

The Cordiality Game and the Game Cordiality Number

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ABSTRACT. The *cordiality game* is played on a graph G by two players, Admirable (A) and Impish (I), who take turns selecting unlabeled vertices of G . Admirable labels the selected vertices by 0 and Impish by 1, and the resulting label on any edge is the sum modulo 2 of the labels of the vertices incident to that edge. The two players have opposite goals: Admirable attempts to minimize the number of edges with different labels as much as possible while Impish attempts to maximize this number. When both Admirable and Impish play their optimal games, we define the *game cordiality number*, $c_g(G)$, as the absolute difference between the number of edges labeled zero and one. Let P_n be the path on n vertices. We show $c_g(P_n) \leq \frac{n-3}{3}$ when $n \equiv 0 \pmod{3}$, $c_g(P_n) \leq \frac{n-1}{3}$ when $n \equiv 1 \pmod{3}$, and $c_g(P_n) \leq \frac{n+1}{3}$ when $n \equiv 2 \pmod{3}$. Furthermore, we show a similar bound, $c_g(T) \leq \frac{|T|}{2}$ holds for any tree T .

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1. Introduction and Definitions

It is natural to study combinatorial functions under the regime of two-person games, often presenting new perspectives to old problems. One such example is the classic cops and robbers formulation of tree-width [24]. More examples include game chromatic number, see [1] and [25], and the more recent game domination number [3], which has bloomed into numerous publications, of which we only list a few, e.g., [4, 15, 17], and the monograph [2]. For more examples of graph-theoretic games, see the extended bibliography [11] for more references.

With this in mind, we define a new game to study a simple “algebraic balance” in graphs, which is motivated from the following definition. A graph G is *cordial* if there exists a labeling of the vertices of G by 0 and 1, where labels on edges are defined as the sum of incident vertex labels modulo 2 such that:

- the absolute difference in the number of vertices labeled 0 and 1 is no more than one;
- the absolute difference in the number of edges labeled 0 and 1 is no more than one.

In other words, a graph is cordial, if there exists a bipartition of its vertex set, such that the size of each part is $\lceil |V(G)|/2 \rceil$ or $\lfloor |V(G)|/2 \rfloor$, and the number of edges that goes across the cut induced by the bipartition is either $\lceil |E(G)|/2 \rceil$ or $\lfloor |E(G)|/2 \rfloor$. Cordial labelings were first defined in [6] and extended to k -cordial

labelings in [16], which includes several difficult and outstanding problems. One particular simple and natural problem is the 3-cordial conjecture.

A graph G is 3-cordial, if there exists a vertex labeling of G satisfying the following two conditions:

- The vertices are labeled by the integers 0, 1, and 2 such that for any integer in $\{0, 1, 2\}$, there is at most one more vertex labeled by one of the integers than of vertices labeled by any other fixed integer.
- If for any edge, the label on that edge is found by summing its incident vertices modulo 3, then there is at most one more edge labeled by one of the integers than of edges labeled by a different fixed integer.

CONJECTURE 1.1 (Hovey [16]). *All graphs are 3-cordial.*

We know of no progress made on this problem in the last thirty years. Some examples of other extensions and variants of cordiality problems from [16] can be seen in [8, 9, 10, 19, 20].

Let $G = (V, E)$ be a graph. The *cordiality game* is played on G by two players, Admirable and Impish, who take turns selecting the unlabeled vertices of G . Admirable labels selected vertices by 0 and Impish labels selected vertices by 1. The labels on edges are then determined by the sum of incident vertex labels modulo 2. Admirable's goal is to produce a labeling of E with the minimum number of different labels while Impish attempts to produce the labels of E with the maximum number of different labels. In other words, after all the vertices are labeled, if we let e_0 be the number of edges labeled by 0 and e_1 be the number of edges labeled by 1, then we define the *discrepancy* to be $d = |e_1 - e_0|$. Then Admirable attempts to minimize d and Impish attempts to maximize d . Note at the end of the game, the vertex labelings induce a balanced bipartition of the graph.

