# A Correction of Pseudo Log-Likelihood Method 

Shi Feng ${ }^{1}$, Nuoya Xiong ${ }^{2}$, Zhijie Zhang ${ }^{3}$, and Wei Chen *4<br>${ }^{1}$ Harvard University, MA, USA<br>${ }^{2}$ Institute for Interdisciplinary Information Sciences, Tsinghua University, Beijing, China<br>${ }^{3}$ School of Mathematics and Statistics, Fuzhou University, Fuzhou, China<br>${ }^{4}$ Microsoft Research, Beijing, China


#### Abstract

Pseudo log-likelihood is a type of maximum likelihood estimation (MLE) method used in various fields including contextual bandits, influence maximization of social networks, and causal bandits. However, in previous literature [Li et al., 2017, Zhang et al., 2022a, Xiong and Chen, 2022, Feng and Chen, 2023, Feng et al., 2023], the log-likelihood function may not be bounded, which may result in the algorithm they proposed not well-defined. In this paper, we give a counterexample that the maximum pseudo log-likelihood estimation fails and then provide a solution to correct the algorithms in [Li et al., 2017, Zhang et al., 2022a, Xiong and Chen, 2022, Feng and Chen, 2023, Feng et al., 2023].


## 1 Problem Description

In [Li et al., 2017, Zhang et al., 2022a, Xiong and Chen, 2022, Feng and Chen, 2023, Feng et al., 2023], the authors use the same maximum likelihood estimation (MLE) method. Suppose $X_{1}, X_{2}, \ldots, X_{d}$ are random variables such that $X_{i} \in[0,1]$. For convenience, we use $\boldsymbol{X}$ to denote the vector ( $X_{1}, X_{2}, \cdots, X_{d}$ ). The function $\mu$ is a monotone increasing and second-order differentiable function from $\mathbb{R}$ to $\mathbb{R}$, and $m(x)$ is defined as:

$$
m(x)= \begin{cases}\int_{0}^{x} \mu\left(x^{\prime}\right) \mathrm{d} x^{\prime} & x \geq 0 \\ -\int_{-x}^{0} \mu\left(x^{\prime}\right) \mathrm{d} x^{\prime} & x<0\end{cases}
$$

There is a set of parameters $\theta_{1}^{*}, \theta_{2}^{*}, \ldots, \theta_{d}^{*}$, which we also denote by vector $\boldsymbol{\theta}^{*}$. Moreover, $Y \in[0,1]$ is the outcome of $X_{1}, X_{2}, \ldots, X_{d}$ such that $\mathbb{E}[Y \mid \boldsymbol{X}]=\mu\left(\boldsymbol{X}^{\top} \boldsymbol{\theta}^{*}\right)$. $Y$ depends on $\boldsymbol{X}^{\top} \boldsymbol{\theta}^{*}$ and its noise term is independent of $\boldsymbol{X}$ and $\boldsymbol{\theta}^{*}$.

There are in total $t$ rounds and in the $i^{t h}$ round, the values of $X_{1}, X_{2}, \ldots, X_{d}$ are $x_{1, i}, x_{2, i}, \ldots, x_{d, i}$, and the value of $Y$ is $y_{i}$. For convenience, we denote the vector $\left(x_{1, i}, x_{2, i}, \ldots, x_{d, i}\right)$ by $\boldsymbol{x}_{i}$. To estimate $\boldsymbol{\theta}^{*}$, the MLE method [Li et al., 2017, Zhang et al., 2022a] takes

$$
\begin{equation*}
\hat{\boldsymbol{\theta}}_{t}=\underset{\boldsymbol{\theta} \in \mathbb{R}^{d}}{\operatorname{argmax}} \sum_{i=1}^{t}\left(y_{i} \boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}-m\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right) \tag{1}
\end{equation*}
$$

as the estimation of $\boldsymbol{\theta}^{*}$. Since $m$ is a second-order differentiable function, when $\hat{\boldsymbol{\theta}}_{t}$ is a maximum of

$$
\sum_{i=1}^{t}\left(y_{i} \boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}-m\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right),
$$

the gradient $\sum_{i=1}^{t}\left(y_{i}-\mu\left(\boldsymbol{x}_{i}^{\top} \hat{\boldsymbol{\theta}}_{t}\right)\right) \boldsymbol{x}_{i}$ should be $\mathbf{0}$. In [Xiong and Chen, 2022, Feng and Chen, 2023, Feng et al., 2023], they solve $\hat{\boldsymbol{\theta}}_{t}$ by the equation $\sum_{i=1}^{t}\left(y_{i}-\mu\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right) \boldsymbol{x}_{i}=\mathbf{0}$.

