathrm{AM}6)$ on M. Relation $(\beta 7)$ is a consequence of relation $(\beta 7)$ on A, $(\beta\mathrm{AM}7)$ and $(\mathrm{Ab}\beta 7)$ on M.

Let us now prove relation $(\beta 8)$. For clarity, we can assume that $s=1$. We want to prove that:

$$\beta_{x,(n)}\left(\beta_{y,\underline{q}}((a_1,m_1),\dots,(a_u,m_u))\right)=\beta_{\sum_\tau\tau\cdot x(y^{\times n}),(n)\diamond(\underline{q})}\left((a_1,m_1),\dots,(a_u,m_u)\right),$$

where τ ranges over $\mathfrak{S}_{(n)\diamond(\underline{q})}/\mathfrak{S}_{(n)}\wr\mathfrak{S}_{\underline{q}}$ in the sum. Here, $x\in\mathcal{P}(n)^{\mathfrak{S}_n}$, $\underline{q}\in\mathrm{Comp}_u(k)$, and $y\in\mathcal{P}(k)^{\mathfrak{S}_{\underline{q}}}$. On the one hand, using the assignment $(**)$,

$$\begin{aligned}&\beta_{x,(n)}\left(\beta_{y,\underline{q}}((a_1,m_1),\dots,(a_u,m_u))\right)\\ &=\beta_{x,(n)}\left(\left(\beta_{y,\underline{q}}(a_1,\dots,a_u),0\right)+\sum_{j=1}^{n}\sum_{l+l'=q_j,l'>0}\left(0,\beta_{y,\underline{q}\circ_j(l,l')}(a_1,\dots,a_j,m_j,a_{j+1},\dots,a_u)\right)\right)\\ &=\left(\beta_{x,(n)}\left(\beta_{y,\underline{q}}(a_1,\dots,a_u)\right),0\right)+\\ &\sum_{\lambda+\lambda'=n,\lambda'>0}\left(0,\beta_{x,(\lambda+\lambda')}\left(\beta_{y,\underline{q}}(a_1,\dots,a_u),\sum_{j=1}^{u}\sum_{l+l'=q_j,l'>0}\beta_{y,\underline{q}\circ_j(l,l')}(a_1,\dots,a_j,m_j,a_{j+1},\dots,a_u)\right)\right).\end{aligned}$$

Using relation $(\beta 8)$ on A and $(\beta\mathrm{M}5)$ on M, we then get,

$$\begin{aligned}&\beta_{x,(n)}\left(\beta_{y,\underline{q}}((a_1,m_1),\dots,(a_u,m_u))\right)=\left(\beta_{\sum_\tau\tau\cdot x(y^{\times n}),(n)\diamond(\underline{q})}(a_1,\dots,a_u),0\right)+\\ &\sum_{j=1}^{u}\sum_{\lambda+\lambda'=n,\lambda'>0}\sum_{l+l'=q_j,l'>0}\left(0,\beta_{x,(\lambda+\lambda')}\left(\beta_{y,\underline{q}}(a_1,\dots,a_u),\beta_{y,\underline{q}\circ_j(l,l')}(a_1,\dots,a_j,m_j,a_{j+1},\dots,a_u)\right)\right)\end{aligned}\tag{5.eq.5}$$

On the other hand, using the assignment $(**)$,

$$\begin{aligned}&\beta_{\sum_\tau\tau\cdot x(y^{\times n}),(n)\diamond(\underline{q})}((a_1,m_1),\dots,(a_u,m_u))=\left(\beta_{\sum_\tau\tau\cdot x(y^{\times n}),(n)\diamond(\underline{q})}(a_1,\dots,a_u),0\right)+\\ &\sum_{j=1}^{u}\sum_{t+t'=nq_j,t'>0}\left(0,\beta_{\sum_\tau\tau\cdot x(y^{\times n}),((n)\diamond(\underline{q}))\circ_j(t,t')}(a_1,\dots,a_j,m_j,a_{j+1},\dots,a_u)\right).\end{aligned}\tag{5.eq.6}$$

Comparing the expressions (5.eq.5) and (5.eq.6), we then have left to prove that in M, for all $j \in [u]$,

$$\sum_{\lambda+\lambda'=n,\lambda'>0}\sum_{l+l'=q_j,l'>0}\beta_{x,(\lambda+\lambda')}\Big(\beta_{y,\underline{q}}(a_1,\dots,a_u),\beta_{y,\underline{q}\circ_j(l,l')}(a_1,\dots,a_j,m_j,a_{j+1},\dots,a_u)\Big)=$$
$$\sum_{t+t'=nq_j,t'>0}\beta_{\sum_\tau \tau\cdot x(y^{\times n}),\big((n)\diamond(\underline{q})\big)\circ_j(t,t')}(a_1,\dots,a_j,m_j,a_{j+1},\dots,a_u).$$

We can, without loss of generality, suppose that $j=u$. Then, we are left to prove that:

$$\sum_{\lambda+\lambda'=n,\lambda'>0}\sum_{l+l'=q_u,l'>0}\beta_{x,(\lambda+\lambda')}\Big(\beta_{y,\underline{q}}(a_1,\dots,a_u),\beta_{y,\underline{q}\circ_u(l,l')}(a_1,\dots,a_u,m_u)\Big)=$$
$$\sum_{t+t'=nq_u,t'>0}\beta_{\sum_\tau \tau\cdot x(y^{\times n}),\big((n)\diamond(\underline{q})\big)\circ_j(t,t')}(a_1,\dots,a_u,m_u).$$

Using relation (βAM8) on the left hand side of the equality gives us:

$$\sum_{\lambda+\lambda'=n,\lambda'>0}\sum_{l+l'=q_u,l'>0}\beta_{x,(\lambda+\lambda')}\Big(\beta_{y,\underline{q}}(a_1,\dots,a_u),\beta_{y,\underline{q}\circ_u(l,l')}(a_1,\dots,a_u,m_u)\Big)=$$
$$\sum_{\lambda+\lambda'=n,\lambda'>0}\sum_{l+l'=q_u,l'>0}\beta_{\sum_\sigma x(y^{\times n}),(\lambda,\lambda')\diamond(\underline{q},\underline{q}\circ_u(l,l'))}(a_1,\dots,a_u,a_1,\dots,a_u,m_u),$$

where σ ranges over $\mathfrak{S}_{(\lambda,\lambda')\diamond(\underline{q},\underline{q}\circ_u(l,l'))}/\mathfrak{S}_\lambda\wr\mathfrak{S}_{\underline{q}}\times\mathfrak{S}_{\lambda'}\wr\mathfrak{S}_{\underline{q}\circ_u(l,l')}$ in the sum.

Denote by $R^{l,l'}_{\lambda,\lambda'}$ the partition of kn into $u+1$ parts such that, for all $i\in[u-1]$, $(R^{l,l'}_{\lambda,\lambda'})_i = \big((n)\diamond(\underline{q})\big)_i$, and such that:

$$(R^{l,l'}_{\lambda,\lambda'})_u=\{k-q_u+\alpha k+\gamma,\alpha\in[\lambda],\gamma\in[q_u]\}\cup\{\lambda k+(k-q_u)+\alpha k+\gamma,\alpha\in[\lambda'],\gamma\in[l]\}.$$
$$(R^{l,l'}_{\lambda,\lambda'})_{u+1}=\{\lambda k+(k-l')+\alpha k+\gamma,\alpha\in[\lambda'],\gamma\in[l']\}.$$

Then, using relation (βA4), and noting that:

$$\sum_{\sigma''\in\mathfrak{S}_{R^{ll'}_{\lambda\lambda'}}/\mathfrak{S}_{(\lambda,\lambda')\diamond(\underline{q},\underline{q}\circ_u(l,l'))}}\sigma\sum_{\mathfrak{S}_{(\lambda,\lambda')\diamond(\underline{q},\underline{q}\circ_u(l,l'))}/\mathfrak{S}_\lambda\wr\mathfrak{S}_{\underline{q}}\times\mathfrak{S}_{\lambda'}\wr\mathfrak{S}_{\underline{q}\circ_u(l,l')}}\sigma x(y^{\times n})=$$
$$\sum_{\sigma'\in\mathfrak{S}_{R^{ll'}_{\lambda\lambda'}}/\mathfrak{S}_\lambda\wr\mathfrak{S}_{\underline{q}}\times\mathfrak{S}_{\lambda'}\wr\mathfrak{S}_{\underline{q}\circ_u(l,l')}}\sigma' x(y^{\times n}),$$

we then get:

$$\sum_{\lambda+\lambda'=n,\lambda'>0}\sum_{l+l'=q_u,l'>0}\beta_{x,(\lambda+\lambda')}\Big(\beta_{y,\underline{q}}(a_1,\dots,a_u),\beta_{y,\underline{q}\circ_u(l,l')}(a_1,\dots,a_u,m_u)\Big)=$$
$$\sum_{\lambda+\lambda'=n,\lambda'>0}\sum_{l+l'=q_u,l'>0}\beta_{\sum_{\sigma'}\sigma' x(y^{\times n}),R^{ll'}_{\lambda\lambda'}}(a_1,\dots,a_u,m_u).$$

We have left to prove the equality:

$$\sum_{\lambda+\lambda'=n,\lambda'>0}\sum_{l+l'=q_u,l'>0}\beta_{\sum_{\sigma'}\sigma' x(y^{\times n}),R^{ll'}_{\lambda\lambda'}}(a_1,\dots,a_u,m_u)$$
$$=\sum_{t+t'=nq_u,t'>0}\beta_{\sum_\tau \tau\cdot x(y^{\times n}),\big((n)\diamond(\underline{q})\big)\circ_j(t,t')}(a_1,\dots,a_u,m_u).$$

But this is a particular case of the relation (βAM9).

Assigning to each A-module M the Beck module $A \ltimes M$ obtained by equipping the vector space $A \oplus M$ with the divided power $\mathcal{P}$-algebra structure given by $(**)$ provides a functor:

$$A \ltimes -\colon A\text{-Mod} \to (\Gamma(\mathcal{P})\text{-Alg}/A)_{\mathrm{ab}}\,,$$

One can readily check that ker and $A \ltimes -$ are inverse to each other. □

Definition 5.3. If M is an A-module, the Beck module $A \ltimes M$ defined above is called the **semidirect product** of A and M.

Let us now turn to the particular case of abelian $\Gamma(\mathcal{P})$-algebras. Abelian group objects in the category of $\Gamma(\mathcal{P})$-algebras are Beck modules over the terminal object. Since $\mathcal{P}$ is reduced, this terminal object is always the zero algebra 0. Then, we obtain the following:

Corollary 5.4. *Abelian group objects in $\Gamma(\mathcal{P})$-algebras are equivalent to abelian $\Gamma(\mathcal{P})$-algebras from Definition 4.1, that is:*

$$(\Gamma(\mathcal{P})\text{-Alg})_{\mathrm{ab}} \cong (\Gamma(\mathcal{P})\text{-Alg})_{\mathrm{Ab}}.$$

Proof. Applying Theorem 5.2 in the case $A = 0$, the data of an abelian group object in $\Gamma(\mathcal{P})$-algebras is equivalent to the data of a module over the $\Gamma(\mathcal{P})$-algebra 0. Such a 0-module M is an abelian $\Gamma(\mathcal{P})$-algebra equipped with operations $\beta_{x,(r_1,\ldots,r_s)}(0,\ldots,0,-)\colon M \to M$. If $s > 1$ and if at least one of the r_i for $i < s$ is non-zero, then relation (βA3) implies that $\beta_{x,(r_1,\ldots,r_s)}(0,\ldots,0,m) = 0$ for all M. If $s > 1$ and $r_i = 0$ for all $i < s$, relation (βA2) shows that $\beta_{x,(r_1,\ldots,r_s)}(0,\ldots,0,m) = \beta_{x,(r_s)}(m)$. Finally, when $s = 1$, relation (βAM7) shows that $\beta_{x,(r_s)}(m) = \beta_x(m)$ is provided by the structure of abelian $\Gamma(\mathcal{P})$-algebra on M. Thus, a 0-module is an abelian $\Gamma(\mathcal{P})$-algebra equipped with additionnal operations which are all trivial. □

6. Universal enveloping algebra

The objective of this section is to show that the category of modules over the $\Gamma(\mathcal{P})$-algebra A is equivalent to the category of left modules over an associative algebra $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$. This result, is contained in Proposition 6.9. In practice, it will be useful to see $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ as the algebra containing all the operations $\beta_x(-)$ in the definition of an abelian $\Gamma(\mathcal{P})$-algebra (see Definition 4.1), and all the operations $\beta_{x,\underline{r}}(a_1,\ldots,a_{s-1},-)$ in the definition of an A-module (see Definition 4.5). Here, the hyphen "$-$" is considered as a placeholder. It is then useful to consider the multiplication μ of the algebra $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ as the composition of operations: if $O_1(-), O_2(-)$ are two operations with one placeholder symbol, then $\mu(O_1(-)\otimes O_2(-))$ represents the operation $O_1(O_2(-))$ obtained by replacing the placeholder in $O_1(-)$ by the operation $O_2(-)$.

For this section, we fix the base field $\mathbb{F}$ of characteristic p.

Definition 6.1. Let $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ be the quotient of the $\mathbb{F}$-vector space spanned by the set of symbols $\beta_{x,\underline{r}}(a_1,\ldots,a_{s-1},-)$, for all $\underline{r} = (r_1,\ldots,r_s)$, such that $r_1+\cdots+r_s = n$, $x \in \mathcal{P}(n)^{\mathfrak{S}_{\underline{r}}}$, and $a_1,\ldots,a_{s-1} \in A$, by the vector subspace spanned by the elements:

(1) $\beta_{\lambda x+y,\underline{r}}(a_1,\ldots,a_{s-1},-)-\big(\lambda\beta_{x,\underline{r}}(a_1,\ldots,a_{s-1},-)+\beta_{y,\underline{r}}(a_1,\ldots,a_{s-1},-)\big)$ for all $x,y \in \mathcal{P}(n)^{\underline{r}}$, $\lambda \in \mathbb{F}$,

(2) $\beta_{x,\underline{r}}(a_1,\ldots,a_{s-1},-) - \beta_{\rho^*\cdot x,\underline{r}^\rho}(a_{\rho^{-1}(1)},\ldots,a_{\rho^{-1}(s-1)},-)$ for all $\rho \in \mathfrak{S}_{s-1}$, where ρ^* denotes the block permutation with blocks of size (r_i) associated to ρ.

(3) $\beta_{x,(0,r_1,r_2,\ldots,r_s)}(a_0,a_1,\ldots,a_{s-1},-) - \beta_{x,(r_1,r_2,\ldots,r_s)}(a_1,\ldots,a_{s-1},-)$.

(4) $\beta_{x,\underline{r}}(\lambda a_1,a_2,\ldots,a_{s-1},m) - \lambda^{r_1}\beta_{x,\underline{r}}(a_1,\ldots,a_{s-1},m) \quad \forall\lambda \in \mathbb{F}$.

(5) For all $\underline{r} \in \mathrm{Comp}_s(n)$ and $\underline{q} \in \mathrm{Comp}_{s'}(s-1)$, the element

$$\beta_{x,\underline{r}}(\underbrace{a_1,\ldots,a_1}_{q_1},\underbrace{a_2,\ldots,a_2}_{q_2},\ldots,\underbrace{a_{s-1},\ldots,a_{s-1}}_{q_{s'}},-) - \beta_{\left(\sum_{\sigma\in\mathfrak{S}_{\underline{q}'\rhd\underline{r}}/\mathfrak{S}_{\underline{r}}}\sigma\cdot x\right),\ \underline{q}'\rhd\underline{r}}(a_1,a_2,\ldots,a_{s-1},-),$$

where $\underline{q}' = (\underline{q},0)$,

(6) $\beta_{x,\underline{r}}(a_0+a_1,\dots,a_{s-1},-)-\left(\sum_{l+l'=r_1}\beta_{x,\underline{r}\circ_1(l,l')}(a_0,a_1,\dots,a_{s-1},-)\right)$,

(7) $\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-)$ for all x such that there exists $q_1+\cdots+q_u=r_s$, $y\in\mathcal{P}(n)^{\mathfrak{S}_{\underline{r}\circ_s\underline{q}}}$, such that $x=\sum_{\sigma\in\mathfrak{S}_{n-r_s}\times\mathfrak{S}_{r_s}/\mathfrak{S}_{n-r_s}\times\mathfrak{S}_{\underline{q}}}\sigma y$, and at least two of the q_js are non-zero.

