CONSTRUCTING ABELIAN VARIETIES FROM RANK 3 GALOIS REPRESENTATIONS WITH REAL TRACE FIELD

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Abstract. Let U/K be a smooth affine curve over a number field and let L be an irreducible rank 3 $\overline{\mathbb{Q}}_\ell$-local system on U with trivial determinant and infinite geometric monodromy around a cusp. Suppose further that L extends to an integral model such that the Frobenius traces are contained in a fixed totally real number field. Then, after potentially shrinking U, there exists an abelian scheme $f\colon B_U \to U$ such that L is a summand of $R^2 f_* \overline{\mathbb{Q}}_\ell(1)$.

The key ingredients are: (1) the totally real assumption implies L admits a square root M; (2) the trace field of M is sufficiently bounded, allowing us to use [KYZ24] to construct an abelian scheme over $U_{\bar{K}}$ geometrically realizing L; and (3) Deligne's weight-monodromy theorem and the Rapoport-Zink spectral sequence, which allow us to pin down the arithmetizations using the total degeneration.

Setup 1. Let X/K be a smooth projective geometrically connected curve over a number field, let $D \subset X$ be a reduced divisor, and set $U := X \setminus D$. Fix a prime ℓ and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. Let L be a rank $r \geq 1$ irreducible $\overline{\mathbb{Q}}_\ell$-local system on U with trivial determinant.

It is well known that there exists an integral model $(\mathfrak{X}, \mathfrak{D})$, defined over $\mathcal{O} := \mathcal{O}_{K,S}$ for S a finite set of primes of $\mathcal{O}_K$, of (X, D), such that L naturally extends to a $\overline{\mathbb{Q}}_\ell$-local system $\mathscr{L}$ on $\mathfrak{U} := \mathfrak{X} \setminus \mathfrak{D}$, still with trivial determinant (see e.g. [Pet23, Proposition 6.1]).

The relative Fontaine–Mazur conjecture, as reformulated by Petrov, predicts that local systems in Setup 1 occur in the cohomology of a smooth projective family after possible restricting to an open $V \subset U$ and twisting by a character of G_K. In this article, we make some progress towards this conjecture when the rank is 3.

Theorem A. *Suppose $r = 3$ and L has infinite geometric monodromy around at least one point of D. Suppose further there exists a totally real number subfield $E \subset \overline{\mathbb{Q}}_\ell$ such that the field generated by Frobenius traces of $\mathscr{L}$ is contained in E. Then after potentially increasing D, there exists an abelian scheme $f\colon B_U \to U$ such that L is a summand of $R^2 f_* \overline{\mathbb{Q}}_\ell(1)$.*

Remark 2. Note that if we do not make any assumption on the trace field being totally real, then the conclusion of Theorem A is false. Indeed, there exist rank 3 hypergeometric local systems (with monodromy Zariski dense in $\mathrm{SL}(3, \overline{\mathbb{Q}}_\ell)$) with bounded Frobenius trace field that do not come from abelian varieties.

An auxiliary result, which may be of indepedent interest, is the following.

Lemma 3. *Let U/K be a smooth curve over a number field with compactification X, let $f\colon Y \to U$ be a smooth projective morphism, and let Q be a rank r summand of $R^{r-1} f_* \overline{\mathbb{Q}}_\ell$. Further suppose that Q is totally degenerating at some $\infty \in X \setminus U$, i.e., the local inertia around ∞ acts with a maximal Jordan block. Then Q has determinant isomorphic to $\overline{\mathbb{Q}}_\ell(-r(r-1)/2)$ up to a finite order character.*

Remark 4. Our assumptions in Lemma 3 include that of Q being of rank r and appearing in weight $r-1$. Note that there must be some relation between these quantities for the conclusion of the lemma, namely that the determinant is a power of cyclotomic (up to a finite order character), to hold. For example, for any Q as in Lemma 3, and χ a CM character appearing in H^1 of a CM abelian variety A_0/K, $Q \otimes \chi$ now has rank r and weight r, and no power of the determinant is a power of cyclotomic.

Proof of Theorem A. We break the proof up into steps.

