Adaptive Boundary Control of the Kuramoto-Sivashinsky Equation Under Intermittent Sensing

M. C. Belhadjoudja, M. Maghenem, E. Witrant, C. Prieur

The authors are with the Université Grenoble Alpes, CNRS, Grenoble-INP, GIPSA-lab, F-38000, Grenoble, France

Abstract

We study in this paper boundary stabilization, in the L^2 sense, of the one-dimensional Kuramoto-Sivashinsky equation subject to intermittent sensing. We assume that we measure the state of this spatio-temporal equation on a given spatial subdomain during certain intervals of time, while we measure the state on the remaining spatial subdomain during the remaining intervals of time. As a result, we assign a feedback law at the boundary of the spatial domain and force to zero the value of the state at the junction of the two subdomains. Throughout the study, the destabilizing coefficient is assumed to be space-dependent and bounded but unknown. Adaptive boundary controllers are designed under different assumptions on the forcing term. In particular, when the forcing term is null, we guarantee global exponential stability of the origin. Furthermore, when the forcing term is bounded and admits a known upper bound, we guarantee input-to-state stability, and only global uniform ultimate boundedness is guaranteed when the upper bound is unknown. Numerical simulations are performed to illustrate our results.

Key words: KS equation, intermittent sensing, boundary control, L^2-stability, adaptive control.

1 Introduction

A fundamental constituent of a control loop is the sensor. Sensors have usually some limitations that need to be taken into account when designing a controller. In particular, for systems evolving in both space and time, an important limitation is related to the spatial measurement range. That is, one should not expect to measure the state at every spatial location for all time.

Systems that evolve in both space and time are usually described by partial differential equations (PDEs). In the physics community, a certain attention has been devoted to the stabilization of some PDEs under a spatially-limited range of sensing. For example, in [29], the authors assume that there is a finite number of sensors located at periodically separated spatial points and that each sensor measures the average of the state over a neighborhood of its location. Under this sensing scenario, a controller is designed and applied at the location of each sensor. A similar study is in [25], where the state is measured at a specific spatial location, and a control action is imposed at that same location. In [11], a time-periodic reset of the state is performed at uniformly separated spatial points, using data collected at these points. The aforementioned results are validated via numerical simulations only. Hence, it is important to study the stability properties of the obtained closed-loop systems. Our goal here is to study the effect of a specific spatially-limited sensing scenario on the stabilization of the Kuramoto-Sivashinsky (KS) equation.

The KS equation was first introduced in late 1970 by Y. Kuramoto [13] and G. Sivashinsky [26,27], and is given by

$$u_t + uu_x + \lambda(x)u_{xx} + u_{xxxx} = 0, \tag{1}$$

where $(x,t) \in (0,1) \times (0,+\infty)$ and, known as the *destabilizing coefficient*, $\lambda : [0,1] \to \mathbb{R}_{>0}$ creates the *anti-diffusion* phenomena. This equation is used to model phase turbulence in reaction-diffusion systems [13], thermo-diffusive instabilities in laminar flame fronts [26,27], fluctuations of fluid films on inclined supports [3,18], and plasma instabilities [14]. The nonhomogeneous version of (1), also called the *Noisy* KS equation (NKS), is given by

$$u_t + uu_x + \lambda(x)u_{xx} + u_{xxxx} = f(x,t), \tag{2}$$

where the right-hand side f, we name as the *forcing term*, accounts for disturbances, noises, and unmodeled dynamics. Equation (2) is used, for example, to model surface erosion via Ion sputtering [15,6], where f in this

Email addresses: mohamed.belhadjoudja@gipsa-lab.fr (M. C. Belhadjoudja), mohamed.maghenem@gipsa-lab.fr (M. Maghenem), emmanuel.witrant@gipsa-lab.fr (E. Witrant), christophe.prieur@gipsa-lab.fr (C. Prieur).

case is a Gaussian white noise that represents the fluctuations in the flux of bombarding particles.

Since the seminal work in [19], various boundary controllers have been developed for (1), under different assumptions on either the destabilizing coefficient λ or the size of the initial condition. When λ is small and unknown, an adaptive boundary controller is designed in [10]. In [5], an integral transformation is proposed to achieve exponential stabilization with an arbitrarily specified decay rate, provided that the initial condition is sufficiently small and that λ does not belong to a set of critical values. In [16], a boundary controller is designed to achieve local output feedback stabilization, independently of the size of λ. Boundary stabilization of (2) is considered in [4], under the assumption that $\lambda \in (0,1)$. The aforementioned boundary controllers either assume λ to be sufficiently small or the initial condition sufficiently close to the origin, in which cases, only boundary measurements are required. In all the aforementioned control references, λ is assumed to be constant. However, in physics and mathematics literature, it is usually space-dependent; see, e.g., [7,1].

Boundary stabilization of (1) (and (2)) independently of the size of λ (and f) and the initial condition using only boundary measurements remains open. To the best of our knowledge, the first work that studied boundary stabilization of (1) under limited (intermittent) sensing range is [20], where it is assumed that, for some $Y \in (0,1)$, the state u is measured at the spatial subdomain $(0,Y)$ during certain time intervals, and that it is measured over the spatial subdomain $(Y,1)$ during the remaining time intervals. The actuators act at three different locations: at $x = 0$, $x = Y$, and at $x = 1$. Feedback controllers are designed at $x = 0$ and at $x = 1$, and a zero state is imposed at $x = Y$. The proposed boundary controllers are shown to guarantee global exponential stability of the origin in the L^2 sense. Moreover, the controllers are shown to remain bounded and converge asymptotically to zero along the closed-loop solutions. However, the destabilizing coefficient λ is assumed to be constant and perfectly known.

In this paper, we generalize the approach in [20] by designing adaptive boundary controllers for the NKS equation under different assumptions on f as well as the available measurements. To start, we revisit the problem studied [20], where the sensing was intermittent and $f = 0$. Different from [20], we assume that λ, $|\lambda|_\infty$, and $|\lambda'|_\infty$ are unknown. As a consequence, we design an adaptive boundary controller guaranteeing global exponential stability of the origin $u := 0$ in the L^2 sense. To obtain this result, we exploit the possibility of achieving exponential convergence of the L^2 norm of the measured state to zero at any desired rate. Roughly speaking, during the time intervals when only u on $(Y,1)$ is measured, we set $u = 0$ at $x = 0$. Hence, if the resulting PDE defined on $(0,Y)$ is unstable, the state u at $(0,Y)$ may grow during these time intervals. To still guarantee convergence of u to zero on the entire domain $(0,1)$, we need to guarantee a sufficiently fast decay of the L^2 norm of u at $(0,Y)$ during the time intervals when u at $(0,Y)$ is measured. To achieve a desirable exponential decay, when λ is unknown, we propose a novel adaptive control strategy. Next, we tackle the case where $f \neq 0$. To illustrate the challenges that intermittent sensing brings, we start considering the case where the state u is measured everywhere and for all time, but both λ and f are completely unknown. Note that from now on, when we say that λ and f are unknown, we mean that $|\lambda|_\infty$, $|\lambda'|_\infty$ and $|f|_\infty$ are unknown. In this case, the actuators act at $x = 0$ and $x = 1$. We set to zero the input at $x = 1$ and design an adaptive feedback controller at $x = 0$ that guarantees global practical attractivity in the L^2 sense. Namely, the solutions converge in the L^2 norm, as $t \to \infty$, to any chosen neighborhood of the origin. Practical attractivity cannot be guaranteed under intermittent sensing. Indeed, we show in the latter case that even if $|f|_\infty$ is known, one can guarantee only input-to-state stability with respect to $|f|_\infty$, in the L^2 sense. Indeed, on the time intervals when we do not measure the state on $(0,Y)$, as we set the control input at $x = 0$ to zero, the state can reach, independently of its history, a constant value that is proportional to the supremum of f. Finally, we consider the most general scenario, where the system is subject to intermittent sensing and λ and f are both unknown. In this case, the only result we can guarantee is global uniform ultimate boundedness in the L^2 sense. Namely, we show the existence of a constant, not necessarily proportional in f and independent of the initial conditions, towards which the closed-loop solutions converge in finite time. The reaching time depends on the L^2 norm of the initial conditions.

A preliminary version of this work is in [2], where the KS equation is considered when λ is unknown and constant. In addition to considering the perturbed case, namely, the NKS equation, deeper discussions, and detailed proofs are included here.

The rest of the paper is organized as follows. In Section 2, we describe the considered intermittent sensing scenario, the boundary control locations, as well as the hypotheses that hold throughout the paper. In Section 3, we present our main results. In Section 4, we present the proofs of our main results. Finally, in Section 5, we illustrate our theoretical results via numerical simulations.

Notation. Depending on the context, a.e. means either almost every or almost everywhere. Let X be a Banach space with a norm $|.|_X$, $p \in \{1,2,...,+\infty\}$, and let $a,b \in \mathbb{R}$. We denote by $L^p(a,b;X)$ the space of measurable functions $u : [a,b] \to X$, with finite p-norm $|.|_p$, where $|u|_p^p := \int_a^b |u(t)|_X^p dt$ if $p < \infty$ and $|u|_\infty := \operatorname{ess\,sup}_{t\in[a,b]} |u(t)|_X$. When $X = \mathbb{R}$, we just write $L^p(a,b)$ instead of $L^p(a,b;\mathbb{R})$. For $k \in \mathbb{N}$, we denote by $H^k(a,b)$ the Sobolev space of functions whose distributional derivatives, up to order k, are in $L^2(a,b)$. For $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathcal{C}^k(a,b)$ the space of

k-times continuously differentiable functions on (a, b). For $(x, t) \mapsto f(x, t)$, the partial derivative of f with respect to t is denoted by $\partial_t f$ and the k^{th} partial derivative with respect to x is denoted by $\partial_x^k f$. We denote the time derivative of a function V by either $\frac{d}{dt}V$ or $\dot{V}$. We also denote the derivative of a function $x \mapsto u(x)$ of a scalar variable by u' and its second derivative by u''. For $(x, t) \mapsto u(x, t)$, we may write $u(x)$ instead of $u(x, t)$. Finally, for $x \in \mathbb{R}$, $\operatorname{sgn}(x) = 1$ if $x > 0$, $= 0$ if $x = 0$ and $= -1$ if $x < 0$.

2 Context: Sensing and Control Locations

2.1 Intermittent Sensing

Consider equations (1) and (2). Following [20], we let $Y \in [0, 1]$ and we assume that $u((0, Y), t)$ is measured during certain intervals of time and that $u((Y, 1), t)$ is measured during the remaining intervals of time. More precisely, we assume the existence of a sequence $\{t_i\}_{i=1}^{\infty} \subset \mathbb{R}_+$, with $t_1 = 0$ and $t_{i+1} > t_i$, such that

S1) $u((0, Y), t)$ is available for all $t \in I_1 := \bigcup_{k=1}^{\infty} [t_{2k-1}, t_{2k})$.
S2) $u((Y, 1), t)$ is available for all $t \in I_2 := \bigcup_{k=1}^{\infty} [t_{2k}, t_{2k+1})$.

Associated with the proposed intermittent sensing scenario, we consider the following dwell-time assumption.

Assumption 1 *There exist four constants $\overline{T}_1$, $\overline{T}_2$, $\underline{T}_1$, $\underline{T}_2 > 0$ such that, for each $k \in \mathbb{N}^*$, we have*

$$\underline{T}_1 \leq t_{2k} - t_{2k-1} \leq \overline{T}_1, \quad \underline{T}_2 \leq t_{2k+1} - t_{2k} \leq \overline{T}_2.$$

2.2 Control Locations

In the case of intermittent sensing, we propose to control (1) and (2) at three different locations: at $x = 0$, $x = Y$, and $x = 1$. We, therefore, assimilate (1) and (2) to a system of two NKS equations interconnected by boundary constraints at $x = Y$. That is, we introduce the system of PDEs

$$\begin{aligned} w_t + ww_x + \lambda w_{xx} + w_{xxxx} &= f \qquad x \in (0, Y), \\ v_t + vv_x + \lambda v_{xx} + v_{xxxx} &= f \qquad x \in (Y, 1), \end{aligned} \tag{3a}$$

$$\begin{aligned} &w(Y) = w_x(Y) = w_x(0) = 0, \\ &v(Y) = v_x(Y) = v_x(1) = 0, \\ &w(0) = u_1, \ v(1) = u_2, \end{aligned} \tag{3b}$$

where (u_1, u_2) are control inputs to be designed.

We define a solution $u : (0, 1) \times \mathbb{R}_+ \to \mathbb{R}$ to (2) subject to the boundary conditions

$$\begin{aligned} &u(Y) = 0 \\ &u_x(0) = u_x(Y) = u_x(1) = 0 \\ &u(0) = u_1, \ u(1) = u_2, \end{aligned} \tag{4}$$

for some (u_1, u_2), as $u(x, t) = w(x, t)$ a.e. on $(0, Y) \times \mathbb{R}_{>0}$ and $u(x, t) = v(x, t)$ a.e. on $(Y, 1) \times \mathbb{R}_{>0}$, with (w, v) a strong solution to (3) subject to the same (u_1, u_2).

Remark 1 We recall that (w, v) is a strong solution to (3) subject to (u_1, u_2) if $w \in L^2\left(I, H^4(0, Y)\right)$ and $v \in L^2\left(I, H^4(Y, 1)\right)$ for every compact set $I \subset \mathbb{R}_{>0}$, and (w, v) verifies (3a) a.e. in space and time and verifies (3b) a.e. in time; see [8, Chapter 9].

Remark 2 A solution to (2) under (4), as previously defined, is not necessarily a strong solution to (2). Indeed, a strong solution to (2) must satisfy $u(\cdot, t) \in H^4(0, 1)$. However, the concatenation of w and v (forming a strong solution to (3)) satisfies, *a priori*, only $u(\cdot, t) \in H^2(0, 1)$, since $w_{xxx}(Y)$ and $v_{xxx}(Y)$ are not necessarily equal. Imposing more regularities at $x = Y$ in (3) would lead to an over-determined system of PDEs.