(8) For all $r_1+\cdots+r_s=n$, $x\in\mathcal{P}(n)^{\mathfrak{S}_{\underline{r}}}$, let $q_1+\cdots+q_u=k$, $y\in\mathcal{P}(k)^{\mathfrak{S}_{\underline{q}}}$, $(a_i)_{i\in[s+u]}\in A^{\times s+u}$, the element:

$$\left(\sum_{t+t'=r_sq_u,t'>0}\beta_{\sum_\tau \tau z,\underline{r}\diamond_s(\underline{q})\circ_{s+u}(t,t')}(a_1,\dots,a_{s+u},-)\right)-\left(\sum_{\lambda+\lambda'=r_s,\lambda'>0}\sum_{l+l'=q_u,l'>0}\beta_{\sum_{\sigma'}\sigma' z,R^{ll'}_{\lambda\lambda'}}(a_1,\dots,a_{s+u},-)\right),$$

where $z=x\left(1_{\mathcal{P}}^{\times n-r_s},y^{\times r_s}\right)\in\mathcal{P}(n+r_s(k-1))$, $R^{ll'}_{\lambda\lambda'}$ is defined in 4.5, where τ ranges over $\mathfrak{S}_{\underline{r}\diamond_s\underline{q}}/\left(\prod_{i=1}^{s-1}\mathfrak{S}_{r_i}\right)\times\mathfrak{S}_{r_s}\wr\mathfrak{S}_{\underline{q}}$ in the sum, and where σ' ranges over $\mathfrak{S}_{R^{ll'}_{\lambda\lambda'}}/\left(\prod_{i=1}^{s-1}\mathfrak{S}_{r_i}\right)\times\mathfrak{S}_\lambda\wr\mathfrak{S}_{\underline{q}}\times\mathfrak{S}_{\lambda'}\wr\mathfrak{S}_{\underline{q}\circ_u(l,l')}$ in the sum.

Notation 6.2. Following Notation 4.6, we will allow the notation:

$$\beta_{x,\underline{r}}(a_1,\dots,a_{i-1},-,a_{i+1},\dots,a_s):=\beta_{\sigma_i^*,\underline{r}^{\sigma_i}}(a_1,\dots,a_{i-1},a_{i+1},\dots,a_s,-).$$

Proposition 6.3. *The vector space $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ is spanned by the symbols $\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-)$ where all the integers in $\underline{r}$ are powers of p, and $\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-)=0$ in $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ if r_s is not a power of p.*

Proof. This is equivalent to Proposition 4.7. □

Viewing the $\mathbb{F}$-vector space structure of $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ as a left $\mathbb{F}$-action, we endow $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ with a right $\mathbb{F}$-action by setting, for all $\lambda\in\mathbb{F}$,

$$\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-)\cdot\lambda=\lambda^{r_s}\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-).$$

Remark 6.4. It is important here to restrict ourselves to the case where r_s is a power of p. Otherwise, the preceding definition is not linear in $\lambda\in\mathbb{F}$. This is made possible by Proposition 6.3.

We then equip $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ with a multiplication:

$$\mu:\mathbb{U}_{\Gamma(\mathcal{P})}(A)_{\mathbb{F}}\otimes_{\mathbb{F}}\mathbb{U}_{\Gamma(\mathcal{P})}(A)\to\mathbb{U}_{\Gamma(\mathcal{P})}(A)$$

by setting:

$$\mu\left(\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-)\otimes\beta_{y,\underline{q}}(b_1,\dots,b_{u-1},-)\right)=\beta_{\sum_\tau\tau\cdot x(1_{\mathcal{P}}^{n-r_s},y^{\times r_s}),\underline{r}\diamond_s\underline{q}_s}(a_1,\dots,a_{s-1},b_1,\dots,b_{u-1},-),$$

where τ ranges over $\mathfrak{S}_{\underline{r}\diamond_s\underline{q}_s}/\left(\prod_{i=1}^{s-1}\mathfrak{S}_{r_i}\right)\times\mathfrak{S}_{r_s}\wr\mathfrak{S}_{\underline{q}}$ in the sum, and extending μ as an $\mathbb{F}$-bimodule homomorphism, which is well-defined in light of the equations in Definition 6.1. Note that we used both the right and left $\mathbb{F}$-actions on $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$, that is: $(x\cdot\lambda)\otimes y=x\otimes(\lambda y)$ in the tensor product.

Lemma 6.5. *The multiplication μ endows $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ with the structure of an associative unital ring.*

Proof. Let V be the 1-dimensional $\mathbb{F}$-vector space spanned by v. Consider the $\Gamma(\mathcal{P})$-algebra B obtained as a quotient of the $\Gamma(\mathcal{P})$-algebra $\Gamma(\mathcal{P},A\oplus V)$ by the $\Gamma(\mathcal{P})$-ideal generated by the two following family of elements:

- $\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},v)$ for all x such that there exists $q_1+\cdots+q_u=r_s$, $y\in\mathcal{P}(n)^{\mathfrak{S}_{r_1}\times\cdots\times\mathfrak{S}_{r_{s-1}}\times\mathfrak{S}_{\underline{q}}}$ such that $x=\sum_{\sigma\in\mathfrak{S}_{n-r_s}\times\mathfrak{S}_{r_s}/\mathfrak{S}_{n-r_s}\times\mathfrak{S}_q}\sigma y$,

- For all $r_1+\cdots+r_s=n$, $x\in\mathcal{P}(n)^{\mathfrak{S}_{\underline{r}}}$, let $q_1+\cdots+q_u=k$, $y\in\mathcal{P}(k)^{\mathfrak{S}_{\underline{q}}}$, $(a_i)_{i\in[s+u]}\in A^{\times s+u}$, the element:

$$\left(\sum_{t+t'=r_sq_u,t'>0}\beta_{\sum_\tau \tau z,\underline{r}\diamond_s(\underline{q})\circ_{s+u}(t,t')}(a_1,\dots,a_{s+u},v)\right)-\left(\sum_{\lambda+\lambda'=r_s,\lambda'>0}\sum_{l+l'=q_u,l'>0}\beta_{\sum_{\sigma'}\sigma' z,R^{ll'}_{\lambda\lambda'}}(a_1,\dots,a_{s+u},v)\right),$$

where $z=x\left(1_{\mathcal{P}}^{\times n-r_s},y^{\times r_s}\right)\in\mathcal{P}(n+r_s(k-1))$, $R^{ll'}_{\lambda\lambda'}$ is defined in 4.5, where τ ranges over $\mathfrak{S}_{\underline{r}\diamond_s\underline{q}}/\left(\prod_{i=1}^{s-1}\mathfrak{S}_{r_i}\right)\times\mathfrak{S}_{r_s}\wr\mathfrak{S}_{\underline{q}}$ in the sum, and where σ' ranges over $\mathfrak{S}_{R^{ll'}_{\lambda\lambda'}}/\left(\prod_{i=1}^{s-1}\mathfrak{S}_{r_i}\right)\times\mathfrak{S}_\lambda\wr\mathfrak{S}_{\underline{q}}\times\mathfrak{S}_{\lambda'}\wr\mathfrak{S}_{\underline{q}\circ_u(l,l')}$ in the sum.

There is a linear map $f\colon\mathbb{U}_{\Gamma(\mathcal{P})}(A)\to B$ such that:

$$f(\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-))=\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},v).$$

Indeed, the relations ($\beta1$) to ($\beta8$) on B, and quotienting by the above family of elements ensure that f is well defined. The map f is clearly injective. Note that, for all pair of elements in $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ of the type:

$$\mathfrak{t}=\lambda\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-),\quad \mathfrak{s}=\lambda'\beta_{y,\underline{q}}(b_1,\dots,b_{u-1},-),$$

One has:

$$f(\mu(\mathfrak{t}\otimes\mathfrak{s}))=\lambda\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},\lambda'\beta_{y,\underline{q}}(b_1,\dots,b_{u-1},v)).$$

So, the associativity of μ is a consequence of the associativity of the composition of operations β in B, which is guaranteed by the associativity of the monad $\Gamma(\mathcal{P})$.

The element $\beta_{1_{\mathcal{P}},(1)}(-)$ is clearly a unit for μ. □

Warning 6.6. The ring $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ is an $\mathbb{F}$-vector space but need *not* be an $\mathbb{F}$-algebra. The multiplication μ satisfies:

$$\begin{aligned}
&\mu\left(\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-)\otimes\left(\lambda\beta_{y,\underline{q}}(b_1,\dots,b_{u-1},-)\right)\right)\\
&=\mu\left(\left(\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-)\cdot\lambda\right)\otimes\beta_{y,\underline{q}}(b_1,\dots,b_{u-1},-)\right)\\
&=\mu\left(\left(\lambda^{r_s}\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-)\right)\otimes\beta_{y,\underline{q}}(b_1,\dots,b_{u-1},-)\right)\\
&=\lambda^{r_s}\mu\left(\left(\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-)\right)\otimes\beta_{y,\underline{q}}(b_1,\dots,b_{u-1},-)\right).
\end{aligned}$$

We will see a concrete example with the enveloping algebra $V(A)=\mathbb{U}_{\Gamma(\mathrm{Com})}(A)$ in Section 9. In the special case $\mathbb{F}=\mathbb{F}_p$, the equality $\lambda^{r_s}=\lambda$ holds (since r_s is a power of p), and thus $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ *is* an $\mathbb{F}$-algebra.

Notation 6.7. We will allow the notation:

$$\beta_{x,\underline{r}}(a_1,\dots,a_{j-1},-,a_{j+1},\dots,a_s):=\beta_{\sigma_jx,\underline{r}^{\sigma_j^*}}(a_1,\dots,a_{j-1},a_{j+1},\dots,a_s,-)\in\mathbb{U}_{\Gamma(\mathcal{P})}(A).$$

Definition 6.8. Denote by $\mathbb{U}_{\mathrm{Ab}}(\Gamma(\mathcal{P}))\subset\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ the vector subspace spanned by the elements $\beta_{x,(n)}(-)$. Note that $\mathbb{U}_{\mathrm{Ab}}(\Gamma(\mathcal{P}))$ is a unital associative subring and does not depend on A.

Note that any abelian $\Gamma(\mathcal{P})$-algebra M can be equipped with a left $\mathbb{U}_{\mathrm{Ab}}(\Gamma(\mathcal{P}))$ action by:

$$\beta_{x,(n)}\otimes m\mapsto\beta_x(m),$$

and conversely, any left $\mathbb{U}_{\mathrm{Ab}}(\Gamma(\mathcal{P}))$-module M has a structure of abelian $\Gamma(\mathcal{P})$-algebra given by:

$$\beta_x(m):=\beta_{x,(n)}(-)\cdot m.$$

Since those assignments are inverse to one another, we obtain:

Proposition 6.9. *The category of abelian $\Gamma(\mathcal{P})$-algebra is equivalent to the category of left modules over $\mathbb{U}_{\mathrm{Ab}}(\Gamma(\mathcal{P}))$.*

We also obtain the straightforward corollary:

Corollary 6.10. *Let V be an $\mathbb{F}$-vector space. Then, $\mathbb{U}_{\mathrm{Ab}}(\Gamma(\mathcal{P})) \otimes V$ is the free abelian $\Gamma(\mathcal{P})$-algebra generated by V.*

Similarly, any A-module M can be equipped with a left $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ action by:

$$\beta_{x,\underline{r}}(a_1, \ldots, a_{s-1}, -) \otimes m \mapsto \beta_{x,\underline{r}}(a_1, \ldots, a_{s-1}, m),$$

and conversely, any left $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$-module M has an A-module structure given by:

$$\beta_{x,\underline{r}}(a_1, \ldots, a_{s-1}, m) := \beta_{x,\underline{r}}(a_1, \ldots, a_{s-1}, -) \cdot m.$$

Once again, those assignments are inverse to one another, and we obtain:

Theorem 6.11. *The category of A-modules is equivalent to the category of left modules over $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$.*

As a straightforward corollary, we also obtain:

Corollary 6.12. *Let V be an $\mathbb{F}$-vector space. Then, $\mathbb{U}_{\Gamma(\mathcal{P})}(A) \otimes V$ is the free A-module generated by V.*

From what precedes, we also obtain:

Corollary 6.13. *Let M be an abelian $\Gamma(\mathcal{P})$-algebra. Then, $\mathbb{U}_{\Gamma(\mathcal{P})}(A) \otimes_{\mathbb{U}_{\mathrm{Ab}(\Gamma(\mathcal{P}))}} M$ is the free A-module generated by M.*

The sequence of morphisms $\mathbb{F} \to \mathbb{U}_{\mathrm{Ab}(\Gamma(\mathcal{P}))} \to \mathbb{U}_{\Gamma(\mathcal{P})}(A)$ then yields a commutative diagram of free/forgetful adjunctions:

$$\begin{tikzcd}
\mathrm{Vect}_{\mathbb{F}} \arrow[r, shift left, "\mathbb{U}_{\mathrm{Ab}(\Gamma(\mathcal{P}))}\otimes -"] \arrow[dr, shift left, "\mathbb{U}_{\Gamma(\mathcal{P})}(A)\otimes -"] & (\Gamma(\mathcal{P})\text{-Alg})_{\mathrm{Ab}} \arrow[l, shift left] \arrow[d, shift left, "\mathbb{U}_{\Gamma(\mathcal{P})}(A)\otimes_{\mathbb{U}_{\mathrm{Ab}(\Gamma(\mathcal{P}))}} -"] \\
& A\text{-Mod} \arrow[ul, shift left] \arrow[u, shift left]
\end{tikzcd}$$

where the right adjoints are usual restriction of scalars.

Recall from [LV12, §12.3.4] the description for the enveloping algebra $\mathbb{U}_{\mathcal{P}}(A)$ associated to a $\mathcal{P}$-algebra (without divided powers) A. It has a set of generators denoted by $\nu(a_1, \ldots, a_k; 1)$ in [LV12], for all $\nu \in \mathcal{P}(k+1)$, $a_1, \ldots, a_k \in A$. Note that the norm map induces a monad morphism $\mathrm{Tr}\colon S(\mathcal{P}) \to \Gamma(\mathcal{P})$ (see [Fre00, §1.1.14]). This map has been translated in terms of divided power operations by the third author in [Iko23, Remark 6.3]. In consequence, we get the following:

Proposition 6.14. *For all $\Gamma(\mathcal{P})$-algebra A, the norm map induces a natural $\mathbb{F}$-linear ring homomorphism $\theta\colon \mathbb{U}_{\mathcal{P}}(A) \to \mathbb{U}_{\Gamma(\mathcal{P})}(A)$, sending $\nu(a_1, \ldots, a_k; 1) \in \mathbb{U}_{\mathcal{P}}(A)$ to $\beta_{\nu,(1,\ldots,1)}(a_1, \ldots, a_k, -) \in \mathbb{U}_{\Gamma(\mathcal{P})}(A)$.*

Note that here, in the expression $\mathbb{U}_{\mathcal{P}}(A)$, we have dropped the forgetful functor $\Gamma(\mathcal{P})\text{-Alg} \to \mathcal{P}\text{-Alg}$ which is induced by the norm map as well.

7. Derivations, Module of Kähler differentials

One of the key notions in the construction of André–Quillen cohomology is the abelianization functor, which is obtained as a left adjoint of the forgetful functor from the category of Beck A-modules to the slice category over A which "forgets" the abelian group structure (see [Qui67, §2.5], [Fra10]). In order to build and study such an abelianization functor, one needs to study morphisms of Beck modules. In the usual algebraic case, these morphisms are naturally identified with a certain set of algebraic derivations (see [Fra10, Bec67]). This justifies the terminology of Beck derivations for the maps $B \to M$ which are obtained by composing a morphism $B \to A \ltimes M$ of Beck modules with the projection onto M. In the case of algebras over an operad, Beck derivations correspond again to a natural notion of algebraic derivations (see [GH00, Proposition 2.2]). In these examples, the set of derivations is then represented by a module $\Omega_{\mathcal{P}}(A)$ called the module of Kähler differentials of A (see [LV12, §12.3.8]). In this section, we identify the set of Beck derivations over a $\Gamma(\mathcal{P})$-algebra, as well as an analogue of the A-module of Kähler differentials in the divided power setting.

