ℓ_1 SPREADING MODELS AND THE FPP FOR CESÀRO MEAN NONEXPANSIVE MAPS

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ABSTRACT. Let K be a nonempty set in a Banach space X. A mapping $T\colon K \to K$ is called $\mathfrak{cm}$-*nonexpansive* if for any sequence $(x_i)_{i=1}^n$ and y in K, one has $\|(1/n)\sum_{i=1}^n Tx_i - Ty\| \le \|(1/n)\sum_{i=1}^n x_i - y\|$. As a subclass of nonexpansive maps, the FPP for such maps is well-established in a great variety of spaces. The main result of this paper is a fixed point result relating $\mathfrak{cm}$-nonexpansiveness, ℓ_1 spreading models and Schauder bases with not-so-large basis constants. As a result, we deduce that every Banach space with weak Banach-Saks property has the fixed point property for $\mathfrak{cm}$-nonexpansive maps.

1. INTRODUCTION

A central question in Banach space theory is which relevant structures can be linked to the geometry of Banach spaces. Of prime relevance are those driven by unconditional bases, although after the celebrated work of Gowers and Maurey [33] one cannot globally rely on them. An alternative strategy has been to seek asymptotic unconditionalities along weakly null sequences through combinatorial techniques. On this concern spreading models have played a crucial role. In [15], as e.g. it is proved that every bounded sequence (x_n) in a Banach space X generates a spreading model, that is, there exist a subsequence (x_{n_i}), a seminormed space $(E, |||\cdot|||)$ (*spreading model of* X) and a sequence $(e_i) \subset E$ such that, for any $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ so that for all $n_0 \le \kappa_1 < \kappa_2 < \cdots < \kappa_n$ and $(a_i)_{i=1}^n \in [-1,1]^n$,

$$\left| \Big\| \sum_{i=1}^n a_i x_{\kappa_i} \Big\| - \Big|\Big|\Big| \sum_{i=1}^n a_i e_i \Big|\Big|\Big| \right| < \varepsilon.$$

The sequence (e_i) (*fundamental sequence*) enjoys the 1-spreading property, that is, it is 1-equivalent to its subsequences. The unit vector bases of c_0 and ℓ_p have this property. It is well known that 1-spreading weakly null sequences are suppression 1-unconditional, and so 2-unconditional. So, as spreading models asymptotically surround bounded sequences it is quite natural to expect them to enjoy the same benefit. As it turns out, this happen when (x_n) is semi-normalized and weakly null [10, Lemma 2]. It is worth noting though that weakly null sequences without unconditional subsequences exists, as witnessed by Maurey and Rosenthal in [48]. We refer to the works [6, 11, 15, 22] concerning spreading models and the behavior of semi-normalized weakly null sequences in Banach spaces.

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1.1. **Unconditionality and metric fixed point theory.** The impact of asymptotic unconditionality in the metric *fixed point property* (FPP) has been a topic of great interest. As a sample, Lin [44, Theorem 3] showed that the absense of ℓ_1 spreading models is crucial to getting the FPP in super-reflexive spaces with a suppression 1-unconditional basis. Recall that X is said to have the FPP (resp. weak-FPP) if for every bounded closed (resp. weakly compact) convex set $C \subset X$, every nonexpansive map $T\colon C \to C$ has a fixed point. The affinities between FPP and semi-normalized weakly null sequences were also studied by García Falset in [25, Theorem 3] through the coefficient

$$R(X) := \sup\{\liminf_n \|x_n + x\|\},$$

where the sup is taken over all $x \in B_X$ and all weakly null sequences $(x_n) \subset B_X$. He proved that $R(X) < 2$ implies weak-FPP. A major feature that obstructs ℓ_1 spreading models is the *Banach-Saks property* (BSP) (cf. [10, 11]). It is said that X has BSP if every bounded sequence (x_n) has a subsequence (y_n) so that $(1/n)\sum_{i=1}^n y_i$ is norm-convergent. In its weak form (weak-BSP) it is predicted that every weakly null sequence contains a Cesàro norm-null subsequence. BSP came up in 1930 when Banach and Saks [8] proved that $L_p[0,1]$, $p \in (1,\infty)$, enjoy it. Clearly BSP implies weak-BSP. Also spaces with BSP are reflexive [51] and super-reflexive spaces have BSP [41]. These properties are closely tied to the geometric notion of B-convexity due to Beck [12]. The space X is called *B-convex* if for some $n \geq 2$ and $\varepsilon > 0$, and all vectors $(x_i)_{i=1}^n$ in X, one has $\|(1/n)\sum_{i=1}^n \epsilon_i x_i\| \leq (1-\varepsilon)\max_{i=1,\dots,n}\|x_i\|$ for some choice of signs $(\epsilon_i)_{i=1}^n \in \{-1,1\}^n$. Beck showed that B-convex spaces properly contain all uniformly convex spaces. Thus Hilbert spaces, ℓ_p and L_p, $p \in (1,\infty)$, are B-convex spaces while ℓ_1, ℓ_∞ and c_0 are not (cf. [29]). Rosenthal [52] showed that B-convexity implies weak-BSP. It is also worth noting that X is B-convex if and only if ℓ_1 is not finitely representable in it (cf. [29]). In particular, super-reflexive spaces are B-convex. Further aspects of B-convexity including geometric characterizations can be found in [12, 29, 30, 34, 36, 37]; see also [24, 26] for its usefulness in metric fixed point problems.

1.2. **Goal: FPP for $\mathfrak{cm}$-nonexpansive maps.** In line with Lin's method [44], García Falset [24] proved that Banach spaces with weak-BSP and with a *strongly* bimonotone basis have the weak-FPP for nonexpansive maps. The main purpose of this paper is to sharply improve this result by considering the class of Cesàro mean nonexpansive ($\mathfrak{cm}$-nonexpansive) maps. In fact, under this scope we are able to remove the *basis* requirement.

Definition 1.1. A nonexpansive map $T\colon C \to C$ is $\mathfrak{cm}$-nonexpansive if

$$\Big\|\frac{1}{n}\sum_{i=1}^n Tx_i - Ty\Big\| \leq \Big\|\frac{1}{n}\sum_{i=1}^n x_i - y\Big\|,$$

for all $n \in \mathbb{N}$ and any $x_1, x_2, \dots, x_n, y \in C$.

The approach we took to prove our main result is also in line with that in [24, 44], but is potentiated by some implicit technicalities. As a result we obtain the theorem stated in the abstract. Let's highlight some examples.

Remark 1.2. It is readily seen that $T\colon C\to C$ is $\mathfrak{cm}$-nonexpansive if and only if T satisfies

$$\Big\|\sum_{i=1}^{n}(Tx_i - Ty_i)\Big\| \leq \Big\|\sum_{i=1}^{n}(x_i - y_i)\Big\|,$$

for all sequences $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ in C.