According to the aforementioned concept of solutions, guaranteeing an L^2-stability property for (2) under (4) is equivalent to guaranteeing the same L^2-stability property for (3).

Our results in the sequel use the following assumption.

Assumption 2 *Each one of the maps f, λ, and λ' has a finite sup norm.*

We stress that the coefficient λ in (3a) and the constants $(\overline{T}_1, \overline{T}_2, \underline{T}_1, \underline{T}_2)$ in Assumption 1 are *unknown.*

3 Results

In this section, we design (u_1, u_2) to guarantee different (L^2-stability) properties for (2) under (4) in the following different situations.

- When $f = 0$ and the sensing is intermittent.
- When $f \neq 0$, $|f|_\infty$ is unknown, and the entire state is measured all the time.
- When $f \neq 0$, $|f|_\infty$ is known, and the sensing is intermittent.
- When $f \neq 0$, $|f|_\infty$ is unknown, and the sensing is intermittent.

Note that a similar control strategy is exploited for the four aforementioned situations. This approach, however, is shown to guarantee different (L^2-stability) properties depending on the considered situation.

3.1 Intermittent Sensing with $f = 0$

We start considering the Lyapunov function candidates

$$V_1(w) := \frac{1}{2}\int_0^Y w(x)^2 dx, \quad V_2(v) := \frac{1}{2}\int_Y^1 v(x)^2 dx. \tag{5}$$

Furthermore, we let

$$(u_1, u_2) := \begin{cases} (\kappa(V_1, w_{xxx}(0), \hat{\theta}_1), 0) & \text{on } I_1, \\ (0, -\kappa(V_2, v_{xxx}(1), \hat{\theta}_2)) & \text{on } I_2. \end{cases} \tag{6}$$

The explicit expression of κ in (6) is given by

$$\kappa(V, \omega, \hat{\theta}) := \begin{cases} -\operatorname{sgn}(\omega) \sqrt[3]{V} & \text{if } |\omega| \geq l(V, \hat{\theta}), \\ -3(3\hat{\theta} + 1) \sqrt[3]{V} & \text{otherwise}, \end{cases} \tag{7}$$

where $l(V,\hat{\theta}) := (1/3)(1+3\hat{\theta})V^{2/3}$ and the parameters $(\hat{\theta}_1, \hat{\theta}_2)$ are dynamically updated according to the following algorithm.

Algorithm 1 *Given Δ_1, Δ_2, and $\sigma > 0$, we dynamically update $(\hat{\theta}_1, \hat{\theta}_2)$ according to the following rules:*

R1) On every interval $[t_{2k-1}, t_{2k}) \subset I_1$, we set $\dot{\hat{\theta}}_2 = 0$. Moreover, if $V_1(t_{2k-1}) > V_1(t_{2k-3}) \exp^{-\sigma(t_{2k-1}-t_{2k-3})}$, we set $\dot{\hat{\theta}}_1 := \Delta_1$; otherwise, we set $\dot{\hat{\theta}}_1 := 0$.
R2) On every interval $[t_{2k}, t_{2k+1}) \subset I_2$, we set $\dot{\hat{\theta}}_1 := 0$. Moreover, if $V_2(t_{2k}) > V_2(t_{2k-2}) \exp^{-\sigma(t_{2k}-t_{2k-2})}$, we set $\dot{\hat{\theta}}_2 := \Delta_2$; otherwise, we set $\dot{\hat{\theta}}_2 := 0$.
R3) On $[t_1, t_3]$, $\hat{\theta}_1 = \hat{\theta}_1(0) \geq 0$ and $\hat{\theta}_2 = \hat{\theta}_2(0) \geq 0$.

In this situation, we show that the closed-loop system enjoys the following property of L^2 global exponential stability (L^2-GES).

Property 1 (L^2-GES) *For each $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exists $\gamma \geq 1$ such that, along the solutions to (3) subject to (6), we have*

$$V_1(t) + V_2(t) \leq \gamma(V_1(0) + V_2(0)) \exp^{-\sigma t} \quad \forall t \geq 0, \tag{8}$$

where $\sigma > 0$ comes from Algorithm 1.

Theorem 1 *Consider system (3) such that Assumption 2 holds with $f = 0$. Consider the intermittent sensing scenario S1)-S2) such that Assumption 1 holds. Let (V_1, V_2) be defined in (5), κ defined in (7), and the parameters $(\hat{\theta}_1, \hat{\theta}_2)$ governed by Algorithm 1. By letting (u_1, u_2) as in (6), we conclude that, for (3) in closed loop,*

- *The L^2-GES property is verified.*
- *For each $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exists $M > 0$ such that, for each $(V_1(0), V_2(0))$, $|\hat{\theta}_1|_\infty, |\hat{\theta}_2|_\infty \leq M$.*
- *The inputs (u_1, u_2) remain bounded and converge asymptotically to zero.*

Sketch of proof: We start by referring to Lemma 2 in the appendix, which shows that along (3) we have

$$\dot{V}_1 \leq \theta_1 V_1 + P(u_1, w_{xxx}(0)), \tag{9}$$
$$\dot{V}_2 \leq \theta_2 V_2 - P(u_2, v_{xxx}(1)), \tag{10}$$

where (θ_1, θ_2) are constant parameters that depend on λ and λ', and

$$P(u, \omega) := \frac{u^3}{3} + u\omega. \tag{11}$$

By setting (u_1, u_2) as (6) and κ as in (7), we obtain

$$\begin{aligned} P(u_1, w_{xxx}(0)) &\leq -\hat{\theta}_1 V_1 \quad \text{on } I_1, \\ -P(u_2, v_{xxx}(1)) &\leq -\hat{\theta}_2 V_2 \quad \text{on } I_2. \end{aligned} \tag{12}$$

Moreover, since $u_1 = 0$ on I_2 and $u_2 = 0$ on I_1, we have

$$\begin{aligned} P(u_1, w_{xxx}(0)) &= 0 \quad \text{on } I_2, \\ P(u_2, v_{xxx}(1)) &= 0 \quad \text{on } I_1. \end{aligned}$$

As a consequence, (V_1, V_2) satisfy

$$\begin{cases} \dot{V}_1 \leq (\theta_1 - \hat{\theta}_1)V_1 \\ \dot{V}_2 \leq \theta_2 V_2 \end{cases} \text{a.e. on } I_1, \qquad \begin{cases} \dot{V}_1 \leq \theta_1 V_1 \\ \dot{V}_2 \leq (\theta_2 - \hat{\theta}_2)V_2 \end{cases} \text{a.e. on } I_2. \tag{13}$$

Roughly speaking, if $(\hat{\theta}_1, \hat{\theta}_2) \to +\infty$ as $t \to +\infty$, then the decrease of V_1 on I_1 (resp. of V_2 on I_2) would be significant enough to compensate for the eventual increase of V_1 on I_2 (resp. of V_2 on I_1). Since the adaptation parameters need to remain bounded, the idea behind Algorithm 1 is to increase $\hat{\theta}_1$ and $\hat{\theta}_2$ and to freeze them whenever a decay of (V_1, V_2) at a desired rate is observed. To detect the decrease of V_1, we compare $V_1(t_{2k-1})$ with $V_1(t_{2k-3})$. The latter two values are available for design since $t_{2k-1} \in [t_{2k-1}, t_{2k}) \subset I_1$ and $t_{2k-3} \in [t_{2k-3}, t_{2k-2}) \subset I_1$. The same reasoning applies for V_2 mutatis-mutandis.

Remark 3 The discontinuity of κ is key to verify (12) while guaranteeing that u_1 and u_2 are bounded in $w_{xxx}(0)$ and $v_{xx}(1)$, respectively. This allows to guarantee input boundedness along the closed-loop solutions provided that the same property holds for $\hat{\theta}_1$, $\hat{\theta}_2$, V_1, and V_2. This discontinuity, however, can lead to the unsuitable *chattering* phenomena. Another discontinuity in (u_1, u_2) is due to transitioning from a time interval in I_1 to a time interval in I_2, and vice versa, see (6). Discontinuous boundary control of PDEs brings challenges both in terms of numerical and practical implementation and the study of well-posedness, only a few works have been done in this direction; see, e.g., [31,17].

Remark 4 Since we do not know a subset of $\mathbb{R}$ containing both λ and f, the adaptive technique based on *batch least-square optimization* [9, Eq. 18] cannot be used here. Furthermore, the adaptive approach in [28], although it does not require any information about λ and f, does not necessarily guarantee asymptotic convergence of V_1 and V_2 to zero under the considered intermittent-sensing scenario. Indeed, using such a technique, we can show that, for a desired $\alpha > 0$, $V_1(t) \leq (V_1(t_{2k}) + |\theta_1(t_{2k}) - \hat{\theta}_1(t_{2k})|) \exp^{-\alpha(t-t_{2k})}$ for all $t \in [t_{2k}, t_{2k+1}) \subset I_1$. If such inequality was verified for all $t \geq t_{2k}$, then asymptotic convergence of V_1 to 0 would follow. The latter conclusion does not necessarily hold in the context of intermittent sensing mainly because $\tilde{\theta}_1 := \theta_1 - \hat{\theta}_1$ is not guaranteed to converge to zero. Finally, the adaptation techniques in [23,21,22] guarantee

$V_1(t) \leq V_1(t_{2k}) \exp^{-\alpha(t-t_{2k})}$ for all $t \in [t_{2k}, t_{2k+1}) \subset I_1$, but not at any desired α. Hence, they cannot be used to compensate for the possible increase of V_1 on I_2.

3.2 Full Sensing, $f \neq 0$, and $|f|_\infty$ Unknown

In this section, we suppose that the state of the NKS equation is measured everywhere and for all time. As a result, we assume that we act only at $x = 0$ and $x = 1$ using the boundary conditions

$$u(0) = u_1, \quad u(1) = u_x(0) = u_x(1) = 0, \tag{14}$$

where u_1 is the control input to be designed.

Consider the Lyapunov function candidate

$$V(u) := \frac{1}{2} \int_0^1 u(x)^2 dx, \tag{15}$$

Using Lemma 2 in the Appendix, with $(w, Y, V) = (u, 1, V_1)$, we conclude that, along the solutions to (2) under (14), we have

$$\dot{V} \leq \theta V + C\sqrt{V} + P(u_1, u_{xxx}(0)) \quad \text{a.e.}, \tag{16}$$

where $\theta := |\lambda'|_\infty + 2\left(|\lambda|_\infty + 1/2\right)\left((|\lambda|_\infty + 1/2) + 12\right)$ and $C := \sqrt{2}|f|_\infty$.

Using the feedback law

$$u_1 := \kappa(V, u_{xxx}(0), \hat{\theta}), \tag{17}$$

where κ is introduced in (7) and $\hat{\theta}$ is an adaptation parameter to be designed later, we obtain

$$\dot{V} \leq (\theta - \hat{\theta})V + C\sqrt{V} \qquad \text{a.e.} \tag{18}$$

Using Young inequality, we obtain that, for each $\epsilon > 0$,

$$\dot{V} \leq \left(\theta + \frac{C^2}{\epsilon} - \hat{\theta}\right) V + \epsilon \quad \text{a.e.} \tag{19}$$

We update the parameter $\hat{\theta}$ as follows.

Algorithm 2 *Given Δ, τ, ϵ, and $\sigma > 0$, the coefficient $\hat{\theta}$ is dynamically updated, on each interval $[k\tau, (k+1)\tau]$ with $k \in \mathbb{N}^*$, according to the following rules:*

R1) For each $t \in [k\tau, (k+1)\tau]$ if

$$V(s) \leq V(k\tau) \exp^{-\sigma(s-k\tau)} + \frac{\epsilon}{\sigma} \quad \forall s \in [k\tau, t],$$

then $\dot{\hat{\theta}}(t) = 0$; otherwise,

$$\hat{\theta}(r) = \hat{\theta}(k\tau) + \Delta \quad \forall r \in [t, (k+1)\tau]. \tag{20}$$

R2) On the interval $[0, \tau)$, we set $\hat{\theta} = \hat{\theta}(0) \geq 0$.

We will show that the closed-loop system enjoys the property of L^2 global practical attractivity (L^2-GpA).

Property 2 (L^2-GpA) *For any $\eta > 0$, we can find (τ, σ, ϵ) such that, for any $\hat{\theta}(0) \geq 0$ and for any $V(0) \geq 0$, we have $\limsup_{t\to+\infty} V(t) \leq \eta$.*

Theorem 2 *Consider equation (2) such that Assumption 2 holds. Consider the boundary conditions in (14) and let the feedback law $u_1 := \kappa$ in (7) with $\hat{\theta}$ updated according to Algorithm 2. Then, for the resulting closed-loop system,*

- *The L^2-GpA property is verified.*
- *For each $\hat{\theta}(0)$, there exists $M > 0$ such that, for each $V(0)$, we have $|\hat{\theta}|_\infty \leq M$.*
- *The input u_1 remains bounded.*

Remark 5 In view of (16), our design of u_1 guarantees that $P(u_1, u_{xxx}(0)) \leq -\hat{\theta}_1 V$, where P is in (11), and thus (18) holds. The latter Lyapunov inequality does not allow us to guarantee L^2-GES due to the term $C\sqrt{V}$, which becomes dominant when V is small. If we attempt to compensate the latter term by designing u_1 such that

$$P(u_1, u_{xxx}(0)) \leq -\hat{\theta}V - \hat{C}\sqrt{V}, \tag{21}$$

where $\hat{C}$ is an adaptation parameter, we would obtain

$$\dot{V} \leq (\theta - \hat{\theta})V + (C - \hat{C})\sqrt{V} \quad \text{a.e.} \tag{22}$$

In this case, if $\hat{C} > C$ and $\hat{\theta} \geq \theta$, we would conclude that

$$\dot{V} \leq -\alpha\sqrt{V} \quad \text{a.e.}, \quad \alpha := \hat{C} - C > 0. \tag{23}$$

Hence, V would reach zero in finite time implying the same for w. However, $w = 0$ is not a solution to the NKS equation when $f \neq 0$. Hence, trying to compensate $C\sqrt{V}$ can lead to an ill-posed closed-loop system.

