THE 2D TODA LATTICE HIERARCHY FOR MULTIPLICATIVE FUNCTIONALS OF SCHUR MEASURES

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Abstract. We prove Fredholm determinants build out from generalizations of Schur measures, or equivalently, arbitrary multiplicative functionals of the original Schur measures are tau-functions of the 2D Toda lattice hierarchy. Our result apply to finite temperature Schur measures, and extends both the result of Okounkov in [12] and of Cafasso-Ruzza in [10] concerning the finite-temperature Plancherel measure. Our proof lies on the semi-infinite wedge formalism and the Boson-Fermion correspondance.

1. Introduction

The goal of this note is to build τ-functions for the 2D Toda lattice hierarchy out of natural generalizations of Schur measures. Schur measures were introduced by Okounkov in [12] and are probability measures on the set of Young diagrams parametrized by two countable sets of parameters $t = (t_1, t_2, \dots)$, $t' = (t'_1, t'_2, \dots) \subset \mathbb{C}$ giving rise to determinantal point processes on $\mathbb{Z} + 1/2$ via the map $\lambda \to \{\lambda_i - i + 1/2\}$ with a correltaion kernel $K_{t,t'}$. It is proved in [12], Theorem 3, that Fredholm determinants $\det(1 - K_{t,t'})_{l^2(n+1/2,n+3/2,\dots)}$ are τ-functions of the 2D Toda lattice hierarchy, where the parameters t and t' play the rôle of the time parameters.

In the case when $t = t' = (L, 0, 0, \dots)$, $L > 0$, the Schur measure is the poissonized Plancherel measure with parameter $L > 0$, giving rise to the determinantal point process with the discrete Bessel kernel, see e.g. [7], [3]. In [10], the authors derive integrable equations for Fredholm determinants of deformations of the Plancherel measure similar to the one obtained in [12], see equation (A.21) there and equation (1.11) in [10]. Such deformations include the case of the finite temperature Bessel kernel as an example, see [1] and [4]. Their proof lies on the fact that such Fredholm determinants may be seen as the expectation of a multiplicative functionnal under the original Plancherel measure, and then exploits the integrable structure of the discrete Bessel kernel and the related Riemann-Hilbert problem techniques.

In the present note, we perform the same deformation to the general Schur measures and prove that Fredholm determinants build out of these deformations are also τ-functions of the 2D Toda lattice hierarchy. Our result is both a generalisation of the ones of [10] and [12]. As in [10], we interpret these Fredholm determinants as expectations of multitplicative functionals under the original Schur measure, but our argument then differs from [10] since the kernel in consideration may not be integrable. So, instead of using Riemann-Hilbert problems techniques, we use the semi-infinite wedge formalism of Okounkov introduced in

[12] and describe our τ-functions in terms of matrix coefficients of operators on the semi-infinite Fock space.

Examples include what we call finite temperature deformations of Schur measures, following the construction from [1] and [4]. We plan to describe in more details other examples such as multi-critical Schur measures and their finite temperature version (see [2]) or applications to the homogeneous stochastic six-vertex model (see [5]) in a forthcoming paper.

We start below by stating our main Theorem, before describing the scheme of our argument in more details.

1.1. **Statement of the result.** Let $t = (t_1, t_2, \dots)$, $t' = (t'_1, t'_2, \dots) \subset \mathbb{C}$ be two infinite sequences of numbers satisfying

$$\sum_{n\geq 1} n|t_n||t'_n| < +\infty.$$

Set

$$Z_{t,t'} := \exp\left(\sum_{n\geq 1} n t_n t'_n\right).$$

Observe that we have

$$Z_{t,t'} = Z_{-t,-t'} = Z_{t',t}. \tag{1}$$

Define the functions of $z \in \mathbb{T}$:

$$\gamma(z,t) := \exp\left(\sum_{n\geq 1} t_n z^n\right),$$

$$J(z;t,t') := \frac{\gamma(z,t)}{\gamma(z^{-1},t')} = \exp\left(\sum_{n\geq 1} t_n z^n - t'_n z^{-n}\right).$$