Example A. Every affine nonexpansive map $T\colon C\to C$ is clearly $\mathfrak{cm}$-nonexpansive.

Remark 1.3. Recall [16] that a Banach space X is called *expand-contract plastic space* (EC-space) if its unit ball B_X is plastic, which means to say that every nonexpansive bijection $T\colon B_X\to B_X$ is an isometry. The list of spaces with EC-property includes strictly convex spaces, c, ℓ_1 and many others (see e.g. [16, 38, 39]).

Example B. Let X be a EC-Banach space. Then every nonexpansive bijection $T\colon B_X\to B_X$ is $\mathfrak{cm}$-nonexpansive. Indeed, since B_X is plastic T is an isometry. Also one has $T(0)=0$ (cf. [16, Theorem 2.3]). By a result of Mankiewicz [47] T extends to a linear isometry in the whole of X. Using the linearity of such an extension it is shown that T is $\mathfrak{cm}$-nonexpansive.

Example C. Let

$$K=\big\{u\in L_1[0,1]\colon 0\leq u\leq 1\big\}$$

and consider the non-affine map $T\colon K\to K$ defined by

$$Tu(t)=\frac{u(t)}{2}\cdot\int_0^t u(s)ds \quad \text{a.e.}\,\text{in}[0,1].$$

Then T is $\mathfrak{cm}$-nonexpansive (cf. Example 6.1 for details). Notice that K is weakly compact and $T(0)=0$.

1.3. **Organization of the paper.** We close this section by describing the content of the paper. Section 2 is devoted to background materials where the notation and some basic facts are recalled. In Section 3 we state and prove our main result. Some consequences and final considerations are provided in Sections 4, 5 and 6.

2. Preliminaries

In this section we set up the background material, referring to [3, 22, 53] for more details.

2.1. **Notation and terminology.** For a Banach space X we denote by B_X its closed unit ball and by X^* its dual space. A sequence (x_i) in X is called *basic* if it is a *Schauder* basis for its closed linear span $[\![x_n]\!]$. For a Schauder basis (e_i) in X denote by $(e_i^*)\subset X^*$ its coefficient functionals. For $n\in\mathbb{N}$ the n-th basis projection is the operator $P_n(x)=\sum_{i=1}^n e_i^*(x)e_i$. The *basis constant* of (e_i) is the number $\mathsf{K}_m=\sup_{n\in\mathbb{N}}\|P_n\|$. For $F\subset\mathbb{N}$, the projection P_F over F is the operator $P_F(x)=\sum_{i\in F}e_i^*(x)e_i$. The number $\mathsf{K}_b=\sup_{m<n}\|P_{[m,n]}\|$ is finite and is called the *bimonotonicity projection constant* of (e_i). The constant $\mathsf{K}_{sb}=\sup_{m<n}\max(\|P_{[m,n]}\|,\|I-P_{[m,n]}\|)$ is

called the *strong bimonotonicity projection constant* of (e_i), where I is the identity operator and $[m,n] = \{m, m+1, \dots, n\}$. The basis (e_i) is said to be monotone (resp. bimonotone, strongly bimonotone) if $\mathsf{K}_m = 1$ (resp. $\mathsf{K}_b = 1$, $\mathsf{K}_{sb} = 1$). It is called unconditional if there is $\mathcal{D} \geq 1$ so that $\| \sum_{i=1}^\infty \epsilon_i a_i e_i \| \leq \mathcal{D} \| \sum_{i=1}^\infty a_i e_i \|$ for all signs $\epsilon_i \in \{\pm 1\}$ and scalars $(a_i)_{i=1}^\infty \in \mathrm{c}_{00}$. The smallest such $\mathcal{D}$ is the *unconditional constant* of (e_i). The Banach-Mazur distance between isomorphic Banach spaces X and Y is defined by $\mathrm{d}_{BM}(X,Y) = \inf \big\{ \|\mathcal{L}\| \|\mathcal{L}^{-1}\| \colon \mathcal{L}\colon X \to Y \text{ is a linear isomorphism} \big\}$. Let us recall that X is said to have the *bounded approximation property* (BAP) if there exists a $\lambda \geq 1$ so that for every $\varepsilon > 0$ and every compact $K \subset X$, one can find an operator $T \in \mathfrak{F}(X)$ so that $\|T\| \leq \lambda$ and $\|Tx - x\| < \varepsilon$ for every $x \in K$, where $\mathfrak{F}(X)$ is the space of finite rank operators on X.

2.2. **Ultrapowers in metric fixed point theory.** Let $\mathcal{U}$ be a free ultrafilter defined on $\mathbb{N}$. Given a sequence $(x_n) \subset X$, the *limit of* (x_n) *through* $\mathcal{U}$ will be denoted by $\lim_{n,\mathcal{U}} x_n$. The ultrapower of X with respect to $\mathcal{U}$ is the Banach space $(X)_{\mathcal{U}}$ given by the quotient space of

$$\ell_\infty(X) = \Big\{ (x_n) \colon x_n \in X \text{ for all } n \in \mathbb{N} \text{ and } \sup_n \|x_n\| < \infty \Big\}$$

by

$$N_{\mathcal{U}} = \Big\{ (x_n) \in \ell_\infty(X) \colon \lim_{n,\mathcal{U}} \|x_n\| = 0 \Big\},$$

equipped with the norm $\|[x_i]_{\mathcal{U}}\|_{(X)_{\mathcal{U}}} = \lim_{i,\mathcal{U}} \|x_i\|$, where $[x_i]_{\mathcal{U}}$ is the element of $(X)_{\mathcal{U}}$ whose representative is (x_n). It is easy to see that X embeds isometrically into $(X)_{\mathcal{U}}$.

A nonempty, bounded closed and convex subset C of X has the FPP if every nonexpansive mapping $T\colon C \to C$ has at lest one fixed point. X is said to have the weak-FPP if every weakly compact convex set C in X has the FPP. Given C a weakly compact convex subset of X and $T\colon C \to C$ a nonexpansive mapping without fixed points, using Zorn's lemma we find a closed convex set $K \subset C$ which is minimal with respect to being invariant under T. Minimal sets have many useful properties (cf. [2, 40]) and they are widely used in the study of weak-FPP. One of them is following lemma due to Goebel [31] and Karlovitz [42].

Lemma 2.1 (Goebel-Karlovitz). *Let K be a minimal weakly compact convex set for a nonexpansive fixed-point free mapping T. Then*

$$\lim_{n\to\infty} \|y_n - x\| = \operatorname{diam} K$$

for all $x \in K$ and any approximate fixed point sequence (y_n) of T.