We define the *game cordiality number*, $c_g(G)$, to be the value of d when both players play optimally. Further, to prove our claimed bounds, we create a variant of the cordiality game where Impish starts rather than Admirable. In this case, the discrepancy, assuming optimal play of both competitors, is denoted $c'_g(G)$, which we call the Impish-starts game cordiality number. Finally, consider a further variant of the cordiality game, in which Impish starts and may pass one time during play, so that Admirable moves twice consecutively at some point in the game. For this game, the discrepancy, assuming optimal play of both parties, is denoted $c_g^*(G)$. Note that $c'_g(G) \leq c_g^*(G)$, as Impish may always choose a strategy that avoids passing his turn.

Our cordiality game can be interpreted as a classic positional game under a small adjustment. First introduced by Chvátal and Erdős [7], the Maker-Breaker game has received significant attention prior in the literature, e.g., see [14, 18]. A family \mathcal{F} of winning subsets of some set X is known to both players prior. The players alternate between selecting previously unselected elements. At the end, Maker wins if he obtains a set of \mathcal{F} , otherwise Breaker wins. For some fixed integer $k \geq 0$, let \mathcal{F}_k be the family of vertex sets which are parts of a balanced bipartition of discrepancy at most k . Note then for the cordiality game, Maker would play the role of Admirable, Breaker taking the role of Impish. Clearly, setting $k = c_g(G)$, a winning strategy for one game translates to the other.

In Section 2, we present an upper bound for $c_g(P_n)$ when P_n is a path on n vertices. We prove

$$c_g(P_n) \leq \begin{cases} \frac{n-3}{3} & n \equiv 0 \pmod{3} \\ \frac{n-1}{3} & n \equiv 1 \pmod{3} \\ \frac{n+1}{3} & n \equiv 2 \pmod{3} \end{cases}$$

In Section 3, we show that for any tree T , $c_g(T) \leq \frac{1}{2}n$. In Section 4, we introduce a variant of the cordiality game and end with some open questions.

We finish this section with some notation. For a graph G , we let ϕ denote the *cut function* for G . That is, for all $S \subseteq V(G)$, $\phi(S) = |\{uv \in E(G) : u \in S, v \notin S\}|$. We let $\text{dis}(S)$ denote the discrepancy of the cut induced by S , where $\text{dis}(S) = \phi(S) - (|E(G)| - \phi(S)) = 2\phi(S) - |E(G)|$. Note then the objective of the cordiality game is $\text{dis}(S)$ where S is the set of vertices labeled 0 or 1. Furthermore, we note the parity of $\text{dis}(S)$ is always the same as that of $|E(G)|$. A *balanced partition* of a graph G is a partition of $V(G)$ into two sets, say S_1 and S_2 , such that $||S_1| - |S_2|| \leq 1$.

2. Paths

The problem of finding the exact value of $c_g(P_n)$ is surprisingly involved. We begin with calculating the game cordiality number for small paths.

LEMMA 2.1. *We have the following:*

- i. $c_g(P_3) = c'_g(P_3) = c_g^*(P_3) = 0$,
- ii. $c_g(P_4) = c'_g(P_4) = c_g^*(P_4) = 1$,
- iii. $c_g(P_5) = c'_g(P_5) = c_g^*(P_5) \leq 2$,
- iv. $c_g(P_6) = c'_g(P_6) = c_g^*(P_6) = 1$.

PROOF. Let $P_n = v_1v_2 \cdots v_n$ and let S, \bar{S} be a balanced bipartition of $V(P_n)$.

In the case $n = 3$, $\text{dis}(S)$ is always even. As $|E(P_3)| = 2$, $\{v_2\}, \{v_1, v_3\}$ is the only bipartition that achieves discrepancy greater than 0. Thus if Admirable plays v_2 first, this guarantees a bipartition of discrepancy zero. In the case Impish plays first, Admirable just needs to avoid playing v_2 . Thus we may conclude $c_g(P_3) = c'_g(P_3) = c_g^*(P_3) = 0$.

When $n = 4$, $\text{dis}(S)$ is always odd. By observation, $\{v_1, v_3\}, \{v_2, v_4\}$ is the only balanced bipartition that achieves discrepancy greater than 1. Clearly if Admirable has a strategy of playing once from each set, regardless of who starts, this guarantees a resulting bipartition of discrepancy 1. Note as $\text{dis}(S) \geq 0$ and odd, we may conclude $c_g(P_4) = c'_g(P_4) = c_g^*(P_4) = 1$.