Definition 7.1. Let M be an A-module, and $B \xrightarrow{pr} A$ be a $\Gamma(\mathcal{P})$-algebra over A. A **Beck derivation**, or simply **derivation** from B to M is a linear map $d\colon B \to M$ such that $pr + d\colon B \to A \ltimes M$ is a morphism of $\Gamma(\mathcal{P})$-algebra. We denote by $\mathrm{Der}_A(B, M)$ the vector space of derivations from B to M. We obtain a bifunctor

$$\mathrm{Der}_A\colon\ (\Gamma(\mathcal{P})\text{-Alg}/A)^{\mathrm{op}} \times A\text{-Mod} \to \mathrm{Vect}_{\mathbb{F}}.$$

Proposition 7.2. *A Beck derivation $d\colon B \to M$ is a linear map such that:*

$$d(\beta_{x,\underline{r}}(b_1,\dots,b_s)) = \sum_{j=1}^{s} \sum_{l+l'=r_j, l'>0} \beta_{x,\underline{r}\circ_j(l,l')}(pr(b_1),\dots,pr(b_j),d(b_j),pr(b_{j+1}),\dots,pr(b_s)).$$

Proof. The map $pr + d$ is a $\Gamma(\mathcal{P})$-algebra if and only if:

$$(pr+d)(\beta_{x,\underline{r}}(b_1,\dots,b_s)) = \beta_{x,\underline{r}}((pr(b_1),d(b_1)),\dots,(pr(b_s),d(b_s)))$$
$$= \left(\beta_{x,\underline{r}}(pr(b_1),\dots,pr(b_s)), \sum_{j=1}^{s}\sum_{l,l'} \beta_{x,\underline{r}\circ_j(l,l')}(pr(b_1),\dots,pr(b_j),d(b_j),pr(b_{j+1}),\dots,pr(b_s)).\right),$$

where l, l' runs over the pairs of non-negative integers such that $l + l' = r_j$ and $l' > 0$, hence the result. □

Definition 7.3. For any $\Gamma(\mathcal{P})$-algebra A, denote by dA the underlying vector space of A. Elements of dA are denoted by da for $a \in A$.

The **module of Kähler differentials** of A is the following coequalizer in the category of A-modules:

$$\mathbb{U}_{\Gamma(\mathcal{P})}(A) \otimes \Gamma(\mathcal{P}, A) \underset{\mathbb{U}_{\Gamma(\mathcal{P})}(A)\otimes ev_A}{\overset{\gamma}{\rightrightarrows}} \mathbb{U}_{\Gamma(\mathcal{P})}(A) \otimes dA \twoheadrightarrow \Omega_{\Gamma(\mathcal{P})}(A)\ ,$$

where $\gamma\colon \mathbb{U}_{\Gamma(\mathcal{P})}(A) \otimes \Gamma(\mathcal{P}, A) \to \mathbb{U}_{\Gamma(\mathcal{P})}(A) \otimes dA$ is given by:

$$\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-) \otimes \lambda\beta_{y,\underline{q}}(b_1,\dots,b_u) \mapsto$$
$$\lambda^{r_s} \sum_{j=1}^{u} \sum_{l+l'=q_j, l'>0} \mu\left(\beta_{x,\underline{r}}(a_1,\dots,a_{s-1},-) \otimes \beta_{y,\underline{q}\circ_j(l,l')}(b_1,\dots,b_j,-,b_{j+1},\dots,b_u)\right) \otimes db_j,$$

and where $ev_A\colon \Gamma(\mathcal{P}, A) \to A$ is the structural $\Gamma(\mathcal{P})$-algebra evaluation map of A.

Let us describe $\Omega_{\Gamma(\mathcal{P})}(A)$ in more detail. For an element of the type $\beta_{x,\underline{r}}(a_1, \dots, a_{a-1}, -) \otimes da \in \mathbb{U}_{\Gamma(\mathcal{P})}(A) \otimes dA$, we will denote by $\beta_{x,\underline{r}}(a_1, \dots, a_{a-1}, da)$ its image in $\Omega_{\Gamma(\mathcal{P})}(A)$. Following Notation 4.6, we will allow the notation:

$$\beta_{x,\underline{r}}(a_1, \dots, a_{i-1}, da, a_{i+1}, \dots, a_s) := \beta_{\sigma_i^*,\underline{r}^{\sigma_i}}(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_s, da).$$

Then, by definition, $\Omega_{\Gamma(\mathcal{P})}(A)$ is an A-module generated by the elements $\beta_{x,\underline{r}}(a_1, \dots, a_{a-1}, da)$ under certain relations, including the following:

$$d\left(\beta_{x,\underline{r}}(a_1, \dots, a_s)\right) = \sum_{i=1}^{s} \sum_{l+l'=r_i, l'>0} \beta_{x,\underline{r}\circ_i(l,l')}(a_1, \dots, a_{j-1}, a_j, da_j, a_{j+1}, \dots, a_s). \tag{7.eq.1}$$

Definition 7.4. The **universal derivation** of A is the linear map $d\colon A \to \Omega_{\Gamma(\mathcal{P})}(A)$ induced by the identity map $A \to dA$.

As a consequence of the relation (7.eq.1), we deduce:

Proposition 7.5. *The universal derivation $d\colon A \to \Omega_{\Gamma(\mathcal{P})}(A)$ is a Beck derivation.*

The following result justifies the term "universal" derivation, and completes the analogy with the classical module of Kähler differentials [LV12, 12.3.19]:

Proposition 7.6. $\Omega_{\Gamma(\mathcal{P})}(A)$ *represents the functor* $\mathrm{Der}(A, -)\colon A\text{-Mod} \to \mathrm{Ab}$ *sending M to the abelian group of derivations from A to M.*

Proof. We have to show that for all A-module M, there is a linear bijection, natural in M:

$$\mathrm{Hom}_{A\text{-Mod}}(\Omega_{\Gamma(\mathcal{P})}(A), M) \cong \mathrm{Der}_A(A, M).$$

Let $f\colon \Omega_{\Gamma(\mathcal{P})}(A) \to M$ be an A-module morphism. Consider the linear map $D\colon A \to M$ given by $D = f \circ d$, where $d\colon A \to \Omega_{\Gamma(\mathcal{P})}(A)$ is the universal derivation. Since the universal derivation is a derivation, and since f is an A-module morphism, D is a derivation. The assignment $f \mapsto D$ yields a linear map $\phi\colon \mathrm{Hom}_{A\text{-Mod}}(\Omega_{\Gamma(\mathcal{P})}(A), M) \to \mathrm{Der}_A(A, M)$, natural in M.

Reciprocally, let $D\colon A \to M$ be a derivation. We can consider D as a linear map $dA \to M$. This extends uniquely into an A-module morphism $\mathbb{U}_{\Gamma(\mathcal{P})}(A) \otimes dA \to M$. The fact that D is a derivation ensures that this passes to the coequalizer into an A-module morphism $f\colon \Omega_{\Gamma(\mathcal{P})}(A) \to M$. This assignment $D \mapsto f$ yields a linear map $\psi\colon \mathrm{Der}_A(A, M) \to \mathrm{Hom}_{A\text{-Mod}}(\Omega_{\Gamma(\mathcal{P})}(A), M)$. It is easy to check that ϕ and ψ are mutually inverse and natural in M. ☐

To conclude this section, let us link this new notion of Kähler differentials for a $\Gamma(\mathcal{P})$-algebra to the usual notion of Kähler differentials for a $\mathcal{P}$-algebra. Recall that $U_{\mathcal{P}}^{\Gamma(\mathcal{P})}\colon \Gamma(\mathcal{P})\text{-Alg} \to \mathcal{P}\text{-Alg}$ denotes the forgetful functor from $\Gamma(\mathcal{P})$-algebras to $\mathcal{P}$-algebras. Then, if A is a $\Gamma(\mathcal{P})$-algebra, $\Omega_{\mathcal{P}}(A) := \Omega_{\mathcal{P}}(U_{\mathcal{P}}^{\Gamma(\mathcal{P})}(A))$ is the usual $U_{\mathcal{P}}^{\Gamma(\mathcal{P})}(A)$-module of Kähler differentials of the $\mathcal{P}$-algebra $U_{\mathcal{P}}^{\Gamma(\mathcal{P})}(A)$, as in [LV12, 12.3.8].

Proposition 7.7. *The map $\theta \otimes dA\colon \mathbb{U}_{\mathcal{P}}(A) \otimes dA \to \mathbb{U}_{\Gamma(\mathcal{P})}(A) \otimes dA$, where θ is defined in Proposition 6.14, induces a map $\theta\colon \Omega_{\mathcal{P}}(A) \to \Omega_{\Gamma(\mathcal{P})}(A)$ given by*

$$\nu(a_1, \dots, da_i, \dots, a_s) \mapsto \beta_{\nu,(1,\dots,1)}(a_1, \dots, da_i, \dots, a_s).$$

We leave to the reader to check that the given map passes to the coequalizers.

8. Quillen cohomology

We now have all the ingredients to describe Quillen (co)homology of $\Gamma(\mathcal{P})$-algebras, as listed in Section 2.

Lemma 8.1. *Let $f\colon B \to A$ be a morphism of $\Gamma(\mathcal{P})$-algebras. Via the equivalence from Theorem 5.2, the pushforward along f is given by $f_!(M) = \mathbb{U}_{\Gamma(\mathcal{P})}(A) \otimes_{\mathbb{U}_{\Gamma(\mathcal{P})}(B)} M$.*

Proof. The universal enveloping algebra $\mathbb{U}_{\Gamma(\mathcal{P})}$ provides a functor $\Gamma(\mathcal{P})$-Alg $\to$ As_+-Alg to unital rings. The ring homomorphism $\mathbb{U}_{\Gamma(\mathcal{P})}(f)\colon \mathbb{U}_{\Gamma(\mathcal{P})}(B) \to \mathbb{U}_{\Gamma(\mathcal{P})}(A)$ makes $\mathbb{U}_{\Gamma(\mathcal{P})}(A)$ into a right $\mathbb{U}_{\Gamma(\mathcal{P})}(B)$-module. Via the equivalence from Theorem 6.11, the pushforward adjunction

$$B\text{-Mod} \underset{f^*}{\overset{f_!}{\rightleftarrows}} A\text{-Mod}$$

corresponds to the classical restriction/extension of scalars along $\mathbb{U}_{\Gamma(\mathcal{P})}(f)$. □

Theorem 8.2. *Let A be a $\Gamma(\mathcal{P})$-algebra.*

(1) The following two functors form an adjoint pair:

$$\Gamma(\mathcal{P})\text{-Alg}/A \underset{A\ltimes -}{\overset{\mathbb{U}_{\Gamma(\mathcal{P})}(A)\otimes_{\mathbb{U}_{\Gamma(\mathcal{P})}(-)}\Omega_{\Gamma(\mathcal{P})}(-)}{\rightleftarrows}} A\text{-Mod}.$$

(2) The adjunction simplicially prolongs to a Quillen pair.

Proof. (1) Via the equivalences from Theorem 5.2 and Theorem 6.11, the statement follows from combining Proposition 7.6, Lemma 8.1, and Lemma 2.4.

(2) Via the equivalence from Theorem 6.11, the right adjoint $A \ltimes -$ is the forgetful functor

$$(\Gamma(\mathcal{P})\text{-Alg}/A)_{\mathrm{ab}} \to \Gamma(\mathcal{P})\text{-Alg}/A.$$

The claim then follows from [Fra15, Proposition 3.40]. □

9. Example: The operad Com

In this section, we apply our general construction for Beck modules, universal enveloping algebra, Beck derivations and Kähler differentials for divided power algebras over the operad Com of associative, commutative (non-unital) algebra. We check that these correspond to the construction obtained by the first author in [Dok09, Dok23].

For readability, we will divide this section into two subsections. In the first one, we recall the definition for classical divided power algebras. We refer the reader to [Car56, Rob65] and [Ber74, §I] for more details. We then review the characterisation of Beck modules, universal enveloping algebra, Beck derivations and Kähler differentials for these objects obtained by the first author [Dok09, Dok23].

In the second subsection, we show how these characterisations correspond to those given in this article for a general operad $\mathcal{P}$, once we set $\mathcal{P} = \mathrm{Com}$.

9.1. Classical definition and state of the art

Definition 9.1. Let (A, I) be a commutative ring together with an ideal $I \subset A$. **A system of divided powers** on I is a collection of maps $\gamma_i \colon I \to A$, where $i \geq 0$, such that the following identities hold:

$$\gamma_0(a) = 1, \tag{9.eq.1}$$

$$\gamma_1(a) = a, \gamma_i(a) \in I,\ i \geq 1, \tag{9.eq.2}$$

$$\gamma_i(a+b) = \sum_{k=0}^{k=i} \gamma_k(a)\gamma_{i-k}(b),\ a, b \in I,\ i \geq 0, \tag{9.eq.3}$$

$$\gamma_i(ab) = a^i \gamma_i(b),\ a \in A,\ i \geq 0, \tag{9.eq.4}$$

$$\gamma_i(a)\gamma_j(a) = \frac{(i+j)!}{i!j!}\gamma_{i+j}(a),\ a \in I,\ i, j \geq 0, \tag{9.eq.5}$$

$$\gamma_i(\gamma_j(a)) = \frac{(ij)!}{i!(j!)^i}\gamma_{ij}(a),\ a \in I,\ i \geq 0,\ j \geq 1. \tag{9.eq.6}$$

We say that (I, γ) is a **PD ideal** of A. We call **divided power ring** the data of a triple (A, I, γ) where A is a ring, and (I, γ) is a PD ideal. A morphism of divided power rings

$$f \colon (A, I, \gamma) \to (B, J, \delta)$$

is a ring homomorphism $f \colon A \to A'$ such that $f(I) \subset J$ and such that $f(\gamma_i(a)) = \delta_i(f(a))$ for all $i \geq 0$ and $a \in I$.

Note that the identities (9.eq.2) and (9.eq.5) imply that $a^n = n!\gamma_n(a)$, where $n \in \mathbf{N}$ and $a \in I$. In particular, in a divided power ring of characteristic 0, $\gamma_n(a) = \frac{a^n}{n!}$ for all $a \in I$. This justifies the name "divided powers". In prime characteristic p, one has $a^p = 0$, for all $a \in I$.

Notation 9.2. Given an augmented $\mathbb{F}$-algebra A with augmentation $\epsilon \colon A \to \mathbb{F}$, denote the augmentation kernel by $A_+ = \ker(A \to \mathbb{F})$.

Given a non-unital $\mathbb{F}$-algebra B, denote by $B^+ = \mathbb{F} \oplus B$ the augmented algebra obtained by formally adjoining a unit.

We follow the notation $A_+ = \ker(A \to \mathbb{F})$ used in [Sou87] and [Dok09], though some authors use a different notation, notably $\overline{A}$ in [LV12, §1.1].

We now fix a field $\mathbb{F}$ of prime characteristic $p \neq 0$. We restrict to the case of divided power rings (A, I, γ) such that A is an augmented $\mathbb{F}$-algebra $A = \mathbb{F} \oplus A_+$, and $I = A_+$. In this setting, Soublin showed that the divided power structure is entirely determined by a map π playing the role of γ_p:

Proposition 9.3 ([Sou87])**.** *The category of divided power augmented $\mathbb{F}$-algebras (A, A_+, γ) is equivalent to the category* p-Com *with objects the pairs (A, π), where A is an augmented algebra with augmentation ideal A_+, and where $\pi \colon A_+ \to A_+$ is a set map satisfying:*

$$a^p = 0,\ a \in A_+, \tag{9.eq.7}$$

$$\pi(a+b) = \pi(a) + \pi(b) + \sum_{k=1}^{k=p-1} \frac{(-1)^k}{k} a^k b^{p-k},\ a, b \in A_+, \tag{9.eq.8}$$

$$\pi(ab) = 0,\ a, b \in A_+, \tag{9.eq.9}$$

$$\pi(\lambda a) = \lambda^p \pi(a),\ a \in A_+, \lambda \in \mathbb{F}, \tag{9.eq.10}$$

and where morphisms $\alpha \colon (A, \pi) \to (A', \pi')$ are homomorphism of augmented algebras $\alpha \colon A \to A'$ such that $\alpha \circ \pi = \pi' \circ \alpha$.

Beck modules

Fix a divided power augmented $\mathbb{F}$-algebra A, or equivalently, an object (A, π) of the category p-Com defined above. Then, the first author obtained the following characterisation for Beck A-modules:

Theorem 9.4 ([Dok09]). *The category of Beck A-modules is equivalent to the category $\mathcal{M}$ whose objects are pairs (M, π) where M is a A-module and $\pi\colon M \to M$ is a p-semilinear map such that $\pi(am) = 0$ for all $a \in A_+$ and $m \in M$ and whose morphisms $(M, \pi) \to (M', \pi')$ are A-module homomorphisms $\alpha\colon M \to M'$ such that $\pi' \circ \alpha = \alpha \circ \pi$.*

Enveloping algebra

Let us now build the universal enveloping algebra of A, which is the representing object for the category of Beck A-modules. Let R_f be the polynomial ring consisting of the set of polynomials $\sum_{i=0}^{i=m} \lambda_i f^i$ where $\lambda_i \in \mathbb{F}$, f is an indeterminate and $f\lambda = \lambda^p f$.