Remark 6 To guarantee the continuity of $u_1 = \kappa(V, u_{xxx}(0), \hat{\theta})$ with respect to its arguments and its boundedness along the closed-loop solutions, one can try to redesign the function κ to be a continuous function solution to $\kappa^3 + 3\kappa u_{xxx}(0) \leq 3\hat{\theta}V$ and guarantee boundedness of the H^1 norm of the state u in closed loop. In this case, input boundedness would follow thanks to Agmon's inequality [12, Lemma A.2]. While this reasoning can apply to the full-sensing case, it cannot be applied to the intermittent-sensing case. Indeed, during the time intervals when w (resp. v) is not measured, we are not able to upper-bound the H^1 norm of w (resp. v), as we have no information on the terms $w_{xx}(\{0, Y\})$ and $w_{xxx}(\{0, Y\})$ (resp. $v_{xx}(\{Y, 1\})$ and $v_{xxx}(\{Y, 1\})$) that affect the H^1 norm of w (resp. v).

3.3 Intermittent Sensing with $f \neq 0$ and $|f|_\infty$ Known

We suppose here that $|f|_\infty \neq 0$ (or an upper bound of $|f|_\infty$) is known. In the context of intermittent sensing, we design (u_1, u_2) according to (6)-(7), with $(\hat{\theta}_1, \hat{\theta}_2)$ therein dynamically updated as follows.

Algorithm 3 *Given $\Delta_1, \Delta_2, \sigma > 0$, we update $(\hat{\theta}_1, \hat{\theta}_2)$ according to the following rules:*

R1) On each interval $[t_{2k-1}, t_{2k}) \subset I_1$, we set $\dot{\hat{\theta}}_2 := 0$. Moreover, if

$$V_1(t_{2k-1}) > V_1(t_{2k-3}) \exp^{-\sigma(t_{2k-1}-t_{2k-3})} + \left(C_1 + \frac{C_1^2}{4}\right) \exp^{(\hat{\theta}_1(t_{2k-3})+1)(t_{2k-1}-t_{2k-3})},$$

we set $\dot{\hat{\theta}}_1 := \Delta_1$; otherwise, we set $\dot{\hat{\theta}}_1 := 0$.

R2) On each interval $[t_{2k}, t_{2k+1}) \subset I_2$, we set $\dot{\hat{\theta}}_1 := 0$. Moreover, if

$$V_2(t_{2k}) > V_2(t_{2k-2}) \exp^{-\sigma(t_{2k}-t_{2k-2})} + \left(C_2 + \frac{C_2^2}{4}\right) \exp^{(\hat{\theta}_2(t_{2k-2})+1)(t_{2k}-t_{2k-2})},$$

we set $\dot{\hat{\theta}}_2 := \Delta_2$; otherwise, we set $\dot{\hat{\theta}}_2 := 0$.

R3) On $[t_1, t_3]$, $\hat{\theta}_1 = \hat{\theta}_1(0) \geq 0$ and $\hat{\theta}_2 = \hat{\theta}_2(0) \geq 0$.

We show here that the closed-loop system enjoys the property of L^2-input-to-state stability (L^2-ISS) with respect to the perturbation f.

Property 3 (L^2-ISS) *For each $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exists $\gamma \geq 1$ and a class $\mathcal{K}$ function Φ such that, for all $t \geq 0$, we have*

$$V_1(t) + V_2(t) \leq \gamma(V_1(0) + V_2(0))e^{-\sigma t} + \Phi(|f|_\infty), \quad (24)$$

where $\sigma > 0$ comes from Algorithm 3.

Theorem 3 *Consider system* (3) *such that Assumption 2 holds. Consider the intermittent sensing scenario in S1)-S2) such that Assumption 1 holds. Let (V_1, V_2) be defined in* (5), *κ be defined in* (7), *and the parameters $(\hat{\theta}_1, \hat{\theta}_2)$ governed by Algorithm 3. By letting (u_1, u_2) as in* (6), *we conclude that, for* (3) *in closed-loop system,*

- *The L^2-ISS property is verified.*
- *For each $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exists $M > 0$ such that for all $(V_1(0), V_2(0))$, we have $|\hat{\theta}_1|_\infty, |\hat{\theta}_2|_\infty \leq M$.*
- *The inputs (u_1, u_2) remain bounded.*

Remark 7 In Section 3.1 and when $f = 0$, to prove that $(V_1, V_2) \to 0$ as $t \to +\infty$, we used the fact that the maximal value of V_1 (resp. V_2) on each interval in I_1 (resp. I_2) is smaller than the value of V_1 (resp. V_2) at the beginning of that interval multiplied by a constant. Moreover, when $f \neq 0$, the maximal value V_1 (resp. V_2) on each interval in I_1 (resp. I_2) additionally depends on $|f|_\infty$. In other words, there are values, depending on $|f|_\infty$, that V_1 (resp. V_2) can reach over each interval in I_1 (resp. I_2) independently on its initial value at the beginning of that interval. As a consequence, we can only try to decrease V_1 on I_1 and V_2 on I_2 significantly enough so that (24) holds.

3.4 Intermittent Sensing With $f \neq 0$ and $|f|_\infty$ Unknown

We design (u_1, u_2) as in (6) with κ as in (7) and $(\hat{\theta}_1, \hat{\theta}_2)$ are updated according to the following algorithm.

Algorithm 4 *Given Δ_1, Δ_2, $\sigma > 0$, the coefficients $(\hat{\theta}_1, \hat{\theta}_2)$ are dynamically updated as follows:*

R1) On every interval $[t_{2k-1}, t_{2k}) \subset I_1$, we set $\dot{\hat{\theta}}_2 := 0$. Moreover, if

$$V_1(t_{2k-1}) > V_1(t_{2k-3}) \exp^{-\sigma(t_{2k-1}-t_{2k-3})} + \left(\hat{\theta}_1(t_{2k-3}) + \frac{\hat{\theta}_1(t_{2k-3})^2}{4}\right) \exp^{(\hat{\theta}_1(t_{2k-3})+1)(t_{2k-1}-t_{2k-3})},$$

we set $\dot{\hat{\theta}}_1 := \Delta_1$; otherwise, we set $\dot{\hat{\theta}}_1 := 0$.

R2) On every interval $[t_{2k}, t_{2k+1}) \subset I_2$, we set $\dot{\hat{\theta}}_1 := 0$. Moreover, if

$$V_2(t_{2k}) > V_2(t_{2k-2}) \exp^{-\sigma(t_{2k}-t_{2k-2})} + \left(\hat{\theta}_2(t_{2k-2}) + \frac{\hat{\theta}_2(t_{2k-2})^2}{4}\right) \exp^{(\hat{\theta}_2(t_{2k-2})+1)(t_{2k}-t_{2k-2})},$$

we set $\dot{\hat{\theta}}_2 := \Delta_2$; otherwise, we set $\dot{\hat{\theta}}_2 := 0$.

R3) On $[t_1, t_3]$, $\hat{\theta}_1 = \hat{\theta}_1(0) \geq 0$ and $\hat{\theta}_2 = \hat{\theta}_2(0) \geq 0$.

We will show that the resulting closed-loop system satisfies the property of L^2 global uniform ultimate boundedness (L^2-GUUB).

Property 4 (L^2-GUUB) *For any $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exists a constant $r > 0$ such that for any $R > 0$, there exists a finite time $T(R) \geq 0$ such that, along the closed-loop solutions, we have*

$$V_1(0) + V_2(0) \leq R \Rightarrow V_1(t) + V_2(t) \leq r \; \forall t \geq T(R). \quad (25)$$

Theorem 4 *Consider system* (3) *such that Assumption 2 holds. Consider the intermittent sensing scenario in S1)-S2) such that Assumption 1 holds. Let (V_1, V_2) be defined in* (5), *κ be defined in* (7), *and $(\hat{\theta}_1, \hat{\theta}_2)$ be governed by Algorithm 4. Then, if we set (u_1, u_2) as in* (6), *for the resulting closed-loop system,*

- *The L^2-GUUB property is verified.*
- *For each $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exists $M > 0$ such that for all $(V_1(0), V_2(0))$, we have $|\hat{\theta}_1|_\infty, |\hat{\theta}_2|_\infty \leq M$.*
- *The control inputs remain bounded.*

Sketch of proof: According to Lemma 2 in the Appendix, along the closed-loop solutions, (V_1, V_2) verify

$$\begin{aligned} \dot{V}_1 &\le \theta_1 V_1 + C_1\sqrt{V}_1 + P(u_1, w_{xxx}(0)) \quad \text{a.e.}, \\ \dot{V}_2 &\le \theta_2 V_2 + C_2\sqrt{V}_2 - P(u_2, v_{xxx}(1)) \quad \text{a.e.}, \end{aligned} \tag{26}$$

where P is in (11) and (C_1, C_2) are constant parameters that depend on $|f|_\infty$; see (90). Now, by setting (u_1, u_2) as in (6), one can show that

$$\begin{cases} \begin{aligned} \dot{V}_1 &\le (\theta_1 + C_1 - \hat{\theta}_1)V_1 + C_1 \\ \dot{V}_2 &\le \qquad \theta_2 V_2 + C_2\sqrt{V_2} \end{aligned} & \text{a.e. on } I_1, \\ \\ \begin{aligned} \dot{V}_1 &\le \qquad \theta_1 V_1 + C_1\sqrt{V_1} \\ \dot{V}_2 &\le (\theta_2 + C_2 - \hat{\theta}_2)V_2 + C_2 \end{aligned} & \text{a.e. on } I_2. \end{cases} \tag{27}$$

Our approach consists of decreasing V_1 on I_1 and V_2 on I_2 significantly enough so that they remain bounded all the time. To do so, we can show that by letting $(\hat{\theta}_1, \hat{\theta}_2) \to +\infty$ as $t \to +\infty$, we can guarantee that V_1 and V_2 remain bounded at the price of using unbounded control. To guarantee input boundedness, the idea behind Algorithm 4 is to increase $\hat{\theta}_1$ (resp. $\hat{\theta}_2$) until the inequality in R1) (resp. R2)) is not verified, and $\hat{\theta}_1$ (resp. $\hat{\theta}_2$) remains constant as long as R1) (resp. R2)) is not verified. One can show that there exists a finite time, from which, the conditions in the inequalities in R1) (resp. R2)) are never verified. Hence, we can conclude boundedness of $\hat{\theta}_1$ (resp. $\hat{\theta}_2$)) and thus boundedness of (u_1, u_2).

Remark 8 In view of (26), we could design (u_1, u_2) such that

$$\begin{aligned} P(u_1, w_{xxx}(0)) &\le -\hat{\theta}_1 V_1 - \hat{C}_1\sqrt{V_1}, \\ -P(u_2, v_{xxx}(1)) &\le -\hat{\theta}_2 V_2 - \hat{C}_2\sqrt{V_2}, \end{aligned}$$

where P is in (11) and $(\hat{C}_1, \hat{C}_2)$ are adaptation parameters. In this case, we would obtain

$$\begin{aligned} \dot{V}_1 &\le (\theta_1 - \hat{\theta}_1)V_1 + (C_1 - \hat{C}_1)\sqrt{V}_1 \quad \text{a.e. on } I_1. \\ \dot{V}_2 &\le (\theta_2 - \hat{\theta}_2)V_2 + (C_2 - \hat{C}_2)\sqrt{V}_2 \quad \text{a.e. on } I_2. \end{aligned}$$

This choice is not suitable for the same ill-posedness reasons described in Remark 5.

Remark 9 In Section 3.4, where $|f|_\infty$ (or its upper bound) was known, we were able to use this knowledge to guarantee the relatively stronger L^2-ISS property. Indeed, the parameters C_1 and C_2, involving $|f|_\infty$, are used in Algorithm 3 to tune the control parameters until convergence of V_1 (resp. V_2) to a neighborhood of the origin, whose size is proportional to C_1 (resp. C_2), is detected. Here, since $|f|_\infty$ is unknown, we are not able to guarantee that V_1 (resp. V_2), although maintained bounded, is proportional to C_1 (resp. C_2) in steady state.

4 Proofs

4.1 Proof of Theorem 1

The proof of Theorem 1 follows in four steps. In Step 1, by setting (u_1, u_2) as in (6), we show that the Lyapunov functions (V_1, V_2) in (5), along the closed-loop solutions, verify the inequalities in (13). In Step 2, we show that $(\hat{\theta}_1, \hat{\theta}_2)$ become constant after some finite time $T > 0$. Furthermore, the maximal values that $(\hat{\theta}_1, \hat{\theta}_2)$ can reach do not exceed the maximum value between $\hat{\theta}_1(0)$ and a constant M_1 and $\hat{\theta}_2(0)$ and a constant M_2, respectively. The constants M_1 and M_2 depend only on $(\sigma, \theta_1, \theta_2, \overline{T}_1, \overline{T}_2, \underline{T}_1, \underline{T}_2, \Delta_1, \Delta_2)$. In Step 3, we show that the function $V_1 + V_2$ verifies the inequality (8), which concludes L^2-GES of the origin. Finally, we use the structure of κ in (7) guaranteeing that $|u_1| \le 3(3|\hat{\theta}_1|_\infty + 1)\sqrt[3]{V_1}$ and $|u_2| \le 3(3|\hat{\theta}_2|_\infty + 1)\sqrt[3]{V_2}$, after proving that $(\hat{\theta}_1, \hat{\theta}_2)$ are bounded and (V_1, V_2) are bounded and converge asymptotically to zero, to conclude that (u_1, u_2) are bounded and converge asymptotically to zero.