Observer that we have

$$J(z;t',t) = J(z^{-1},-t,-t') \tag{2}$$

and

$$J(z;-t,-t') = J(z;t,t')^{-1} \tag{3}$$

Define the sequence $\mathbb{Z} \ni k \mapsto J_k(t,t')$ via the generating Laurent series

$$J(z;t,t') = \sum_{k\in\mathbb{Z}} J_k(t,t') z^k.$$

From equations (2) and (3), we have

$$J_{-k}(-t,-t') = J_k(t',t) \tag{4}$$

and

$$\sum_{k\in\mathbb{Z}} J_k(-t,-t') z^k = J(z;t;t')^{-1}. \tag{5}$$

Let $\mathbb{Z}' := \mathbb{Z} + 1/2$ be the set of half-integers and let $\sigma : \mathbb{Z}' \to [0,1]$ be a function such that

$$\sum_{k \in \mathbb{Z}' \cap (-\infty, 0)} \sigma(k) < +\infty. \tag{6}$$

We assume that $J(z; t, t')$ is analytic in a neighbourhood of the unit circle. We form the kernel

$$K_{t,t',\sigma}(x, y) = \sum_{k \in \mathbb{Z}'} \sigma(k) J_{x+k}(t, t') J_{-y-k}(-t, -t'). \tag{7}$$

From equations (4) and (5), the kernel $K_{t,t',\sigma}$ is invariant under the transformation $t \leftrightarrow t'$. Moreover, from the analyticity assumption of $J(z; t, t')$ and from the condition (6) for the function σ, we have

Proposition 1.1. *The restriction of $K_{t,t',\sigma}$ on $l^2\{n+1/2, n+3/2, \dots\}$ is trace class for any $n \in \mathbb{Z}$.*

The above proposition allows to consider, for $n \in \mathbb{Z}$, the Fredholm determinant

$$\tau_n(t, t'; \sigma) := Z_{t,t'} \det(1 - K_{t,t',\sigma})_{l^2\{n+1/2, n+3/2, \dots\}}. \tag{8}$$

When the function σ is the indicator function of the positive half integer $\mathbf{1}_{\mathbb{Z}_{>0}}$, we write simply $K_{t,t'} := K_{t,t',\mathbf{1}_{\mathbb{Z}_{>0}}}$. We recall in section 2 below that the kernel $K_{t,t'}$ is the correlation kernel of the determinantal point process given by the image of the Schur measure with parameters t and t'. As announced, the Fredholm determinant (8) is the expectation of a multiplicative functional for the latter Schur measure, see section 2 for definitions and notation:

Proposition 1.2. *Let $\mathbb{P}_{t,t'}$ be the Schur measure with parameters t, t'. Then, we have*

$$\tau_n(t, t'; \sigma) = Z_{t,t'} \mathbb{E}_{\mathbb{P}_{t,t'}} \left[\prod_{x \in \mathfrak{S}_0(\lambda)} (1 - \sigma(x - n)) \right].$$

Our main result is the following.

Theorem 1.3. *The functions $\tau_n(t, t'; \sigma)$ are τ-functions of the 2D Toda lattice hierarchy, in the sense that they satisfy bilinear Hirota equations:*

$$\begin{aligned} &[z^{l-m}]\gamma(z^{-1}, -2s')\tau_{m+1}(t + s, t' + s' + \{z\}; \sigma)\tau_l(t - s, t' - s' - \{z\}; \sigma) \\ &\qquad = [z^{m-l}]\gamma(z^{-1}, 2s)\tau_m(t + s - \{z\}, t + s'; \sigma)\tau_{l+1}(t - s + \{z\}, t - s'; \sigma), \end{aligned} \tag{9}$$

where $s = (s_1, s_2, \dots)$, $s' = (s'_1, s'_2, \dots) \subset \mathbb{C}$ are sequences satisfying

$$\sum_{n \geq 1} n |s_n| |s'_n| < +\infty,$$

and

$$\{z\} = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \dots \right),$$

and where $[z^k]F(z)$ is the coefficient in z^k in the Laurent series expansion of $F(z)$.

Observe that (9) is a partial differential equations in the parameters t and t', namely the one from [13], Theorem 1.11. Indeed, Taylor formula implies that, for a smooth function $f(x) = f(x_1, x_2, \dots)$, we have

$$f(x + \{z\}) = \sum_{n\geq 0} \mathrm{p}_n \left(\partial_{x_1}, \frac{\partial_{x_2}}{2}, \dots \right) (f) z^n,$$

where the polynomial $\mathrm{p}(a_1, a_2, \dots)$ is defined by

$$\exp\left(\sum_{k\geq 1} a_k z^k \right) = \sum_{n\geq 0} \mathrm{p}_n(a_1, a_2, \dots) z^n.$$

1.2. **Proof's strategy and organisation of the paper.** The proof of Theorem 1.3 first lies on the fact that the function $\tau_n(t, t'; \sigma)$ can be seen as the average of a multiplicative functional under a Schur measure, see Theorem 2.1 and Proposition 1.2. We recall the necessary material on Schur measures in section 2, where we also describe their finite temperature generalizations and see them as determinantal point processes with correlation kernel $K_{t,t',\sigma}$ for a particular choice of the function σ, see Theorem 2.2.