In the case $n = 5$, $\text{dis}(S)$ is always even. As $|E(P_5)| = 4$, the only balanced bipartition that achieves discrepancy greater than 2 is $\{v_1, v_3, v_5\}, \{v_2, v_4\}$. Again if Admirable takes a strategy of playing once from each set, this is a strategy guaranteeing $c_g(P_5) = c'_g(P_5) = c_g^*(P_5) \leq 2$.

We now work towards understanding our game variants on P_6 . By parity, $\text{dis}(S)$ is odd and $\text{dis}(S) \geq 1$ for any balanced bipartition S, \bar{S} . There are only two choices for S that achieve discrepancy 5, namely $\{v_1, v_3, v_5\}$ and its complement $\{v_2, v_4, v_6\}$. We now observe the balanced bipartitions which achieve discrepancy 3. To do so, we would either need exactly 1 or 4 edges crossing the cut induced by the bipartition. The former case only occurs with S being $\{v_1, v_2, v_3\}$ or $\{v_4, v_5, v_6\}$. The latter case occurs only with S being $\{v_2, v_3, v_5\}, \{v_2, v_4, v_5\}, \{v_1, v_4, v_6\}$, or $\{v_1, v_3, v_6\}$. Note then for Admirable to obtain discrepancy 1 in the cordiality game, there is in total

8 “bad labelings” to avoid. We list them as follows.

$$\begin{aligned} &\{v_1, v_3, v_5\}, \quad \{v_2, v_4, v_6\}, \quad \{v_1, v_2, v_3\}, \quad \{v_4, v_5, v_6\}, \\ &\{v_2, v_3, v_5\}, \quad \{v_2, v_4, v_5\}, \quad \{v_1, v_4, v_6\}, \quad \{v_1, v_3, v_6\}. \end{aligned}$$

We now give a strategy for Admirable which guarantees a balanced cut of discrepancy 1, i.e., $c_g(P_6) = 1$. We may assume Admirable plays on v_1 first. Note then there is only four bad labelings for Admirable to avoid now, $\{v_1, v_3, v_5\}$, $\{v_1, v_2, v_3\}$, $\{v_1, v_4, v_6\}$, or $\{v_1, v_3, v_6\}$. Note v_3 is in three of these bad sets. As Admirable plays first and $|P_6|$ is even, we may suppose Admirable never plays v_3 . Thus to avoid the final bad set $\{v_1, v_4, v_6\}$, on Admirable’s second move, we may assume Admirable plays on either v_2 or v_5 . This guarantees that Admirable labels a set S such that $\text{dis}(S) = 1$.

We now give a strategy for Admirable in which Impish plays first. In the case where Impish is also allowed to pass his turn to Admirable once, the analysis is almost the same which we note at the end. By symmetry, we may suppose Impish plays v_1, v_2 or v_3 .

If Impish plays on $\{v_1, v_3\}$ on his first move, then Admirable plays $\{v_1, v_3\}$ for his first move as well. Note then the only bad labeling for Admirable to avoid now is $\{v_1, v_4, v_6\}$ or $\{v_2, v_3, v_5\}$ depending on his first move choice. This can clearly be achieved. If Impish plays v_2 , then Admirable plays v_5 . Note then the only bad labelings to avoid are $\{v_4, v_5, v_6\}$ and $\{v_1, v_3, v_5\}$. To avoid these, if Impish plays on $\{v_4, v_6\}$ on his second move, then Admirable plays on $\{v_4, v_6\}$. In a similar manner, if Impish plays on $\{v_1, v_3\}$ on his second move, then Admirable plays on $\{v_1, v_3\}$. If Impish chooses to pass on his second turn, Admirable simply plays on $\{v_1, v_3\}$ on his second move and then $\{v_4, v_6\}$ on his third move. Thus we may conclude $c'_g(P_6) = c_g^*(P_6) = 1$. □

We now prove an upper bound for all paths.