We define the ring $V(A)$ as the ring whose underlying $\mathbb{F}$-vector space is the tensor product $V(A) := A \otimes_{\mathbb{F}} R_f$ and the multiplication is given by:

$$
\begin{aligned}
&(a \otimes 1)(a' \otimes 1) = (aa' \otimes 1), \;\; a, a' \in A,\\
&(1 \otimes q)(1 \otimes q') = (1 \otimes qq'), \;\; q, q' \in R_f,\\
&(a \otimes 1)(1 \otimes f) = (a \otimes f), \; a \in A,\\
&(1 \otimes f)(a \otimes 1) = 0, \; a \in A_+,\\
&(1 \otimes f)(\lambda \otimes 1) = (\lambda^p \otimes f), \; \lambda \in \mathbb{F}.
\end{aligned}
$$

Then, the first author obtained the following:

Theorem 9.5 ([Dok09]). *The category of Beck A-modules is equivalent to the category of left $V(A)$-modules.*

Beck derivations

Let $A' \in p\text{-Com}/A$ be a divided powers algebra over A and (M, π) a Beck A-module. The first author obtained the following:

Proposition 9.6 ([Dok09]). *The abelian group of Beck derivations of A' into M is given by*

$$\mathrm{Der}_p(A', (M, \pi)) = \{d \in \mathrm{Der}(A', M) |\, d(\pi(a)) = \pi(d(a)) - a^{p-1} d(a),\, a \in A'_+\}.$$

Kähler differentials

We then get the following characterisation for the module of Kähler differentials:

Theorem 9.7 ([Dok09]). *The module of Kähler differentials for the augmented divided power $\mathbb{F}$-algebra A is the $V(A)$-module $\Omega_{\text{p-Com}}(B)$ with the following presentation: the generators are the symbols da for $a \in A$, and the relations are*

$$d(\lambda a + \mu b) = \lambda da + \mu db, \tag{9.eq.11}$$

$$d(ab) = adb + bda, \tag{9.eq.12}$$

$$d(\pi(c)) = f dc - c^{p-1} dc, \tag{9.eq.13}$$

where $a, b \in A$, $\lambda, \mu \in \mathbb{F}$, and $c \in A_+$.

9.2. Operadic point of view

Let us now show how we recover these notions from Sections5, 6 and 7. Recall that the operad Com of non-unital commutative algebras is defined as a symmetric sequence by

$$\mathrm{Com}(n) = \begin{cases} T_n, & \text{if } n > 0,\\ \mathbf{0}, & \text{if } n = 0, \end{cases}$$

where T_n denotes the trivial representation of dimension 1 generated by an element we denote by X_n, and with partial compositions given by $X_n \circ_i X_k = X_{n+k-1}$. We get the following:

Theorem 9.8 ([Fre00])**.** *A* $\Gamma(\mathrm{Com})$*-algebra is a (non-unital) associative, commutative algebra* B *equipped, for all* $i > 0$*, with a set map* $\gamma_i \colon B \to B$ *satisfying the relations* (9.eq.2) *to* (9.eq.6) *from Definition 9.1.*

Note that there is a slight abuse of notation here: relation (9.eq.3) should be replaced by:

$$\gamma_i(a+b) = \gamma_i(a) + \gamma_i(b) + \sum_{k=1}^{k=i-1} \gamma_k(a)\gamma_{i-k}(b).$$

The category of $\Gamma(\mathrm{Com})$-algebras is then equivalent to the category of divided power augmented $\mathbb{F}$-algebras. Indeed, if B is a $\Gamma(\mathrm{Com})$-algebra, then $B^+ = \mathbb{F} \oplus B$ is equipped with a unique structure of divided power augmented $\mathbb{F}$-algebra (B^+, B, δ) such that $\delta_i(b) = \gamma_i(b)$ for all $i > 0$ and $b \in B$. Conversely, if (A, A_+, γ) is an augmented divided power $\mathbb{F}$-algebra, then, the collection of maps γ_i for $i > 0$ equips A_+ with a structure of $\Gamma(\mathrm{Com})$-algebra as above. Following Soublin's theorem, a $\Gamma(\mathrm{Com})$-algebra is equivalently characterised as a (non-unital) associative, commutative algebra B equipped with a map $\pi \colon B \to B$ satisfying relations (9.eq.7) to (9.eq.10) of Proposition 9.3.

Let us now use the characterisation of $\Gamma(\mathrm{Com})$-algebra from Theorem 3.3. Following [Iko20], if B is a $\Gamma(\mathrm{Com})$-algebra, its multiplication is given by $\beta_{X_2,(1,1)}$, and π, which represents the divided power operation γ_p, is given by $\beta_{X_p,(p)}$.

Abelian algebras, Beck modules

Following Definition 4.1, an abelian $\Gamma(\mathrm{Com})$-algebra M is a vector space M endowed with a trivial multiplication $mm' = 0$ and a semilinear map $\pi \colon M \to M$ satisfying $\pi(\lambda m) = \lambda^p \pi(m)$.

Following Definition 4.5, a B-module M becomes a module M over the commutative algebra B, endowed with a trivial multiplication, and a semilinear map $\pi \colon M \to M$ satisfying $\pi(\lambda m) = \lambda^p \pi(m)$ and such that $\pi(bm) = 0$. This last property comes from relation (βAM8), noticing that $|\mathfrak{S}_{(p)\diamond(1,1)}/\mathfrak{S}_p \wr (\mathfrak{S}_{(1,1)})| = p!$, and using ($\beta$AM6):

$$\begin{aligned}
\pi(bm) &= \beta_{X_p,(p)}(\beta_{X_2,(1,1)}(b,m)),\\
&= \beta_{p!X_{2p},(p,p)}(b,m),\\
&= p!\beta_{X_{2p},(p,p)}(b,m) = 0.
\end{aligned}$$

Enveloping algebra

From Definition 6.1 the universal enveloping algebra $\mathbb{U}_{\Gamma(\mathrm{Com})}(B)$ is spanned by symbols

$$\beta_{X_n,\underline{r}}(b_1, \ldots, b_{s-1}, -) \quad \text{with} \quad r_1 + \cdots + r_s = n.$$

We can reduce this generating family: using the relations of Definition 6.1, and the same reasoning as in [Iko20, §3.3] and [Sou87], $\mathbb{U}_{\Gamma(\mathrm{Com})}(B)$ is spanned by symbols $\beta_{X_1,(1)}(-)$ (the unit), $\beta_{X_2,(2)}(b,-)$ for $b \in B$, and $\beta_{X_p,(p)}(-)$.

We can then show that there is an isomorphism between $\mathbb{U}_{\Gamma(\mathrm{Com})}(B)$ and $V(B^+)$, where $V(B)$ was defined in Section 9.1. This isomorphism sends $\beta_{X_1,(1)}(-)$ to $1 \otimes 1$, $\beta_{X_2,(1,1)}(b,-) \in \mathbb{U}_{\Gamma(\mathrm{Com})}(B)$ to $b \otimes 1$, and $\beta_{X_p,(p)}(-) \in \mathbb{U}_{\Gamma(\mathrm{Com})}(B)$ to $1 \otimes f$.

Note that B injects into $\mathbb{U}_{\Gamma(\mathrm{Com})}(B)$, by identifying $b \in B$ to $\beta_{X_2,(1,1)}(b,-)$. For a B-module (i.e., a $\mathbb{U}_{\Gamma(\mathrm{Com})}(B)$-module) M, we will denote by $bm := \beta_{X_2,(1,1)}(b,m) = \beta_{X_2,(1,1)}(b,-) \cdot m$, and $\pi(m) := \beta_{X_p,(p)}(m)$, for $b \in B$, $m \in M$.

Beck derivations

Following Proposition 7.2, a Beck derivation $d\colon B \to M$ is a linear map $d\colon B \to M$ satisfying $d(ab) = adb + bda$ and

$$d(\pi(b)) = \sum_{i=1}^{p} \beta_{X_p,(p-i,i)}(b, d(b)).$$

Note that, for all i such that $2 \leq i \leq p-1$,

$$X_p = \frac{1}{i!} i! X_p = \frac{1}{i!} \sum_{\tau \in \mathfrak{S}_{p-i} \times \mathfrak{S}_i / \mathfrak{S}_{p-i} \times \mathfrak{S}_1^{\times i}} \tau X_p,$$

and so,

$$\begin{aligned}\beta_{X_p,(p-i,i)}(b, db) &= \beta_{\frac{1}{i!} \sum_{\tau \in \mathfrak{S}_{p-i} \times \mathfrak{S}_i / \mathfrak{S}_{p-i} \times \mathfrak{S}_1^{\times i}} \tau X_p,(p-i,i)}(b, db) \\ &= \frac{1}{i!} \beta_{\sum_{\tau \in \mathfrak{S}_{p-i} \times \mathfrak{S}_i / \mathfrak{S}_{p-i} \times \mathfrak{S}_1^{\times i}} \tau X_p,(p-i,i)}(b, db),\end{aligned}$$

and according to relation (βM4), $\beta_{\sum_{\tau \in \mathfrak{S}_{p-i} \times \mathfrak{S}_i / \mathfrak{S}_{p-i} \times \mathfrak{S}_1^{\times i}} \tau X_p,(p-i,i)}(b, db) = 0$. So,

$$\begin{aligned}d(\pi(b)) &= \beta_{X_p,(p-1,1)}(b, db) + \beta_{X_p,(0,p)}(b, db), \\ &= \beta_{X_2,(1,1)}(\beta_{X_{p-1},(p-1)}(b), db) + \beta_{X_p,(p)}(db), \\ &= \frac{b^{p-1}}{(p-1)!} db + \pi(db).\end{aligned}$$

Finally, since in characteristic p, $(p-1)! = -1$ (see Wilson's Theorem [DF04, §13.5 Exercise 6]), one has:

$$d(\pi(b)) = \pi(d(b)) - b^{p-1} d(b).$$

We recover the characterisation of Beck derivations given in [Dok09].

Kähler Differentials

Following Definition 7.3, the module of Kähler differentials $\Omega_{\Gamma(\mathrm{Com})}(B)$ of B is the $\mathbb{U}_{\Gamma(\mathrm{Com})}(B)$-module generated by elements db for $b \in B$, linear in b, under the relations:

$$d(ab) = d(\beta_{X_2,(1,1)}(a, b)) = \beta_{X_2,(1,1)}(-, b) \otimes da + \beta_{X_2,(1,1)}(a, -) \otimes db = bda + adb,$$

$$d(\pi(b)) = d(\beta_{X_p,(p)}(b)) = \sum_{i=1}^{p} \beta_{X_p,(p-i,i)}(b, -) \otimes db.$$

For this last relation, note that the term corresponding to $i = 1$ is:

$$\beta_{X_p,(p-1,1)}(b, -) \otimes b = \beta_{X_2,(1,1)}(\beta_{X_{p-1},(p-1)}(b), -) \otimes db = \frac{1}{(p-1)!} b^{p-1} db = -b^{p-1} db.$$

the term corresponding to $i = p$ is

$$\beta_{X_p,(0,p)}(b, -) \otimes b = \beta_{X_p,(p)}(-) \otimes db = \pi(db).$$

For all $i \in \{2, \ldots, p-1\}$, we have again $X_p = i \frac{1}{i} X_p = \sum_{\sigma \in \mathfrak{S}_{p-i} \times \mathfrak{S}_i / \mathfrak{S}_{p-i} \times \mathfrak{S}_1 \times \mathfrak{S}_{i-1}} \frac{1}{i} X_p$. So, $\beta_{X_p,(p-i,i)}(b, -) = \sum_{\sigma \in \mathfrak{S}_{p-i} \times \mathfrak{S}_i / \mathfrak{S}_{p-i} \times \mathfrak{S}_1 \times \mathfrak{S}_{i-1}} \frac{1}{i} X_p$ is one of the elements described in Definition 6.1 in point (7), and so, is equal to 0 in $\mathbb{U}_{\Gamma(\mathrm{Com})}(B)$. Finally, this last relation reads:

$$d(\pi(b)) = \pi(db) - b^{p-1} db.$$

We can show that this module of Kähler differentials $\Omega_{\Gamma(\mathrm{Com})}(B)$ is isomorphic to the module of Kähler differentials $\Omega_{\text{p-Com}}(B^+)$ of the augmented divided power $\mathbb{F}$-algebra B^+. This isomorphism is transparent on elements db, adb, and sends $\pi(db)$ to $f \cdot db$.

10. Example: The operad Lie

In this section, we apply our general construction for Beck modules, universal enveloping algebra, Beck derivations and Kähler differentials for divided power algebras over the operad Lie of Lie algebra. We check that these correspond to the construction obtained by the first author in [Dok04]. For readability, we will divide this section into two subsections: in the first one, we recall the definition for restricted Lie algebras. We refer the reader to [Jac62] for more details. We then review the characterisation of Beck modules, Beck derivations for these objects obtained by the first author [Dok04].

In the second subsection, we show how these characterisations correspond to those given in this article for a general operad $\mathcal{P}$, once we set $\mathcal{P} = \mathrm{Lie}$.

10.1. Classical definition and state of the art

We suppose that $\mathbb{F}$ is a field of prime characteristic $p \neq 0$.

Definition 10.1 ([Jac62, §V.7])**.** A **restricted Lie algebra** $L = (L, (-)^{[p]})$ over $\mathbb{F}$ is a Lie algebra over $\mathbb{F}$ together with a map $(-)^{[p]}\colon L \to L$ called the p-map such that the following relations hold

$$(\alpha l)^{[p]} = \alpha^p\, l^{[p]} \tag{10.eq.1}$$

$$[l, l'^{[p]}] = [\cdots [l, \underbrace{l'], l'], \cdots, l'}_{p}] \tag{10.eq.2}$$

$$(l + l')^{[p]} = l^{[p]} + l'^{[p]} + \sum_{i=1}^{p-1} s_i(l, l') \tag{10.eq.3}$$

where $is_i(l, l')$ is the coefficient of λ^{i-1} in $ad^{p-1}_{\lambda l + l'}(l)$. Here, $ad_l\colon L \to L$ denotes the adjoint representation given by $ad_l(l') := [l', l]$, $l, l' \in L$, $\alpha \in \mathbb{F}$. A Lie algebra homomorphism $f\colon L \to L'$ is called **restricted** if $f(l^{[p]}) = f(l)^{[p]}$. We denote by RLie the category of restricted Lie algebras over $\mathbb{F}$.

Example 10.2. Let A be any associative algebra over a field $\mathbb{F}$. We denote by A_{Lie} the induced Lie algebra with the bracket given by $[l, l'] := ll' - l'l$, for all $l, l' \in A$. Then $(A_{\mathrm{Lie}}, (-)^{[p]})$ is a restricted Lie algebra where $(-)^{[p]}$ is the p-th power $l \mapsto l^p$. Thus, there is a functor

$$(-)_{\mathrm{RLie}}\colon \mathrm{As} \to \mathrm{RLie}$$

from the category of associative algebras to the category of restricted Lie algebras.

Example 10.3. Let G be an algebraic group over $\mathbb{F}$. The associated Lie algebra $Lie(\mathrm{G})$ of G is endowed with the structure of restricted Lie algebra [Bor91, §I.3] [Wat79, §12.1].

Beck modules

Let L be a Lie algebra over $\mathbb{F}$. A **Lie module** over L is a $\mathbb{F}$-vector space M equipped with a $\mathbb{F}$-bilinear map $L \otimes_{\mathbb{F}} M \to M\colon\ l \otimes m \mapsto lm$ such that

$$[l, l']m = l(l'm) - l'(lm), \text{ for all } l, l' \in L \text{ and } m \in M.$$

Definition 10.4. Let L be a restricted Lie algebra over $\mathbb{F}$. A Lie L-module M is called **restricted** if $l^{[p]}m = (\underbrace{l(\cdots(l(l}_{p}\, m)\cdots)$.

Let M be a restricted L-module. We denote by M^L, the following L-submodule of M:

$$M^L = \{m \in M : lm = 0 \text{ for all } l \in L\}.$$

The first author obtained the following characterisation for Beck L-modules.

Theorem 10.5 ([Dok04]). *The category of Beck L-modules is equivalent to the category $\mathcal{M}$ whose objects are pairs (M, f) where M is a restricted L-module and $f\colon M \to M^L$ is a p-semilinear map from M into its submodule of invariants M^L and whose morphisms $(M_1, f_1) \to (M_2, f_2)$ are L homomorphisms $\alpha\colon M_1 \to M_2$ such that $f_2 \circ \alpha = \alpha \circ f_1$.*

When no confusion arises, we will always denote by $f\colon M \to M$ the p-semilinear map of a Beck L-module M.

Enveloping algebra

Let us now build the universal enveloping algebra of L, which is the representing object for the category of Beck L-modules. Let $L \in \mathrm{RLie}$ be a restricted Lie algebra and $U(L)$ its usual enveloping algebra [Jac62, §V.1]. We first recall the construction of the restricted enveloping algebra $u(L)$ which is a representing object for restricted L-modules.

Definition 10.6. We denote by $u(L)$ the quotient of the algebra $U(L)$ by the relations $l^p - l^{[p]}$ for $l \in L$, which we call the **restricted enveloping algebra** of L.