We further recall the semi-infinite wedge formalism of Okounkov in section 3, where we see in Propositions 3.1 and 3.2 how to generate solutions of the bilinear Hirota equations (9) from that theory.

We conclude the proof in section 4 by seeing that the functions $\tau_n(t, t'; \sigma)$ fit into this formalism, see Lemma 4.1.

2. On Schur measures and finite temperature Schur measures

2.1. **Schur measures.** A partition, or equivalently a Young diagram, is a non-increasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ of non-negative integers that is eventually zero. The length of a Young diagram is the index of its last non-zero entry and is denoted by $l(\lambda)$. The set of all Young diagrams is denoted by $\mathbb{Y}$. To a Young diagram and a an integer $n \in \mathbb{Z}$, we associate a subset of $\mathbb{Z}'$ by

$$\lambda \mapsto \mathfrak{S}_n(\lambda) := \{\lambda_i - i + 1/2 + n\}. \tag{10}$$

We let Λ be the algebra over $\mathbb{C}$ of symmetric functions. It is spanned by the Newton power sums defined by

$$p_k(\mathrm{x}_1, \mathrm{x}_2, \dots) = \sum_{i=1}^{+\infty} \mathrm{x}_i^k, \quad k = 1, 2, \dots$$

The Schur functions s_λ, $\lambda \in \mathbb{Y}$, are defined by the Jacobi-Trudi Formula

$$s_\lambda := \det(h_{\lambda_i - i + j})_{i,j=1}^N, \quad N \geq l(\lambda),$$

where h_k is the complete homogeneous function:

$$h_k(\mathrm{x}_1, \mathrm{x}_2, \dots) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} \mathrm{x}_{i_1} \cdots x_{i_k}, \quad k \in \mathbb{Z},$$

with the convention that $h_0 = 1$ and $h_k = 0$ for $k < 0$.

For $\lambda \in \mathbb{Y}$, we write $s_\lambda(t)$ for the image of the Schur function s_λ given by the algebra morphism from Λ to $\mathbb{C}$ defined by

$$p_k \mapsto \frac{1}{k} t_k, \quad k = 1, 2, \ldots$$

Definition 2.1. The Schur measure with parameters t, t' is the probability measure $\mathbb{P}_{t,t'}$ on $\mathbb{Y}$ given by

$$\mathbb{P}_{t,t'}(\lambda) := Z_{t,t'}^{-1} s_\lambda(t) s_\lambda(t').$$

2.2. **Finite temperature Schur measures.** The Schur measures may be generalized as follows (see Borodin, Betea-Bouttier). For Young diagrams $\mu, \lambda \in \mathbb{Y}$, we write $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for all $i = 1, 2 \ldots$, and denote by $\lambda \setminus \mu$ the sequence of non-negative integers $(\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots)$. The skew Schur function indexed by the skew diagram $\lambda \setminus \mu$ is defined by

$$s_{\lambda \setminus \mu} = \det(h_{\lambda_i - i - \mu_j + j})_{i,j=1}^N, \quad N \geq l(\lambda).$$

Definition 2.2. For $u \in [0, 1]$, and sets of parameters $t, t' \subset \mathbb{C}$, the *finite temperature Schur measure* $\mathbb{P}_{u,t,t'}$ with parameters u, t, t' is the probability measure on $\mathbb{Y}$ defined as

$$\mathbb{P}_{u,t,t'}(\lambda) = Z_{u,t,t'}^{-1} \sum_{\mu \subset \lambda} u^{|\mu|} s_{\lambda/\mu}(t) s_{\lambda \setminus \mu}(t) s_{\lambda \setminus \mu}(t'), \quad \lambda \in \mathbb{Y}.,$$

where $Z_{u,t,t'}$ is the normalizing constant.

Observe that for $u = 0$, one recovers the definition of Schur measures.