Step 1: Under Lemma 2 in the Appendix, (V_1, V_2) verify for almost everywhere the inequalities

$$\dot{V}_1 \le \theta_1 V_1 + \frac{u_1^3}{3} + u_1 w_{xxx}(0), \tag{28}$$

$$\dot{V}_2 \le \theta_2 V_2 - \frac{u_2^3}{3} - u_2 v_{xxx}(1). \tag{29}$$

By setting $u_1 = 0$ on I_2 and $u_2 = 0$ on I_1, we obtain

$$\dot{V}_1 \le \theta_1 V_1 \quad \text{a.e. on } I_2, \tag{30}$$

$$\dot{V}_2 \le \theta_2 V_2 \quad \text{a.e. on } I_1. \tag{31}$$

Thus, it remains to show that

$$\dot{V}_1 \le (\theta_1 - \hat{\theta}_1)V_1 \quad \text{a.e. on } I_1, \tag{32}$$

$$\dot{V}_2 \le (\theta_2 - \hat{\theta}_2)V_2 \quad \text{a.e. on } I_2. \tag{33}$$

Let us prove inequality (32), the proof of (33) being similar. To do so, we distinguish between two cases. When $|w_{xxx}(0)| \ge l(V_1, \hat{\theta}_1)$, then

$$\frac{u_1^3}{3} + u_1 w_{xxx}(0) \le \frac{V_1}{3} - \sqrt[3]{V_1} l(V_1, \hat{\theta}_1) \le -\hat{\theta}_1 V_1.$$

Otherwise, we have

$$\begin{aligned} \frac{u_1^3}{3} + u_1 w_{xxx}(0) &\le -3(3\hat{\theta}_1 + 1)^3 V_1 \\ &\quad + 3(3\hat{\theta}_1 + 1)\sqrt[3]{V_1} l(V_1, \hat{\theta}_1) \le -\hat{\theta}_1 V_1. \end{aligned}$$

Step 2: We first show that $(\hat{\theta}_1, \hat{\theta}_2)$ become constants after a finite time $T > 0$. Indeed, proceeding by contradiction, we assume that there is no such a $T > 0$ such that $\dot{\hat{\theta}}_1(t) =$

$\dot{\hat{\theta}}_2(t) = 0$ for all $t \geq T$. This means, according to R1)-R2) in Algorithm 1, that there exists an infinite number of time intervals, each one having a length greater or equal than $\min\{\underline{T}_1, \underline{T}_2\}$, on which, $\hat{\theta}_1$ and $\hat{\theta}_2$ are linearly increasing, thus, $\lim_{t\to\infty} \hat{\theta}_1(t) = \lim_{t\to\infty} \hat{\theta}_2(t) = \infty$. It follows that there must exists $k' > 1$ such that, for all $k \geq k'$, we have $\tilde{\theta}_1(t_{2k-3}) < 0$, $\tilde{\theta}_2(t_{2k-2}) < 0$, and

$$\tilde{\theta}_1(t_{2k-3})\underline{T}_1 + \theta_1\overline{T}_1 \leq -\sigma(\overline{T}_1 + \overline{T}_2), \tag{34}$$
$$\tilde{\theta}_2(t_{2k-2})\underline{T}_2 + \theta_2\overline{T}_2 \leq -\sigma(\overline{T}_1 + \overline{T}_2). \tag{35}$$

As a consequence, for all $k \geq k'$, we have

$$\begin{aligned}\tilde{\theta}_1(t_{2k-3})(t_{2k-2} - t_{2k-3}) + \theta_1(t_{2k-1} - t_{2k-2})& \\ \leq -\sigma(t_{2k-1} - t_{2k-3}),& \quad (36)\\ \tilde{\theta}_2(t_{2k-2})(t_{2k-1} - t_{2k-2}) + \theta_2(t_{2k} - t_{2k-1})& \\ \leq -\sigma(t_{2k} - t_{2k-2}).& \quad (37)\end{aligned}$$

On the other hand, using Grönwall-Bellman inequality, we find that, for all $k \geq k'$,

$$\begin{aligned}&V_1(t_{2k-1}) \leq V_1(t_{2k-3}) \exp^{\int_{t_{2k-3}}^{t_{2k-2}} \tilde{\theta}_1(t)dt + \theta_1(t_{2k-1}-t_{2k-2})} \\ &\leq V_1(t_{2k-3}) \exp^{\tilde{\theta}_1(t_{2k-3})(t_{2k-2}-t_{2k-3}) + \theta_1(t_{2k-1}-t_{2k-2})}, \quad (38)\\ &V_2(t_{2k}) \leq V_2(t_{2k-2}) \exp^{\int_{t_{2k-2}}^{t_{2k-1}} \tilde{\theta}_2(t)dt + \theta_2(t_{2k}-t_{2k-1})} \\ &\leq V_2(t_{2k-2}) \exp^{\tilde{\theta}_2(t_{2k-2})(t_{2k-1}-t_{2k-2}) + \theta_2(t_{2k}-t_{2k-1})}. \quad (39)\end{aligned}$$

By combining (36), (37), (38), and (39), we conclude that, for all $k \geq k'$, we have

$$V_1(t_{2k-1}) \leq V_1(t_{2k-3}) \exp^{-\sigma(t_{2k-1}-t_{2k-3})}, \tag{40}$$
$$V_2(t_{2k}) \leq V_2(t_{2k-2}) \exp^{-\sigma(t_{2k}-t_{2k-2})}. \tag{41}$$

From the design of $\hat{\theta}_1$ and $\hat{\theta}_2$, the inequalities in (40)-(41) imply that $\hat{\theta}_1$ is constant for all $t \geq t_{2k'-1}$ and $\hat{\theta}_2$ is constant for all $t \geq t_{2k'}$, which yields to a contradiction.

Now, we show the existence of M_1, $M_2 > 0$ (functions of $\sigma, \theta_1, \theta_2, \overline{T}_1, \overline{T}_2, \underline{T}_1, \underline{T}_2, \Delta_1, \Delta_2$) such that

$$|\hat{\theta}_1|_\infty \leq \max\{M_1, \hat{\theta}_1(0)\}, \; |\hat{\theta}_2|_\infty \leq \max\{M_2, \hat{\theta}_2(0)\}. \tag{42}$$

Let us suppose that there exists $T \geq 0$ such that

$$\hat{\theta}_1(T) = \theta_1 + \frac{\sigma\left(\overline{T}_1 + \overline{T}_2\right) + \theta_1\overline{T}_1}{\underline{T}_1}. \tag{43}$$

We can always pick such a T to be in I_1, as $\hat{\theta}_1$ is constant over each interval in I_2. Thus, we let $T \in [t_{2k'-3}, t_{2k'-2}) \subset I_1$ for some $k' > 1$. Using R1), we conclude that $0 \leq \dot{\hat{\theta}}(t) \leq \Delta_1$ for all $t \geq 0$. As a result,

$$\hat{\theta}_1(T) \leq \hat{\theta}_1(t_{2k'+1}) \leq \hat{\theta}_1(T) + 2\Delta_1\overline{T}_1.$$

At the same time, under (43) and the monotonicity of $\hat{\theta}_1$, we conclude that

$$V_1(t) \leq V_1(t_{2k'-1}) \exp^{-\sigma(t-t_{2k'-1})} \quad \forall t \geq t_{2k'+1}.$$

According to R1), the latter inequality implies that $\dot{\hat{\theta}}_1(t) = 0$ for all $t \geq t_{2k'+1}$. As a result,

$$\hat{\theta}_1(t) \leq \hat{\theta}_1(T) + 2\Delta_1\overline{T}_1 \quad \forall t \geq t_{2k'+1}. \tag{44}$$

Finally, if there is no T such that (43) is satisfied, then

$$\hat{\theta}_1(t) \leq \theta_1 + \frac{\sigma\left(\overline{T}_1 + \overline{T}_2\right) + \theta_1\overline{T}_1}{\underline{T}_1} \qquad \forall t \geq 0. \tag{45}$$

Using the same argument for $\hat{\theta}_2$, we conclude that (42) is verified with

$$M_1 := \theta_1 + \frac{\sigma\left(\overline{T}_1 + \overline{T}_2\right) + \theta_1\overline{T}_1}{\underline{T}_1} + 2\Delta_1\overline{T}_1, \tag{46}$$
$$M_2 := \theta_2 + \frac{\sigma\left(\overline{T}_1 + \overline{T}_2\right) + \theta_2\overline{T}_2}{\underline{T}_2} + 2\Delta_2\overline{T}_2. \tag{47}$$

Step 3: To analyze $V_1 + V_2$, we define the sequences $\{T_i\}_{i=0}^\infty$ and $\{T'_i\}_{i=1}^\infty$ such that $T_i := t_{2i+1}$ and $T'_i := t_{2i}$. In particular, we note that, for all $i \in \mathbb{N}$, we have

$$\underline{T}_1 + \underline{T}_2 \leq T_{i+1} - T_i \leq \overline{T}_1 + \overline{T}_2, \tag{48}$$
$$\underline{T}_1 + \underline{T}_2 \leq T'_{i+2} - T'_{i+1} \leq \overline{T}_1 + \overline{T}_2. \tag{49}$$

Since $(\hat{\theta}_1, \hat{\theta}_2)$ are nondecreasing and become constant after some finite time, we conclude on the existence of at most a finite number of intervals $[t_{2k-1}, t_{2k}) \subset I_1$, on which $\hat{\theta}_1$ may increase. On the latter intervals, we know that V_1 is governed by $\dot{V}_1 \leq \theta_1 V_1$, while on the remaining intervals V_1 does not verify the inequality in R1), implying its exponential decay. The same reasoning applies to $\hat{\theta}_2$ and V_2. More precisely, for each $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, we can find $N_1^*, N_2^* \in \mathbb{N}$ such that, for each locally absolutely continuous solution (V_1, V_2) to (13), there exists two finite increasing sub-sequences $\{i_1, i_2, ..., i_{N_1^*}\} \subset \mathbb{N}$ and $\{j_1, j_2, ..., j_{N_2^*}\} \subset \mathbb{N}^*$ such that

- For each $i \in \{i_1, i_2, ..., i_{N_1^*}\}$, we have

$$V_1(T_{i+1}) \leq V_1(T_i) \exp^{\theta_1(T_{i+1}-T_i)}. \tag{50}$$

- For each $j \in \{j_1, j_2, ..., j_{N_2^*}\}$, we have

$$V_2(T'_{j+1}) \leq V_2(T'_j) \exp^{\theta_2(T'_{j+1}-T'_j)}. \quad (51)$$

- For each $i \in \mathbb{N}/\{i_1, i_2, ..., i_{N_1^*}\}$, we have

$$V_1(T_{i+1}) \leq V_1(T_i) \exp^{-\sigma(T_{i+1}-T_i)}. \quad (52)$$

- For each $j \in \mathbb{N}^*/\{j_1, j_2, ..., j_{N_2^*}\}$, we have

$$V_2(T'_{j+1}) \leq V_2(T'_j) \exp^{-\sigma(T'_{j+1}-T'_j)}. \quad (53)$$

Using Lemma 4 in the appendix, while replacing (V, M, ψ, N^*) therein by $(V_1, 0, \theta_1, N_1^*)$, we conclude that, for each $i \geq 0$,

$$V_1(T_i) \leq \exp^{(\theta_1+\sigma)N_1^*(\overline{T}_1+\overline{T}_2)} V_1(0) \exp^{-\sigma T_i}. \quad (54)$$

Similarly, using Lemma 4 in the appendix, while replacing $(V, M, \psi, \{T_i\}_{i=1}^{\infty}, N^*)$ therein by $(V_2, 0, \theta_2, \{T'_i\}_{i=1}^{\infty}, N_2^*)$, we conclude that, for each $i \geq 0$,

$$V_2(T'_i) \leq \exp^{(\theta_2+\sigma)N_2^*(\overline{T}_1+\overline{T}_2)+\theta_2\overline{T}_1} V_2(0) \exp^{-\sigma T'_i}. \quad (55)$$

Let $t \in [T_i, T_{i+1}]$, we have

$$V_1(t) \leq V_1(T_i) \exp^{\theta_1(\overline{T}_1+\overline{T}_2)}. \quad (56)$$

By combining (54) and (56), and using the fact that $t - T_i \leq \overline{T}_1 + \overline{T}_2$, we obtain

$$V_1(t) \leq \exp^{(\theta_1+\sigma)(N_1^*+1)(\overline{T}_1+\overline{T}_2)} V_1(0) \exp^{-\sigma t}. \quad (57)$$

Similarly, for $t \in [T'_i, T'_{i+1}]$, we have

$$V_2(t) \leq \exp^{(\theta_2+\sigma)(N_2^*+1)(\overline{T}_1+\overline{T}_2)+\theta_2\overline{T}_1} V_2(0) \exp^{-\sigma t}. \quad (58)$$

Hence, (8) follows by summing up (57) and (58).

4.2 Proof of Theorem 2

We first show that $\hat{\theta}$ admits an upperbound that does not depend on $V(0)$. As a result, since $\hat{\theta}$ is nondecreasing and, when it increases, it does so according to (20), we conclude that $\hat{\theta}$ becomes constant after some finite time $T \geq 0$. Next, we analyze the Lyapunov function candidate V in (15) and show that, after a finite time, V starts decaying exponentially towards a neighborhood of the origin, whose size is proportional to ϵ. Hence, we conclude L^2-GpA. Finally, using boundedness of $(V, \hat{\theta})$ and the structure of the feedback law κ, boundedness of the control input u_1 follows.