(1) First of all, we claim that it is sufficient to prove the theorem after replacing K by a finite extension and U_K with a (geometrically connected) finite étale cover. Indeed, suppose we have a finite étale map $V \to U$ and an abelian scheme $g\colon B_V \to V$ such that $L(-1)|_V$ is a summand of $R^2 g_* \overline{\mathbb{Q}}_\ell$. Then we claim that the Weil restriction $f\colon A_U := \mathfrak{Res}_U^V B_V \to U$, an abelian scheme on U, will satisfy the conclusion of the theorem. The two relevant facts we need: $R^2 g_* \overline{\mathbb{Q}}_\ell$ is semi-simple by [Fal+92, Ch. 6, Sec. 3, Theorem I on p. 211], and if $H \subset G$ is a finite index subgroup and V a representation of H, then $\mathrm{Ind}_H^G(\bigwedge^2 V) \hookrightarrow \bigwedge^2 \mathrm{Ind}_H^G(V)$.

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(2) Fix $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$ with residue field $\mathbb{F}_q$ and set $\mathscr{L}_\mathfrak{p} := \mathscr{L}|_{\mathfrak{U}_\mathfrak{p}}$. First of all, we claim that the monodromy of $\mathscr{L}_\mathfrak{p}$ is contained in $\mathrm{SO}(3, \overline{\mathbb{Q}}_\ell)$. To see this, fix an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and set $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}_\ell)$ to be the transport of structure of complex conjugation.

[Laf02, Théorème VII.6] implies that every Frobenius eigenvalue of $\mathscr{L}_\mathfrak{p}$ is a q-Weil number of weight 0 and is contained in a CM number field. Therefore the Frobenius trace field (see [KL23, Definition 1.1.4] for the definition) of $\mathscr{L}_\mathfrak{p}$ is contained in a CM number field. Hence σ acts by complex conjugation on the traces of Frobenius conjugacy classes. Let ${}^\sigma(\mathscr{L}_\mathfrak{p})$ denote the σ-companion of $\mathscr{L}_\mathfrak{p}$, which exists again by [Laf02, Théorème VII.6]. Then ${}^\sigma(\mathscr{L}_\mathfrak{p}) \cong \mathscr{L}_\mathfrak{p}$ by the totally real assumption. On the other hand, as $\mathscr{L}_\mathfrak{p}$ is pure of weight 0, it follows that ${}^\sigma(\mathscr{L}_\mathfrak{p}) \cong \mathscr{L}_\mathfrak{p}^\vee$. Therefore there is an equivariant pairing $\mathscr{L}_\mathfrak{p} \otimes \mathscr{L}_\mathfrak{p} \to \overline{\mathbb{Q}}_\ell$, which is moreover non-degenerate as $\mathscr{L}_\mathfrak{p}$ is irreducible. As $\mathscr{L}_\mathfrak{p}$ has odd rank, this must be a symmetric pairing, i.e. the monodromy lies in $\mathrm{SO}(3, \overline{\mathbb{Q}}_\ell)$.

(3) There is an isogeny $\mathrm{SL}(2, \overline{\mathbb{Q}}_\ell) \to \mathrm{SO}(3, \overline{\mathbb{Q}}_\ell)$, given by the second symmetric square of the defining representation. Therefore, after possibly extending K and replacing U by a finite étale cover, we may assume $L \cong \mathrm{Sym}^2 M$, with M an irreducible rank 2 $\overline{\mathbb{Q}}_\ell$-local system on U, with trivial determinant. Note that M still has infinite monodromy around a point of D. Moreover, by [Pet23, Proposition 6.1], after potentially enlarging S, M canonically extends to a $\overline{\mathbb{Q}}_\ell$ local system $\mathscr{M}$ on $\mathfrak{U}$ with trivial determinant.

(4) Enlarge S to contain 2. We claim that there exists a finite étale map $\mathcal{O} \to \mathcal{O}'$, a curve $\mathfrak{V}/\mathcal{O}'$, and a finite étale morphism $\mathfrak{V} \to \mathfrak{U}$ such that the stable trace field (defined in [KL23, Definition 1.1.4]) of $\mathscr{M}|_\mathfrak{V}$ is contained in E. Indeed, let $V \to U$ be a cover that trivializes the 2-torsion of the Jacobian. Let us prove that $\mathscr{M}|_\mathfrak{V}$ has stable trace field contained in E.