Recall that (y_n) approximate fixed points of T if

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0.$$

The set $[K]_{\mathcal{U}} = \{[v_i]_{\mathcal{U}} \in (X)_{\mathcal{U}} \colon v_i \in K \, \forall i \in \mathbb{N}\}$ is the ultrapower of K, and the ultrapower mapping of T is the map $[T]_{\mathcal{U}}\colon [K]_{\mathcal{U}} \to [K]_{\mathcal{U}}$ defined by $[T]_{\mathcal{U}}([v_i]_{\mathcal{U}}) = [T(v_i)]_{\mathcal{U}}$. Notice that the fixed point set of $[T]_{\mathcal{U}}$ is nonempty and consists of all classes $[v_i]_{\mathcal{U}} \in [K]_{\mathcal{U}}$ for which (v_i) is an approximate fixed point sequence.

In order to reach the main purpose of this paper it will suffice to consider the quotient space $[X] := \ell_\infty(X)/c_0(X)$ endowed with the quotient norm given by $\|[v_i]\| := \limsup_{i\to\infty} \|v_i\|$, where $[v_i]$ denotes the equivalent class of $(v_i) \in \ell_\infty(X)$. Under this perspective, we shall also denote $[K] := \{[v_i] \in [X] \colon v_i \in K \, \forall i \in \mathbb{N}\}$ and $[T] \colon [K] \to [K]$ the map defined by $[T]([v_i]) = [T(v_i)]$. For further aspects regarding FPP we refer the reader to [2, 27, 40].

The following result is due to Lin [44].

Lemma 2.2 (Lin). *Let K be a weakly compact convex set in a Banach X and T a nonexpansive mapping on K. Assume that K is minimal for T and $\mathcal{M} \subseteq [K]$ is a nonempty closed convex subset such that $[T](\mathcal{M}) \subseteq \mathcal{M}$. Then*

$$\sup\big\{\|[v_i] - x\| \colon [v_i] \in \mathcal{M}\big\} = \operatorname{diam} K \quad \text{for all } x \in K.$$

3. Main result

We start this section by recording the following weakened notion of bounded approximation (cf. [53, Definition 18.1]) that will be used in our result.

Definition 3.1 (EAB). Let X be a Banach space. We say that X has an extended approximative basis (EAB) if there is a net of finite rank operators $(P_\alpha)_{\alpha\in\mathfrak{D}}$ on X such that

$$x = \lim_{\alpha\in\mathfrak{D}} P_\alpha(x) \ \ (x \in X) \quad \text{and} \quad \sup_{\alpha\in\mathfrak{D}} \|P_\alpha\| < \infty.$$

If $(P_\alpha)_{\alpha\in\mathfrak{D}}$ can be chosen with $\sup_\alpha \|P_\alpha\| \le \lambda$ for some $\lambda \ge 1$, then X is said to have an extended λ-approximative basis (λ-EAB).

It is easy to see that separable Banach spaces with BAP have λ-EAB.

Lemma 3.2. *Let X be a Banach space with a λ-EAB $(P_\alpha)_{\alpha\in\mathfrak{D}}$. Assume that K is a separable subset of X and (y_n) is a semi-normalized weakly null sequence in K. Then there exist an increasing sequence $(\alpha_{m_i})_{i\ge 0}$ in $\mathfrak{D}$ and a basic subsequence (x_{n_i}) of (y_n) such that*

(1) $\lim_{i\to\infty} \big\|P_{\alpha_{m_i}}\big((1/N)\sum_{k=1}^N x_{n_{i+k}}\big)\big\| = 0$ *for all $N \in \mathbb{N}$.*
(2) $\lim_{i\to\infty} \|x_{n_i} - P_{\alpha_{m_i}}(x_{n_i})\| = 0$.
(3) $\lim_{i\to\infty} \|x - P_{\alpha_{m_i}}(x)\| = 0$ *for all $x \in K$.*

Proof. By a result of Bessaga and Pełczyński [13] there is a basic subsequence (x_n) of (y_n). Let $G = \overline{\operatorname{span}}(K)$ (the closed linear span of K). Then G is separable. By the proof of Theorem 18.2 in [53, p. 606] there exist an increasing sequence $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \dots$ in $\mathfrak{D}$ and a separable subspace F of X containing G so that $(P_{\alpha_n})_{n\in\mathbb{N}}$ is an extended λ-approximative basis for F. That is, $\|P_{\alpha_n}\| \le \lambda$ for all $n \ge 1$ and

$$z = \lim_{n\to\infty} P_{\alpha_n}(z) \quad (z \in F). \tag{3.1}$$

We now employ standard gliding hump method. As (x_n) is weakly null and P_{α_1} has finite rank there must exist $n_1 \in \mathbb{N}$ so that $\|P_{\alpha_1}(x_n)\| < 1/2^2$ for all $n \ge n_1$. Since $x_{n_1} \in F$ by (3.1) there is $m_1 > n_1$ so that $\|x_{n_1} - P_{\alpha_{m_1}}(x_{n_1})\| < 1/2^2$. As $(x_n)_{n\ge m_1}$ is weakly null and $P_{\alpha_{m_1}}$ has finite rank, there is $n_2 > m_1$ so that $\|P_{\alpha_{m_1}}(x_n)\| < 1/2^3$ for all $n \ge n_2$. Again by (3.1) we deduce that

$\|x_{n_2} - P_{\alpha_{m_2}}(x_{n_2})\| < 1/2^3$ for some $m_2 > n_2$. Continuing in this manner we obtain an increasing sequence $(\alpha_{m_i})_{i\geq 0}$ in $\mathcal{D}$, $m_0 = 1$, and a subsequence (x_{n_i}) such that for all $i \in \mathbb{N}$, $\|P_{\alpha_{m_i}}(x_n)\| < 1/2^{i+1}$ for all $n \geq n_{i+1}$ and $\|x_{n_i} - P_{\alpha_{m_i}}(x_{n_i})\| < 1/2^{i+1}$. The first set of inequalities clearly shows

$$\left\| P_{\alpha_{m_i}}\left(\frac{1}{N}\sum_{k=1}^{N} x_{n_{i+k}}\right)\right\| \leq \frac{1}{2^i} \quad (\forall N \in \mathbb{N}),$$

and this proves (*1*). Also, the second set of inequalities implies (*2*). Finally, we note that (*3*) follows directly from (3.1). The proof is complete. □

Lemma 3.3. *Let K be a minimal weakly compact convex set for a fixed point free $\mathfrak{cm}$-nonexpansive mapping $T\colon K \to K$. Assume that $\mathcal{M}$ is a closed convex subset of $[K]$. If $\mathcal{M}$ is nonempty and $[T]$-invariant then*

$$\sup_{[v_i]\in\mathcal{M}} \left\| \frac{1}{N}\sum_{k=1}^{N} [v_{i+k}] - x \right\| = \operatorname{diam} K \quad \forall x \in K\, \forall N \in \mathbb{N}.$$