Let us show that

$$\hat{\theta}(t) \leq \max\{\theta + C^2/\epsilon + \sigma + \Delta, \hat{\theta}(0)\} \quad \forall t \geq 0. \quad (59)$$

To prove (59), we first suppose that $\hat{\theta}(0) \leq \theta + C^2/\epsilon + \sigma$. As a result, either

$$\hat{\theta}(t) \leq \theta + C^2/\epsilon + \sigma \quad \forall t \geq 0.$$

Otherwise, in view of R1), there exist $k^* \geq 1$ and $t \in [(k^*-1)\tau, k^*\tau)$ such that

$$\theta + C^2/\epsilon + \sigma < \hat{\theta}(t) \leq \theta + C^2/\epsilon + \sigma + \Delta.$$

Using (19), we then conclude that

$$V(s) \leq V(t) \exp^{-\sigma(s-t)} + \frac{\epsilon}{\sigma} \quad \forall s \in [t, k^*\tau].$$

This implies that $\hat{\theta}$ is constant on $[t, k^*\tau]$. Since $\hat{\theta}$ is nondecreasing, it follows that (19) is verified for all $s \geq t$ and thus

$$V(s) \leq V(t) \exp^{-\sigma(s-t)} + \frac{\epsilon}{\sigma} \qquad \forall s \geq t,$$

which in turn implies that

$$\hat{\theta}(s) \leq \hat{\theta}(t) \leq \theta + C^2/\epsilon + \sigma + \Delta \qquad \forall s \geq t.$$

If $\hat{\theta}(0) > \theta + C^2/\epsilon + \sigma$, we use the fact that $\hat{\theta}$ is nondecreasing to conclude that the inequality in (19) is verified for all $s \geq 0$ and thus

$$V(s) \leq V(0) \exp^{-\sigma(s-t)} + \frac{\epsilon}{\sigma} \qquad \forall s \geq t,$$

which in turn implies that

$$\hat{\theta}(s) \leq \hat{\theta}(0) \qquad \forall s \geq t.$$

To analyze the function V, we let $k \geq 0$ such that $\hat{\theta}$ is constant on $[k\tau, +\infty)$. As a result, for each $t \in [k\tau, +\infty)$, there exists $n \geq 0$ such that $t \in [(k+n)\tau, (k+1+n)\tau)$. Now, according to R1), we have

$$\begin{aligned} V(t) \leq V((k+n)\tau) \exp^{-\sigma(t-(k+n)\tau)} + \frac{\epsilon}{\sigma} \\ \forall t \in [(k+n)\tau, (k+1+n)\tau]. \end{aligned} \quad (60)$$

Moreover, by continuity of V and according to R1), we conclude that, for each $i \in \{1, ..., n\}$, we have

$$V((k+i)\tau) \leq V((k+i-1)\tau) \exp^{-\sigma\tau} + \frac{\epsilon}{\sigma}. \quad (61)$$

We show next that

$$V((k+n)\tau) \leq V(k\tau)\exp^{-\sigma n\tau} + \left(\frac{1}{1-\exp^{-\sigma\tau}}\right)\frac{\epsilon}{\sigma}. \tag{62}$$

Indeed, the latter inequality combined with (60) allows us to conclude L^2-GpA.

To prove (62), we note that

$$\begin{aligned} V((k+n)\tau) &\leq V((k+n-1)\tau)\exp^{-\sigma\tau} + \frac{\epsilon}{\sigma} \\ &\leq V((k+n-2)\tau)\exp^{-2\sigma\tau} + \frac{\epsilon}{\sigma}\left(1+\exp^{-\sigma\tau}\right) \\ &\leq V(k\tau)\exp^{-\sigma n\tau} + \frac{\epsilon}{\sigma}\left(\sum_{j=0}^{n}\exp^{-j\sigma\tau}\right). \end{aligned}$$

Finally, (62) follows using the fact that

$$\sum_{j=0}^{n}\exp^{-j\sigma\tau} \leq \sum_{j=0}^{\infty}\exp^{-j\sigma\tau} \leq \frac{1}{1-\exp^{-\sigma\tau}}.$$

4.3 Proof of Theorem 3

The proof follows in four steps. In Step 1, by setting (u_1, u_2) as in (6), we show that the Lyapunov functions (V_1, V_2) in (5), along the closed-loop system, satisfy the inequalities in (27). In Step 2, we prove that $(\hat{\theta}_1, \hat{\theta}_2)$ become constant after some $T > 0$. Furthermore, $(\hat{\theta}_1, \hat{\theta}_2)$ never exceed $\max\{\hat{\theta}_1(0), M_1\}$ and $\max\{\hat{\theta}_2(0), M_2\}$, respectively, where $M_1, M_2 > 0$ depend on $(\sigma, \theta_1, \theta_2, C_1, C_2, \overline{T}_1, \overline{T}_2, \underline{T}_1, \underline{T}_2, \Delta_1, \Delta_2)$. In Step 3, we analyze the function $V_1 + V_2$ and show that the L^2-ISS property with respect to f is verified. Finally, using the structure of κ and the fact that $(\hat{\theta}_1, \hat{\theta}_2, V_1, V_2)$ are bounded to conclude that (u_1, u_2) remain bounded.

Step 1: According to Lemma 2 in the Appendix, (V_1, V_2) verify almost everywhere the inequalities

$$\dot{V}_1 \leq \theta_1 V_1 + C_1\sqrt{V}_1 + \frac{u_1^3}{3} + u_1 w_{xxx}(0), \tag{63}$$

$$\dot{V}_2 \leq \theta_2 V_2 + C_2\sqrt{V}_2 - \frac{u_2^3}{3} - u_2 v_{xxx}(1). \tag{64}$$

By setting $u_1 = 0$ on I_2, and $u_2 = 0$ on I_1, we therefore obtain

$$\dot{V}_1 \leq \theta_1 V_1 + C_1\sqrt{V_1} \quad \text{a.e. on } I_2, \tag{65}$$

$$\dot{V}_2 \leq \theta_2 V_2 + C_2\sqrt{V_2} \quad \text{a.e. on } I_1. \tag{66}$$

Next, due to our choice of κ, we obtain

$$u_1^3 + 3u_1 w_{xxx}(0) \leq -3\hat{\theta}_1 V_1 \quad \text{on } I_1, \tag{67}$$

$$-u_2^3 - 3u_2 v_{xxx}(1) \leq -3\hat{\theta}_2 V_2 \quad \text{on } I_2. \tag{68}$$

Using the inequalities $\sqrt{V_1} \leq V_1 + 1$ and $\sqrt{V_2} \leq V_2 + 1$, we find

$$\dot{V}_1 \leq (\theta_1 + C_1 - \hat{\theta}_1)V_1 + C_1 \quad \text{a.e. on } I_1, \tag{69}$$

$$\dot{V}_2 \leq (\theta_2 + C_2 - \hat{\theta}_2)V_2 + C_2 \quad \text{a.e. on } I_2. \tag{70}$$

Step 2: We show that $(\hat{\theta}_1, \hat{\theta}_2)$ is constant after some $T \geq 0$ using contradiction. Assume that there is not such a finite time $T \geq 0$ such that

$$\dot{\hat{\theta}}_1(t) = \dot{\hat{\theta}}_2(t) = 0 \qquad \forall t \geq T.$$

As a result, according to R1)-R3) in Algorithm 3,

$$\lim_{t\to\infty}\hat{\theta}_1(t) = \lim_{t\to\infty}\hat{\theta}_2(t) = \infty.$$

Therefore, there exists $T \geq 0$ such that, for all $t \geq T$, we have

$$\hat{\theta}_1(t) \geq \theta_1 + C_1 + \frac{(\theta_1+1)\overline{T}_2 + \sigma(\overline{T}_1 + \overline{T}_2)}{\underline{T}_1} + 1, \tag{71}$$

$$\hat{\theta}_2(t) \geq \theta_2 + C_2 + \frac{(\theta_2+1)\overline{T}_1 + \sigma(\overline{T}_1 + \overline{T}_2)}{\underline{T}_2} + 1. \tag{72}$$

Let $k > 1$ be such that $t_{2k-3} \geq T$. Using (71), we obtain

$$V_1(t_{2k-2}) \leq V_1(t_{2k-3})\exp^{-\sigma(t_{2k-2}-t_{2k-3})} + C_1. \tag{73}$$

Moreover, using Lemma 3, we find

$$\begin{aligned} V_1(t_{2k-1}) &\leq V_1(t_{2k-2})\exp^{(\theta_1+1)(t_{2k-1}-t_{2k-2})} \\ &\quad + (C_1^2/4)\exp^{(\hat{\theta}_1(t_{2k-3})+1)(t_{2k-1}-t_{2k-2})}. \end{aligned}$$

By combining the latter inequality to (73), we obtain

$$\begin{aligned} V_1(t_{2k-1}) &\leq V_1(t_{2k-3})\exp^{-\sigma(t_{2k-1}-t_{2k-3})} + \\ &\quad \left(C_1 + C_1^2/4\right)\exp^{(\hat{\theta}_1(t_{2k-3})+1)(t_{2k-1}-t_{2k-3})}, \end{aligned}$$

which implies, according to R1) in Algorithm 3, that $\dot{\hat{\theta}}_1(t) = 0$ for all $t \geq t_{2k-1}$. We show in a similar way that $\dot{\hat{\theta}}_2(t) = 0$ for all $t \geq t_{2k}$, which leads to a contradiction.

To show (42), we suppose the existence of $T \geq 0$ such that

$$\hat{\theta}_1(T) = \theta_1 + C_1 + \frac{(\theta_1+1)\overline{T}_2 + \sigma(\overline{T}_1 + \overline{T}_2)}{\underline{T}_1} + 1.$$

Then we follow the exact same steps as in the proof of

Theorem 1 to conclude that we can take

$$M_1 := \theta_1 + C_1 + \frac{(\theta_1+1)\overline{T}_2 + \sigma(\overline{T}_1+\overline{T}_2)}{\underline{T}_1} + 1 + 2\Delta_1\overline{T}_1, \tag{74}$$

$$M_2 := \theta_2 + C_2 + \frac{(\theta_2+1)\overline{T}_1 + \sigma(\overline{T}_1+\overline{T}_2)}{\underline{T}_2} + 1 + 2\Delta_2\overline{T}_2. \tag{75}$$

Step 3: To study the function V_1+V_2, we introduce the sequences $\{T_i\}_{i=0}^{\infty}$ and $\{T'_i\}_{i=1}^{\infty}$, such that $T_i := t_{2i+1}$ and $T'_i := t_{2i}$.

Since $(\hat{\theta}_1, \hat{\theta}_2)$ are nondecreasing and become constant after some finite time, we conclude the existence of at most a finite number of intervals $[t_{2k-1}, t_{2k}) \subset I_1$, on which, $\hat{\theta}_1$ may increase. On the latter intervals, we know that V_1 is governed by the inequality in Lemma 3, while on the remaining intervals V_1 does not verify the inequality in R1). The same reasoning applies to $\hat{\theta}_2$ and V_2. More precisely, for each initial conditions $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exist two integers $N_1^*, N_2^* \in \mathbb{N}$ such that, for each locally absolutely continuous solution (V_1, V_2) to (27), there exist two finite increasing subsequences $\{i_1, i_2, ..., i_{N_1^*}\} \subset \mathbb{N}$ and $\{j_1, j_2, ..., j_{N_2^*}\} \subset \mathbb{N}^*$ such that

- For each $i \in \{i_1, i_2, ..., i_{N_1^*}\}$, we have

$$V_1(T_{i+1}) \leq \left(V_1(T_i) + \frac{C_1^2}{4}\right)\exp^{(M_1+1)(T_{i+1}-T_i)}.$$

- For each $j \in \{j_1, j_2, ..., j_{N_2^*}\}$, we have

$$V_2(T'_{j+1}) \leq \left(V_2(T'_j) + \frac{C_2^2}{4}\right)\exp^{(M_2+1)(T'_{j+1}-T'_j)}.$$

- For each $i \in \mathbb{N}/\{i_1, i_2, ..., i_{N_1^*}\}$, we have

$$V_1(T_{i+1}) \leq V_1(T_i)\exp^{-\sigma(T_{i+1}-T_i)} + \left(C_1 + \frac{C_1^2}{4}\right)\exp^{(M_1+1)(T_{i+1}-T_i)}.$$

- For each $j \in \mathbb{N}^*/\{j_1, j_2, ..., j_{N_2^*}\}$, we have

$$V_2(T'_{j+1}) \leq V_2(T'_j)\exp^{-\sigma(T'_{j+1}-T'_i)} + \left(C_2 + \frac{C_2^2}{4}\right)\exp^{(M_2+1)(T'_{j+1}-T'_j)}.$$

Using Lemma 4 in the appendix, while replacing (V, M, ψ, N^*) therein by (V_1, C_1, M_1+1, N_1^*), we obtain the inequality

$$V_1(T_i) \leq \gamma_1 V_1(0)\exp^{-\sigma T_i} + \Phi_1(C_1) \quad \forall i \in \mathbb{N}, \tag{76}$$

for some $\gamma_1 > 0$ and $\Phi_1 \in \mathcal{K}$. Similarly, using Lemma 4 in the appendix, while replacing $(V, M, \psi, N^*, \{T_i\}_{i=1}^{\infty})$ therein by $(V_2, C_2, M_2+1, N_2^*, \{T'_i\}_{i=1}^{\infty})$, we obtain

$$V_2(T'_i) \leq \gamma_2 V_2(0)\exp^{-\sigma T'_i} + \Phi_2(C_2) \quad \forall i \in \mathbb{N}^*, \tag{77}$$

for some $\gamma_2 > 0$ and $\Phi_2 \in \mathcal{K}$. As a consequence, for each $t \in [T_i, T_{i+1}]$, we have

$$V_1(t) \leq \gamma_1 \exp^{(\theta_1+1+\sigma)(\overline{T}_1+\overline{T}_2)} V_1(0)\exp^{-\sigma t} + \left[\Phi_1(C_1) + \frac{C_1^2}{4(\theta_1+1)}\right]\exp^{(\theta_1+1)(\overline{T}_1+\overline{T}_2)}. \tag{78}$$

Similarly, for each $t \in [T'_i, T'_{i+1}]$, we have

$$V_2(t) \leq \gamma_2 \exp^{(\theta_2+2+\sigma)(\overline{T}_1+\overline{T}_2)} V_2(0)\exp^{-\sigma t} + \left[\Phi_2(C_2) + \frac{C_2^2}{4(\theta_2+1)}\right]\exp^{(\theta_2+1)(\overline{T}_1+\overline{T}_2)}. \tag{79}$$

Defining

$$\gamma := \max\left\{\gamma_1 \exp^{(\theta_1+1+\sigma)(\overline{T}_1+\overline{T}_2)}, \gamma_2 \exp^{(\theta_2+2+\sigma)(\overline{T}_1+\overline{T}_2)}\right\}$$

$$\Phi(|f|_\infty) := \left[\Phi_1(C_1) + \frac{C_1^2}{4(\theta_1+1)}\right]\exp^{(\theta_1+1)(\overline{T}_1+\overline{T}_2)} + \left[\Phi_2(C_2) + \frac{C_2^2}{4(\theta_2+1)}\right]\exp^{(\theta_2+1)(\overline{T}_1+\overline{T}_2)},$$

and combining (78) and (79), we obtain, for all $t \geq 0$,

$$V_1(t) + V_2(t) \leq \gamma(V_1(0)+V_2(0))\exp^{-\sigma t} + \Phi(|f|_\infty). \tag{80}$$

4.4 Proof of Theorem 4

According to the first step in the proof of Theorem 3, along the closed-loop solutions of (3) with (u_1, u_2) given by (6) with (7), (V_1, V_2) verify (27). The rest of the proof follows in three steps. In Step 1, we prove that $(\hat{\theta}_1, \hat{\theta}_2)$ become constant after some $T > 0$. Furthermore, $(\hat{\theta}_1, \hat{\theta}_2)$ never exceed $\max\{\hat{\theta}_1(0), M_1\}$ and $\max\{\hat{\theta}_2(0), M_2\}$, respectively, where $M_1, M_2 > 0$ depend on $(\sigma, \theta_1, \theta_2, C_1, C_2, \overline{T}_1, \overline{T}_2, \underline{T}_1, \underline{T}_2, \Delta_1, \Delta_2)$. In Step 2, we analyze the function $V_1 + V_2$ and show the L^2-GUUB property. Finally, as a last step, we use the structure of κ and the fact that $(\hat{\theta}_1, \hat{\theta}_2, V_1, V_2)$ are bounded to conclude that (u_1, u_2) remain bounded.