Fix a prime $\mathfrak{p}'$ of $\mathcal{O}'$ lying over a prime $\mathfrak{p}$ of $\mathcal{O}$, let the trace field of $\mathscr{M}_\mathfrak{p}$ be $F \supset E$, and consider the embeddings $\mathcal{S} := \{\sigma\colon F \hookrightarrow \overline{\mathbb{Q}}_\ell\}$ that fix E elementwise. Then, for each $\sigma \in \mathcal{S}$, the companion ${}^\sigma(\mathscr{M}_\mathfrak{p})$ has second symmetric square isomorphic to L, as the operation of companions commutes with tensorial constructions [KP22, Proof of Lemma 2.5]. Hence for each $\sigma \in \mathcal{S}$, there exists a 2-torsion rank 1 local system R on $\mathfrak{U}_\mathfrak{p}$ such that ${}^\sigma(\mathscr{M}_\mathfrak{p}) \cong \mathscr{M}_\mathfrak{p} \otimes R$. Therefore we have ${}^\sigma(\mathscr{M}_\mathfrak{p})|_{\mathfrak{V}_{\mathfrak{p}'}} \cong \mathscr{M}_\mathfrak{p}|_{\mathfrak{V}_{\mathfrak{p}'}} \otimes \psi$, with ψ being a 2-torsion rank 1 local system of the base $\mathrm{Spec}(\mathcal{O}'/\mathfrak{p}')$.

Therefore, all of the local systems ${}^\sigma(\mathscr{M}_\mathfrak{p})|_{\mathfrak{V}_{\mathfrak{p}'}}$ become isomorphic after a quadratic extension of the base $\mathcal{O}'/\mathfrak{p}'$, which implies that the stable trace field of $\mathscr{M}|_\mathfrak{V}$ is contained in E. The upshot is that, replacing U by V, we may assume that the stable trace field of M is bounded.

(5) The boundedness of the stable trace field of M implies, using the argument of [KYZ24] (using the moduli space $\mathcal{H}$, not the more refined moduli spaces $\mathcal{H}_i$ or $\mathcal{H}_\infty$ of *loc. cit.*) that there exists a principally polarized abelian scheme $\bar{f}\colon A_{U_{\bar{K}}} \to U_{\bar{K}}$ such that $R^1\bar{f}_*\overline{\mathbb{Q}}_\ell$ has $M|_{U_{\bar{K}}}$ as a summand.[1] As the abelian scheme $A_{U_{\bar{K}}} \to U_{\bar{K}}$ is rigid (see Section 4 of *loc. cit.*), it follows from [Fal83, Theorem 2] that the summand $M|_{U_{\bar{K}}}$ is cut out by an element of $End(A_{U_{\bar{K}}}) \otimes \overline{\mathbb{Q}}_\ell$. Therefore, after replacing K by a finite extension (using that the endomorphism ring of an abelian scheme is a finitely free $\mathbb{Z}$-module), it follows that there exists an abelian scheme $f : A \to U$ such that the cohomology has a rank 2 factor M' with the following property: $M_{U_{\bar{K}}} \cong M'_{U_{\bar{K}}}$. It now suffices to show that there exists a finite étale cover $V \to U$ such that $M|_V \simeq M'|_V(1/2)$ for some choice of $\overline{\mathbb{Q}}_\ell(1/2)$: indeed, this implies that $L|_V \simeq (\mathrm{Sym}^2 M)|_V \simeq \mathrm{Sym}^2(M'|_V)(1)$, and by construction the latter appears in $R^2 f_*\overline{\mathbb{Q}}_\ell(1)$.

(6) By Lemma 3, M' has determinant equal to $\overline{\mathbb{Q}}_\ell(-1)$ up to a finite order character, and we conclude that

$$M \simeq M'(1/2) \otimes \chi$$

for any choice of $\overline{\mathbb{Q}}_\ell(1/2)$, and χ some finite order rank one local system on U. Passing to the finite cover $V \to U$ which trivializes χ gives $M|_V \simeq M'|_V$ as desired.

□

Proof of Lemma 3. Since the conclusion of our lemma is insensitive to passing to finite covers of U, we may assume that Q has maximal unipotent degeneration around some cusp and moreover the map $f\colon Y \to U$ has strict semistable reduction.