Theorem 1 in [Li et al., 2017], Theorem 3 in [Zhang et al., 2022b], and Lemma 1 in [Feng and Chen, 2023] provide a guarantee for the distance from $\hat{\boldsymbol{\theta}}_{t}$ to $\boldsymbol{\theta}^{*}$ when $\hat{\boldsymbol{\theta}}_{t}$ exists. However, none of these previous literature

[^0][Li et al., 2017, Zhang et al., 2022a, Xiong and Chen, 2022, Feng and Chen, 2023, Feng et al., 2023] discusses the existence of $\hat{\boldsymbol{\theta}}_{t}$, i.e., whether $\sum_{i=1}^{t}\left(y_{i} \boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}-m\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right)$ could tend to positive infinity when some elements of $\boldsymbol{\theta}$ tend to infinity. If $\hat{\boldsymbol{\theta}}_{t}$ does not exist, the deductions in all the previous literature collapse. In this short paper, we present a counterexample to show that $\hat{\boldsymbol{\theta}}_{t}$ may not exist under the conditions of the previous papers [Li et al., 2017, Zhang et al., 2022a, Xiong and Chen, 2022, Feng and Chen, 2023, Feng et al., 2023] and provide a solution to address this issue. For consistency, we will use the notations in this section throughout this paper. Readers can correlate the notations with those of previous papers.

## 2 Counter-Examples

In this section, we present a counterexample to demonstrate the incorrectness of the default assumption that $\hat{\boldsymbol{\theta}}_{t}$ exists. In [Li et al., 2017], $X_{1}, \ldots, X_{d}, Y$ are continuous variables in $[0,1]$. In our counterexample, we let the link function be $\mu(x)=\frac{1}{2+2 e^{-x}}$, which satisfies Assumptions 1 and 2 in [Li et al., 2017]. For convenience, we denote $\sum_{i=1}^{t}\left(y_{i} \boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}-m\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right)$ by $H(\boldsymbol{\theta})$.

When $x_{1, i}=x_{2, i}=\cdots=x_{d, i}>0$ and $y_{i}>\frac{1}{2}$ for $i=1,2, \ldots, t$, the $j^{\text {th }}$ element of the gradient of $H(\boldsymbol{\theta})$ satisfies:

$$
\begin{aligned}
\frac{\partial H(\boldsymbol{\theta})}{\partial \theta_{j}}=\left(\sum_{i=1}^{t}\left(y_{i}-\mu\left(\boldsymbol{x}_{i}^{\top} \hat{\boldsymbol{\theta}}_{t}\right)\right) \boldsymbol{x}_{i}\right)_{j} & =\sum_{i=1}^{t}\left(y_{i}-\mu\left(\boldsymbol{x}_{i}^{\top} \hat{\boldsymbol{\theta}}_{t}\right)\right) x_{j, i} \\
& >\sum_{i=1}^{t}\left(\frac{1}{2}-\frac{1}{2+2 e^{-\boldsymbol{x}_{i}^{\top} \hat{\boldsymbol{\theta}}_{t}}}\right) x_{j, i} \\
& >0
\end{aligned}
$$

This occurs with a nonzero probability ${ }^{1}$, and it indicates that $\lim _{\theta_{1} \rightarrow+\infty, \ldots, \theta_{d} \rightarrow+\infty} H(\boldsymbol{\theta})=+\infty$ and thus Eq.(1) does not have a solution. This is a contradiction, and therefore, the proof of Theorem 1 in [Li et al., 2017] collapses. ${ }^{2}$

In [Zhang et al., 2022a, Xiong and Chen, 2022, Feng and Chen, 2023, Feng et al., 2023], $X_{1}, \ldots, X_{d}, Y$ are binary variables. In our counterexample, we let $\mu(x)=1-e^{-x}$, satisfying all assumptions in these papers. When $x_{1, i}=x_{2, i}=\cdots=x_{d, i}=1$ and $y_{i}=1$ for $i=1,2, \ldots, t$, the $j^{\text {th }}$ element of the gradient of $H(\boldsymbol{\theta})$ satisfies:

$$
\begin{aligned}
\frac{\partial H(\boldsymbol{\theta})}{\partial \theta_{j}}=\left(\sum_{i=1}^{t}\left(y_{i}-\mu\left(\boldsymbol{x}_{i}^{\top} \hat{\boldsymbol{\theta}}_{t}\right)\right) \boldsymbol{x}_{i}\right)_{j} & =\sum_{i=1}^{t}\left(y_{i}-\mu\left(\boldsymbol{x}_{i}^{\top} \hat{\boldsymbol{\theta}}_{t}\right)\right) x_{j, i} \\
& >\sum_{i=1}^{t} e^{-\boldsymbol{x}_{\boldsymbol{i}}^{\top} \hat{\boldsymbol{\theta}}_{t}}>0 .
\end{aligned}
$$