This construction provides a functor $u\colon \mathrm{RLie} \to \mathrm{As}$.

Theorem 10.7 ([Jac62, §V.7]). *The category of restricted L-modules is equivalent to the category of $u(L)$-modules.*

Following N. Jacobson (see [Jac62, §V.2]), the functor u is part of an adjunction

$$u\colon \mathrm{RLie} \rightleftarrows \mathrm{As}\colon (-)_{\mathrm{RLie}}$$

Following [Dok04], denote by $w(L)$ the $\mathbb{F}$-vector space $R_f \otimes_{\mathbb{F}} u(L)$, where R_f is the polynomial ring on one indeterminate f as in Section 9. Then, $w(L)$ equipped with a ring structure such that $R_f \to w(L)$ and $u(L) \to w(L)$ are ring homomorphisms, and such that:

$$(f \otimes 1)(1 \otimes l) := f \otimes l \quad \text{and} \quad (1 \otimes l)(f \otimes 1) := 0$$

for all $l \in L$. The first author obtained the following characterization of Beck modules, which yields $\mathbb{U}_{\Gamma(\mathrm{Lie})}(L) = w(L)$.

Theorem 10.8 ([Dok04]). *The category of Beck L-modules is equivalent to the category of left $w(L)$-modules.*

Lemma 10.9. *The ring homomorphism $\theta\colon \mathbb{U}_{\mathrm{Lie}}(L) \to \mathbb{U}_{\Gamma(\mathrm{Lie})}(L)$ from Proposition 6.14 is the composite*

$$U(L) \twoheadrightarrow u(L) \to w(L)$$

where $U(L) \twoheadrightarrow u(L)$ is the quotient map and $u(L) \to w(L)$ is the ring homomorphism $x \mapsto 1 \otimes x$.

Beck derivations

Let M be a Lie L-module, a **derivation** of L into M is a $\mathbb{F}$-linear map $D\colon L \to M$ such that the Leibniz formula holds

$$D([l, l']) = lD(l') - l'D(l)$$

for all $l, l' \in L$. The set of such derivations is denoted by $\mathrm{Der}(L, M)$. Let $L' \in RLie/L$ be a restricted Lie algebra over L, and (M, f) a Beck L-module. Then, the first author obtained the following

Proposition 10.10 ([Dok04]). *The abelian group of Beck derivations of L' into M is given by:*

$$\mathrm{Der}_p(L', (M, f)) := \{d \in \mathrm{Der}(L', M) : d(l^{[p]}) = \underbrace{l \cdots l}_{p-1} dl + f(d(l)),\ l \in L'\}$$

Kähler differentials

What we call the module of Kähler differentials of the Lie algebra L is, by analogy with the case of commutative algebras, the L-Beck module $\Omega_{\mathrm{RLie}}(L)$ which represents the functor of Beck derivations. It follows from Theorem 10.8, Proposition 10.10 and [Par68, Lemma 2.1] that the module $\Omega_{\mathrm{RLie}}(L)$ of Kähler differentials is nothing else than the $w(L)$-module $C(L)$ considered by Pareigis in [Par68], cf. [Dok04, §1.3].

Theorem 10.11 ([Par68])**.** *The module of Kähler differentials $\Omega_{\mathrm{RLie}}(L)$ for the restricted Lie algebra L is the $w(L)$-module with the following presentation: the generators are the symbols dl for $l \in L$, and the relations are:*

$$d(\lambda l + \mu l') = \lambda dl + \mu dl', \tag{10.eq.4}$$

$$d\left([l, l']\right) = l dl' - l' dl, \tag{10.eq.5}$$

$$d\left(l^{[p]}\right) = f(dl) + l^{p-1} dl. \tag{10.eq.6}$$

10.2. Operadic point of view

Let us now show how we recover these notions from Sections 5, 6 and 7. Recall that Lie is the operad generated by a binary operation $[-,-] \in \mathrm{Lie}(2)$ satisfying $(1\,2) \cdot [-,-] = -[-,-]$ and the Jacobi relation:

$$[-,-] \circ_2 [-,-] + (1\ 2\ 3) \cdot ([-,-] \circ_2 [-,-]) + (1\ 3\ 2) \cdot ([-,-] \circ_2 [-,-]) = 0.$$

We fix a field $\mathbb{F}$ of prime characteristic $p \neq 0$.

Theorem 10.12 ([Fre00])**.** *The category* $\Gamma(\mathrm{Lie})$-Alg *coincides with the category* RLie *of restricted Lie algebras.*

We now use the characterisation of $\Gamma(\mathrm{Lie})$-algebras using Theorem 3.3. Following [Iko23, Example 6.6.c)], if L is a $\Gamma(\mathrm{Lie})$-algebra, the Lie bracket on L is given by $\beta_{[-,-],(1,1)}$, and the p-map is given by $\beta_{F_p,(p)}$, where $F_p \in \mathrm{Lie}(p)^{\mathfrak{S}_p}$ is the element:

$$\sum_{\sigma} \underbrace{[-,-] \circ_1 [-,-] \circ_1 \cdots \circ_1 [-,-]}_{p-1} \cdot \sigma,$$

where the sum runs over the $\sigma \in \mathfrak{S}_p$ such that $\sigma(1) = 1$. Note that relation (10.eq.2) then reads:

$$\beta_{[-,-],(1,1)}(\beta_{1_{\mathcal{P}},(1)}, \beta_{F,(p)}) = \beta_{\sum_\sigma \sigma[-,-]\circ_1\cdots\circ_1[-,-],(1,p)}, \tag{10.eq.1}$$

where σ ranges over $\mathfrak{S}_1 \times \mathfrak{S}_p$ in the sum, which can also be written $\beta_{[-,-],(1,1)}(\beta_{1_{\mathcal{P}},(1)}, \beta_{F,(p)}) = \beta_{F_{p+1},(1,p)}$.

Abelian algebras, Beck modules

Following Definition 4.1, an abelian $\Gamma(\mathrm{Lie})$-algebra M is a vector space M endowed with a trivial Lie bracket $[m, m'] = 0$ and a semilinear map $f\colon M \to M$ satisfying $f(\lambda m) = \lambda^p f(m)$.

Following Definition 4.5, an L-module M becomes a vector space M equipped with a semilinear map $f\colon M \to M$ and an action of L that we denote $[l, m]$ for $l \in L$, $m \in M$ such that $[l, f(m)] = 0$. This last relation comes from the following computation:

$$\begin{aligned}[l, m^{[p]}] &= \beta_{[-,-],(1,1)}(l, \beta_{F,(p)}(m)),\\ &= \beta_{\sum_\sigma \sigma[-,-]\circ_1\cdots\circ_1[-,-],(1,p)}(l, m),\end{aligned}$$

Here we used the relation (10.eq.1). Now, using relation (βM4) of Definition 4.5, this is equal to 0.

Enveloping algebra

Using the same reasoning as in [Fre00, Theorem 1.2.5], and the relations of Definition 6.1, we can show that the universal enveloping algebra $\mathbb{U}_{\Gamma(\mathrm{Lie})}(L)$ is generated by symbols $\beta_{[-,-],(1,1)}(l,-)$ for $l \in L$, $\beta_{F_p,(p)}(-)$, and a unit $\beta_{1_{\mathrm{Lie}},(1)}(-)$. We can then build an isomorphism between $\mathbb{U}_{\Gamma(\mathrm{Lie})}(L)$ and the universal enveloping algebra $w(L)$ defined in Section 10.1. This isomorphism sends $\beta_{[-,-],(1,1)}(l,-) \in \mathbb{U}_{\Gamma(\mathrm{Lie})}(L)$ to $1 \otimes l$, and $\beta_{F_p,(p)}(-) \in \mathbb{U}_{\Gamma(\mathrm{Lie})}(L)$ to $f \otimes 1$.

Note that L injects into $\mathbb{U}_{\Gamma(\mathrm{Lie})}(L)$, by identifying $l \in L$ to $\beta_{[-,-],(1,1)}(l,-)$. For an L-module (i.e., a $\mathbb{U}_{\Gamma(\mathrm{Lie})}(L)$-module) M, we will denote by $[l,m] := \beta_{[-,-],(1,1)}(l,m) = \beta_{[-,-],(1,1)}(l,-)\cdot m$, and $f(m) := \beta_{F_p,(p)}(m) = \beta_{F_p,(p)}(-)\cdot m$ for $l \in L$, $m \in M$.

Beck derivations

Following Proposition 7.2, a Beck derivation $d\colon L \to M$ is a linear map $d\colon L \to M$ satisfying $d\left([l,l']\right) = \beta_{[-,-],(1,1)}(l,dl') + \beta_{[-,-],(1,1)}(dl,l') = [l,dl'] - [l',dl]$, and

$$d\left(l^{[p]}\right) = \sum_{i=1}^{p} \beta_{F_p,(p-i,i)}(l,dl).$$

Note that, since $F_p \in Lie(p)^{\mathfrak{S}_p}$, for all i such that $2 \leq i \leq p-1$,

$$F_p = \frac{1}{i!} i! F_p = \frac{1}{i!} \sum_{\tau \in \mathfrak{S}_{p-i}\times\mathfrak{S}_i/\mathfrak{S}_{p-i}\times\mathfrak{S}_1^{\times i}} \tau F_p,$$

and so,

$$\begin{aligned}\beta_{F_p,(p-i,i)}(l,dl) &= \beta_{\frac{1}{i!}\sum_{\tau \in \mathfrak{S}_{p-i}\times\mathfrak{S}_i/\mathfrak{S}_{p-i}\times\mathfrak{S}_1^{\times i}} \tau F_p,(p-i,i)}(l,dl)\\ &= \frac{1}{i!}\beta_{\sum_{\tau \in \mathfrak{S}_{p-i}\times\mathfrak{S}_i/\mathfrak{S}_{p-i}\times\mathfrak{S}_1^{\times i}} \tau F_p,(p-i,i)}(l,dl),\end{aligned}$$

and according to relation (βM4), $\beta_{\sum_{\tau \in \mathfrak{S}_{p-i}\times\mathfrak{S}_i/\mathfrak{S}_{p-i}\times\mathfrak{S}_1^{\times i}} \tau F_p,(p-i,i)}(l,dl) = 0$.

Kähler differentials

Following Definition 7.3, the module of Kähler differentials $\Omega_{\Gamma(\mathrm{Lie})}(L)$ of L is the $\mathbb{U}_{\Gamma(\mathrm{Lie})}(L)$-module generated by elements dl, $l \in L$, under the relations:

$$d(\lambda l + \mu l') = \lambda dl + \mu dl',$$

$$d([l,l']) = d(\beta_{[-,-],(1,1)}(l,l')) = \beta_{[-,-],(1,1)}(-,l') \otimes dl + \beta_{[-,-],(1,1)}(l,-) \otimes dl' = [l,dl'] - [l',dl],$$

$$d(f(l)) = d(\beta_{F_p,(p)}(l)) = \sum_{i=1}^{p} \beta_{F_p,(p-i,i)}(b,-) \otimes db.$$

To compute the term corresponding to $i=1$, note that, by [Fre00, Remark 1.2.8], $\beta_{F_p,(p-1,1)}(l,-)\cdot dl = \beta_{F_p,(p-1,1)}(l,dl) = [\underbrace{l,\dots[l}_{p-1},dl]\dots]$. the term corresponding to $i=p$ is

$$\beta_{F_p,(0,p)}(l,-)\otimes l = \beta_{F_p,(p)}(-)\otimes dl = fdl.$$

For all $i \in \{2,\dots,p-1\}$, note that again, $F_p = i\frac{1}{i}F_p = \sum_{\sigma \in \mathfrak{S}_{p-i}\times\mathfrak{S}_i/\mathfrak{S}_{p-i}\times\mathfrak{S}_1\times\mathfrak{S}_{i-1}} \frac{1}{i}F_p$. So, $\beta_{F_p,(p-i,i)}(l,-) = \beta_{\sum_{\sigma \in \mathfrak{S}_{p-i}\times\mathfrak{S}_i/\mathfrak{S}_{p-i}\times\mathfrak{S}_1\times\mathfrak{S}_{i-1}} \frac{1}{i}F_p,(p-i,i)}(l,-)$ is one of the elements described in Definition 6.1 in point (7), and so, is equal to 0 in $\mathbb{U}_{\Gamma(\mathrm{Lie})}(L)$. Finally, this last relation reads:

$$d(fl) = f(dl) + [\underbrace{l,\dots[l}_{p-1},dl]\dots].$$

Finally, $\Omega_{\Gamma(\mathrm{Lie})}(L)$ is indeed isomorphic to the module $\Omega_{\mathrm{RLie}}(L)$ defined in Section 10.1.

11. Comparisons

In this section, we identity some adjunctions involving $\Gamma(\mathcal{P})$-algebras and check that they induce comparisons on Quillen cohomology. The next sections will focus on examples.

Lemma 11.1. *Let $\mathcal{C}$ be a cocomplete closed symmetric monoidal category and $\mathcal{P}$ an operad in $\mathcal{C}$.*

(1) The free $\mathcal{P}$-algebra monad $S(\mathcal{P})\colon \mathcal{C} \to \mathcal{C}$ preserves reflexive coequalizers and filtered colimits.
(2) The forgetful functor $U^{\mathcal{P}}_{\mathcal{C}}\colon \mathcal{P}\text{-Alg} \to \mathcal{C}$ creates reflexive coequalizers and filtered colimits.

Proof. The first part is proved for instance in [Rez96, Proposition 2.3.5]. The second part follows from the first part and [Bor94, Proposition 4.3.2]. □

Lemma 11.2. *The forgetful functor $U^{\Gamma(\mathcal{P})}_{\mathcal{P}}\colon \Gamma(\mathcal{P})\text{-Alg} \to \mathcal{P}\text{-Alg}$ preserves and reflects regular epimorphisms.*

Proof. Both categories are monadic over $\mathrm{Vect}_{\mathbb{F}}$, as illustrated in the diagram of forgetful functors

$$\begin{tikzcd} \Gamma(\mathcal{P})\text{-Alg} \arrow[r, "U^{\Gamma(\mathcal{P})}_{\mathcal{P}}"] \arrow[dr, "U^{\Gamma(\mathcal{P})}_{\mathbb{F}}"'] & \mathcal{P}\text{-Alg} \arrow[d, "U^{\mathcal{P}}_{\mathbb{F}}"] \\ & \mathrm{Vect}_{\mathbb{F}}. \end{tikzcd}$$

A regular epimorphism $q\colon X \twoheadrightarrow Y$ is the coequalizer of its kernel pair $X \times_Y X \rightrightarrows X$, which is a reflexive pair, with common section the diagonal $X \to X \times_Y X$. By Lemma 11.1, the functor $U^{\mathcal{P}}_{\mathbb{F}}$ preserves and reflects reflexive coequalizers, hence also regular epimorphisms.

In $\mathrm{Vect}_{\mathbb{F}}$, all regular epimorphisms (namely the surjective maps) split, assuming the axiom of choice. Thus any functor $\mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}$ preserves regular epimorphisms, and the functor $U^{\Gamma(\mathcal{P})}_{\mathbb{F}}$ preserves and reflects regular epimorphisms, by [Bor94, Theorem 4.3.5]. □

Remark 11.3. Working over a more general base commutative ring k instead of a field $\mathbb{F}$, the endofunctor $\Gamma(\mathcal{P})\colon \mathrm{Mod}_k \to \mathrm{Mod}_k$ need *not* preserve regular epimorphisms.

For example, take $k = \mathbb{Z}$ and the operad $\mathcal{P}$ in $\mathrm{Mod}_{\mathbb{Z}} = \mathrm{Ab}$ generated by one binary operation $\mu \in \mathcal{P}(2)$ subject to the relation $\mu \cdot (12) = -\mu$. Then $\mathcal{P}$ is a reduced operad with $\mathcal{P}(2) = \mathbb{Z}_\sigma$, which denotes $\mathbb{Z}$ with Σ_2-action by the sign. Consider the quotient map of abelian groups $q\colon \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$. We compute the Σ_2-fixed points

$$(\mathcal{P}(2) \otimes \mathbb{Z}^{\otimes 2})^{\Sigma_2} \cong (\mathbb{Z}_\sigma \otimes \mathbb{Z}_{\mathrm{triv}})^{\Sigma_2} = 0$$
$$(\mathcal{P}(2) \otimes (\mathbb{Z}/2)^{\otimes 2})^{\Sigma_2} \cong ((\mathbb{Z}/2)_\sigma)^{\Sigma_2} = \mathbb{Z}/2.$$

Thus the degree 2 summand of the map of abelian groups

$$\Gamma(\mathcal{P}, q)\colon \Gamma(\mathcal{P}, \mathbb{Z}) \to \Gamma(\mathcal{P}, \mathbb{Z}/2)$$

is the map $0 \to \mathbb{Z}/2$, which is not surjective.