By Cebotarev density, it suffices to prove the claim after reduction mod $\mathfrak{p}$ for almost all primes [2]. Let ∞ be a cusp of $U_\mathfrak{p}$, around which Q is totally degenerating, with uniformizing parameter z_∞. It suffices to address the formal local picture around $k((z_\infty))$, where k is the field of definition of ∞. More precisely, if V is the $G_{k((z_\infty))}$-representation corresponding to $Q|_{\mathrm{Spec}(k((z_\infty)))}$, we will show that $\det V$ is isomorphic to $\overline{\mathbb{Q}}_\ell(-r(r-1)/2)$, up to a finite order character.

[1]This argument is also indicated in Remark 1.9 of *loc. cit.* Here is a sketch. After inverting finitely many primes, $\mathcal{H}/\mathcal{O}$ of Definition 3.1 of *loc. cit.* is finite flat by Section 4. Moreover, $\mathcal{H}$ and has mod p points for infinitely many p by Section 2; hence has a characteristic 0 point. Then the prime-to-p specialization isomorphism of π_1 implies the desired result. See also Footnote 5 of *loc. cit.*

[2]using also the fact that, for an ℓ-adic local field F and a number field K, an almost everywhere unramified character $\lambda : G_K \to \mathcal{O}_F^\times$ is of finite order iff the Frobenius elements at almost all primes have finite (though a priori unbounded) orders

The maximal unipotent degeneration assumption implies the monodromy filtration Fil_i^N (see e.g. [Sch12, Definition/Proposition 9.2] for a definition) has associated graded satisfying

$$\dim_{\overline{\mathbb{Q}}_\ell} \mathrm{gr}^N_{-(r-1)+2k} = 1, \dim_{\overline{\mathbb{Q}}_\ell} \mathrm{gr}^N_{-(r-1)+2k+1} = 0$$

for $k = 0, \cdots, r-1$; furthermore, the monodromy operator

$$N : \mathrm{gr}^N_{-(r-1)+2k+2} \to \mathrm{gr}^N_{-(r-1)+2k} \tag{1}$$

is a G_k-equivariant isomorphism.

Moreover $Q^I = \mathrm{Fil}^N_{-(r-1)}$ is one dimensional, where $I \subset G_{k((z_\infty))}$ denotes the inertia subgroup. Let $\Phi \in G_k$ denote a geometric Frobenius element. The ℓ-adic Steenbrink spectral sequence of Rapoport-Zink ([RZ82, Satz 2.10]) implies that Φ acts on Q^I via an algebraic integer; on the other hand, the weight-monodromy conjecture, in this setting a theorem of Deligne [Del80, Corollaire 1.8.5], implies that it is pure of weight 0, and we deduce that Φ acts on Q^I by a root of unity ζ. By Equation (1) and the commutation relation $N\Phi = q\Phi N$, the Φ-action on $\mathrm{gr}^N_{-(r-1)+2k}$ is given by $q^k\zeta$, and hence the Φ-action on $\det V$ is given by $\zeta^r q^{1+\cdots+(r-1)} = \zeta^r q^{\frac{(r-1)r}{2}}$, as desired. □

Remark 5. The hypothesis of Theorem A may be weakend to the following: the Frobenius trace field of L is bounded and there exists a prime $\mathfrak{p}$ of $\mathcal{O}$ such that the Frobenius trace field of $\mathscr{L}_\mathfrak{p}$ is totally real. Indeed, the hypothesis will imply that the geometric monodromy of L lands in $\mathrm{SO}(3, \overline{\mathbb{Q}}_\ell) \subset \mathrm{SL}(3, \overline{\mathbb{Q}}_\ell)$. Then we claim that for any other prime $\mathfrak{p}'$, the arithmetic monodromy of $\mathscr{L}_{\mathfrak{p}'}$ must also be orthogonal; indeed, this follows from the fact that the geometric monodromy group is a normal subgroup of the arithmetic monodromy group and that the group $\mathrm{SL}(3, \overline{\mathbb{Q}}_\ell)$ is almost simple.

Remark 6. Step (5) of the proof of Theorem A, and in particular Lemma 3 obviates the use of the more delicate moduli spaces $\mathcal{H}_k$ in [KYZ24, Proof of Theorem 1.4], c.f. Definition 3.5 and Lemma 3.6 of *loc. cit.*

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