Proof. The proof borrows ideas from the proof of Lemma 2.2. Now take $(\tilde{w}_s)_{s\in\mathbb{N}}$ to be an approximate fixed point sequence for $[T]$ in $\mathcal{M}$. Fix $x \in K$ and $N \in \mathbb{N}$. Write $\tilde{w}_s = [w_i^{(s)}]$. As in [2, p. 74] we will show that $\operatorname{diam} K$ is the only cluster point of the sequence $(\|(1/N)\sum_{k=1}^{N}[w_{i+k}^{(s)}] - x\|)_s$. Without loss of generality, we can assume that the limit

$$d := \lim_{s\to\infty} \|(1/N)\sum_{k=1}^{N}[w_{i+k}^{(s)}] - x\|$$

exists. Write $\delta_s = \|\tilde{w}_s - [T](\tilde{w}_s)\|$ and let (ε_j) a null sequence in $(0,1)$. Then for every $j \in \mathbb{N}$ there exists an integer $s_j > j$ so that

$$A_j^s = B_j^s \cap C_j^s$$

is infinite for all $s > s_j$, where

$$B_j^s = \left\{ m \in \mathbb{N}\colon \left\| \frac{1}{N}\sum_{k=1}^{N} w_{m+k}^{(s)} - x \right\| \leq d + 2\varepsilon_j \right\}$$

and

$$C_j^s = \left\{ m \in \mathbb{N}\colon \max_{1\leq k\leq N} \|w_{m+k}^{(s)} - T(w_{m+k}^{(s)})\| \leq \varepsilon_j + \delta_s \right\}.$$

Thus we can find strictly increasing sequences of integers $(s(j))_j$ and $(m(j))_j$ in $\mathbb{N}$ such that

$$\left\| \frac{1}{N}\sum_{k=1}^{N} w_{m(j)+k}^{(s(j))} - x \right\| \leq d + 2\varepsilon_j \tag{3.2}$$

and

$$\|w_{m(j)+k}^{(s(j))} - T(w_{m(j)+k}^{(s(j))})\| \leq \varepsilon_j + \delta_{s(j)},$$

for all $j \in \mathbb{N}$ and $k = 1, \ldots, N$. Thus, for each $k = 1, \ldots, N$, $(w_{m(j)+k}^{(s(j))})_j$ is an approximate fixed point sequence for T in K.

By passing to a subsequence (if needed) we may assume that the sequence $(w^{(s(j))}_{m(j)+k})_j$ weakly converges to some $z_k \in K$, for $k = 1, \dots, N$. Set

$$y_{j+k} := w^{(s(j))}_{m(j)+k} \quad \text{for } j \in \mathbb{N}, \ \ k = 1, \dots, N.$$

We now claim that

$$\limsup_{j\to\infty} \left\| \frac{1}{N} \sum_{k=1}^{N} y_{j+k} - x \right\| = \mathrm{diam} K \quad \forall x \in K \, \forall N \in \mathbb{N}. \tag{3.3}$$

Indeed, for $N \in \mathbb{N}$ define $\alpha_N \colon K \to \mathbb{R}_+$ by

$$\alpha_N(x) = \limsup_{j\to\infty} \left\| \frac{1}{N} \sum_{k=1}^{N} y_{j+k} - x \right\| \quad x \in K.$$

It is easy to see that $\alpha_N(Tx) \leq \alpha_N(x)$ for all $x \in K$. Also, notice that α_N is convex. Hence, since K is minimal, α_N is constant. So, for some $c_N \in \mathbb{R}_+$ one has $\alpha_N(x) = c_N$ for all $x \in K$. Notice that $\big((1/N)\sum_{k=1}^{N} y_{j+k}\big)_j$ weakly converges to $(1/N)\sum_{k=1}^{N} z_k$, so as $\|\cdot\|$ is weak lower semi-continuous we have

$$\left\| \frac{1}{N} \sum_{k=1}^{N} z_k - x \right\| \leq \limsup_{j\to\infty} \left\| \frac{1}{N} \sum_{k=1}^{N} y_{j+k} - x \right\| = c_N,$$

for any $x \in K$. Set $z := (1/N)\sum_{k=1}^{N} z_k$. It follows that

$$\mathrm{diam} K = \sup_{x\in K} \|z - x\| \leq \limsup_{j\to\infty} \left\| \frac{1}{N} \sum_{k=1}^{N} y_{j+k} - x \right\| \leq \mathrm{diam} K.$$

This proves the claim.

Therefore by (3.2) we deduce that

$$\mathrm{diam} K \leq d$$

and this easily proves the lemma. □

We are now ready to state and prove the main theorem of this paper.

Main Theorem. *Let Z be a Banach space having a λ-EAB $(P_\alpha)_{\alpha\in\mathfrak{D}}$ with $\lambda < 2$. Assume that X is a subspace of Z that fails the weak-FPP for $\mathfrak{cm}$-nonexpansive maps. Then X has a spreading model isomorphic to ℓ_1.*

Proof. Let $\|\cdot\|$ denote the norm of Z. By assumption there exist a weakly compact convex set $C \subset X$ and a $\mathfrak{cm}$-nonexpansive map $T \colon C \to C$ with no fixed points. Let K be a minimal weakly compact convex set $K \subset C$ with positive diameter such that $T(K) \subseteq K$. By standard arguments we may assume that $0 \in K$, $\mathrm{diam} K = 1$ and that there is an approximate fixed point sequence $(x_n) \subset K$ of T weakly converging to 0. By Lemma 2.1,

$$\lim_{n\to\infty} \|x_n - y\| = 1 \quad (\forall\, y \in K). \tag{3.4}$$

Moreover, it is well-known and easy to check that weakly compact convex minimal invariant sets are separable. Taken all these facts for granted, we can apply Lemma 3.2 in conjunction with Brunel-Sucheston's result [15] to get a basic subsequence (x_{n_i}) of (x_n) generating a spreading model $\mathbb{E}$, and an increasing sequence (α_{m_i}) in $\mathcal{D}$ such that properties *(1)*–*(3)* are satisfied. For $i \in \mathbb{N}$ set $R_i = I - P_{\alpha_{m_i}}$.