Step 1: We first show that $(\hat{\theta}_1, \hat{\theta}_2)$ become constant after some $T \geq 0$ using contradiction. Namely, we assume that there is not such a $T \geq 0$ such that, for all $t \geq T$, $\dot{\hat{\theta}}_1(t) = \dot{\hat{\theta}}_2(t) = 0$. As a result, according to R1)-R3) in Algorithm 4, we conclude that $\lim_{t\to\infty}\hat{\theta}_1(t) = \lim_{t\to\infty}\hat{\theta}_2(t) = \infty$.

Therefore, there exists $T \geq 0$ such that, for all $t \geq T$, (71) and (72) hold. Let $k > 1$ be such that $t_{2k-3} \geq T$. We have

$$V_1(t_{2k-2}) \leq V_1(t_{2k-3}) \exp^{-\sigma(t_{2k-2}-t_{2k-3})} + \hat{\theta}_1(t_{2k-3}). \tag{81}$$

Moreover, using Lemma 3, we obtain

$$V_1(t_{2k-1}) \leq V_1(t_{2k-2}) \exp^{(\theta_1+1)(t_{2k-1}-t_{2k-2})} + \frac{\hat{\theta}_1(t_{2k-3})^2}{4} \exp^{(\hat{\theta}_1(t_{2k-3})+1)(t_{2k-1}-t_{2k-2})}.$$

By combining the latter inequality to (81), we obtain

$$V_1(t_{2k-1}) \leq V_1(t_{2k-3}) \exp^{-\sigma(t_{2k-1}-t_{2k-3})} + \left(\hat{\theta}_1(t_{2k-3}) + \frac{\hat{\theta}_1(t_{2k-3})^2}{4}\right) \exp^{(\hat{\theta}_1(t_{2k-3})+1)(t_{2k-1}-t_{2k-3})},$$

which implies, according to R1) in Algorithm 4, that $\dot{\hat{\theta}}_1(t) = 0$ for all $t \geq t_{2k-1}$. We show in a similar way that $\dot{\hat{\theta}}_2(t) = 0$ for all $t \geq t_{2k}$, which leads to a contradiction.

To find the constants M_1 and M_2 verifying (42), we follow the same steps as in the proof of Theorem 1. That is, we can show that (42) holds with M_1, M_2 as in (74) and (75), respectively.

Step 2: To study the function $V_1 + V_2$, we first define the sequences $\{T_i\}_{i=0}^{\infty}$ and $\{T'_i\}_{i=1}^{\infty}$, such that $T_i := t_{2i+1}$ and $T'_i := t_{2i}$. As in the proof of Theorem 3, for each initial condition $(\hat{\theta}_1(0), \hat{\theta}_2(0))$, there exist two integers $N_1^*, N_2^* \in \mathbb{N}$ such that, for each locally absolutely continuous solution (V_1, V_2) to (27), there exist two finite increasing subsequences $\{i_1, i_2, ..., i_{N_1^*}\} \subset \mathbb{N}$ and $\{j_1, j_2, ..., j_{N_2^*}\} \subset \mathbb{N}^*$ such that

- For each $i \in \{i_1, i_2, ..., i_{N_1^*}\}$, we have $V_1(T_{i+1}) \leq \left(V_1(T_i) + \frac{M_1^2}{4}\right) \exp^{(M_1+1)(T_{i+1}-T_i)}$.
- For each $j \in \{j_1, j_2, ..., j_{N_2^*}\}$, we have $V_2(T'_{j+1}) \leq \left(V_2(T'_j) + \frac{M_2^2}{4}\right) \exp^{(M_2+1)(T'_{j+1}-T'_j)}$.
- For each $i \in \mathbb{N}/\{i_1, i_2, ..., i_{N_1^*}\}$, we have $V_1(T_{i+1}) \leq V_1(T_i) \exp^{-\sigma(T_{i+1}-T_i)} + \left(M_1 + \frac{M_1^2}{4}\right) \exp^{(M_1+1)(T_{i+1}-T_i)}$.
- For each $j \in \mathbb{N}^*/\{j_1, j_2, ..., j_{N_2^*}\}$, we have $V_2(T'_{j+1}) \leq V_2(T'_j) \exp^{-\sigma(T'_{j+1}-T'_j)} + \left(M_2 + \frac{M_2^2}{4}\right) \exp^{(M_2+1)(T'_{j+1}-T'_j)}$.

Using Lemma 4 in the appendix, while replacing (V, M, ψ, N^*) therein by (V_1, M_1, M_1+1, N_1^*), we obtain the inequality

$$V_1(T_i) \leq \gamma_1 V_1(0) \exp^{-\sigma T_i} + \Phi_1(M_1) \quad \forall i \in \mathbb{N}, \tag{82}$$

for some $\gamma_1 > 0$ and $\Phi_1 \in \mathcal{K}$. Similarly, using Lemma 4 in the appendix, while replacing $(V, M, \psi, N^*, \{T_i\}_{i=1}^{\infty})$ therein by $(V_2, M_2, M_2+1, N_2^*, \{T'_i\}_{i=1}^{\infty})$, we obtain

$$V_2(T'_i) \leq \gamma_2 V_2(0) \exp^{-\sigma T'_i} + \Phi_2(M_2) \quad \forall i \in \mathbb{N}^*, \tag{83}$$

for some $\gamma_2 > 0$ and $\Phi_2 \in \mathcal{K}$.

As a consequence, for each $t \in [T_i, T_{i+1}]$,

$$V_1(t) \leq \gamma_1 \exp^{(\theta_1+1+\sigma)(\overline{T}_1+\overline{T}_2)} V_1(0) \exp^{-\sigma t} + \left[\Phi_1(M_1) + \frac{M_1^2}{4(\theta_1+1)}\right] \exp^{(\theta_1+1)(\overline{T}_1+\overline{T}_2)}. \tag{84}$$

Moreover, for each $t \in [T'_i, T'_{i+1}]$,

$$V_2(t) \leq \gamma_2 \exp^{(\theta_2+2+\sigma)(\overline{T}_1+\overline{T}_2)} V_2(0) \exp^{-\sigma t} + \left[\Phi_2(M_2) + \frac{M_2^2}{4(\theta_2+1)}\right] \exp^{(\theta_2+1)(\overline{T}_1+\overline{T}_2)}. \tag{85}$$

Defining

$$\gamma := \max\left\{\gamma_1 \exp^{(\theta_1+1+\sigma)(\overline{T}_1+\overline{T}_2)}, \gamma_2 \exp^{(\theta_2+2+\sigma)(\overline{T}_1+\overline{T}_2)}\right\},$$

and

$$\Phi := \left[\Phi_1(M_1) + \frac{M_1^2}{4(\theta_1+1)}\right] \exp^{(\theta_1+1)(\overline{T}_1+\overline{T}_2)} + \left[\Phi_2(M_2) + \frac{M_2^2}{4(\theta_2+1)}\right] \exp^{(\theta_2+1)(\overline{T}_1+\overline{T}_2)},$$

and summing (84) and (85), which are valid for all $t \geq 0$, we obtain

$$V_1(t) + V_2(t) \leq \gamma(V_1(0) + V_2(0)) \exp^{-\sigma t} + \Phi. \tag{86}$$

Let $r := \Phi + \epsilon$, where ϵ is any positive constant, and suppose that $V_1(0) + V_2(0) \leq R$. We conclude that $V_1(t) + V_2(t) \leq \gamma R \exp^{-\sigma t} + \Phi$ for all $t \geq 0$. Hence, to guarantee that $\gamma R \exp^{-\sigma t} + \Phi \leq r$, it is sufficient to have $t \geq T(R) := \frac{1}{\sigma} \log\left(\frac{\gamma R}{\epsilon}\right)$.

5 Numerical Results

In this section, we propose to illustrate our results in Theorems 1 and 4 via simulations.

5.1 Numerical Scheme

The used numerical scheme is based on the mesh-free collocation method using radial basis functions (RBF)s [30]. We estimate $w_{xxx}(0)$ and $w_x(0)$ using the Euler forward scheme and $v_{xxx}(1)$ and $v_x(1)$ using the Euler backward scheme. The Lyapunov function candidates (V_1, V_2) are approximated using Riemannian sums. Furthermore, we select multiquadric RBFs, which depend on a shape parameter $c > 0$. We set $c := 0.4$ in all the simulations.

5.2 *The Different Parameters*

We select the initial time to be $t_o := 0$, the final time to be $t_f := 8 \times 10^{-3}$, and the time step $\Delta t := 10^{-7}$. We select 10 uniformly separated collocation points on the interval $[0, Y]$ (with $Y := 0.5$) ranging from $x_o := 0$ to $x_9 := Y$. We select the same number of collocation points on the interval $[Y, 1]$. We select the anti-diffusion parameter $\lambda := 4\pi^2/0.25 + 50$, for which, the linearization of (1) under $u(0) = u(1) = u_x(0) = u_x(1) = 0$ is unstable [19]. The initial condition is $u(x, 0) := u_o := -A(\cos(4\pi x) - 1)$ for all $x \in (0, 1)$, where $A > 0$ is the amplitude to be selected. The sequences I_1 and I_2 are given by

$$I_1 = [0, 1) \cup [2, 2.8) \cup [3.9, 5) \cup [5.5, 6.5) \cup [7, 7.6) \times 10^{-3},$$
$$I_2 = [1, 2) \cup [2.8, 3.9) \cup [5, 5.5) \cup [6.5, 7) \cup [7.6, 8) \times 10^{-3}.$$

5.3 *Closed-Loop Response When $f = 0$*

Starting from the initial condition u_o with $A := 3$, we simulate the solution to (1) under (4), (u_1, u_2) as in (6), and the adaptation parameters starting from $\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0$ and updated according to Algorithm 1 with $\Delta_1 = \Delta_2 = 0.01$ and $\sigma = 100$. The obtained solution is depicted in Figure 1. Along such a solution, the Lyapunov functions (V_1, V_2) are depicted in Figure 2. The obtained simulations confirm the stability statement in Theorem 1. In particular, on the time intervals where $u_1 = 0$, we observe an exponential growth in V_1. This growth is compensated on the intervals where $u_1 \neq 0$. Similarly, on the time intervals where $u_2 = 0$, we notice an exponential growth in V_2, which is compensated on the intervals where $u_2 \neq 0$. In Figure 3, we plot the evolution of the adaptation parameters $(\hat{\theta}_1, \hat{\theta}_2)$. We observe the existence of an adaptation phase. Indeed, according to Figure 2, V_1 does not decrease on the time interval $[0, 2)$. Its decrease starts on the time interval $[2, 2.8)$ and corresponds, according to Figure 3, to an increase in the value of $\hat{\theta}_1$. Note also that, as expected, $\hat{\theta}_1$ and $\hat{\theta}_2$ increase until they become constant, which corresponds to a decay of V_1 and V_2 at the prescribed rate of $\sigma = 100$. It is interesting to observe that $\hat{\theta}_1$ and $\hat{\theta}_2$ do not exceed θ_1 and θ_2. Indeed, according to Figure 3, we have $|\hat{\theta}_1|_\infty = 80$ and $|\hat{\theta}_2|_\infty = 50$ while, according to Lemma 2, the nominal parameters are $\theta_1 = \theta_2 = 106880$. This illustrates some conservatism of our Lyapunov-based approach.