This also occurs with a nonzero probability and it indicates that $\lim _{\theta_{1} \rightarrow+\infty, \ldots, \theta_{d} \rightarrow+\infty} H(\boldsymbol{\theta})=+\infty$ and thus Eq.(1) does not have a solution. This is a contradiction, and therefore, the proofs of Theorem 3 in [Zhang et al., 2022a] and Lemma 1 in [Feng and Chen, 2023] collapses. Furthermore, the regret analysis in [Zhang et al., 2022a, Xiong and Chen, 2022, Feng and Chen, 2023, Feng et al., 2023] collapse. ${ }^{3}$

## 3 The Solution

In this section, we provide a feasible solution to promise the existence of $\hat{\boldsymbol{\theta}}_{t}$, which fixes the issues in the analysis of the five previous papers we mentioned in Section 1. Before our solution, one should notice that the constraint on $\mu$ in all five previous papers is Assumptions 1 and 2 in [Li et al., 2017].

Assumption 1 (Assumption 1 in [Li et al., 2017]). $\kappa:=\inf _{\boldsymbol{x} \in[0,1]^{d},\left\|\boldsymbol{\theta}-\boldsymbol{\theta}^{*}\right\| \leq 1} \mu^{\prime}\left(\boldsymbol{x}^{\boldsymbol{\top}} \boldsymbol{\theta}\right)>0 .{ }^{4}$

[^1]Assumption 2 (Assumption 2 in [Li et al., 2017]). Function $\mu$ is twice differentiable. Its first and second-order derivatives are upper-bounded by $L_{\mu}$ and $M_{\mu}$, respectively.

In summary, our solution is replacing function $\mu$ by another function $h$ in the algorithm such that the following conditions hold: $\lim _{x \rightarrow+\infty} h(x)=+\infty, \lim _{x \rightarrow-\infty} h(x)=-\infty, h$ is monotone increasing and twice differentiable, $h$ satisfies Assumptions 1 and 2, and when $x$ is in the range $\left[-\sum_{i=1}^{d} \operatorname{ReLU}\left(-\theta_{i}^{*}\right), \sum_{i=1}^{d} \operatorname{ReLU}\left(\theta_{i}^{*}\right)\right]^{5}$, $h(x)=\mu(x)$.

We firstly prove that $\mu$ can be converted to a monotone increasing function $g$ satisfying $\lim _{x \rightarrow+\infty} g(x)=+\infty$. If we already have $\lim _{x \rightarrow+\infty} \mu(x)=+\infty$, we can directly let $g \equiv \mu$. Otherwise, $\mu$ has an upper bound. In [Zhang et al., 2022a, Xiong and Chen, 2022, Feng and Chen, 2023, Feng et al., 2023], $\theta_{i}^{*} \in[0,1]$ so each $\theta_{i}^{*}$ is bounded. In [Li et al., 2017], since $\mu$ is upper-bounded, we know that $\lim _{x \rightarrow+\infty} \mu^{\prime}(x)=0$. According to Assumption 1, $\theta_{i}^{*} \leq \max \left\{x: \mu^{\prime}(x)=\kappa\right\}-1 .{ }^{6}$ Therefore, $\sum_{i=1}^{d} \operatorname{ReLU}\left(\theta_{i}^{*}\right)$ is also upper-bounded in this case, we denote the upper bound by $U$. We find a $x^{*} \geq U+d$ such that $\mu^{\prime \prime}\left(x^{*}\right)<0$. If $x^{*}$ does not exist, we know that $\mu(x) \geq \mu(U+d)+\mu^{\prime}(U+d)(x-(U+d))$ when $x \geq U+d$, which is contradictory to $\mu$ is upper-bounded. Hence, when $\mu$ is upper-bounded, we define the conversion as

$$
g(x)=\left\{\begin{array}{ll}
\mu(x) & x \leq x^{*} \\
\mu\left(x^{*}\right)+\frac{\mu^{\prime}\left(x^{*}\right)^{2}}{\mu^{\prime \prime}\left(x^{*}\right)} \ln \left(-\frac{\mu^{\prime}\left(x^{*}\right)}{\mu^{\prime \prime}\left(x^{*}\right)}\right)-\frac{\mu^{\prime}\left(x^{*}\right)^{2}}{\mu^{\prime \prime}\left(x^{*}\right)} \ln \left(x-x^{*}-\frac{\mu^{\prime}\left(x^{*}\right)}{\mu^{\prime \prime}\left(x^{*}\right)}\right) & x>x^{*}
\end{array} .\right.
$$