In what follows we shall denote by $\vert\!\vert\!\vert \cdot \vert\!\vert\!\vert$ the norm of $\mathbb{E}$ and by $\mathbf{e} = (e_k)$ its fundamental sequence. Recall that for all scalars $(a_i)_{i=1}^m$ in $\mathbb{R}$ one has

$$\Big\vert\!\Big\vert\!\Big\vert \sum_{i=1}^m a_i e_i \Big\vert\!\Big\vert\!\Big\vert = \lim_{s_1\to\infty} \lim_{s_2\to\infty} \dots \lim_{s_m\to\infty} \Big\| \sum_{i=1}^m a_i x_{n_{s_i}} \Big\|.$$

It follows from this and (3.4) that $\mathbf{e}$ is normalized. In what follows we will use the following easily verified fact (cf. [2, (4.18), p. 84] for its ultrafilter version):

$$\Big\vert\!\Big\vert\!\Big\vert \sum_{k=1}^m a_k e_k \Big\vert\!\Big\vert\!\Big\vert = \Big\| \sum_{k=1}^m a_k [x_{n_{i+k}}] \Big\|, \tag{3.5}$$

for all scalars $(a_k)_{k=1}^m \subset \mathbb{R}$. From now on, and when there is no confusion, we will suppress the index $(X)_{\mathcal{U}}$ in the notation for $\| \cdot \|_{(X)_{\mathcal{U}}}$. Now pick a constant $d > 0$ so that

$$d < \frac{2-\lambda}{2(1+\lambda)}. \tag{3.6}$$

We now proceed by following the general ideas of [44] and [24], but with a different approach. Fix $m \in \mathbb{N}$ and set

$$\mathcal{M}_m = \left\{ [v_i] \in [K] : \begin{array}{r} \exists x \in K \text{ so that } \Big\| \dfrac{1}{m} \displaystyle\sum_{k=1}^m [v_{i+k}] - x \Big\| \leq d, \\ \text{and } \dfrac{1}{m} \displaystyle\sum_{k=1}^m \| [v_{i+k}] - [x_{n_{i+k}}] \| \leq \dfrac{1}{2} \end{array} \right\}.$$

It is clear $\mathcal{M}_m$ is closed convex. Let's prove that $[T](\mathcal{M}_m) \subset \mathcal{M}_m$. Fix any $[v_i] \in \mathcal{M}_m$ and take $x \in K$ satisfying the inequalities

$$\Big\| \frac{1}{m} \sum_{k=1}^m [v_{i+k}] - x \Big\| \leq d \quad \text{and} \quad \frac{1}{m} \sum_{k=1}^m \| [v_{i+k}] - [x_{n_{i+k}}] \| \leq \frac{1}{2}.$$

Note that $Tx \in K$ and $[T]([v_i]) \in [K]$. Therefore as T is $\mathfrak{cm}$-nonexpansive and $\|x_{n_i} - T(x_{n_i})\| \to 0$ we have

$$\Big\| \frac{1}{m} \sum_{k=1}^m [T v_{i+k}] - Tx \Big\| \leq d \quad \text{and} \quad \frac{1}{m} \sum_{k=1}^m \| [T v_{i+k}] - [x_{n_{i+k}}] \| \leq \frac{1}{2}.$$

This shows that $[T]([v_i]) \in \mathcal{M}_m$ and hence $[T](\mathcal{M}_m) \subset \mathcal{M}_m$, as desired.

Let us denote by $\tilde{I} = [I]$ and $\tilde{Q} = [R_i]$ the ultrapower mappings of the operators I and R_i's, respectively. By construction one has $\tilde{I} - \tilde{Q} = [P_{\alpha_{m_i}}]$, so

$$\|\tilde{Q}\| \leq 1 + \lambda \quad \text{and} \quad \|\tilde{I} - \tilde{Q}\| \leq \lambda.$$

In addition, by Lemma 3.2 we have

$$\Big\| (\tilde{I} - \tilde{Q}) \Big(\frac{1}{m} \sum_{k=1}^m [x_{n_{i+k}}] \Big) \Big\| = 0.$$

Now fix any $[v_i] \in \mathcal{M}_m$ and take $x \in K$ so that

$$\Big\| \frac{1}{m} \sum_{k=1}^{m} [v_{i+k}] - x \Big\| \leq d.$$

Note that $\|\tilde{Q}(x)\| = 0$. Hence

$$\begin{aligned} \frac{1}{m} \sum_{k=1}^{m} [v_{i+k}] &= \tilde{Q}\Big(\frac{1}{m} \sum_{k=1}^{m} [v_{i+k}] \Big) + (\tilde{I} - \tilde{Q}) \Big(\frac{1}{m} \sum_{k=1}^{m} [v_{i+k}] \Big) \\ &= \tilde{Q}\Big(\frac{1}{m} \sum_{k=1}^{m} [v_{i+k}] - x \Big) + (\tilde{I} - \tilde{Q}) \Big(\frac{1}{m} \sum_{k=1}^{m} ([v_{i+k}] - [x_{n_{i+k}}]) \Big), \end{aligned}$$

from which we get

$$\begin{aligned} \Big\| \frac{1}{m} \sum_{k=1}^{m} [v_{i+k}] \Big\| &\leq \Big\| \tilde{Q}\Big(\frac{1}{m} \sum_{k=1}^{m} [v_{i+k}] - x \Big) \Big\| + \Big\| (\tilde{I} - \tilde{Q}) \Big(\frac{1}{m} \sum_{k=1}^{m} [v_{i+k}] - [x_{n_{i+k}}] \Big) \Big\| \\ &\leq d(1+\lambda) + \frac{\lambda}{2}. \end{aligned}$$

It then follows from (3.6) that

$$\sup_{[v_i] \in \mathcal{M}_m} \Big\| \frac{1}{m} \sum_{k=1}^{m} [v_{i+k}] \Big\| \leq d(1+\lambda) + \frac{\lambda}{2} < 1.$$

Therefore by Lemma 3.3 we infer that $\mathcal{M}_m = \emptyset$ for all $m \in \mathbb{N}$.

As we will see below, the idea behind the way we define the set $\mathcal{M}_m$ is aligned with the need to force the generation of ℓ_1 spreading models. To see this, for each $m \in \mathbb{N}$ consider the class

$$[\vartheta_i] = \frac{1}{2} [x_{n_i}].$$

It is easy to see that

$$\frac{1}{m} \sum_{k=1}^{m} \|[\vartheta_{i+k}] - [x_{n_{i+k}}]\| \leq \frac{1}{2}.$$

It follows therefore that

$$\left\| \frac{1}{m} \sum_{k=1}^{m} [x_{n_{i+k}}] \right\| \geq 2d.$$

It then follows from this and (3.5) that

$$\left\| \frac{1}{m} \sum_{k=1}^{m} e_k \right\| \geq 2d.$$

Since (x_{n_i}) is semi-normalized and weakly null (e_i) is a non-trivial spreading model. Consequently the previous inequality combined with [1, Proposition 6.4] shows that (e_i) is equivalent to the unit basis of ℓ_1. This completes the proof of the theorem. $\square$

4. Immediate consequences

In this section we explore some immediate consequences of the main result of this work. We first consider the following localized version of the notion of weak Banach-Saks property.