2.3. **Schur measures and finite temperature Schur measures as determinantal point processes.** One of the main results of Okounkov in [12] is the following Theorem.

Theorem 2.1. *For any finite set $X = \{x_1, \ldots, x_m\} \subset \mathbb{Z}'$, we have*

$$\mathbb{P}_{t,t'}(X \subset \mathfrak{S}_0(\lambda)) = \det\left(K_{t,t',\mathbb{1}_{\mathbb{Z}'_{>0}}}(x_i, x_j)\right)_{i,j=1}^m,$$

where the correlation kernel $K_{t,t',\mathbb{1}_{\mathbb{Z}'_{>0}}}$ is given by (7):

$$K_{t,t',\mathbb{1}_{\mathbb{Z}'_{>0}}}(x, y) = \sum_{k \in \mathbb{Z}',\ k > 0} J_{x+k}(t, t') J_{-y-k}(-t, -t').$$

The above result has been generalized by Borodin in [4] (see also [1]) to finite temperature Schur measures as follows.

Theorem 2.2. *Let $u \in [0, 1)$ and consider the parameters*

$$t = \frac{1}{1-u}(\tilde{t}_1, \tilde{t}_2, \ldots), \quad t' = \frac{1}{1-u}(\tilde{t}'_1, \tilde{t}'_2, \ldots),$$

for fixed $\tilde{t}_k, \tilde{t}'_k$, $k = 1, 2, \ldots$. Let $\mathbb{P}_{u,\tilde{t},\tilde{t}'}$ be the finite temperature Schur measure with parameters $u, \tilde{t}, \tilde{t}'$. Let c be a $\mathbb{Z}$-valued random variable defined on some probability space (Ω, P)

such that

$$P(c=n)=\frac{u^{\frac{n^2}{2}}}{\theta_3(1,u)},$$

where

$$\theta_3(1,u)=\sum_{n\in\mathbb{Z}}u^{\frac{n^2}{2}}.$$

Then we have for any finite set $X=\{x_1,\dots x_m\}\subset\mathbb{Z}'$,

$$\mathbb{P}_{u,\tilde{t},\tilde{t}'}\otimes P(X\subset\mathfrak{S}_c(\lambda))=\det\left(K_{t,t',\sigma_u}(x_i,x_j)\right)_{i,j=1}^m,$$

where K_{t,t',σ_u} *is given by (7) with*

$$\sigma_u(k)=\frac{1}{1-u^k},\quad k\in\mathbb{Z}'.$$

3. Semi-infinite wedge formalism

3.1. **Definition.** Let $\Lambda(\mathbb{Z}')$ be the set of all subsets $S\subset\mathbb{Z}'$ such that $|S^+|:=|\mathbb{Z}'_{>0}\cap S|<+\infty$ and $|S^-|:=|\mathbb{Z}'_{<0}\setminus(S\cap\mathbb{Z}'_{<0})\,|<+\infty$. The fermionic Fock space is by definition the Hilbert space freely spanned by $\Lambda(\mathbb{Z}')$ For $S\in\Lambda(\mathbb{Z}')$, we denote by v_S the vector indexed by S and write

$$v_S:=\underline{s_1}\wedge\underline{s_2}\wedge...$$

where $S=\{s_1>s_2>...\}$. The Hilbert space structure on the fermionic Fock space $\Lambda^{\frac{\infty}{2}}(\mathbb{Z}')$ comes from the inner product $\langle .,.\rangle$ such that v_S, $S\in\Lambda(\mathbb{Z}')$ is an orthonormal basis.

For an integer $n\in\mathbb{Z}$, let $\Lambda_n(\mathbb{Z}')$ be the set of all $S\in\Lambda(\mathbb{Z}')$ such that $|S^+|-|S^-|=n$. We have

$$\Lambda(\mathbb{Z}')=\sqcup_{n\in\mathbb{Z}}\Lambda_n(\mathbb{Z}'),$$

and the canonical inclusion $\iota:\Lambda(\mathbb{Z}')\hookrightarrow\Lambda^{\frac{\infty}{2}}V$ gives rise to a direct sum decomposition $\Lambda^{\frac{\infty}{2}}V$, called the charge decomposition:

$$\Lambda^{\frac{\infty}{2}}V=\oplus_{n\in\mathbb{Z}}\Lambda_n^{\frac{\infty}{2}}V$$

where $\Lambda_n^{\frac{\infty}{2}}V=\iota(\Lambda_n(\mathbb{Z}'))$.