Lemma 1. By doing the conversion above, we can replace function $\mu$ by $g$ such that $\lim _{x \rightarrow+\infty} g(x)=+\infty, g$ is monotone increasing and twice differentiable, $g$ satisfies Assumptions 1 and 2, and when $x$ is in the range $\left[-\sum_{i=1}^{d} \operatorname{ReLU}\left(-\theta_{i}^{*}\right), \sum_{i=1}^{d} \operatorname{ReLU}\left(\theta_{i}^{*}\right)\right], g(x)=\mu(x)$.

Proof. When $\mu$ is not upper-bounded, we let $g \equiv \mu$ so the claim is proved. When $\mu$ is upper-bounded, during the producing process of $Y$, the input of $\mu$ is $\boldsymbol{X} \cdot \boldsymbol{\theta}^{*}$, which is in the range $\left[-\sum_{i=1}^{d} \operatorname{ReLU}\left(-\theta_{i}^{*}\right), \sum_{i=1}^{d} \operatorname{ReLU}\left(\theta_{i}^{*}\right)\right] \subseteq$ $(-\infty, U]$. Hence, when we replace $\mu$ with $g$, the joint conditional distribution of $Y$ on $\boldsymbol{X}$ is not impacted.

Moreover, we can compute that

$$
g^{\prime}(x)= \begin{cases}\mu^{\prime}(x) & x \leq x^{*} \\ -\frac{\mu^{\prime}\left(x^{*}\right)^{2}}{\mu^{\prime \prime}\left(x^{*}\right)\left(x-x^{*}-\frac{\mu^{\prime}\left(x^{*}\right)}{\mu^{\prime \prime}\left(x^{*}\right)}\right)} & x>x^{*}\end{cases}
$$

and

$$
g^{\prime \prime}(x)= \begin{cases}\mu^{\prime \prime}(x) & x \leq x^{*} \\ \frac{\mu^{\prime}\left(x^{*}\right)^{2}}{\mu^{\prime \prime}\left(x^{*}\right)\left(x-x^{*}-\frac{\mu^{\prime}\left(x^{*}\right)}{\mu^{\prime \prime}\left(x^{*}\right)}\right)^{2}} & x>x^{*} .\end{cases}
$$

Therefore, we have $\lim _{x \rightarrow x^{*}+} g(x)=\mu\left(x^{*}\right)$ and $\lim _{x \rightarrow x^{*}-} g(x)=\mu\left(x^{*}\right)$. Hence, $g$ is continuous. Moreover, $\lim _{x \rightarrow x^{*}+} g^{\prime}(x)=\mu^{\prime}\left(x^{*}\right)=\lim _{x \rightarrow x^{*}-} g^{\prime}(x)$ and $\lim _{x \rightarrow x^{*+}} g^{\prime \prime}(x)=\mu^{\prime \prime}\left(x^{*}\right)=\lim _{x \rightarrow x^{*-}} g^{\prime \prime}(x)$, so $g(x)$ is twice differentiable and $g^{\prime \prime}$ is continuous.

Now we only need to verify Assumptions 1 and 2. Firstly, when $x>x^{*}$, we have $g^{\prime}(x)<g^{\prime}\left(x^{*}\right)=\mu^{\prime}\left(x^{*}\right) \leq L \mu$ and $g^{\prime \prime}(x)<g^{\prime \prime}\left(x^{*}\right)=\mu^{\prime \prime}\left(x^{*}\right) \leq M_{\mu}$, so Assumption 2 holds. Secondly, $\max _{\boldsymbol{x} \in[0,1]^{d},\left\|\boldsymbol{\theta}-\boldsymbol{\theta}^{*}\right\| \leq 1} \boldsymbol{x} \cdot \boldsymbol{\theta} \leq U+d \leq x^{*}$, so the conversion does not impact the value of $\kappa$. The claim then follows.