Since $\Gamma(\mathcal{P})$-Alg and $\mathcal{P}$-Alg are algebraic categories, Lemma 11.2 ensures that the adjunction

$$F^{\Gamma(\mathcal{P})}_{\mathcal{P}}\colon \mathcal{P}\text{-Alg} \rightleftarrows \Gamma(\mathcal{P})\text{-Alg}\colon U^{\Gamma(\mathcal{P})}_{\mathcal{P}} \tag{11.eq.1}$$

gives rise to the comparison diagrams described in [Fra15, Theorem 4.7].

Another source of comparisons will be given by morphisms of operads, as we now describe.

Lemma 11.4. *Let $f\colon \mathcal{P} \to \mathcal{Q}$ be a morphism between reduced operads in $\mathrm{Vect}_{\mathbb{F}}$. Consider the diagram of four adjunctions*

$$\begin{tikzcd}
\mathcal{P}\text{-Alg} \arrow[r, shift left, "f_!"] \arrow[d, shift right, swap, "F_{\mathcal{P}}^{\Gamma(\mathcal{P})}"] & \mathcal{Q}\text{-Alg} \arrow[l, shift left, "f^*"] \arrow[d, shift right, swap, "F_{\mathcal{Q}}^{\Gamma(\mathcal{Q})}"] \\
\Gamma(\mathcal{P})\text{-Alg} \arrow[r, shift left, "f_!"] \arrow[u, shift right, swap, "U_{\mathcal{P}}^{\Gamma(\mathcal{P})}"] & \Gamma(\mathcal{Q})\text{-Alg}. \arrow[l, shift left, "f^*"] \arrow[u, shift right, swap, "U_{\mathcal{Q}}^{\Gamma(\mathcal{Q})}"]
\end{tikzcd}$$

(1) The right adjoints commute, and therefore the left adjoints commute (up to natural isomorphism).

(2) Both restriction functors f^* preserves and reflect regular epimorphisms.

Proof. (1) Via the explicit description of $\Gamma(\mathcal{P})$-algebras in Theorem 3.3, the restriction functor $f^*\colon \Gamma(\mathcal{Q})\text{-Alg} \to \Gamma(P)\text{-Alg}$ can be described as follows. Consider a $\Gamma(\mathcal{Q})$-algebra A with operations $\beta_{y,\underline{r}}\colon A^{\times s} \to A$, given for all $\underline{r} = (r_1, \ldots, r_s)$ and $y \in \mathcal{Q}(n)^{\Sigma_{\underline{r}}}$ with $n = r_1 + \cdots + r_s$. Its restriction f^*A has the same underlying $\mathbb{F}$-vector space A, with $\Gamma(\mathcal{P})$-algebra structure given by the operations

$$\beta_{x,\underline{r}} = \beta_{f(x),\underline{r}}$$

for all $x \in \mathcal{P}(n)^{\Sigma_{\underline{r}}}$ and $\underline{r}$ as above. Note that the map on the arity n parts $f\colon \mathcal{P}(n) \to \mathcal{Q}(n)$ is Σ_n-equivariant, hence restricts to fixed point subspaces $f\colon \mathcal{P}(n)^{\Sigma_{\underline{r}}} \to \mathcal{Q}(n)^{\Sigma_{\underline{r}}}$.

(2) As observed in the proof of Lemma 11.2, in all four categories, regular epimorphisms are preserved and reflected by the forgetful functor to $\mathrm{Vect}_{\mathbb{F}}$. More concretely, they are the morphisms whose underlying map of vector spaces is surjective. □

Next, we want to compare Quillen cohomology of $\Gamma(\mathcal{P})$-algebras and $\mathcal{P}$-algebras. Start with a $\Gamma(\mathcal{P})$-algebra A and consider its underlying $\mathcal{P}$-algebra $U_{\mathcal{P}}^{\Gamma(\mathcal{P})}A$, also denoted A when the context indicates the category. The diagram of adjunctions in [Fra15, §4.2.2] specializes to

$$\begin{tikzcd}
\mathcal{P}\text{-Alg}/A \arrow[r, shift left, "\mathrm{Ab}_A"] \arrow[d, shift right, swap, "\epsilon_{A!}F_{\mathcal{P}}^{\Gamma(\mathcal{P})}"] & (\mathcal{P}\text{-Alg}/A)_{\mathrm{ab}} \arrow[l, shift left, "U_A"] \arrow[d, shift right, swap, "\epsilon_{A\#}\tilde{F}_{\mathcal{P}}^{\Gamma(\mathcal{P})}"] \\
\Gamma(\mathcal{P})\text{-Alg}/A \arrow[r, shift left, "\mathrm{Ab}_A"] \arrow[u, shift right, swap, "U_{\mathcal{P}}^{\Gamma(\mathcal{P})}"] & (\Gamma(\mathcal{P})\text{-Alg}/A)_{\mathrm{ab}}\,. \arrow[l, shift left, "U_A"] \arrow[u, shift right, swap, "U_{\mathcal{P}}^{\Gamma(\mathcal{P})}"]
\end{tikzcd}$$

Using the identification of Beck modules and Kähler differentials in [LV12, Propositions 12.3.8 and 12.3.13] for the top row and Theorem 6.11 and Proposition 7.6 for the bottom row, the diagram becomes

$$\begin{tikzcd}
\mathcal{P}\text{-Alg}/A \arrow[r, shift left, "\mathbb{U}_{\mathcal{P}}A\otimes_{\mathbb{U}_{\mathcal{P}}(-)}\Omega_{\mathcal{P}}(-)"] \arrow[d, shift right, swap, "\epsilon_{A!}F_{\mathcal{P}}^{\Gamma(\mathcal{P})}"] & \mathbb{U}_{\mathcal{P}}A\text{-Mod} \arrow[l, shift left, "A\ltimes -"] \arrow[d, shift right, swap, "\theta_!"] \\
\Gamma(\mathcal{P})\text{-Alg}/A \arrow[r, shift left, "\mathbb{U}_{\Gamma(\mathcal{P})}A\otimes_{\mathbb{U}_{\Gamma(\mathcal{P})}(-)}\Omega_{\Gamma(\mathcal{P})}(-)"] \arrow[u, shift right, swap, "U_{\mathcal{P}}^{\Gamma(\mathcal{P})}"] & \mathbb{U}_{\Gamma(\mathcal{P})}A\text{-Mod}. \arrow[l, shift left, "A\ltimes -"] \arrow[u, shift right, swap, "\theta^*"]
\end{tikzcd} \tag{11.eq.2}$$

Here $\theta\colon \mathbb{U}_{\mathcal{P}}A \to \mathbb{U}_{\Gamma(\mathcal{P})}A$ is the $\mathbb{F}$-linear ring homomorphism described in Proposition 6.14, θ^* denotes restriction of scalars along θ, and $\theta_!$ denotes extension of scalars $\theta_!(M) = \mathbb{U}_{\Gamma(\mathcal{P})}A \otimes_{\mathbb{U}_{\mathcal{P}}A} M$. Applying [Fra15, Propositions 4.13 and 4.14] yields the following.

Proposition 11.5. *Let A be a $\Gamma(\mathcal{P})$-algebra.*

(1) There is a natural (up to homotopy) comparison of cotangent complexes

$$\mathbf{L}_A^{\mathcal{P}} \to \theta^*\mathbf{L}_A^{\Gamma(\mathcal{P})} \tag{11.eq.3}$$

in simplicial $\mathbb{U}_{\mathcal{P}}A$-modules.

(2) *For each degree $n \geq 0$, there is a natural comparison map of $\mathbb{U}_{\mathcal{P}}A$-modules*

$$\mathrm{HQ}_n^{\mathcal{P}}(A) \to \theta^* \mathrm{HQ}_n^{\Gamma(\mathcal{P})}(A)$$

from Quillen homology of A as a $\mathcal{P}$-algebra to Quillen homology of A as a $\Gamma(P)$-algebra.

(3) *For M a left $\mathbb{U}_{\Gamma(\mathcal{P})}A$-module, there is a natural comparison map of abelian groups*

$$\mathrm{HQ}^n_{\Gamma(\mathcal{P})}(A; M) \to \mathrm{HQ}^n_{\mathcal{P}}(A; \theta^* M) \tag{11.eq.4}$$

from Quillen cohomology of A as a $\Gamma(\mathcal{P})$-algebra to Quillen cohomology of A as a $\mathcal{P}$-algebra.

(4) *If the map of simplicial $\mathbb{U}_{\Gamma(\mathcal{P})}A$-modules*

$$\theta_! \mathbf{L}_A^{\mathcal{P}} \to \mathbf{L}_A^{\Gamma(\mathcal{P})}$$

adjunct to the map (11.eq.3) *is a weak equivalence, then the comparison in Quillen cohomology* (11.eq.4) *is an isomorphism in all degrees n.*

We can describe the comparison map of cotangent complexes (11.eq.3) more explicitly.

Proposition 11.6. *(1) For any $\Gamma(\mathcal{P})$-algebra A, there is a natural map of $\mathbb{U}_{\mathcal{P}}A$-modules*

$$\Omega_{\mathcal{P}}(A) \to \theta^* \Omega_{\Gamma(\mathcal{P})}(A)$$

given by

$$\mu(a_1, \ldots, da_i, \ldots, a_n) \mapsto \beta_{\mu(1,\ldots,1)}(a_1, \ldots, da_i, \ldots, a_n)$$

for $\mu \in \mathcal{P}(n)$, $a_j \in A$.

(2) *More generally, for any morphism of $\Gamma(\mathcal{P})$-algebras $g \colon B \to A$, there is a natural map of $\mathbb{U}_{\mathcal{P}}A$-modules*

$$\begin{array}{ccc} \mathbb{U}_{\mathcal{P}}A \otimes_{\mathbb{U}_{\mathcal{P}}B} \Omega_{\mathcal{P}}(B) & \longrightarrow & \theta^* \left(\mathbb{U}_{\Gamma(\mathcal{P})}A \otimes_{\mathbb{U}_{\Gamma(\mathcal{P})}B} \Omega_{\Gamma(\mathcal{P})}(B) \right) \\ \| & & \| \\ \mathrm{Ab}_{UA}(UB \to UA) & & \theta^* \, \mathrm{Ab}_A(B \to A) \end{array} \tag{11.eq.5}$$

given by

$$\mu(a_1, \ldots, db_i, \ldots, a_n) \mapsto \beta_{\mu(1,\ldots,1)}(a_1, \ldots, db_i, \ldots, a_n)$$

for $\mu \in \mathcal{P}(n)$, $a_j \in A$ ($j \neq i$), and $b_i \in B$.

(3) *Given a cofibrant replacement $C_\bullet \xrightarrow{\sim} A$ in simplicial $\Gamma(\mathcal{P})$-algebras, the comparison maps* (11.eq.5) *for $C_n \to A$ in simplicial degree $n \geq 0$ yield the comparison map of cotangent complexes $\mathbf{L}_A^{\mathcal{P}} \to \theta^* \mathbf{L}_A^{\Gamma(\mathcal{P})}$ from* (11.eq.3).

Proof. First, we show that the underlying vector space of $\mathbb{U}_{\mathcal{P}}A \otimes_{\mathbb{U}_{\mathcal{P}}B} \Omega_{\mathcal{P}}(B)$ is spanned by the elements of the form $\mu(a_1, \ldots, db_i, \ldots, a_n)$. The tensor product $\mathbb{U}_{\mathcal{P}}A \otimes_{\mathbb{U}_{\mathcal{P}}B} \Omega_{\mathcal{P}}(B)$ is spanned by elements of the type:

$$\mu(a_1, \ldots, a_{i-1}, -, a_{i+1} \ldots, a_n) \otimes_{\mathbb{U}_{\mathcal{P}}B} \nu(b_1, \ldots, db_j, \ldots, b_k),$$

With $\mu \in \mathcal{P}(n)$, $\nu \in \mathcal{P}(k)$, $a_u \in A$ and $b_v \in B$. However, noting that this element is equal to:

$$\mu(a_1, \ldots, a_{i-1}, -, a_{i+1} \ldots, a_n) \otimes_{\mathbb{U}_{\mathcal{P}}B} \nu(b_1, \ldots, b_{j-1}, -, b_{j+1}, \ldots, b_k) \cdot db_j,$$

tensoring over $\mathbb{U}_{\mathcal{P}}B$ means that this element is identified with:

$$\mu(a_1, \ldots, a_{i-1}, -, a_{i+1} \ldots, a_n) \cdot \nu(g(b_1), \ldots, g(b_{j-1}), -, g(b_{j+1}), \ldots, g(b_k)) \otimes_{\mathbb{U}_{\mathcal{P}}B} db_j$$

which is equal to:

$$\mu \circ_i \nu(a_1, \ldots, a_{i-1}, g(b_1), \ldots, g(b_{j-1}), -, g(b_{j+1}), \ldots, g(b_k), a_{i+1} \ldots, a_n) \otimes_{\mathbb{U}_{\mathcal{P}}B} db_j.$$

We now see that $\mathbb{U}_{\mathcal{P}}A \otimes_{\mathbb{U}_{\mathcal{P}}B} \Omega_{\mathcal{P}}(B)$ is spanned by elements of the form:

$$\mu(a_1, \ldots, a_{i-1}, -, a_{i+1} \ldots, a_n) \otimes_{\mathbb{U}_{\mathcal{P}}B} db_i,$$

Similarly, one shows that $\mathbb{U}_{\Gamma(\mathcal{P})}A \otimes_{\mathbb{U}_{\Gamma(\mathcal{P})}B} \Omega_{\Gamma(\mathcal{P})}(B)$ is spanned by elements of the type:

$$\beta_{\mu,\underline{r}}(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_s) \otimes_{\mathbb{U}_{\Gamma(\mathcal{P})}B} db_i,$$

which we denote by $\beta_{\mu,\underline{r}}(a_1, \ldots, db_i, \ldots, a_s)$. The expression of the comparison map is then induced by that of the norm map θ from Proposition 6.14. □

Here the left adjoint induced on Beck modules $\theta_!\colon \mathbb{U}_{\mathcal{P}}A\text{-Mod} \to \mathbb{U}_{\Gamma(\mathcal{P})}A\text{-Mod}$ is described in terms of the ring homomorphism θ, but one might hope to describe it in terms of the original left adjoint $F_{\mathcal{P}}^{\Gamma(\mathcal{P})}\colon \mathcal{P}\text{-Alg} \to \Gamma(\mathcal{P})\text{-Alg}$. However, the left adjoint $F_{\mathcal{P}}^{\Gamma(\mathcal{P})}$ does *not always* pass to Beck modules in the sense of [Fra15, Definition 3.29]. For example, we will see, in the case of divided power algebras, that $F_{\mathrm{Com}}^{\Gamma(\mathrm{Com})}$ does not pass to Beck modules (see Proposition 12.3), while in the case of restricted Lie algebras, $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}$ does pass to Beck modules (see Proposition 13.4).

12. Divided power algebras versus commutative algebras

In this section, we take the operad $\mathcal{P} = \mathrm{Com}$ and analyze the effect of the free-forget adjunction $\mathrm{Com}\text{-Alg} \rightleftarrows \Gamma(\mathrm{Com})\text{-Alg}$.

We denote by $\mathrm{Com}_{\mathrm{aug}}$ the category of augmented $\mathbb{F}$-algebras and use the equivalence $\mathrm{Com}\text{-Alg} \cong \mathrm{Com}_{\mathrm{aug}}$, as well as the equivalence $\Gamma(\mathrm{Com})\text{-Alg} \cong \text{p-Com}$ from Proposition 9.3. P. Berthelot defined the notion of PD envelope of an ideal. In particular, it follows from [Ber74, §2.3] that the forgetful functor

$$U_{\mathrm{Com}}^{\Gamma(\mathrm{Com})}\colon \text{p-Com} \to \mathrm{Com}_{\mathrm{aug}}$$

admits a left adjoint functor

$$F_{\mathrm{Com}}^{\Gamma(\mathrm{Com})}\colon \mathrm{Com}_{\mathrm{aug}} \to \text{p-Com}$$

given by $F_{\mathrm{Com}}^{\Gamma(\mathrm{Com})}(A) = \hat{A}$, where $\hat{A}$ denotes the PD envelope of the augmentation ideal A_+ of A. We denote by η the unit of this adjunction, which is a homomorphism of augmented algebras $\eta_A\colon A \to \hat{A}$.