Definition 4.1. A subset K of a Banach space X is said to have the weak-BSP (or, that is a weak-BS set) if whenever $(x_n)_n \subset K$ weakly converges to x then there exists a subsequence $(y_i)_i$ of (x_n) such that the sequence of averages $((1/n)\sum_{i=1}^n y_i)_n$ is norm-convergent to x.

Cleary if X has weak-BSP then every weakly compact subset of X has weak-BSP.

Let us recall (see e.g. [46, Definition 1]) that if a sequence $(x_n)_n$ weakly converges to some $x \in X$, then it is said to generate an ℓ_1-spreading model when there is $\delta > 0$ such that

$$\left\| \sum_{n\in S} a_n(x_n - x) \right\| \geq \delta \sum_{n\in S} |a_n| \tag{4.1}$$

for every $S \subseteq \mathbb{N}$ with $\#S \leq \min S$ and every sequence of scalars $(a_n)_{n\in S}$.

Remark 4.2. The proof of [46, Theorem 2.4] shows that if K is a weakly compact set with weak-BSP then no weakly-convergent sequence in K can generate an ℓ_1-spreading model.

Therefore our main result can be rephrased as follows:

Theorem 4.3 (Main Theorem)**.** *Let Z be a Banach space having a λ-EAB $(P_\alpha)_{\alpha\in\mathfrak{D}}$ with $\lambda < 2$. Assume that C is a weakly compact convex subset of Z that fails the fixed point property for $\mathfrak{cm}$-nonexpansive maps. Then C fails the weak-BSP.*

The first important consequence is the following fixed point theorem.

Theorem 4.4. *Every weakly compact convex weak-BS subset of a Banach space X has the fixed point property for $\mathfrak{cm}$-nonexpansive maps.*

Proof. Let $C \subset X$ be a weakly compact convex subset with weak-BSP. Since C is separable, so is its closed linear space. Therefore, there is no loss of generality in assuming that X is itself separable. By a result of Banach and Mazur [9] X embeds isometrically into $C[0,1]$ which, as is well known, has a monotone Schauder basis. For $n \in \mathbb{N}$ let P_n denote the corresponding n-th basis projection. Clearly $\{P_n\}_{n\in\mathbb{N}}$ is a 1-EAB for $C[0,1]$. Since weak-BSP is invariant under isometries, C can be seen as a subset of $Z = C[0,1]$ enjoying the weak-BSP. By Theorem 4.3 the result follows. ☐

Remark 4.5. It is worth noting that $L_1[0,1]$ has a monotone basis and (due to Alspach's example [4]) fails the weak-FPP for isometric maps. In contrast, by a result of Szlenk [54] $L_1[0,1]$ has the weak-BSP and consequently has the weak-FPP for $\mathfrak{cm}$-nonexpansive maps. As a result $\mathfrak{cm}$-nonexpansiveness is a sharp condition in our result.

An open question in metric fixed point theory is whether super-reflexive Banach spaces have FPP. Our next result settles this problem for the class of $\mathfrak{cm}$-nonexpansive maps.

Remark 4.6. Recall [35, Definition 3] that a Banach space X is called *super-reflexive* if every Banach space Y that is finitely representable in X is reflexive. Recall also that given Banach spaces X and Y, Y is said to be *finitely representable* in X if for any $\varepsilon > 0$ and any finite dimensional subspace N of Y there exist a subspace M of X and an isomorphism $\mathfrak{L}\colon N \to M$ such that $(1-\varepsilon)\|y\| \leq \|\mathfrak{L}y\| \leq (1+\varepsilon)\|y\|$ for all $y \in N$.

Theorem 4.7. *Let X be a super-reflexive Banach space. Then X has the fixed point property for $\mathfrak{cm}$-nonexpansive maps.*

Proof. As we have seen before super-reflexive spaces are B-convex, and hence have the weak-BSP. By Theorem 4.4 the result follows. □

Corollary 4.8. *Let X be a Banach space isomorphic to a super-reflexive space. Then X has the fixed point property for $\mathfrak{cm}$-nonexpansive maps.*

Proof. Super-reflexivity is invariant under isomorphisms (cf. [35]). □

Corollary 4.9. *Every Banach space isomorphic to a uniformly convex space has the fixed point property for $\mathfrak{cm}$-nonexpansive maps.*

Proof. A uniformly convex space is uniformly non-square and a uniformly non-square space is super-reflexive (cf. [35]). □

The following corollaries follow immediately from above results.

Corollary 4.10. *Every Banach space isomorphic to ℓ_p with $1 < p < \infty$ has the fixed point property for $\mathfrak{cm}$-nonexpansive maps.*

Corollary 4.11. *Every Banach space isomorphic to a Hilbert space has the fixed point property for $\mathfrak{cm}$-nonexpansive maps.*

5. Relation to existing results

The FPP has been studied since the 1965s, especially in connection with geometric properties of Banach spaces. The earlier contributions of Browder [14], Göhde [32] and Kirk [43] were determining and inspiring. Note that FPP and weak-FPP are equivalent in reflexive spaces. The spaces ℓ_p, $L_p[0,1]$ with $1 < p < \infty$ have FPP. In contrast, c_0, ℓ_1 and $L_1[0,1]$ fail it. However, they have weak-BSP. Another important result is that a bounded, closed convex subset K of c_0 has the FPP if and only if K is weakly compact. One direction is due to Maurey [49], while the other was proved by Dowling, Lennard and Turett [21]. Two further significant results are due to Lin [44] and Domínguez Benavides [18]. Lin proved that ℓ_1 admits an equivalent norm $|||\cdot|||$ for which $(\ell_1, |||\cdot|||)$ has the FPP, giving the first example of a nonreflexive Banach space with this property. Domínguez Benavides proved every Banach space which can be embedded in $\mathrm{c}_0(\Gamma)$ can be equivalently renormed so as to have FPP. However whether reflexive spaces have FPP remains open. Our main result shows that if one intends to use Maurey's ultrapower techniques to attack this problem, a deeper understanding on the impact of ℓ_1-spreading models is still needed.

5.1. **Further applications and questions.** In the sequel we show how the main theorem connects to existing works.