We secondly prove that $g$ can be converted to a monotone increasing function $h$ satisfying $\lim _{x \rightarrow+\infty} h(x)=$ $+\infty$ and simultaneously, $\lim _{x \rightarrow-\infty} h(x)=-\infty$. If we already have $\lim _{x \rightarrow-\infty} \mu(x)=-\infty$, we can directly let $h \equiv g$. Otherwise, $\mu$ has a lower bound. In [Zhang et al., 2022a, Xiong and Chen, 2022, Feng and Chen, 2023, Feng et al., 2023], $\theta_{i}^{*} \in[0,1]$ so each $\theta_{i}^{*}$ is bounded. In [ Li et al., 2017], since $\mu$ is lower-bounded, we know that $\lim _{x \rightarrow-\infty} \mu^{\prime}(x)=0$. According to Assumption 1, $\theta_{i}^{*} \geq \min \left\{x: \mu^{\prime}(x)=\kappa\right\}+1 .{ }^{7}$ Therefore, $-\sum_{i=1}^{d} \operatorname{ReLU}\left(-\theta_{i}^{*}\right)$ is also lower-bounded in this case, we denote the lower bound by $L$. We find a $x^{* *} \leq L-d$ such that $\mu^{\prime \prime}\left(x^{* *}\right)>0$.

[^2]If $x^{* *}$ does not exist, we know that $\mu(x) \leq \mu(L-d)+\mu^{\prime}(L-d)(x-(L-d))$ when $x \leq L-d$, which is contradictory to $\mu$ is lower-bounded. Hence, when $\mu$ is lower-bounded, We define the conversion as

$$
h(x)=\left\{\begin{array}{ll}
g(x) & x \geq x^{* *} \\
g\left(x^{* *}\right)-\frac{g^{\prime}\left(x^{* *}\right)^{2}}{g^{\prime \prime}\left(x^{* *}\right)} \ln \left(-\frac{g^{\prime}\left(x^{* *}\right)}{g^{\prime \prime}\left(x^{* *}\right)}\right)+\frac{g^{\prime}\left(x^{* *}\right)^{2}}{g^{\prime \prime}\left(x^{* *}\right)} \ln \left(-x+x^{* *}+\frac{g^{\prime}\left(x^{* *}\right)}{g^{\prime \prime}\left(x^{* *}\right)}\right) & x<x^{* *}
\end{array} .\right.
$$

Lemma 2. By doing the conversion above, we can replace function $\mu$ by $h$ such that $\lim _{x \rightarrow+\infty} h(x)=+\infty$, $\lim _{x \rightarrow-\infty} h(x)=-\infty, h$ is monotone increasing and twice differentiable, $h$ satisfies Assumptions 1 and 2, and when $x$ is in the range $\left[-\sum_{i=1}^{d} \operatorname{ReLU}\left(-\theta_{i}^{*}\right), \sum_{i=1}^{d} \operatorname{ReLU}\left(\theta_{i}^{*}\right)\right], h(x)=\mu(x)$.

Proof. When $\mu$ is not lower-bounded, we let $h \equiv g$ so the claim is proved by Lemma 1. When $\mu$ is lower-bounded, during the producing process of $Y$, the input of $\mu$ is $\boldsymbol{X} \cdot \boldsymbol{\theta}^{*}$, which is in the range $\left[-\sum_{i=1}^{d} \operatorname{ReLU}\left(-\theta_{i}^{*}\right), \sum_{i=1}^{d} \operatorname{ReLU}\left(\theta_{i}^{*}\right)\right] \subseteq$ $[-L,+\infty)$. Hence, combining Lemma 1, when we replace $\mu$ with $h$, the joint conditional distribution of $Y$ on $\boldsymbol{X}$ is not impacted.

Moreover, we can compute that

$$
h^{\prime}(x)= \begin{cases}g^{\prime}(x) & x \geq x^{* *} \\ \frac{g^{\prime}\left(x^{* *}\right)^{2}}{g^{\prime \prime}\left(x^{* *}\right)\left(-x+x^{* *}+\frac{g^{\prime}\left(x^{* *}\right)}{g^{\prime \prime}\left(x^{* *}\right)}\right)} & x<x^{* *},\end{cases}
$$

and

$$
h^{\prime \prime}(x)= \begin{cases}g^{\prime \prime}(x) & x \geq x^{* *} \\ \frac{g^{\prime}\left(x^{* *}\right)^{2}}{g^{\prime \prime}\left(x^{* *}\right)\left(x-x^{* *}-\frac{g^{\prime}\left(x^{* *}\right)}{g^{\prime \prime}\left(x^{* *}\right)}\right)^{2}} & x<x^{* *} .\end{cases}
$$

Therefore, we have $\lim _{x \rightarrow x^{* *+}} h(x)=g\left(x^{* *}\right)$ and $\lim _{x \rightarrow x^{* *-}} h(x)=g\left(x^{* *}\right)$. Hence, $h$ is continuous. Moreover, $\lim _{x \rightarrow x^{* *+}} h^{\prime}(x)=g^{\prime}\left(x^{* *}\right)=\lim _{x \rightarrow x^{* *-}} h^{\prime}(x)$ and $\lim _{x \rightarrow x^{* *+}} h^{\prime \prime}(x)=g^{\prime \prime}\left(x^{* *}\right)=\lim _{x \rightarrow x^{* *-}} h^{\prime \prime}(x)$, so $h(x)$ is twice differentiable and $h^{\prime \prime}$ is continuous.