Let A be a divided power algebra, equivalently, an object of p-Com. Since Beck modules over an augmented algebra B are just B-modules, A is the object $\mathbb{U}_{\mathrm{Com}}(A)$ representing Beck modules over the underlying augmented algebra of A. Recall from Section 9 that $V(A) = A \otimes_{\mathbb{F}} R_f$ represents Beck modules over A. The $\mathbb{F}$-linear ring homomorphism $\theta\colon A \to V(A)$ is given by $\theta(a) = a \otimes 1$. Denote by Ω^1_A the usual Kähler differentials over the underlying augmented algebra, whereas $\Omega_{\text{p-Com}}(A)$ was described in Theorem 9.7. The diagram of adjunctions (11.eq.2) specializes to

$$\begin{tikzcd}
\mathrm{Com}_{\mathrm{aug}}/A \arrow[r, shift left, "A\otimes_{(-)}\Omega^1_{(-)}"] \arrow[d, shift right, "\epsilon_{A!}F_{\mathrm{Com}}^{\Gamma(\mathrm{Com})}"'] & A\text{-Mod} \arrow[l, shift left, "A\ltimes -"] \arrow[d, shift right, "\theta_!"'] \\
\text{p-Com}/A \arrow[r, shift left, "V(A)\otimes_{V(-)}\Omega_{\text{p-Com}}(-)"] \arrow[u, shift right, "U_{\mathrm{Com}}^{\Gamma(\mathrm{Com})}"'] & V(A)\text{-Mod}. \arrow[l, shift left, "A\ltimes -"] \arrow[u, shift right, "\theta^*"']
\end{tikzcd}$$

Let $\mathfrak{M}$ be a $V(A)$-module. By Theorem 9.5 the $V(A)$-module $\mathfrak{M}$ is associated to a pair (M, π), where by M we denote $\mathfrak{M}$ viewed as A-module and $\pi\colon M \to M$ is a p-semilinear map such that $\pi(am) = 0$ holds for all $a \in A_+$ and $m \in M$. Equivalently, the $V(A)$-module $\mathfrak{M}$ is associated to an abelian group object $A \oplus_p M \to A$ in $(\text{p-Com}/A)_{\mathrm{ab}}$, where $A \oplus_p M$ is the semidirect product in Com of A and M together with the map

$$\pi(a, m) = (\pi(a), \pi(m) - a^{p-1}m),\ a \in A_+,\ m \in M.$$

Via Theorem 9.5, restriction of scalars along $\theta\colon A \to V(A)$ is the functor sending a pair (M, π) to M, forgetting the p-semilinear map π.

Lemma 12.1. *Let $A \in \mathrm{Com}_{aug}$ be an augmented algebra, and M be a $V(\hat{A})$-module. Then the functor*

$$\eta_A^* U_{\mathrm{Com}}^{\Gamma(\mathrm{Com})} \colon (\text{p-Com}/\hat{A})_{\mathrm{ab}} \to (\mathrm{Com}_{\mathrm{aug}}/A)_{\mathrm{ab}}$$

is given by

$$\eta_A^* U_{\mathrm{Com}}^{\Gamma(\mathrm{Com})}(M) = {}_A M$$

where ${}_A M$ denotes M with the A-module structure induced by restriction of scalars along the morphism $\eta_A \colon A \to \hat{A}$.

Proof. In the composite of functors

$$\begin{tikzcd}
(\text{p-Com}/\hat{A})_{\mathrm{ab}} \arrow[r, "U_{\mathrm{Com}}^{\Gamma(\mathrm{Com})}"] \arrow[d, "\cong", "\text{Thm. 9.5}"'] & (\mathrm{Com}_{\mathrm{aug}}/U\hat{A})_{\mathrm{ab}} \arrow[r, "\eta_A^*"] \arrow[d, "\cong"] & (\mathrm{Com}_{\mathrm{aug}}/A)_{\mathrm{ab}} \arrow[d, "\cong"] \\
V(\hat{A})\text{-Mod} \arrow[r, "\theta^*"] & \hat{A}\text{-Mod} \arrow[r, "\eta_A^*"] & A\text{-Mod}
\end{tikzcd}$$

the first step is restriction of scalars along the ring homomorphism $\theta \colon \hat{A} \to V(\hat{A})$. The second step is restriction of scalars along the ring homomorphism $\eta_A \colon A \to \hat{A}$, since this is how pullbacks of Beck modules are computed in commutative algebras. □

Proposition 12.2. *(1) Let $A \in \mathrm{Com}_{\mathrm{aug}}$ an augmented commutative algebra and M be a $V(\hat{A})$-module. Then there is a comparison map*

$$\mathrm{HQ}^*_{\text{p-Com}}(\hat{A}; M) \to \mathrm{HQ}^*_{\mathrm{Com}}(A; {}_A M).$$

(2) Let $B \in$ p-Com be a divided power algebra and M be a $V(B)$-module. Then there is a comparison map

$$\mathrm{HQ}^*_{\text{p-Com}}(B; M) \to \mathrm{HQ}^*_{\mathrm{Com}}(B; \theta^* M).$$

Proof. Part (1) follows from [Fra15, Proposition 4.12] and Lemma 12.1. Part (2) is a specialization of Proposition 11.5 (3) to the operad $P = \mathrm{Com}$. □

We now show that the p-envelope functor does not pass to Beck modules:

Proposition 12.3. *In the case $\mathbb{F} = \mathbb{F}_2$, the functor $F_{\mathrm{Com}}^{\Gamma(\mathrm{Com})}$ that freely adjoins divided power operations does* not *pass to Beck modules.*

Proof. Take the commutative (unital) $\mathbb{F}_2$-algebra $A = \mathbb{F}_2$ and the A-module $M = \mathbb{F}_2$ whose generator (only non-zero element) we denote by x. Viewing the A-module as a square-zero split extension $pr \colon A \oplus M \twoheadrightarrow A$, apply the functor $F_{\mathrm{Com}}^{\Gamma(\mathrm{Com})}$ to obtain the split epimorphism of divided power algebras

$$\hat{pr} \colon \widehat{A \oplus M} \twoheadrightarrow \hat{A}.$$

Then, $\widehat{A \oplus M}$ is the free divided power algebra on one generator $\Gamma(x)$, that is, its underlying vector space is isomorphic to the vector space of polynomials $\mathbb{F}_2[x]$, but the multiplication is induced by $x^n * x^m = \binom{m+n}{n} x^{n+m}$. The divided power algebra $\hat{A}$ is still equal to $\mathbb{F}_2$. The kernel $K = \ker(\hat{pr})$ is equal to the subalgebra of $\Gamma(x)$ of non-constant polynomials. In K, we have for example, $x * x^2 = 3x^3 = x^3$. So K is not a square-zero algebra, thus the split epimorphism $\hat{pr}$ does not yield a Beck module. □

To conclude this section, we will specify the comparison maps of Proposition 11.6 to the case of divided power algebras. Let $g \colon B \to A$ be a morphism of non-unital divided power algebras. On the one hand,

$$\mathbb{U}_{\mathrm{Com}} A \otimes_{\mathbb{U}_{\mathrm{Com}} B} \Omega_{\mathrm{Com}}(B) = A^+ \otimes_{B^+} \Omega^1_{B^+}$$

is spanned by the elements adb for $a \in A$, $b \in B$, under relations expressing the fact that d is a (linear) A^+-derivation, that is:

$$\begin{cases} d(\lambda b + b') = \lambda db + db' \\ d(bb') = g(b)db' + g(b')db \end{cases}$$

and the action of A^+ is given by the multiplication in A ($a \cdot a'db = (aa')db$). On the other hand,

$$\mathbb{U}_{\Gamma(\mathrm{Com})}A \otimes_{\mathbb{U}_{\Gamma(\mathrm{Com})}B} \Omega_{\Gamma(\mathrm{Com})}(B) = V(A^+) \otimes_{V(B^+)} \Omega_{\text{p-Com}}(B)$$

is spanned by elements $af^k db$ for $a \in A$, $b \in B$ and $k \in \mathbb{N}$, under relation expressing the fact that d is a Beck A-derivation in p-Com, that is, we also have

$$d\gamma_p(b) = fdb - g(b)^{p-1}db.$$

The action of $V(A^+)$ is again given by the multiplication in $V(A^+)$, in particular, $(a \otimes f^k) \cdot a' f^{k'} db = aa'^{p^k} f^{k+k'} db$. The comparison map

$$\mathbb{U}_{\mathrm{Com}}A \otimes_{\mathbb{U}_{\mathrm{Com}}B} \Omega_{\mathrm{Com}}(B) \to \mathbb{U}_{\Gamma(\mathrm{Com})}A \otimes_{\mathbb{U}_{\Gamma(\mathrm{Com})}B} \Omega_{\Gamma(\mathrm{Com})}(B)$$

from Proposition 11.6 is simply defined by $adb \mapsto af^0 db$.

13. Restricted Lie algebras versus Lie algebras

Let $\mathbb{F}$ be a field of prime characteristic p. Let $(H, (-)^{[p]})$ be a restricted Lie algebra. Then by Theorem 10.8 a $w(H)$-module M is an associated to a pair $({}_{u(H)}M, f)$, where ${}_{u(H)}M$ is M viewed as a $u(H)$-module and $f\colon {}_{u(H)}M \to {}_{u(H)}M^H$ is a p-semilinear map. Equivalently, the $w(H)$-module M is associated to the abelian group object

$$H \ltimes_{f\ u(H)} M \to H$$

in $(\mathrm{RLie}/L)_{\mathrm{ab}}$, where $H \ltimes_{f\ u(H)} M$ denotes the semidirect product in RLie of H by ${}_{u(H)}M$. In particular,

$$H \ltimes_{f\ u(H)} M = \{(h, m),\ h \in H,\ m \in M\}.$$

The Lie bracket is given by

$$[(h, m), (h', m')] = ([h, h'], hm' - h'm)$$

and the p-map is given by

$$(h, m)^{[p]} = (h^{[p]}, \underbrace{h \cdots h}_{p-1} m + f(m)).$$

The notion of p-envelope of a Lie algebra has been studied in detail in [SF88, §2.5]. Let L be a Lie algebra over $\mathbb{F}$ and $U(L)$ its enveloping algebra. The p-envelope $\hat{L}$ of a Lie algebra L is the Lie subalgebra of $U(L)_{\mathrm{Lie}}$ which contains L and all iterated associative p-th powers. Thus, $\hat{L}$ is a restricted Lie subalgebra of $U(L)_{\mathrm{RLie}}$. We note that L is an ideal in $\hat{L}$. In [Mil75], A. A. Mil'ner proves that $\hat{L}$ has the following universal property: for all restricted Lie algebras $A \in \mathrm{RLie}$ and all Lie algebra homomorphisms $f\colon L \to A$, there is exactly one restricted Lie algebra homomorphism $\hat{f}\colon \hat{L} \to A$ such that $\hat{f} \circ i = f$; see [SF88, §2.5, Theorem 2.5.2]. We then deduce that $\hat{L} \cong F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L)$, where L is a Lie algebra. It also follows from the universal properties of $U(L)$ and $u(\hat{L})$ that they are isomorphic.

Denote by η the unit of the adjunction $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})} \dashv U_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}$ and ι the unit of the adjunction $u \dashv (-)_{\mathrm{RLie}}$. From the foregoing discussion, for L a Lie algebra, we get a Lie algebra homomorphism $\eta_L\colon L \to \hat{L}$, and a restricted Lie homomorphism $\iota_{\hat{L}}\colon \hat{L} \to u(\hat{L}) \cong U(L)$.

Lemma 13.1. *Let L be a Lie algebra, and M be a $w(\hat{L})$-module. Then the functor*

$$\eta_L^* U_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})} \colon (\mathrm{RLie}/\hat{L})_{\mathrm{ab}} \to (\mathrm{Lie}/L)_{\mathrm{ab}}$$

is given by

$$\eta_L^* U_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(M) = {}_{U(L)}M.$$

Proof. Let

$$\hat{L} \ltimes_{f\ u(\hat{L})} M \to \hat{L}$$

be an abelian group object in $(\mathrm{RLie}/\hat{L})_{\mathrm{ab}}$. We have the pullback diagram

$$\begin{tikzcd} L \times_{\hat{L}} (\hat{L} \ltimes_{f\ u(\hat{L})} M) \arrow[r] \arrow[d] & \hat{L} \ltimes_{f\ u(\hat{L})} M \arrow[d] \\ L \arrow[r, "\eta_L"] & \hat{L}. \end{tikzcd}$$

Since $L \times_{\hat{L}} (\hat{L} \ltimes_{f\ u(\hat{L})} M)$ is spanned by the elements $(l, (l, m))$ with $l \in L$ and $m \in {}_{u(\hat{L})}M$, we get:

$$\eta_L^* U_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L \ltimes_{f\ u(\hat{L})} M \to \hat{L}) = L \times_{\hat{L}} (\hat{L} \ltimes_{f\ u(\hat{L})} M) \to L,$$

so, $\psi \colon L \times_{\hat{L}} (\hat{L} \ltimes_{f\ u(\hat{L})} M) \to L$ is an abelian group object in $(\mathrm{Lie}/L)_{\mathrm{ab}}$. The kernel of ψ is isomorphic to ${}_{U(L)}M$. $\square$

Proposition 13.2. *(1) Let L be a Lie algebra and M a $w(\hat{L})$-module. Then there is a comparison map*

$$\mathrm{HQ}^*_{\mathrm{RLie}}(\hat{L}; M) \to \mathrm{HQ}^*_{\mathrm{Lie}}(L; {}_{U(L)}M).$$

(2) Let H be a restricted Lie algebra and M a $w(H)$-module. Then there is a comparison map

$$\mathrm{HQ}^*_{\mathrm{RLie}}(H; M) \to \mathrm{HQ}^*_{\mathrm{Lie}}(H; \theta^* M)$$

where $\theta \colon U(H) \to w(H)$ is the ring homomorphism described in Lemma 10.9.

Proof. Part (1) follows from [Fra15, Proposition 4.12] and Lemma 13.1. Part (2) is a specialization of Proposition 11.5 (3) to the operad $P = \mathrm{Lie}$. $\square$

Let us now show that the p-envelope functor passes to Beck modules. We will need the following observation:

Remark 13.3. Let H be a restricted Lie algebra and $x_i \in H$. By Jacobson's formula on p-th powers we have:

$$\left(\sum_i x_i \right)^{[p]} - \left(\sum_i x_i^{[p]} \right) \in [H, H].$$

Let L be a Lie algebra with basis $(e_i)_{i \in I}$ over a field $\mathbb{F}$. From the previous formula it follows by induction that the elements of $\hat{L}$ are of the type $\sum_{i \in I, n_i \geq 0} \mathbb{F} e_i^{p^{n_i}} \in U(L)$.

Proposition 13.4. *For any field $\mathbb{F}$ of characteristic p, the p-envelope functor $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}$ that freely adjoins a p-map passes to Beck modules.*

Proof. For a Lie algebra L, the restricted Lie algebra $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L)$ is spanned by elements of the type $l^{[p^k]}$, for $l \in L$ and $k \in \mathbb{N}$, where $(-)^{[p^k]}$ represents the k-th iteration of the p-map in $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L)$. Let $L \ltimes M \to L$ be a Beck module over L, where M is an L-module. Consider the induced split epimorphism $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L \ltimes M) \to F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L)$. By [SF88, §2.5, Proposition 5.3], $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}$ preserves both surjections and injections. In particular, $l^{[p^k]} \mapsto (l, 0)^{[p^k]}$ and $m^{[p^k]} \mapsto (0, m)^{[p^k]}$ induce split injections $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L) \hookrightarrow F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L \ltimes M)$ and $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(M) \hookrightarrow F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L \ltimes M)$.

Under these injections, the abelian restricted Lie algebra $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(M)$ is a restricted $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L)$-module. If f is the p-map in $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(M)$, then $(F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(M), f)$ is a $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L)$ Beck module.