THEOREM 2.2. *For any integer $n \geq 3$,*

$$c_g(P_n) \leq \begin{cases} \frac{n-3}{3} & n \equiv 0 \pmod{3} \\ \frac{n-1}{3} & n \equiv 1 \pmod{3} \\ \frac{n+1}{3} & n \equiv 2 \pmod{3} \end{cases}$$

PROOF. Let $P_n = v_1 v_2 \cdots v_n$. We now proceed by induction on n . By Lemma 2.1, we may suppose that the theorem holds for all paths of order less than $n > 6$ and consider P_n . We describe a strategy for Admirable which, regardless of the moves Impish makes, will produce the claimed upper bounds.

Let P be the subpath of P_n induced on v_1, \dots, v_{n-6} and P' be the P_6 subpath of P induced by v_{n-5}, \dots, v_n . We will prove the inequality

$$c_g(P_n) \leq \begin{cases} 2\lfloor \frac{n}{6} \rfloor - 1 & n \equiv 0 \pmod{6} \\ 2\lfloor \frac{n}{6} \rfloor & n \equiv 1, 3 \pmod{6} \\ 2\lfloor \frac{n}{6} \rfloor + 1 & n \equiv 2, 4 \pmod{6} \\ 2\lfloor \frac{n}{6} \rfloor + 2 & n \equiv 5 \pmod{6} \end{cases}$$

which can be easily reduced to the claimed statement.

Suppose n is odd. Admirable begins by playing an optimal game on P . From that point, Admirable follows Impish between P and P' . If Impish plays on P , Admirable continues playing an optimal game on P . If Impish plays on P' , then

Admirable follows with a next optimal move on P' . This strategy results in an Impish-starts game on P' and an Admirable-starts game on P . Recall by Lemma 2.1, $c'_g(P') = c'_g(P_6) = 1$. We apply the induction hypothesis on P and consider that the edge between P and P' , $v_{n-6}v_{n-5}$, as an unknown label. Since $n - 6 \pmod{6} = n \pmod{6}$, the values in the proposed upper bound of $c_g(P_n)$ follows by adding 2 to the bounds for $c_g(P_{n-6})$.

Next, we suppose that n is even. The strategy for Admirable is the same as the previous case. Admirable begins by playing an optimal game on P . From that point, Admirable follows Impish between P and P' . If Impish plays on P , Admirable continues playing an optimal game on P . If Impish plays on P' , Admirable follows with a next optimal move on P' . Due to the parity of n in this case, this strategy allows for several variants of the cordiality game. If Impish completes a game on P before moving to P' , then the game on P' becomes an Admirable-starts game. If Impish plays P' before completing the game on P , then the game on P' becomes an Impish-starts game. If Impish does not finish the game on P' , returns to P , and completes the game there, then Admirable must make the next move on P' , which is equivalent to playing an Impish-starts game on P' where Impish may pass once. Regardless of Impish strategy, by Lemma 2.1, $c_g(P_6) = c'_g(P_6) = c_g^*(P_6) = 1$. Thus by induction on P , and treating $v_{n-6}v_{n-5}$ as an unknown label, this produces the claimed upper bounds for $c_g(P_n)$ as before. \square

For a lower bound, our methods are ineffective, and so from some empirical considerations ($n \leq 12$), we ask the following speculative question.

OPEN QUESTION 1. *Does $c_g(P_n)$ assume the values 0 and 1 for infinitely many integers n ?*

3. Trees

In Theorem 2.2, if our input path P_n is long, i.e. $n \geq 6$, we break the instance into two smaller paths, P_{n-6} and P_6 , and then induct with a strategy of Admirable following Impish. There are challenges to extending this proof strategy to an arbitrary tree T . A subtree B of a tree T is a *branch*, if there exists a subtree T' of T such that $B \cup T' = T$ and $|V(T') \cap V(B)| = 1$. Ideally, we would like to find a small branch of constant size to break the instance into two smaller trees. If we do so naively, we no longer have control on what this branch looks like, in particular, the branch may be of odd order. Using the strategy of Admirable following Impish, Impish may play the odd ordered branch entirely with some vertices still unlabeled in the remainder of the tree, which would lead to Admirable playing twice in-a-row on the remainder of the tree. Thus, to avoid this potential difficulty in analysis, we consider how to remove a branch that is small enough to label, yet also of even order.