($\mathfrak{a}$) Let X be a Banach space. For $k \in \mathbb{N}$ denote by $s_k(X)$ the supremum of the set of numbers $\varepsilon \in [0,2]$ for which there exist points $x_1, \dots, x_{k+1}$ in B_X with

$$\min\Big\{\|x_i - x_j\| : i \neq j\Big\} \geq \varepsilon.$$

In [26] García Falset, Llorens Fuster and Mazcuñán Navarro introduced the geometric coefficient $\tilde{\varepsilon}_0^k(X)$ given by

$$\tilde{\varepsilon}_0^k(X) = \sup\Big\{\varepsilon \in [0, s_k(X)) : \tilde{\delta}^k(\varepsilon) = 0\Big\}$$

where $\tilde{\delta}^k \colon [0, s_k(X)) \to [0,1]$ is defined as follows:

$$\tilde{\delta}^k(\varepsilon) = \inf\left\{1 - \Big\|\frac{1}{k+1}\sum_{i=1}^{k+1} x_i\Big\| : \{x_i\}_{i=1}^{k+1} \subset B_X,\, \min_{i\neq j}\|x_i - x_j\| \geq \varepsilon\right\}.$$

They then proved that if X has a strongly bimonotone basis and $\tilde{\varepsilon}_0^k(X) < 2$ for some $k \in \mathbb{N}$, then X has the weak-FPP. A careful analysis of their proof shows that the assumption $\tilde{\varepsilon}_0^k(X) < 2$ *for some* $k \in \mathbb{N}$ actually implies X is B-convex. Therefore in view of Theorem 4.4 we have:

Theorem 5.1. *Let X be a Banach space such that $\tilde{\varepsilon}_0^k(X) < 2$ for some $k \in \mathbb{N}$. Then X has the weak-FPP for $\mathfrak{cm}$-nonexpansive maps.*

($\mathfrak{b}$) As mentioned before Lin [45] gave the first example of a *non*-reflexive Banach space with FPP. Inspired by Lin's approach many authors have provided several other renorming fixed-point techniques (e.g. see [17]). In [34] James build an example of a non-reflexive B-convex space, solving an open question from [36]. Theorem 4.4 shows that the quoted space is an example of a non-reflexive Banach space that has the weak-FPP for $\mathfrak{cm}$-nonexpansive maps without any renorming procedure.

($\mathfrak{c}$) B-convexity is not equivalent to the weak-FPP. Indeed, by the results of Maurey [49] and Dowling, Lennard and Turett [21], a closed bounded convex subset of c_0 has the FPP if and only if it is weakly compact. It is also an open question whether c_0 has the weak-FPP under equivalent renormings.

($\mathfrak{e}$) Among the *non* B-convex reflexive spaces with FPP (see [7]) are the Cesàro sequence spaces ces_p,$p \in (1,\infty)$, of real sequences $x = (a_k)$ so that

$$\|x\|_{c(p)} = \left(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n}|a_k|\Big)^p\right)^{1/p} < \infty.$$

($\mathfrak{f}$) The *characteristic of convexity* of a Banach space X is the number

$$\varepsilon_0(X) = \sup\{\varepsilon \in [0,2] : \delta_X(\varepsilon) = 0\}$$

where $\delta_X(\varepsilon)$ denotes the usual Clarkson *modulus of convexity* of X. The space X is said to be *uniformly non-square* whenever $\varepsilon_0(X) < 2$. In [28] García Falset, Llorens Fuster and Mazcuñán Navarro solved a long-standing problem in the theory by proving that all uniformly non-square Banach spaces have FPP. It is worth pointing out that spaces with such property are super-reflexive.

($\mathfrak{g}$) Another (apparently still open) problem in the theory asks whether $X \oplus Y$ has the weak-FPP provided that X and Y are two Banach spaces enjoying this property. There are plenty of works addressing this problem (see [55] and references therein). Let Z be a finite dimensional normed space $(\mathbb{R}^n, \|\cdot\|_Z)$. We shall write $(X_1 \oplus \cdots \oplus X_n)_Z$ for the Z-direct sum of the Banach spaces $X_1, \ldots, X_n$ equipped with the norm

$$\|(x_1, \ldots, x_n)\| = \|\big(\|x_1\|_{X_1}, \ldots, \|x_n\|_{X_n}\big)\|_Z$$

whenever $x_i \in X_i$ for each $i = 1, \ldots, n$. In [29, Lemma 11], Giesy proved that $(X_1 \oplus \cdots \oplus X_n)_{\ell_1^n}$ is B-convex if and only if $X_1, \ldots, X_n$ are. Thus:

Corollary 5.2. *If $X = (X_1 \oplus \cdots \oplus X_n)_{\ell_1^n}$ is the ℓ_1^n-direct sum of B-convex Banach spaces then X has the weak-FPP for $\mathfrak{cm}$-nonexpansive maps.*

($\mathfrak{h}$) For our next consideration we need a bit of notation. Let (Ω, Σ, μ) be a complete σ-finite measure space and let $p\colon \Omega \to [1, +\infty]$ be a measurable function. Next consider the vector space $\mathcal{X}$ of all measurable functions $g\colon \Omega \to \mathbb{R}$. For $g \in \mathcal{X}$ define the modular

$$\rho(g) := \int_{\Omega_f} |g(t)|^{p(t)} d\mu + \operatorname{ess\,sup}_{p^{-1}(\{+\infty\})} |g(t)|,$$

where $\Omega_f := \{t \in \Omega\colon p(t) < \infty\}$. The Variable Lebesgue Space (VLS) ([19, Definition 2.1]) is the space $L^{p(\cdot)}(\Omega)$ endowed with the Luxemburg norm

$$\|g\| = \inf\Big\{\alpha > 0\colon \rho\Big(\frac{g}{\alpha}\Big) \leq 1\Big\} \text{ for } g \in \mathcal{X}_\rho.$$

In [19, Theorem 2.5] T. Domínguez Benavides and M. Japón established the following relevant characterization of reflexivity for VLSs.

Theorem 5.3 (Domínguez Benavides-Japón)**.** *Let (Ω, Σ, μ) be an arbitrary σ-finite measure space and let $p\colon \Omega \to [1, +\infty]$ be a measurable function. The following conditions are equivalent:*

i) $L^{p(\cdot)}(\Omega)$ is reflexive.
ii) $L^{p(\cdot)}(\Omega)$ contains no isomorphic copy of ℓ_1.
iii) Let $\Omega^* := \Omega \setminus p^{-1}(\{1, +\infty\})$. Then $1 < p_-(\Omega^*) \leq p_+(\Omega^*) < +\infty$ and $p^{-1}(\{1, +\infty\})$ is essentially formed by finitely many atoms at most.

Let us remark that their proof that *iii)* implies *i)* shows a bit more, they in fact proved that $L^{p(\cdot)}(\Omega)$ is isomorphic to a uniformly convex space. Thus, as B-convexity is invariant under isomorphisms ([29, Corollary 6]), we may deduce that $L^{p(\cdot)}(\Omega)$ is reflexive if and only if it is B-convex. As a scholium of this fact by Theorem 4.4 we have:

Corollary 5.4. *Let (Ω, Σ, μ) be an arbitrary σ-finite measure space and let $p\colon \Omega \to [1, +\infty]$ be a measurable function. If $L^{p(\cdot)}(\Omega)$ is reflexive then it has the fixed point property for $\mathfrak{cm}$-nonexpansive maps.*

We note that this result yields a slight generalization of [19, Theorem 4.2], as no further condition is needed on the function $p(\cdot)$. As highlighted in [19, Theorem 3.3], see also comments in p. 12, one can find plenty of examples of nonreflexive VLSs that still have the weak-FPP. For instance, $L^{1+x}([0, 1])$

is one of them. All these remarks naturally lead us to ask whether there is a reflexive *non* B-convex subspace of $L^{p(\cdot)}(\Omega)$.