We then get an injection $g\colon F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L) \ltimes_f F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(M) \hookrightarrow F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L \ltimes M)$ given by $(l^{[p^{k_1}]}, m^{[p^{k_2}]}) \mapsto (l,0)^{[p^{k_1}]} + (0,m)^{[p^{k_2}]}$. From Remark 13.3, we deduce that g is in fact an isomorphism, and so, $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L \ltimes M \to L)$ has the structure of an $F_{\mathrm{Lie}}^{\Gamma(\mathrm{Lie})}(L)$ Beck module. □

To conclude this section, we will specify the comparison maps of Proposition 11.6 to the case of restricted Lie algebra. Let $g\colon L \to H$ be a morphism of restricted lie algebra. Then, on the one hand, $\mathbb{U}_{\mathrm{Lie}} L \otimes_{\mathbb{U}_{\mathrm{Lie}} H} \Omega_{\mathrm{Lie}}(H)$ is spanned by the elements ldh for $l \in L$, $h \in H$, under relations expressing the fact that d is a (linear) L-derivation:

$$\begin{cases} d(\lambda h + h') = \lambda dh + dh' \\ d([h,h']) = g(h)dh' - g(h')dh \end{cases}$$

and the action of L is given by the bracket in L ($l \cdot l'dh = [l,l']$). On the other hand, $\mathbb{U}_{\Gamma(\mathrm{Lie})} L \otimes_{\mathbb{U}_{\Gamma(\mathrm{Lie})} H} \Omega_{\Gamma(\mathrm{Lie})}(H) = w(L) \otimes_{w(H)} \Omega_{\mathrm{RLie}}(H)$ is spanned by elements $f^k ldh$ for $l \in U(L)$, $h \in H$ and $k \in \mathbb{N}$, under relation expressing the fact that d is a Beck L-derivation in RLie, that is, we also have:

$$d(h^{[p]}) = \underbrace{g(h) \cdots g(h)}_{p-1} dh + fdh$$

and the action of $w(L)$ is again given by the multiplication in $w(L)$. In particular, for $l \in L$, $(f^k l) \cdot f^{k'} l' dh = 0$ as soon as $k' > 0$. The comparison map

$$\mathbb{U}_{\mathrm{Lie}} L \otimes_{\mathbb{U}_{\mathrm{Lie}} H} \Omega_{\mathrm{Lie}}(H) \to \mathbb{U}_{\Gamma(\mathrm{Lie})} L \otimes_{\mathbb{U}_{\Gamma(\mathrm{Lie})} H} \Omega_{\Gamma(\mathrm{Lie})}(H)$$

from Proposition 11.6 is simply defined by $ldh \mapsto f^0 ldh$.

14. Associative algebras versus restricted Lie algebras

Let $\mathbb{F}$ be a field of prime characteristic p. Denote by η the unit of the adjunction $u \vdash (-)_{\mathrm{RLie}}$. Let $(L, (-)^{[p]})$ be a restricted Lie algebra and let M be an $u(L)$-bimodule. We denote by ${}_{u(L)}M$ the left $u(L)$-module obtained from M by the action $l \cdot m := lm - ml$, where the dotless notation is for the left and right bimodule actions on M. Let $A \in \mathrm{As}$ be an associative algebra. We recall that the category of Beck A-modules is equivalent to the category of A-bimodules (see [Bar96, §2.1]).

Lemma 14.1. *Let L be a restricted Lie algebra, and M be a $u(L)$-bimodule. Then the functor*

$$\eta_L^*(-)_{\mathrm{RLie}}\colon (\mathrm{As}\,/u(L))_{\mathrm{ab}} \to (\mathrm{RLie}\,/L)_{\mathrm{ab}}$$

is given by

$$\eta_L^*(-)_{\mathrm{RLie}}(M) = {}_{u(L)}M.$$

Proof. Consider the associative algebra obtained by semidirect product $u(L) \ltimes M$, whose multiplication is given by $(u,m)(u',m') = (uu', um' + mu')$. The category of Beck $u(L)$-modules is equivalent to the category of $u(L)$-bimodules. Under this equivalence, the $u(L)$-bimodule M is associated to the abelian group object $u(L) \ltimes M \to u(L)$ in $(As/u(L))_{\mathrm{ab}}$. The Lie bracket in $(u(L) \ltimes M)_{\mathrm{RLie}}$ is given by

$$[(u,m),(u',m')] = (uu' - u'u, um' + mu' - u'm - m'u),$$

and the p-map is given by

$$(u,m) \mapsto (u,m)^p.$$

Therefore the p-map on elements of the form $(0, m)$ is zero. We have a restricted Lie algebra homomorphism $\eta_L \colon L \to u(L)_{\mathrm{RLie}}$, and a pullback functor $\eta_L^* \colon \mathrm{RLie}/u(L)_{\mathrm{RLie}} \to \mathrm{RLie}/L$. We get a pullback diagram

$$\begin{array}{ccc} L \times_{u(L)_{\mathrm{RLie}}} (u(L) \ltimes M)_{\mathrm{RLie}} & \longrightarrow & (u(L) \ltimes M)_{\mathrm{RLie}} \\ \downarrow & & \downarrow \\ L & \xrightarrow{\eta_L} & u(L)_{\mathrm{RLie}}, \end{array}$$

and $L \times_{u(L)_{\mathrm{RLie}}} (u(L) \ltimes M)_{\mathrm{RLie}}$ is spanned by elements $(l, (l, m))$ for $l \in L$ and $m \in M$. The Lie bracket of this restricted Lie algebra is given by

$$[(l, (l, m)), (l', (l', m)] = \left([l, l'], [(l, m), (l', m')]\right),$$

and the p-map is given by

$$(l, (l, m)) \mapsto (l^{[p]}, (l, m)^p).$$

Therefore,

$$\eta_L^*(-)_{\mathrm{RLie}}(u(L) \ltimes M \to u(L)) = \eta_L^*((u(L) \ltimes M)_{\mathrm{RLie}} \to u(L)_{\mathrm{RLie}}) = L \times_{u(L)_{\mathrm{RLie}}} (u(L) \ltimes M)_{\mathrm{RLie}} \to L,$$

and $\phi \colon L \times_{u(L)_{\mathrm{RLie}}} (u(L) \ltimes M)_{\mathrm{RLie}} \to L$ is an abelian group object in $(RLie/L)_{\mathrm{ab}}$. The action of L on $M := \ker \phi$ is given by

$$l \cdot (0, (0, m)) = [(l, (l, 0)), (0, (0, m))] = ([l, 0], [(l, 0), (0, m)]) = (0, (0, lm - ml)).$$

With this action, the module M is the restricted Lie module ${}_{u(L)}M$. The p-map on M is trivial, $m \mapsto 0$. □

Theorem 14.2. *Let L be a restricted Lie algebra and M a $u(L)$-bimodule. Then there is an isomorphism*

$$\mathrm{HQ}^*_{\mathrm{As}}(u(L), M) \cong \mathrm{HQ}^*_{\mathrm{RLie}}(L, {}_{u(L)}M).$$

Proof. The restricted enveloping algebra functor u preserves weak equivalences (see [Pri70, 2.8]). Thus by [Fra15, Proposition 4.12] the comparison map between Quillen cohomology in both categories is an isomorphism. □

For an associative algebra A over a field, Quillen cohomology agrees with Hochschild cohomology up to a shift:

$$\mathrm{HQ}^n_{\mathrm{As}}(A; M) \cong \mathrm{HH}^{n+1}(A; M) \quad \text{for } n > 0$$

and a small change in degree 0 [Qui70, Proposition 3.6].

15. Good triple of operads and comparison maps in Quillen (co)homology

J.-L. Loday defined and studied generalized bialgebras and triple of operads in [Lod08]. In this section we prove a comparison isomorphism theorem for Quillen cohomology in the context of good triple of operads. We suppose that the ground field $\mathbb{F}$ has characteristic zero. In that case, if $\mathcal{P}$ is an operad then the norm map $\mathrm{Tr} \colon S(\mathcal{P}) \to \Gamma(\mathcal{P})$ is a natural isomorphism (see [Fre00]). Therefore a $\Gamma(\mathcal{P})$-algebra doesn't carry more structure than a $\mathcal{P}$-algebra.

Let $\mathcal{A}, \mathcal{C}$ be two algebraic operads. The primitive part of a coalgebra C over the operad $\mathcal{C}$ is defined by

$$Prim\, C := \{x \in C \mid \delta(x) = 0, \text{ for any generating cooperation } \delta\}.$$

There is a filtration on C given by

$$F_r(C) = \{x \in C \mid \delta(x) = 0, \text{ for any } \delta \in \mathcal{C}(n),\ n > r\}.$$

We note that $F_1(C)$ is the primitive part of C. The above filtration is called the primitive filtration. A coalgebra C is connected (or conilpotent) if $C = \cup_{r\geq 1} F_r C$. A $\mathcal{C}^c$-$\mathcal{A}$-bialgebra H is a vector space which is a $\mathcal{A}$-algebra and $\mathcal{C}$-coalgebra such that the operations of $\mathcal{A}$ and cooperations of $\mathcal{C}$ acting on H satisfy some compatibility relations.

Let $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ be a good triple of operads. Then $\mathcal{P}$ denotes the primitive operad $Prim_{\mathcal{C}} \mathcal{A}$. The primitive operad is a suboperad of $\mathcal{A}$ and there is an induced functor

$$G\colon \mathcal{A}\text{-Alg} \to \mathcal{P}\text{-Alg}$$

which is a forgetful functor in a sense that the composition

$$\mathcal{A}\text{-Alg} \xrightarrow{G} \mathcal{P}\text{-Alg} \to \mathrm{Vect}_{\mathbb{F}}$$

is the forgetful functor

$$\mathcal{A}\text{-Alg} \to \mathrm{Vect}_{\mathbb{F}}.$$

The functor G admits a left adjoint functor

$$U\colon \mathcal{P}\text{-Alg} \to \mathcal{A}\text{-Alg}$$

called the universal enveloping functor. Let η be the unit of this adjunction. If P is a $\mathcal{P}$-algebra then there is a $\mathcal{P}$-algebra morphism $\eta_P\colon P \to GU(P)$. Moreover, the universal enveloping algebra $U(L)$ of a $\mathcal{P}$-algebra L is a connected $\mathcal{C}^c$-$\mathcal{A}$-bialgebra and $Prim\, U(L) = L$.

Let $\mathcal{Z} = \mathcal{A}/(\bar{\mathcal{P}})$ be the quotient operad of $\mathcal{A}$ by the ideal generated by the (nontrivial) primitive operations. Then J.-L. Loday proved a generalised Poincaré—Birkhoff—Witt theorem. In particular, it is proved in [Lod08, Theorem 3.1.4] that for any $\mathcal{P}$-algebra L there is an isomorphism of $\mathcal{Z}$-algebras

$$\mathcal{Z}(L) \to gr\, U(L).$$

Let L, L' be two simplicial $\mathcal{P}$-algebras. A morphism $f\colon L \to L'$ is a weak equivalence if it satisfies $\pi_*(f)\colon \pi_*(L) \simeq \pi_*(L')$.

Proposition 15.1. *Let $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ be a good triple of operads. Then the universal enveloping functor $U\colon \mathcal{P}\text{-Alg} \to \mathcal{A}\text{-Alg}$ preserves weak equivalences.*

Proof. If L be a simplicial $\mathcal{P}$-algebra then $U(L)$ is a simplicial $\mathcal{A}$-algebra. Using Dold-Kan correspondence we denote by $U(L)$ the associated chain complex. The primitive filtration on the universal enveloping algebra makes $U(L)$ a filtered complex. By Loday's generalised Poincaré—Birkhoff—Witt theorem the associated spectral sequence $E_r(U(L))$ of $U(L)$ satisfies

$$E_0(U(L)) \simeq \mathcal{Z}(L)$$

and

$$E_1(U(L)) = H_*(\mathcal{Z}(L)).$$

Let L, L' be two simplicial $\mathcal{P}$-algebras and $f\colon L \to L'$ is a weak equivalence. The morphism $U(f)\colon U(L) \to U(L')$ preserves filtrations and it is induced a morphism of spectral sequences

$$E_r(U(L)) \to E'_r(U(L')).$$

Since f is a weak equivalence it follows by a theorem of Dold [Dol58] that

$$H_*(\mathcal{Z}(L)) \simeq H_*(\mathcal{Z}(L')).$$

Therefore

$$E_1 U(f)\colon E_1(U(L)) \to E_1(U(L'))$$

is an isomorphism. The filtrations of $U(L)$ and $U(L')$ are complete and bounded below. Hence the spectral sequences converge and the induced map $E_\infty U(f)$ is an isomorphism. It follows that U preserves weak equivalences. $\square$

Let $\mathcal{T}$ be a operad and S a $\mathcal{T}$-algebra. The category S-modules over $\mathcal{T}$ is equivalent to the category of abelian group objects of $\mathcal{T}\text{-Alg}/S$ i.e the category of Beck S-modules in the category of $\mathcal{T}$-algebras (see [LV12], [GH00]). Hence the category of $U(P)$-modules over the operad $\mathcal{A}$ is equivalent to the category of Beck $U(P)$-modules. Under this equivalence the $U(P)$-module M is associated to the abelian group object $U(P) \ltimes M \to U(P)$, where by $U(P) \ltimes M$ we denote the semidirect product of $U(P)$ by M in the category of $\mathcal{A}$-algebras. We have the following pullback diagram in the category of $\mathcal{P}$-algebras,

$$\begin{tikzcd}
P \times_{GU(P)} G(U(P) \ltimes M) \arrow[r] \arrow[d] & G(U(P) \ltimes M) \arrow[d] \\
P \arrow[r, "\eta_P"] & GU(P)
\end{tikzcd}$$

and

$$0 \to M \to P \times_{GU(P)} G(U(P) \ltimes M) \to P \to 0$$

is an abelian extension in the category of $\mathcal{P}$-algebras. Therefore is induced on M a structure of P-module over the operad $\mathcal{P}$ which we denote by ${}_PM$.

Theorem 15.2. *Let $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ be a good triple of operads. Let P be an $\mathcal{P}$-algebra and M a Beck $U(P)$-module. Then we have the following isomorphism*

$$\mathrm{HQ}^*_{\mathcal{A}\text{-Alg}}(U(P), M) \simeq \mathrm{HQ}^*_{\mathcal{P}\text{-Alg}}(P, {}_PM).$$

Proof. By Proposition 15.1 we have that the enveloping algebra functor U preserves weak equivalences. Then the theorem follows from [Fra15, Proposition 4.12]. □

If we consider the good triple $(\mathrm{Com}, \mathrm{As}, \mathrm{Lie})$ then by Theorem 15.2 we recover in characteristic zero a well know result (see [CE56, §XIII, Theorem 5.1]). In particular, let L be a Lie algebra over $\mathbb{F}$ and M a $U(L)$-bimodule. By Theorem 15.2 we have the following isomorphism

$$\mathrm{HQ}^*_{\mathrm{As}\text{-Alg}}(U(L), M) \cong \mathrm{HQ}^*_{\mathrm{Lie}\text{-Alg}}(L, {}_LM).$$

where ${}_LM$ is M viewed as a left Lie L-module via the action $l \cdot m := lm - ml$ for all $l \in L$ and $m \in M$. Moreover, the Quillen cohomology for the category of associative algebras is shifted Hochschild cohomology of associative algebras and Quillen cohomology of Lie algebras is shifted Chevalley–Eilenberg cohomology of Lie algebras (cf. [Bar96]). In other words we have

$$\mathrm{HH}^*_{Hoch}(U(L), M) \cong H^*_{\mathrm{Lie}}(L, {}_LM).$$

In [Lod01] J.-L. Loday defined the notion of dendriform algebra which dichotomizes the notion of associative algebra. A dendriform algebra H is a vector space over $\mathbb{F}$ equipped with two binary operations $\prec, \succ : H \otimes H \to H$ such that

$$(x \prec y) \prec z = x \prec (y * z)$$
$$(x \succ y) \prec z = x \succ (y \prec z)$$
$$(x * y) \succ z = x \succ (y \succ z)$$

where $x * y = x \succ y + x \prec y$ and $x, y, z \in H$. The product $*$ is associative. One can notice that a dendriform algebra is an associative algebra whose product splits into two binary operations which satisfy the above identities. Dendriform algebras are Koszul dual to diassociative algebras (see [LV12]). Besides, M. Gerstenhaber and A. Voronov in [GV95] introduced the notion of brace algebras. We denote by $\mathcal{D}$ the operad associated to dendriform algebras and by $\mathcal{B}$ the operad associated to brace algebras.

M. Ronco in [Ron02] proves a Milnor—Moore type theorem for dendriform algebras. In particular, there is a good triple of operads $(\mathrm{As}, \mathcal{D}, \mathcal{B})$. The concept of bimodule over a dendriform algebra is defined by M. Aguiar in [Agu04]. Cohomology of dendriform algebras with coefficients in bimodules has been studied by A. Das in [Das22].

Let D be a dendriform algebra. The notion of Beck D-module is equivalent to the notion of bimodule over D. By Theorem 15.2 we have the following.

Proposition 15.3. *Let B be a $\mathcal{B}$-algebra and M a $U_{dend}(B)$-bimodule, where $U_{dend}(B)$ denote the enveloping dendriform algebra of the brace algebra B. Then we have the following isomorphism*

$$\mathrm{HQ}^*_{\mathcal{D}\text{-Alg}}(U_{dend}(B), M) \cong \mathrm{HQ}^*_{\mathcal{B}\text{-Alg}}(B, {}_B M).$$

Remark 15.4. In prime characteristic, for classical bialgebras we proved Theorem 14.2. We notice that the primitives of a classical bialgebra is a restricted Lie algebra i.e. a $\Gamma(Lie)$-algebra. If one wants extend this result to the context of generalized bialgebras, the right framework seems to be the category of $\Gamma(\mathcal{P})$-algebras.

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