THEOREM 3.1. *For any tree T of order n ,*

$$c_g(T) \leq \frac{1}{2}n.$$

PROOF. We proceed by induction on the order n . Note that the theorem holds trivially for trees of order 1, 2, and 3, so we suppose the statement is true for all trees of order less than $n > 3$ and consider a tree T of order n .

We describe Admirable's strategies and calculate discrepancies for various possible branches of T . We will then show that our bound holds in every instance.

Let B be a branch of T , let T' be a subtree of T such that $B \cup T' = T$ and $|V(T') \cap V(B)| = 1$, and let v be the vertex which is both in B and T' . Finally, we let S be the subgraph of T induced by the vertices of $B - v$.

Throughout the proof, we will assume Admirable begins playing an optimal game on T' , and then following Impish between T' and S depending on where Impish plays. In the case Impish played on T' , and T' has no remaining unlabeled vertices, Admirable will begin playing on S if possible. We also will choose B such that $|S|$ is always even. We remark that as $|T'| + |S| = |T|$, we have

- If $|T'|$ is even, then the game on S is either an Admirable-start game or an Impish-start game where Impish may pass up to one time during play.
- If $|T'|$ is odd, then the game on S is an Impish-start game.

Note in the case $|T'|$ is even, S is an Admirable-start game if and only if every vertex of T' has been previously played.

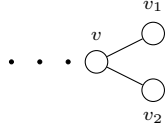
CASE 1. *Branch B isomorphic to P_5 with end-vertex v in T'*

If T contains a branch isomorphic to $P_5 = (v, v_1, v_2, v_3, v_4)$, where $\deg(v_i) = 2$ for $i = 1, 2, 3$ and $\deg(v_4) = 1$, in T , then we define $T' = T - \{v_1, \dots, v_4\}$ and let S be the subgraph of T induced by v_1, v_2, v_3, v_4 . By Lemma 2.1, $c_g(P_4) = c'_g(P_4) = c_g^*(P_4) = 1$, we can apply the induction hypothesis to T' and consider the edge between T' and S as one with an unknown label, to produce the bound

$$c_g(T) \leq c_g(T') + 2 \leq \frac{1}{2}(n - 4) + 2 = \frac{1}{2}n.$$

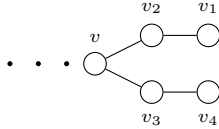
■

CASE 2. *Branch B isomorphic to P_3 with central vertex v in T' .*



Let the vertices of B be labeled as the above figure, and let S be the subgraph induced by $\{v_1, v_2\}$. As Admirable plays first on T' , and follows Impish between T' and S , playing on S first if and only if T' is completely labeled, the edges vv_1 and vv_2 will always have different labels. Thus the claim follows by induction on T' . ■

CASE 3. *Branch B isomorphic to $P_5 = \{v_1, v_2, v, v_3, v_4\}$ with vertex v in T' .*



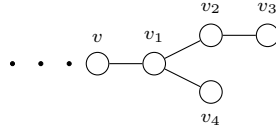
Let B be labeled as the above figure and let S be the subgraph induced by $\{v_1, v_2, v_3, v_4\}$. We now give strategies for Admirable on S that produces a discrepancy of at most 2 for the edges induced by the branch B , regardless of the value of the label on v . Note this occurs if both Admirable and Impish have played on $\{v_2, v_3\}$.

To see this, vv_2 and vv_3 must have different signs. Thus we have discrepancy at most 2 from edges v_1v_2 and v_3v_4 .

We begin with the Admirable-start games. We may suppose Admirable plays v_2 , and then plays on $\{v_1, v_4\}$. This guarantees both players played on $\{v_2, v_3\}$.