Now we only need to verify Assumptions 1 and 2. Firstly, when $x<x^{* *}$, we have $h^{\prime}(x)<h^{\prime}\left(x^{* *}\right)=g^{\prime}\left(x^{* *}\right) \leq$ $L_{\mu}$ and $h^{\prime \prime}(x)<h^{\prime \prime}\left(x^{* *}\right)=g^{\prime \prime}\left(x^{* *}\right) \leq M_{\mu}$, so Assumption 2 holds. Secondly, $\min _{\boldsymbol{x} \in[0,1]^{d},\left\|\boldsymbol{\theta}-\boldsymbol{\theta}^{*}\right\| \leq 1} \boldsymbol{x} \cdot \boldsymbol{\theta} \geq L-d \geq$ $x^{* *}$, so the conversion does not impact the value of $\kappa$. Until now, the claim has been proven.

Hence, we can replace $\mu$ by $h$ in the MLE method in (1) without impacting the data distribution and our requirements on $\mu$. Finally, we prove that by using $h$ in (1), the existence of $\hat{\boldsymbol{\theta}}_{t}$ is promised. let

$$
m_{h}(x)= \begin{cases}\int_{0}^{x} h\left(x^{\prime}\right) \mathrm{d} x^{\prime} & x \geq 0 \\ -\int_{-x}^{0} h\left(x^{\prime}\right) \mathrm{d} x^{\prime} & x<0\end{cases}
$$

Lemma 3. When $h$ is monotone increasing, $\lim _{x \rightarrow+\infty} h(x)=+\infty$, and $\lim _{x \rightarrow-\infty} h(x)=-\infty$, the maximum of $H_{h}(\boldsymbol{\theta})=\sum_{i=1}^{t}\left(y_{i} \boldsymbol{x}_{i}^{\boldsymbol{\top}} \boldsymbol{\theta}-m_{h}\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right)$ exists.

Proof. We only need to prove that

$$
H_{h}(\boldsymbol{\theta})=\sum_{i=1}^{t}\left(y_{i} \boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}-m_{h}\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right)
$$

is a concave function with respect to $\boldsymbol{\theta}$ and $\lim _{(\boldsymbol{\theta})_{j} \rightarrow \infty} H_{h}(\boldsymbol{\theta})=-\infty$ or $\frac{\partial H_{h}(\boldsymbol{\theta})}{\partial(\boldsymbol{\theta})_{j}} \equiv 0$ for all $j \in[d]$, which implies that $H$ has a maximal point. Firstly, we know that

$$
\frac{\partial^{2} m_{h}(x)}{\partial x^{2}}=h^{\prime}(x)>0
$$

so $m_{h}$ is a convex function. Therefore, for any vectors $\boldsymbol{\theta}_{1}, \boldsymbol{\theta}_{2} \in \mathbb{R}^{d}$ and $\lambda \in[0,1]$, we have

$$
\begin{aligned}
m_{h}\left(\boldsymbol{x}_{i}^{\top}\left(\lambda \boldsymbol{\theta}_{1}+(1-\lambda) \boldsymbol{\theta}_{2}\right)\right) & \left.=m_{h}\left(\lambda \boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}_{1}+(1-\lambda) \boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}_{2}\right)\right) \\
& \leq \lambda m_{h}\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}_{1}\right)+(1-\lambda) m_{h}\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}_{2}\right),
\end{aligned}
$$

so $m_{h}\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)$ is also a convex function with respect to $\boldsymbol{\theta}$ and the Hessian matrix $\mathbf{H}\left[m_{h}\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right]$ of $m_{h}\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)$ with respect to $\boldsymbol{\theta}$ should be positive semidefinite. Now we can compute the Hessian matrix $\mathbf{H}\left[H_{h}(\boldsymbol{\theta})\right]$ as

$$
\mathbf{H}\left[H_{h}(\boldsymbol{\theta})\right]=\sum_{i=1}^{t}\left(-\boldsymbol{x}_{i}^{\top} \boldsymbol{x}_{i} \cdot \mathbf{H}\left[m_{h}\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right]\right) .
$$

Hence, $\mathbf{H}\left[H_{h}(\boldsymbol{\theta})\right]$ is negative semidefinite because multiplying a positive semidefinite matrix by a negative scalar preserves the semidefiniteness. Thus $H$ is a concave function with respect to $\boldsymbol{\theta}$.