5.4 *Closed-Loop Response When $f \neq 0$*

Next, we set $f(t) := 12 \times 10^3 \sin(20 \times 10^3 t)$. We simulate the solution to (2) under (4), (u_1, u_2) as in (6), and the adaptation parameters starting from $\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0$ and updated according to Algorithm 4 with $\Delta_1 = \Delta_2 = 0.01$ and $\sigma = 100$. In Figure 4, we plot $V_1 + V_2$ for $A \in \{3, 5, 7\}$. The conclusions of Theorem 4 are in agreement with Figure 4. Indeed, we observe the finite-time

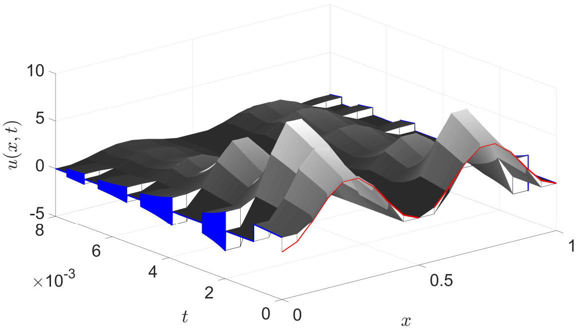

Fig. 1. The KS response to (4), (6), and Algorithm 1

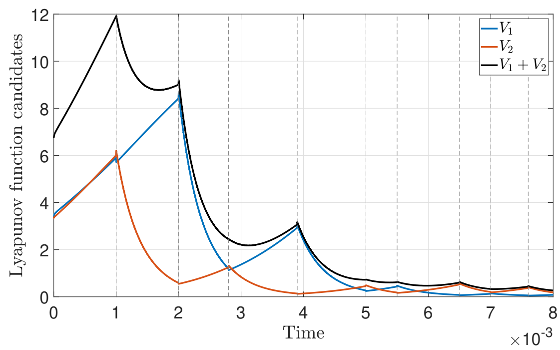

Fig. 2. Lyapunov functions along the KS response under (4), (6), and Algorithm 1

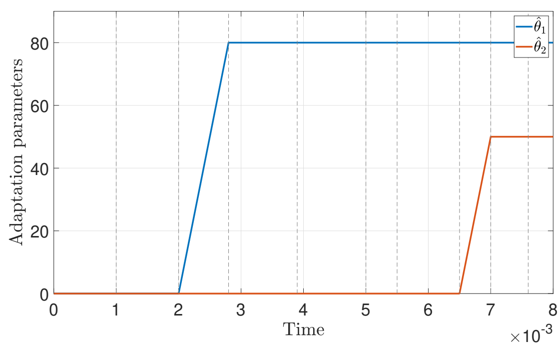

Fig. 3. The adaptation parameters $(\hat{\theta}_1, \hat{\theta}_2)$ constructed according to Algorithm 1

convergence of $V_1 + V_2$ to a bound that is uniform despite the different initial conditions. The evolution of the adaptation parameters $(\hat{\theta}_1, \hat{\theta}_2)$ is the same as in Figure 3 (i.e. we obtain the same plot). In particular, $|\hat{\theta}_1|_\infty = 80$, and $|\hat{\theta}_2|_\infty = 50$, which means that $\hat{\theta}_1$ and $\hat{\theta}_2$ do not exceed the nominal parameters $\theta_1 = \theta_2 = 106880$.

6 Conclusion

We proposed an adaptive boundary-control approach to stabilize the origin of the NKS equation under different assumptions on the perturbation f and the mea-

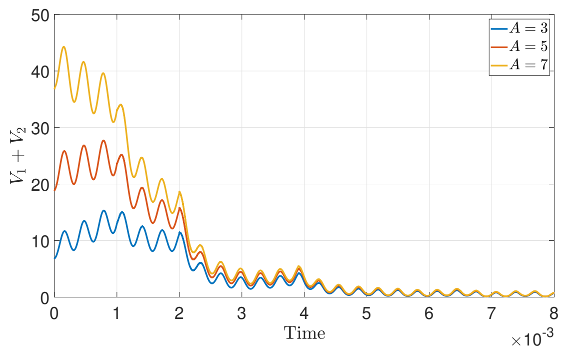

Fig. 4. $V_1 + V_2$ along the NKS response to (4), (6), and Algorithm 4, for different initial conditions

surements available. In particular, under the considered sensing scenario, we guaranteed L^2-GES when $f = 0$, L^2-ISS with respect to f when $|f|_\infty \neq 0$ is known, and L^2-GUUB when $|f|_\infty \neq 0$ is unknown. Note that we do not constrain λ to be constant or in a specific region and we only require $|\lambda|_\infty$ and $|\lambda'|_\infty$ to be bounded. In the future, it is of interest to study well-posedness of the obtained closed-loop system in each case. This may require different tools than those in existing PDE control literature. Furthermore, it would be desirable to propose an alternative to our discontinuous feedback law that is continuous and guaranteed to remain bounded along the closed-loop solutions. Finally, it would be of interest to consider a scanning sensing scenario, in which, we suppose that Y evolves with time and that we measure the state, and/or control the PDE, only around Y.

References

[1] L. Baudouin, E. Cerpa, E. Crépeau, and A. Mercado. Lipschitz stability in an inverse problem for the Kuramoto–Sivashinsky equation. *Applicable Analysis*, 92(10):2084–2102, 2013.

[2] M. C Belhadjoudja, M. Maghenem, E. Witrant, and C. Prieur. Adaptive stabilization of the Kuramoto-Sivashinsky equation subject to intermittent sensing. *In Proceedings of the 2023 American Control Conference, 1608-1613, San Diego, CA, USA.*

[3] D-J. Benny. Long waves in liquid film. *J. Math. and Phys.*, 45:150–155, 1966.

[4] H. Cao, D. Lu, L. Tian, and Y. Cheng. Boundary control of the Kuramoto-Sivashinsky equation with an external excitation. *International Journal of Nonlinear Science*, 1(2):67–81, 2006.

[5] J-M. Coron and Q. Lü. Fredholm transform and local rapid stabilization for a Kuramoto–Sivashinsky equation. *Journal of Differential Equations*, 259(8):3683–3729, 2015.

[6] R. Cuerno, H. A. Makse, S. Tomassone, S. T. Harrington, and H. E. Stanley. Stochastic model for surface erosion via ion sputtering: Dynamical evolution from ripple morphology to rough morphology. *Phys. Rev. Lett.*, 75:4464–4467, 1995.

[7] D. Gajardo, A. Mercado, and J. C. Muñoz. Identification of the anti-diffusion coefficient for the linear Kuramoto-Sivashinsky equation. *Journal of Mathematical Analysis and Applications*, 495(2):124747, 2021.

[8] D. Gilbarg, N. S. Trudinger, D. Gilbarg, and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224. Springer, 1977.

[9] I. Karafyllis, M. Kontorinaki, and M. Krstić. Adaptive control by regulation-triggered batch least squares. *IEEE Trans. on Automatic Control*, 65(7):2842–2855, 2019.

[10] T. Kobayashi. Adaptive stabilization of the Kuramoto-Sivashinsky equation. *Internat. J. Systems Sci.*, 33:3:175–180, 2002.

[11] L. Kocarev, Z. Tasev, and U. Parlitz. Synchronizing spatiotemporal chaos of partial differential equations. *Physical Review Letters*, 79(1):51, 1997.

[12] M. Krstić. On global stabilization of Burgers' equation by boundary control. *Systems & Control Letters*, 37:123–141, 1999.

[13] Y. Kuramoto. Diffusion-induced chaos in reaction systems. *Progress of Theoretical Physics Supplement*, 64:346–367, 1978.

[14] R. LaQuey, S. Mahajan, P. Rutherford, and W. Tang. Nonlinear saturation of the trapped-ion mode. *Physical Review Letters*, 33:391–394, 1975.

[15] K. B. Lauritsen, R. Cuerno, and H. A. Makse. Noisy Kuramoto-Sivashinsky equation for an erosion model. *Physical Review E*, 54:3577–3580, 1996.

[16] H. Lhachemi. Local output feedback stabilization of a nonlinear Kuramoto-Sivashinsky equation. *IEEE Trans. on Automatic Control*, pages 1–6, 2023.

[17] T. Liard, I. Balogoun, S. Marx, and F. Plestan. Boundary sliding mode control of a system of linear hyperbolic equations: a Lyapunov approach. *Automatica*, 135:109964, 2022.

[18] S-P. Lin. Finite amplitude side-band stability of a viscous fluid. *J. Fluid Mech.*, 63:417–429, 1974.

[19] W-J. Liu and M. Krstić. Stability enhancement by boundary control in the Kuramoto–Sivashinsky equation. *Nonlinear Analysis: Theory, Methods & Applications*, 43(4):485–507, 2001.

[20] M. Maghenem, C. Prieur, and E. Witrant. Boundary control of the Kuramoto-Sivashinsky equation under intermittent data availability. *In Proceedings of the 2022 American Control Conference, 2227-2232, Atlanta, GA, USA.*

[21] D. E. Miller. Near optimal LQR performance for a compact set of plants. *IEEE Trans. on Automatic Control*, 51(9):1423–1439, 2006.

[22] D. E. Miller and J. R. Vale. Pole placement adaptive control with persistent jumps in the plant parameters. *Mathematics of Control, Signals, and Systems*, 26(2):177–214, 2014.

[23] D. E. Miller and E. J. Davison. An adaptive controller which provides Lyapunov stability. *IEEE Trans. on Automatic Control*, 34(6):599–609, 1989.

[24] D. S. Mitrinovic, J. E. Pecaric, and A. M. Fink. *Inequalities involving functions and their integrals and derivatives.* Kluwer Academic Publishers, Dordrecht/Boston/London, 1991.

[25] P. Parmananda. Generalized synchronization of spatiotemporal chemical chaos. *Phys. Rev. E*, 56:1595–1598, 1997.

[26] G. Sivashinsky. Nonlinear analysis of hydrodynamic instability in laminar flames—I. Derivation of basic equations. *Acta Astronautica*, 4(11-12):1177–1206, 1977.

[27] G. Sivashinsky. On flame propagation under conditions of stoichiometry. *SIAM Journal on Applied Mathematics*, 39(1):67–82, 1980.

[28] Y. Song, K. Zhao, and M. Krstić. Adaptive control with exponential regulation in the absence of persistent excitation. *IEEE Trans. on Automatic Control*, 62(5):2589–2596, 2016.

[29] Z. Tasev, L. Kocarev, L. Junge, and U. Parlitz. Synchronization of Kuramoto–Sivashinsky equations using spatially local coupling. *International Journal of Bifurcation and Chaos*, 10(04):869–873, 2000.

[30] M. Uddin, S. Haq, and S. ul Islam. A mesh-free numerical method for solution of the family of Kuramoto–Sivashinsky equations. *Applied Mathematics and Computations. 212*, pages 458–469, 2009.

[31] J-M. Wang, J-J. Liu, B. Ren, and J. Chen. Sliding mode control to stabilization of cascaded heat PDE–ODE systems subject to boundary control matched disturbance. *Automatica*, 52:23–34, 2015.

Appendix

We recall a key inequality that links the L^2-norm of a function to the L^2-norms of its first- and second-order derivatives.

Lemma 1 ([24], page 84) *Given $u \in C^2(a,b)$ and $\epsilon > 0$, we have*

$$\int_a^b (u')^2 dx \leq \left[\frac{1}{\epsilon} + \frac{12}{(b-a)^2}\right] \int_a^b u^2 dx + \epsilon \int_a^b (u'')^2 dx. \tag{87}$$

By virtue of Lemma 1, we are able to establish the following intermediate result.

Lemma 2 *Consider system* (3) *such that Assumption 2 holds. Then, along* (3), *the Lyapunov functions* (V_1, V_2) *in* (5) *satisfy*

$$\dot{V}_1 \leq \theta_1 V_1 + C_1\sqrt{V_1} + \frac{u_1^3}{3} + u_1 w_{xxx}(0), \tag{88}$$

$$\dot{V}_2 \leq \theta_2 V_2 + C_2\sqrt{V_2} - \frac{u_2^3}{3} - u_2 v_{xxx}(1), \tag{89}$$

where

$$C_1 := \sqrt{2Y}|f|_\infty, \; C_2 := \sqrt{2(1-Y)}|f|_\infty, \tag{90}$$

and (θ_1, θ_2) are given by

$$\theta_1 := |\lambda'|_\infty + 2\left(|\lambda|_\infty + \frac{1}{2}\right)\left(\left(|\lambda|_\infty + \frac{1}{2}\right) + \frac{12}{Y^2}\right),$$
$$\theta_2 := |\lambda'|_\infty + 2\left(|\lambda|_\infty + \frac{1}{2}\right)\left(\left(|\lambda|_\infty + \frac{1}{2}\right) + \frac{12}{(1-Y)^2}\right).$$

Proof. By differentiating V_1 along (3), we obtain

$$\begin{aligned}\dot{V}_1 &= \int_0^Y w(x)w_t(x)dx \\ &= \int_0^Y w(x)\left[-w(x)w_x(x) - \lambda(x)w_{xx}(x)\right. \\ &\left.-w_{xxxx}(x) + f(x)\right]dx.\end{aligned} \tag{91}$$

Note that $-3\int_0^Y w(x)^2 w_x(x)dx = -w(Y)^3 + w(0)^3$. Furthermore, using integration by part, we obtain

$$\begin{aligned}&-\int_0^Y w(x)w_{xxxx}(x)dx \\ &= -\left[w(x)w_{xxx}(x)\right]_0^Y + \int_0^Y w_x(x)w_{xxx}(x)dx \\ &= -\left[w(x)w_{xxx}(x)\right]_0^Y + \left[w_x(x)w_{xx}(x)\right]_0^Y - \int_0^Y w_{xx}(x)^2 dx.\end{aligned}$$

Using the boundary conditions $w_x(0) = w_x(Y) = 0$, we obtain

$$\begin{aligned}\int_0^Y w(x)w_{xxxx}(x)dx &= \left[w(x)w_{xxx}(x)\right]_0^Y - \int_0^Y w_{xx}(x)^2 dx \\ &= -u_1 w_{xxx}(0) + \int_0^Y w_{xx}(x)^2 dx.\end{aligned}$$

Next, we note that

$$\begin{aligned}&-\int_0^Y \lambda(x)w(x)w_{xx}(x)dx = -[\lambda(x)w(x)w_x(x)]_{x=0}^{x=1} \\ &+\int_0^Y \lambda(x)w_x(x)^2 dx + \int_0^Y \lambda'(x)w(x)w_x(x)dx.\end{aligned}$$

Using the boundary conditions $w_x(0) = w_x(Y) = 0$ and Young inequality, we find

$$\begin{aligned}-\int_0^Y \lambda(x)w(x)w_{xx}(x)dx \leq &\left(|\lambda|_\infty + \frac{1}{2}\right)\int_0^Y w_x(x)^2 dx \\ &+ |\lambda'|_\infty^2 V_1.\end{aligned}$$

Finally, using Cauchy-Schwarz inequality, we obtain

$$\int_0^Y w(x)f(x)dx \leq |f|_\infty \int_0^Y |w(x)|dx \leq C_1\sqrt{V_1}, \tag{92}$$

where C_1 is defined in (90). As a consequence, we obtain

$$\begin{aligned}\dot{V}_1 \leq &|\lambda'|_\infty^2 V_1 + \left(|\lambda|_\infty + \frac{1}{2}\right)\int_0^Y w_x(x)^2 dx \\ &- \int_0^Y w_{xx}(x)^2 dx + C_1\sqrt{V_1} + \frac{u_1^3}{3} + u_1 w_{xxx}(0).\end{aligned}$$

Invoking Lemma 1 with $\epsilon := 1/(|\lambda|_\infty + \frac{1}{2})$, we find

$$\left(|\lambda|_\infty + \frac{1}{2}\right)\int_0^Y w_x^2 dx - \int_0^Y w_{xx}^2 dx \leq (\theta_1 - |\lambda'|_\infty)V_1,$$

which proves inequality (88). We show inequality (89) in a similar way.