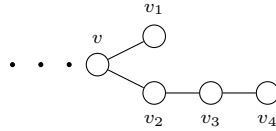
For an Impish-starts game, if Impish plays on $\{v_2, v_3\}$, then Admirable immediately follows by playing on $\{v_2, v_3\}$ as well. If Impish plays on $\{v_1, v_4\}$, then Admirable again immediately follows by playing on $\{v_1, v_4\}$ as well. Note this strategy is robust against the case where Impish is also allowed to pass his turn. ■

CASE 4. *Branch B is illustrated below with vertex v in T' .*



Let the vertices of B be labeled as in the figure, and let S be the subgraph induced by $\{v_1, v_2, v_3, v_4\}$. We give a strategy for Admirable which produces a discrepancy of at most 2 for the edges induced by B , regardless of the value of the label on v . Note that $v_4v_1v_2v_3$ induces a path, thus it is sufficient to bound the discrepancy of the edges v_4v_1, v_1v_2, v_2v_3 by one. This is true by Lemma 2.1. ■

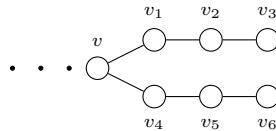
CASE 5. *Branch B is isomorphic to $P_5 = \{v_1, v, v_2, v_3, v_4\}$ with vertex v in T' .*



Let the vertices of B be labeled as the figure above, and let S be the subgraph induced by $\{v_1, v_2, v_3, v_4\}$. As in the prior case, we give a strategy for Admirable on S that produces a discrepancy of at most 2 for B , regardless of the value of the label on v . Note this occurs if Admirable plays once on $S_1 = \{v_1, v_2\}$ and once on $S_2 = \{v_3, v_4\}$.

For Admirable-start games, we first play v_1 and then Admirable on his second move plays on $\{v_3, v_4\}$. When Impish plays first, if Impish plays on S_i for $i \in \{1, 2\}$, then Admirable follows by playing on S_i as well. In the case Impish is allowed to pass and does so on his second move, Admirable plays in a way so that he plays on both S_1 and S_2 . ■

CASE 6. *Branch B is isomorphic to $P_7 = \{v_1, v_2, v_3, v, v_4, v_5, v_6\}$ with vertex v in T' .*



Let the vertices of B be labeled as above and S be the subgraph induced by $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Again, we give a strategy for Admirable on S that produces a discrepancy of at most 2 for the branch B , regardless of label on v . To achieve this, we describe the possible winning labelings for Admirable. In the first case, suppose both Admirable and Impish play on $\{v_1, v_4\}$. Conditioned on this constraint, by inspection, the sets inducing cuts of discrepancy at most 2 on S are the following,

$$\begin{aligned} &\{v_1, v_2, v_5\}, \quad \{v_1, v_2, v_6\}, \quad \{v_1, v_3, v_6\}, \quad \{v_1, v_5, v_6\}, \\ &\{v_2, v_3, v_4\}, \quad \{v_2, v_4, v_5\}, \quad \{v_3, v_4, v_5\}, \quad \{v_3, v_4, v_6\}. \end{aligned}$$

As the edges vv_1 and vv_4 must have different labelings, if Admirable labels one of the above sets, this would be considered a winning labeling. Now consider the other case, either only Admirable or Impish play on $\{v_1, v_4\}$. Under this assumption, Admirable must guarantee the edges induced by S have discrepancy 0 under his labeling. By inspection, the following are such labelings,

$$\{v_1, v_2, v_4\}, \quad \{v_1, v_4, v_5\}, \quad \{v_2, v_3, v_6\}, \quad \{v_3, v_5, v_6\}.$$

Thus, in all cases, we will give a strategy for Admirable for playing on S that guarantees one of the above twelve labelings.

We first handle Admirable-start games on S . Admirable begins by playing v_1 . First suppose Impish does not play v_6 on their first move. Then on Admirable's second move, he plays v_6 . For Admirable third move, he avoids playing v_4 . As Admirable played first and $|S|$ is even, this is always possible. The possible labelings for Admirable are

$$\{v_1, v_2, v_6\}, \quad \{v_1, v_3, v_6\}, \quad \{v_1, v_5, v_6\},$$

all of which are one of the possible twelve winning labelings. Now suppose Impish plays v_6 on his first move. On Admirable's second move, they play v_2 . If Impish does not play v_4 on his second turn (or he passes), Admirable would then play v_4 , Admirable would play v_4 , obtaining the set $\{v_1, v_2, v_4\}$, a winning set. If Impish plays on v_4 on his second move, Admirable would then play v_5 , which would guarantee the winning set $\{v_1, v_2, v_5\}$.