Now for any $j \in[d]$, we prove that $\lim _{\left(\boldsymbol{\theta}_{X}\right)_{j} \rightarrow+\infty} H_{h}\left(\boldsymbol{\theta}_{X}\right)=-\infty$ and $\lim _{\left(\boldsymbol{\theta}_{X}\right)_{j} \rightarrow-\infty} H_{h}\left(\boldsymbol{\theta}_{X}\right)=-\infty$ or $\frac{\partial H_{h}(\boldsymbol{\theta})}{\partial(\boldsymbol{\theta})_{j}} \equiv 0$. Firstly, we have

$$
\begin{aligned}
\frac{\partial H_{h}(\boldsymbol{\theta})}{\partial(\boldsymbol{\theta})_{j}} & =\sum_{i=1}^{t}\left(y_{i}\left(\boldsymbol{x}_{i}\right)_{j}-\left(\boldsymbol{x}_{i}\right)_{j} m_{h}^{\prime}\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right) \\
& =\sum_{i=1}^{t}\left(y_{i}\left(\boldsymbol{x}_{i}\right)_{j}-\left(\boldsymbol{x}_{i}\right)_{j} h\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right) .
\end{aligned}
$$

If $\left(\boldsymbol{x}_{i}\right)_{j}=0$ for all $i \in[t]$, we have $\frac{\partial H_{h}(\boldsymbol{\theta})}{\partial(\boldsymbol{\theta})_{j}} \equiv 0$. Otherwise, we have

$$
\begin{aligned}
\lim _{\left(\boldsymbol{\theta}_{X}\right)_{j} \rightarrow+\infty} \frac{\partial H_{h}(\boldsymbol{\theta})}{\partial(\boldsymbol{\theta})_{j}} & =\lim _{\left(\boldsymbol{\theta}_{X}\right)_{j} \rightarrow+\infty} \sum_{i=1}^{t}\left(y_{i}\left(\boldsymbol{x}_{i}\right)_{j}-\left(\boldsymbol{x}_{i}\right)_{j} h\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right) \\
& =\lim _{\left(\boldsymbol{\theta}_{X}\right)_{j} \rightarrow+\infty} \sum_{i=1}^{t}\left(\boldsymbol{x}_{i}\right)_{j}\left(y_{i}-h\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right) \\
& =-\infty, \quad\left(\lim _{\left(\boldsymbol{\theta}_{X}\right)_{j} \rightarrow+\infty} h\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)=+\infty\right)
\end{aligned}
$$

which indicates that $\lim _{\left(\boldsymbol{\theta}_{X}\right)_{j} \rightarrow+\infty} H_{h}\left(\boldsymbol{\theta}_{X}\right)=-\infty$. Also, we have

$$
\begin{aligned}
\lim _{\left(\boldsymbol{\theta}_{X}\right)_{j} \rightarrow-\infty} \frac{\partial H_{h}(\boldsymbol{\theta})}{\partial(\boldsymbol{\theta})_{j}} & =\lim _{\left(\boldsymbol{\theta}_{X}\right)_{j} \rightarrow-\infty} \sum_{i=1}^{t}\left(y_{i}\left(\boldsymbol{x}_{i}\right)_{j}-\left(\boldsymbol{x}_{i}\right)_{j} h\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right) \\
& =\lim _{\left(\boldsymbol{\theta}_{X}\right)_{j} \rightarrow-\infty} \sum_{i=1}^{t}\left(\boldsymbol{x}_{i}\right)_{j}\left(y_{i}-h\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right) \\
& =+\infty, \quad\left(\lim _{\left(\boldsymbol{\theta}_{X}\right)_{j} \rightarrow-\infty} h\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)=-\infty\right)
\end{aligned}
$$

which indicates that $\lim _{\left(\boldsymbol{\theta}_{X}\right)_{j} \rightarrow-\infty} H_{h}\left(\boldsymbol{\theta}_{X}\right)=-\infty$.
Therefore, we have proved that $H_{h}(\boldsymbol{\theta})$ has at least one global maximum, which indicates that the equation has at least one solution.