Lemma 3 *Let $V : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a locally absolutely continuous solution to the differential inequality*

$$\dot{V} \leq \theta V + C\sqrt{V} \quad \text{a.e. on } [0,T] \subset \mathbb{R}_{\geq 0}, \tag{93}$$

where $\theta, C \geq 0$ are constants. Then, for any constant $\delta > 0$ we have, for all $t \in [0,T]$,

$$V(t) \leq V(0)\exp^{(\theta+\delta)t} + \frac{C^2/4\delta}{(\theta+\delta)}\left(\exp^{(\theta+\delta)t} - 1\right). \tag{94}$$

Proof. Let $\delta > 0$ and consider the function $f(V) := \sqrt{V} - \frac{\delta V}{C} - \frac{C}{4\delta}$. By differentiating f, we find for all $V > 0$, $f'(V) = \frac{1}{2\sqrt{V}} - \frac{\delta}{C}$. As a result, the function f is strictly increasing on $[0, C^2/(4\delta^2)]$ and strictly decreasing on $[C^2/(4\delta^2), \infty)$. Moreover, $f(0) = -C/4\delta$, and $f(C^2/(4\delta^2)) = 0$. Therefore, for all $V \geq 0$, we have $f(V) \leq 0$. As a consequence, we can rewrite (93) as $\dot{V} \leq \theta V + C\sqrt{V} \leq (\theta+\delta)V + \frac{C^2}{4\delta}$. By integrating this inequality from 0 to t, (94) follows.

Lemma 4 *Let the function $V : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be locally absolutely continuous, let a sequence $\{T_i\}_{i=0}^{\infty}$ and $\underline{T}, \overline{T} > 0$ such that $T_0 = 0$ and $\overline{T} \geq T_{i+1} - T_i \geq \underline{T}$ $\forall i \in \mathbb{N}$. Let $\{i_1, i_2, ..., i_{N^*}\} \subset \mathbb{N}$, with $N^* \in \mathbb{N}$, and let (M, ψ, σ) be nonnegative constants. Assume that*

- *For each $i \in \{i_1, i_2, ..., i_{N^*}\}$,*

$$V(T_{i+1}) \leq \left(V(T_i) + \frac{M^2}{4}\right)\exp^{\psi(T_{i+1}-T_i)}. \tag{95}$$

- *For each $i \in \mathbb{N}/\{i_1, i_2, ..., i_{N^*}\}$,*

$$\begin{aligned} V(T_{i+1}) &\leq V(T_i)\exp^{-\sigma(T_{i+1}-T_i)} \\ &\quad + \left(M + \frac{M^2}{4}\right)\exp^{\psi(T_{i+1}-T_i)}. \end{aligned} \tag{96}$$

Then, there exists $\gamma \geq 1$ and $\Phi \in \mathcal{K}$ such that

$$V(T_i) \leq \gamma V(0)\exp^{-\sigma T_i} + \Phi(M) \qquad \forall i \in \mathbb{N}.$$

Proof. To prove the Lemma, it is enough to show that

$$V(T_i) \leq \exp^{(\sigma+\psi)N^*\overline{T}}\left(V(0)\exp^{-\sigma T_i} + \eta(i)\right) \quad \forall i \in \mathbb{N}, \tag{97}$$

where, for each $i \in \mathbb{N}$,

$$\begin{aligned} \eta(i) &:= \left(\sum_{k=0}^{i}\exp^{-k\sigma\underline{T}}\right)\frac{4M+M^2}{4}\exp^{\psi\overline{T}} \\ &\leq \left(\frac{1}{1-\exp^{-\sigma\underline{T}}}\right)\frac{4M+M^2}{4}\exp^{\psi\overline{T}}. \end{aligned}$$

To prove (97), it is sufficient to show that

$$V(T_i) \leq \exp^{(\sigma+\psi)N(i)\overline{T}}\left(V(0)\exp^{-\sigma T_i} + \eta(i)\right) \quad \forall i \in \mathbb{N}, \tag{98}$$

where $N(i) := \text{card}\{[T_j, T_{j+1}] : j+1 \leq i,\ j \in \{i_1, i_2, ..., i_{N^*}\}\}$ is the number of time intervals $[T_j, T_{j+1}]$, $j \in \{i_1, i_2, ..., i_{N^*}\}$, prior to T_i, which satisfies $N(i) \leq N^*$ for all $i \in \mathbb{N}$. To show (98), we proceed by recurrence. Indeed, for $i = 0$, the inequality in (98) is trivially satisfied. Suppose now that the inequality in (98) is verified for $i \in \mathbb{N}$ and let us show that it is also verified for $i+1$.

Note that either $N(i+1) = N(i)$ or $N(i+1) = N(i)+1$. If $N(i+1) = N(i)$ then, using (96), we obtain

$$\begin{aligned} V(T_{i+1}) &\leq V(T_i)\exp^{-\sigma(T_{i+1}-T_i)} \\ &\quad + \left(M + \frac{M^2}{4}\right)\exp^{(\hat{\theta}+1)(T_{i+1}-T_i)} \\ &\leq \exp^{(\sigma+\psi)N(i)\overline{T}} V(0)\exp^{-\sigma T_{i+1}} \\ &\quad + \left(\eta(i)\exp^{-\sigma(T_{i+1}-T_i)}\right)\exp^{(\sigma+\psi)N(i)\overline{T}} \\ &\quad + \left(M + \frac{M^2}{4}\right)\exp^{\psi(T_{i+1}-T_i)}. \end{aligned}$$

Using the fact that

$$\eta(i)\exp^{-\sigma(T_{i+1}-T_i)} \leq \left[\sum_{k=1}^{i+1}\exp^{-k\sigma\underline{T}}\right]\left(M + \frac{M^2}{4}\right)\exp^{\psi\overline{T}},$$

we obtain

$$\begin{aligned} &V(T_{i+1}) \leq \exp^{(\sigma+\psi)N(i)\overline{T}} V(0)\exp^{-\sigma T_{i+1}} \\ &\quad + \left[\sum_{k=1}^{i+1}\exp^{-k\sigma\underline{T}}\right]\left(M + \frac{M^2}{4}\right)\exp^{\psi\overline{T}}\exp^{(\sigma+\psi)N(i)\overline{T}} \\ &\quad + \left(M + \frac{M^2}{4}\right)\exp^{\psi(T_{i+1}-T_i)}. \end{aligned}$$

It implies that

$$\begin{aligned} &V(T_{i+1}) \leq \exp^{(\sigma+\psi)N(i)\overline{T}} V(0)\exp^{-\sigma T_{i+1}} \\ &\quad + \left[\sum_{k=1}^{i+1}\exp^{-k\sigma\underline{T}}\right]\left(M + \frac{M^2}{4}\right)\exp^{\psi\overline{T}}\exp^{(\sigma+\psi)N(i)\overline{T}} \\ &\quad + \left(M + \frac{M^2}{4}\right)\exp^{\psi\overline{T}}\exp^{(\sigma+\psi)N(i)\overline{T}}. \end{aligned}$$

Combining the latter two terms, we obtain

$$\begin{aligned} &V(T_{i+1}) \leq \exp^{(\sigma+\psi)N(i)\overline{T}} V(0)\exp^{-\sigma T_{i+1}} \\ &\quad + \left[\sum_{k=0}^{i+1}\exp^{-k\sigma\underline{T}}\right]\left(M + \frac{M^2}{4}\right)\exp^{\psi\overline{T}}\exp^{(\sigma+\psi)N(i)\overline{T}}. \end{aligned}$$

Finally, since $N(i) = N(i+1) \leq N^*$ and $\eta(i+1) = \left[\sum_{k=0}^{i+1} \exp^{-k\sigma \underline{T}}\right] \left(M + \frac{M^2}{4}\right) \exp^{\psi \overline{T}}$, we obtain

$$\begin{aligned} V(T_{i+1}) \leq \exp^{(\sigma+\psi)N(i+1)\overline{T}} V(0) \exp^{-\sigma T_{i+1}} \\ + \eta(i+1) \exp^{(\sigma+\psi)N(i)\overline{T}}. \end{aligned}$$

If $N(i+1) = N(i) + 1$, we use (95) to conclude that

$$\begin{aligned} V(T_{i+1}) &\leq \left(V(T_i) + \frac{M^2}{4}\right) \exp^{\psi \overline{T}} \\ &\leq \exp^{(\sigma+\psi)N(i)\overline{T}} \exp^{\psi \overline{T}} \\ &\times \left(V(0) \exp^{-\sigma T_i} + \eta(i) + \exp^{-(\sigma+\psi)N(i)\overline{T}} \frac{M^2}{4}\right). \end{aligned}$$

Now, using the fact that $N(i+1) = N(i) + 1$, we obtain

$$\begin{aligned} & V(T_{i+1}) \leq \exp^{(\sigma+\psi)N(i+1)\overline{T}} \exp^{\psi \overline{T}} \exp^{-(\sigma+\psi)\overline{T}} \\ & \times \left(V(0) \exp^{-\sigma T_i} + \eta(i) + \exp^{-(\sigma+\psi)N(i)\overline{T}} \frac{M^2}{4}\right) \\ & \leq \exp^{(\sigma+\psi)N(i+1)\overline{T}} \exp^{\psi \overline{T}} \\ & \times \left(V(0) \exp^{-\sigma T_i} + \eta(i) \exp^{-(\sigma+\psi)\overline{T}} + \frac{M^2}{4}\right). \end{aligned}$$

We finish by showing $\eta(i) \exp^{-(\sigma+\psi)\overline{T}} + M^2/4 \leq \eta(i+1)$. Indeed, we note that

$$\begin{aligned} & \eta(i) \exp^{-(\sigma+\psi)\overline{T}} + \frac{M^2}{4} \\ & \leq \eta(i) \exp^{-\sigma \overline{T}} \exp^{-\psi \overline{T}} + \left(M + \frac{M^2}{4}\right) \exp^{\psi \overline{T}} \\ & \leq \left(\sum_{k=1}^{i+1} \exp^{-k\sigma \underline{T}}\right) \left(M + \frac{M^2}{4}\right) \exp^{\psi \overline{T}} \\ & + \left(M + \frac{M^2}{4}\right) \exp^{\psi \overline{T}} \\ & = \eta(i+1). \end{aligned}$$

PLACE PHOTO HERE

Mohamed Camil Belhadjoudja received a Control-Engineer and MS degrees from the National Polytechnic School, Algiers, Algeria, in 2021, and a MS degree in Control Theory from the Université Grenoble Alpes, Grenoble, France, in 2022. He is currently a Ph.D. student at the Grenoble Image Parole Signal Automatique Laboratory (GIPSA-lab), Grenoble, France. His research interests include nonlinear partial differential equations, model-free control, and time-delay systems.

PLACE PHOTO HERE

Mohamed Maghenem received his Control-Engineer degree from the Polytechnical School of Algiers, Algeria, in 2013, his PhD degree on Automatic Control from the University of Paris-Saclay, France, in 2017. He was a Postdoctoral Fellow at the Electrical and Computer Engineering Department at the University of California at Santa Cruz from 2018 through 2021. M. Maghenem holds a research position at the French National Centre of Scientific Research (CNRS) since January 2021. His research interests include control systems theory (linear, non-linear, and hybrid) to ensure (stability, safety, reachability, synchronisation, and robustness); with applications to mechanical systems, power systems, cyber-physical systems, and some partial differential equations.

PLACE PHOTO HERE

Emmanuel Witrant obtained a B.Sc. in Aerospace Eng. from Georgia Tech in 2001 and a Ph.D. in Automatic Control from Grenoble University in 2005. He joined the Physics department of Univ. Grenoble Alpes and GIPSA-lab as an Associate Professor in 2007 and became Professor in 2020. His research interest is focused on finding new methods for modeling and control of inhomogeneous transport phenomena (of information, energy, gases...), with real-time and/or optimization constraints. Such methods provide new results for automatic control (time-delay and distributed systems), controlled thermonuclear fusion (temperature/magnetic flux profiles in tokamak plasma), environmental sciences (atmospheric history of trace gas from ice cores measurements) and Poiseuille's flows (ventilation control in mines, car engines and intelligent buildings).

PLACE PHOTO HERE

Christophe Prieur graduated in Mathematics from the Ecole Normale Supérieure de Cachan, France in 2000. He received the Ph.D. degree in 2001 in Applied Mathematics from Université Paris-Sud, France. From 2002 he was a researcher at CNRS at the laboratory SATIE, Cachan, France, and at the LAAS, Toulouse, France (2004–2010). In 2010 he joined the Gipsa-lab, Grenoble, France where he is currently a senior researcher at CNRS (since 2011). His current research interests include nonlinear control theory, hybrid systems, and control of partial differential equations, with applications including navigation and object tracking, fluid dynamics, and fusion control. He is an IMA Fellow, and an IEEE Fellow.