We now consider Impish-start games. By symmetry, we may assume Impish plays on $\{v_1, v_2, v_3\}$. First suppose Impish plays v_1 . Admirable would then play v_3 . If Impish does not play v_4 (or passes) for his second move, Admirable plays v_4 for his second move. Regardless of Admirable's third move, he would obtain one of the following labelings,

$$\{v_2, v_3, v_4\}, \quad \{v_3, v_4, v_5\}, \quad \{v_3, v_4, v_6\},$$

all of which are winning. So suppose Impish plays v_4 on his second move. From here Admirable would play v_6 on his second move. Irrespective of Admirable's third move, he obtains one of the following winning labelings,

$$\{v_2, v_3, v_6\}, \quad \{v_3, v_5, v_6\}.$$

Now suppose Impish plays v_2 on his first move. Admirable would then follow by playing v_5 . Regardless of how Impish plays (or passes), Admirable can guarantee he labels exactly one vertex from $\{v_1, v_3\}$ and exactly one vertex from $\{v_4, v_6\}$. The possible labelings Admirable may obtain are

$$\{v_1, v_4, v_5\} \quad \{v_1, v_5, v_6\} \quad \{v_3, v_4, v_5\} \quad \{v_3, v_5, v_6\},$$

all of which are winning.

Now suppose Impish plays on v_3 for his first move, Admirable would then follow by playing v_1 . If Impish does not play v_5 or passes on their second move, Admirable plays v_5 . The possible final labels for Admirable in this case would be

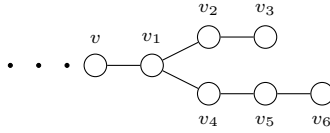
$$\{v_1, v_2, v_5\} \quad \{v_1, v_4, v_5\} \quad \{v_1, v_5, v_6\}$$

all of which are winning. If Impish plays v_5 on this second move, Admirable would then play v_2 . The possible final game states for Admirable are

$$\{v_1, v_2, v_6\} \quad \{v_1, v_2, v_4\}$$

which are both winning. ■

CASE 7. *Branch B is illustrated below with vertex v in T' .*



Let the vertices of B be labeled as pictured above and let S be the subgraph induced by $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Note that S induces a subgraph isomorphic to P_6 . By Lemma 2.1, the optimal discrepancy was shown to be 1 for P_6 , taking into the account the edge between v and v_1 in B produces a discrepancy of at most 2. ■

We now finish the proof by showing that T has a branch B as described as in the prior seven cases. By Theorem 2.2, we may assume T has a vertex of degree at least three. Let T' be the tree obtained from T by suppressing all degree two vertices and let T'' be the tree obtained from T' by deleting all leaves of T' . Let $u \in V(T'')$ be a leaf of T'' . Note then $d_{T'}(u) = d_T(u) \geq 3$, as $u \in V(T')$ and u was not a leaf of T' . Thus, u must be adjacent to at least two leaves in T' , as $d_{T''}(u) \leq 1$. Furthermore, if $|V(T'')| = 1$, we have that $d_{T''}(u) = 0$ and u is adjacent to at least three leaves in T' .

We now observe u in T . Let Q_1, \dots, Q_k be the components of $T - u$ which are paths. If $|V(T'')| = 0$, then u is adjacent to at least three leaves in T' , thus $d_T(u) = k \geq 3$. Otherwise, $|V(T'')| > 1$, $k \geq 2$, and by our choice of u , $d_T(u) = k + 1 \geq 3$.

First suppose there exists i such that $|Q_i| \geq 4$. Then there exists some $v \in V(Q_i) \cup \{u\}$ such that v is the endpoint of a branch B isomorphic to P_5 . Thus by Case 1, we may suppose $|Q_i| \leq 3$ for all i . By Cases 2, 3, and 6, for each $j \in \{1, 2, 3\}$, there can be at most one i such that $|Q_i| = j$. By Case 5, there cannot be two distinct i, i' such that $|Q_i| = 1$ and $|Q_{i'}| = 3$. Note this implies $k = 2$, and without loss of generality, $|Q_1| = 2$ and either $|Q_2| = 1$ or $|Q_2| = 3$. Note this implies $d_T(u) = 3$ as well. Depending on the order of Q_2 , either Case 4 or Case 7 is applicable, thus the claim follows. □

We do not believe that the upper bound in Theorem 3.1 is sharp. In fact, we posit that paths are the worst case for the cordiality game.