In conclusion, by combining Lemmas 1,2 , and 3, we arrive at the following main theorem:
Theorem 1 (Main Theorem). Consider a monotone-increasing and twice differentiable function $\mu$ that satisfies Assumptions 1 and 2. By transforming $\mu$ into the function $h$ as described above, we ensure that $h$ remains monotone-increasing and twice differentiable. Furthermore, $h$ preserves the same mapping in the range of input $\boldsymbol{x}^{\boldsymbol{\top}} \boldsymbol{\theta}^{*}$ and conforms to Assumptions 1 and 2. Most importantly, the maximum of $H_{h}(\boldsymbol{\theta})=$ $\sum_{i=1}^{t}\left(y_{i} \boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}-m_{h}\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right)$ exists.

Proof. According to Lemmas 1 and 2, it is established that $h$ remains monotone-increasing and twice differentiable. Additionally, these lemmas confirm that $h$ preserves the same mapping in the range of input $\boldsymbol{x}^{\boldsymbol{\top}} \boldsymbol{\theta}^{*}$ and conforms to Assumptions 1 and 2. Furthermore, Lemmas 1 and 2 demonstrate that $\lim _{x \rightarrow+\infty} h(x)=+\infty$ and $\lim _{x \rightarrow-\infty} h(x)=-\infty$, satisfying the condition of Lemma 3. Therefore, we can conclude that the maximum of $H_{h}(\boldsymbol{\theta})=\sum_{i=1}^{t}\left(y_{i} \boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}-m_{h}\left(\boldsymbol{x}_{i}^{\top} \boldsymbol{\theta}\right)\right)$ indeed exists.

## 4 Conclusion

In this paper, we identify and address an issue present in a type of Maximum Likelihood Estimation (MLE) method commonly employed in previous studies. Individuals intending to use similar methods in the future should be cautious of the same or analogous issues. Furthermore, one might consider a more intuitive and efficient solution to ensure the existence of $\hat{\boldsymbol{\theta}}_{t}$, rather than artificially constructing a new function.

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[^0]:    *Corresponding author: weic@microsoft.com

[^1]:    ${ }^{1}$ In [Li et al., 2017], the first $\tau$ rounds of Algorithm 1 are random.
    ${ }^{2}$ Even when $\boldsymbol{\theta}$ is bounded, this is still a contradiction because the proof of Theorem 1 in [Li et al., 2017] uses $\nabla_{\boldsymbol{\theta}} H(\boldsymbol{\theta})=\mathbf{0}$.
    ${ }^{3}$ In [Xiong and Chen, 2022], only the regret analysis of Algorithm 1 uses the MLE method in Eq.(1) and collapses due to this counterexample.
    ${ }^{4} \mathrm{Li}$ et al. [2017] use a weaker version with $\|\boldsymbol{x}\| \leq 1$. The other four papers use the current stronger version.

[^2]:    ${ }^{5} \operatorname{ReLU}(x)=\max \{0, x\}$ and $\boldsymbol{X}^{\top} \boldsymbol{\theta}^{*}$ has to be in this interval.
    ${ }^{6}$ Otherwise, when $x_{j}=0$ if $j \neq i, x_{i}=1$ and $\theta_{i}^{*}>\max \left\{x: \mu^{\prime}(x)=\kappa\right\}-1, \theta_{i}=\theta_{i}^{*}+1>\max \left\{x: \mu^{\prime}(x)=\kappa\right\}$, we have $\boldsymbol{x}^{\boldsymbol{\top}} \boldsymbol{\theta}>\max \left\{x: \mu^{\prime}(x)=\kappa\right\}$ and thus $\mu^{\prime}\left(\boldsymbol{x}^{\boldsymbol{\top}} \boldsymbol{\theta}\right)<\kappa$, which is a contradiction to Assumption 1.
    ${ }^{7}$ Otherwise, when $x_{j}=0$ if $j \neq i, x_{i}=1$ and $\theta_{i}^{*}<\min \left\{x: \mu^{\prime}(x)=\kappa\right\}+1, \theta_{i}=\theta_{i}^{*}-1<\min \left\{x: \mu^{\prime}(x)=\kappa\right\}$, we have $\boldsymbol{x}^{\boldsymbol{\top}} \boldsymbol{\theta}<\min \left\{x: \mu^{\prime}(x)=\kappa\right\}$ and thus $\mu^{\prime}\left(\boldsymbol{x}^{\boldsymbol{\top}} \boldsymbol{\theta}\right)<\kappa$, which is a contradiction to Assumption 1.