(i) It is known that $C[0,1]$ fails the weak-BSP. Moreover as $C[0,1]$ is a universal separable Banach space, it fails the weak-FPP for nonexpansive maps. Notice however that $C[0,1]$ contains a weakly compact convex set with weak-BSP (cf. [23, 46]). We do not know whether there is a weakly compact convex set in $C[0,1]$ which fails the FPP for $\mathfrak{cm}$-nonexpansive maps.

6. Appendix

Example 6.1. Let $K = [\mathbf{0}, \mathbf{1}]$ denote the order interval defined in Example **C**. Denote by $\|\cdot\|_1$ the standard L_1-norm. Let us prove that the map $T\colon K \to K$ defined a.e. in $[0,1]$ by

$$Tu(t) = \frac{1}{2}u(t)\cdot \int_0^t u(s)ds$$

is $\mathfrak{cm}$-nonexpansive. Let $(u_i)_{i=1}^n$ and $(v_i)_{i=1}^n$ be arbitrary n-tuples of functions in K. Then

$$\begin{aligned}2\sum_{i=1}^n (Tu_i - Tv_i)(t) &= \sum_{i=1}^n \Big(u_i(t)\cdot\int_0^t u_i ds - v_i(t)\cdot \int_0^t v_i ds\Big)\\ &= \sum_{i=1}^n (u_i(t)-v_i(t))\cdot \int_0^t u_i ds + \sum_{i=1}^n v_i(t)\cdot\int_0^t (u_i - v_i)ds.\end{aligned}$$

It follows that

$$\begin{aligned}2\Big\|\sum_{i=1}^n (Tu_i - Tv_i)\Big\|_1 &= \int_0^1 \Big|\sum_{i=1}^n (Tu_i - Tv_i)\Big| dt\\ &\leq \int_0^1 \Big|\sum_{i=1}^n (u_i(t)-v_i(t))\cdot\int_0^t u_i ds\Big| dt\\ &\qquad + \int_0^1\Big|\sum_{i=1}^n v_i(t)\cdot\int_0^t (u_i-v_i)ds\Big| dt\\ &\leq \Big\|\sum_{i=1}^n (u_i - v_i)\Big\|_1 + \int_0^1\Big|\int_0^t \sum_{i=1}^n v_i(t)\cdot(u_i - v_i)ds\Big|dt.\end{aligned}$$

Claim.

$$\int_0^1\Big|\int_0^t\sum_{i=1}^n v_i(t)\cdot (u_i - v_i)ds\Big|dt \leq \int_0^1 \max_{1\leq i\leq n} v_i(t)dt\cdot \Big\|\sum_{i=1}^n (u_i - v_i)\Big\|_1.$$

To see this for $t\in[0,1]$ set

$$\Delta_t^1 := \int_0^t \sum_{i=1}^n v_i(t)\cdot(u_i - v_i)ds \quad\text{and}\quad \Delta_t^2 := -\Delta_t^1.$$

Then

$$\int_0^1\Big|\int_0^t\sum_{i=1}^n v_i(t)\cdot(u_i - v_i)ds\Big|dt = \int_{\{t\colon\,\Delta_t^1\geq 0\}}\Delta_t^1 dt + \int_{\{t\colon\,\Delta_t^1<0\}}\Delta_t^2 dt.$$

Note that

$$\Delta_t^1 = \int_{P_t} \sum_{i=1}^n v_i(t)\cdot(u_i - v_i)ds + \int_{N_t} \sum_{i=1}^n v_i(t)\cdot(u_i - v_i)ds,$$

where $P_t = \{s \in [0,t] \colon u_i - v_i \geq 0\}$ and $N_t = [0,1] \setminus P_t$. Similarly we have

$$\Delta_t^2 = \int_{P_t} \sum_{i=1}^n v_i(t)\cdot(v_i - u_i)ds + \int_{N_t} \sum_{i=1}^n v_i(t)\cdot(v_i - u_i)ds.$$

It follows that

$$\Delta_t^1 \leq \max_{1\leq i\leq n} v_i(t) \int_{P_t} \sum_{i=1}^n (u_i - v_i)ds + \min_{1\leq i\leq n} v_i(t) \int_{N_t} \sum_{i=1}^n (u_i - v_i)ds$$

and

$$\Delta_t^2 \leq \min_{1\leq i\leq n} v_i(t) \int_{P_t} \sum_{i=1}^n (v_i - u_i)ds + \max_{1\leq i\leq n} v_i(t) \int_{N_t} \sum_{i=1}^n (v_i - u_i)ds.$$

These estimates imply

$$\Delta_t^1 \leq \max_{1\leq i\leq n} v_i(t) \int_{P_t} \Big| \sum_{i=1}^n (u_i - v_i)\Big| ds \leq \max_{1\leq i\leq n} v_i(t) \Big\| \sum_{i=1}^n (u_i - v_i)\Big\|_1$$

and (similarly)

$$\Delta_t^2 \leq \max_{1\leq i\leq n} v_i(t) \int_{N_t} \Big| \sum_{i=1}^n (u_i - v_i)\Big| ds \leq \max_{1\leq i\leq n} v_i(t) \Big\| \sum_{i=1}^n (u_i - v_i)\Big\|_1.$$

Consequently

$$\begin{aligned}\int_0^1 \Big| \int_0^t \sum_{i=1}^n v_i(t)\cdot(u_i - v_i)ds\Big| dt &= \int_{\{t\colon \Delta_t^1 \geq 0\}} \Delta_t^1 dt + \int_{\{t\colon \Delta_t^1 < 0\}} \Delta_t^2 dt \\ &\leq \Big\| \sum_{i=1}^n (u_i - v_i)\Big\|_1 \left(\int_{\{t\colon \Delta_t^1 \geq 0\}} \max_{1\leq i\leq n} v_i(t)dt + \int_{\{t\colon \Delta_t^1 < 0\}} \max_{1\leq i\leq n} v_i(t)dt \right)\end{aligned}$$

which proves the claim. It follows therefore that

$$\Big\| \sum_{i=1}^n (Tu_i - Tv_i)\Big\|_1 \leq \frac{1}{2}\left(1 + \int_0^1 \max_{1\leq i\leq n} v_i(t)dt\right) \Big\| \sum_{i=1}^n (u_i - v_i)\Big\|_1.$$

□

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