CONJECTURE 3.2. *For any tree T of order n , $c_g(T) \leq c_g(P_n)$.*

4. The Balance Game

While studying cordiality game strategies, it becomes evident that Admirable cannot simply greedily label edges 0 or 1, as Impish can decide to “play into” this strategy and further imbalance the edge labelings. As a curiosity, one may wonder what happens if each player instead adopts a greedy objective of trying to maximize the number of edges with a particular label. To consider this proposition, we introduce a new game.

Let G be a graph with vertices V and edges E . The *balance game* is played in a similar fashion as the cordiality game, but with a modified discrepancy d . We remove the absolute value from our previous definition of d and define the (signed) discrepancy of the balance game to be $d = e_1 - e_0$. The objectives of Admirable and Impish are still to minimize and maximize d , respectively. However, these objectives can now be interpreted as Admirable and Impish attempting to maximize the number of 0s and 1s, respectively, in the labeling of E . We define the *game balance number*, $b_g(G)$, to be the value of d when both players play optimally. As in the cordiality game, this game too can be interpreted as an instance of a Maker-Breaker game. For $k \geq 0$, let \mathcal{F}_k be the family of vertex sets S such that S and $V \setminus S$ is a balanced partition of V and $2\phi(S) - |E(G)| \leq k$. Again, Maker plays the role of Admirable, and Breaker plays the role of Impish.

Notice that for any graph G , Admirable can choose to play according to the optimal cordiality game strategy on G . This will produce a balance game discrepancy of $\pm c_g(G)$, depending on whether 0s or 1s form a majority of the edge labels. Hence, we have $b_g(G) \leq c_g(G)$. Our proof method for paths translates to the balance game as the following theorem illustrates.

THEOREM 4.1. *For any integer $n \geq 2$, $b_g(P_n) \geq 0$.*

PROOF. Let $P_n = v_1 v_2 \cdots v_n$. Note the theorem follows if $n = 2$. We proceed by induction on $n \geq 3$. Let P be the subpath of P_n induced on v_1, v_2, \dots, v_{n-2} and P' be the P_2 subpath of P induced by v_{n-1}, v_n . We present a strategy for Impish that, regardless of the moves Admirable makes, will produce the desired bound.

If Admirable plays on P , Impish continues by playing the optimal game on P . If Admirable plays on P' , then Impish plays the remaining vertex on P' . If Admirable finishes a game on P before moving to P' , Impish plays v_n . Regardless of the parity of n , this results in an Admirable-starts game on P and the edge $v_{n-1}v_n$ being labeled 1. We apply the induction hypothesis on P and consider that the edge between P and P' , $v_{n-2}v_{n-1}$, has an unknown label, which produces the claimed lower bound of 0. \square

We are not aware of any graph that violates this lower bound. We leave as an open question on whether there exists a graph G with an accompanied Admirable strategy which produces a negative discrepancy.

OPEN QUESTION 2. *For any graph G , does $b_g(G) \geq 0$?*

As an aside, the balance game has an interesting physical interpretation through the well-known Ising Model from statistical physics, e.g., see [12, Section 1.4.2] for the relevant definitions. Two players assign spin states, one assigning “up”, the other player “down”, to the remaining unassigned particles in alternating turns, competing over the energy of the final configuration at game completion. Open Question 2, reinterpreted, states: does the optimal strategy for the game keeps the Hamiltonian of the model nonnegative (with the assumption of an external magnetic field being absent)?

The complexity of Maker-Breaker games is well-studied, see for example [5, 21, 23]. As the Maker-Breaker instances of the cordiality game and balance game have a large amount of combinatorial structure on the associated set system, this leads to the following natural question.

OPEN QUESTION 3. *Are the corresponding Maker-Breaker games to the cordiality and balance game PSPACE-complete?*

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