Recall that, for each $n\in\mathbb{Z}$, one embeds $\mathbb{Y}$ into $\Lambda_n(\mathbb{Z}')$ by the map

$$\lambda\mapsto\mathfrak{S}_n(\lambda):=\{\lambda_i-i+1/2+n,\ i=1,2,\dots\}.$$

We write $v_\lambda:=v_{\mathfrak{S}_0(\lambda)}$. We also use the following notation: for $n\in\mathbb{Z}$, set

$$v_n:=\underline{n-1/2}\wedge\underline{n-3/2}\wedge...\in\Lambda_n^{\frac{\infty}{2}}V.$$

Observe that $v_0=v_\emptyset$, where $\emptyset$ is the empty Young diagram.

3.2. **Operators on $\Lambda^{\frac{\infty}{2}}V$.**

Definition 3.1. For $k \in \mathbb{Z}'$, define the creation operator ψ_k as being the operator of exterior multiplication by $\underline{k}$: if $S = \{s_1 > s_2 > ...\}$,

$$\psi_k v_S = \underline{k} \wedge \underline{s_1} \wedge \underline{s_2} \wedge ... = \begin{cases} 0 & \text{if } k \in S \\ (-1)^i \underline{s_1} \wedge ... \wedge \underline{s_i} \wedge \underline{k} \wedge \underline{s_{i+1}} \wedge ... & \text{if } s_i > k > s_{i+1}. \end{cases}$$

For $k \in \mathbb{Z}'$, define the anihilation operator ψ_k^* by:

$$\psi_k^* v_S = \begin{cases} 0 & \text{if } k \notin S \\ (-1)^{i-1} \underline{s_1} \wedge ... \wedge \underline{s_{i-1}} \wedge \underline{s_{i+1}} \wedge ... & \text{if } k = s_i \end{cases}$$

The operator ψ_k^* is the adjoint operator of ψ_k.

We have

$$\psi_k \psi_k^* v_S = \begin{cases} v_S & \text{if } k \in S \\ 0 & \text{if } k \notin S \end{cases}$$

and

$$\psi_k^* \psi_k v_S = \begin{cases} v_S & \text{if } k \notin S \\ 0 & \text{if } k \in S \end{cases}$$

We have the following anti-commutation relations:

$$\psi_k \psi_l^* + \psi_l^* \psi_k = \delta_{kl}, \quad \psi_k \psi_l + \psi_l \psi_k = 0, \quad \psi_k^* \psi_l^* + \psi_l^* \psi_k^* = 0.$$

The charge operator C can be defined by

$$C := \sum_{k \in \mathbb{Z}', k > 0} \psi_k \psi_k^* - \psi_{-k}^* \psi_{-k},$$

and we have

$$\ker(C - nI) = \Lambda_n^{\frac{\infty}{2}} V.$$

The shift operator is defined by

$$R v_S = \underline{s_1 + 1} \wedge \underline{s_2 + 1} \wedge \dots, \quad S = \{s_1 > s_2 > \dots\} \in \Lambda(\mathbb{Z}').$$

We introduce the bosonic operators on $\Lambda^{\frac{\infty}{2}}V$ as follows: for $n \in \mathbb{Z} \setminus \{0\}$, we define the operator α_n by:

$$\alpha_n := \sum_{k \in \mathbb{Z}'} \psi_{k-n} \psi_k^*$$

We introduce the generating series

$$\psi(z) := \sum_{i \in \mathbb{Z}'} \psi_i z^i, \quad \psi^*(w) := \sum_{j \in \mathbb{Z}'} \psi_j^* w^{-j}.$$

We have the following commutation relations

$$[\alpha_n, \alpha_m] = n\delta_{n,-m}, \quad [\alpha_n, \psi(z)] = z^n \psi(z), \quad [\alpha_n, \psi^*(w)] = -w^n \psi^*(w).$$

The vertex operators $\Gamma_\pm(t)$ are defined by

$$\Gamma_\pm(t) = \exp\left(\sum_{n\geq 1} t_n \alpha_{\pm n}\right).$$

They satisfy

$$\Gamma_+(t) v_m = v_m \tag{11}$$

and the commutation relations

$$\Gamma_+(t)\Gamma_-(t') = Z_{t,t'}\Gamma_-(t')\Gamma_+(t) \tag{12}$$

$$\Gamma_\pm(t)\psi(z) = \gamma(z^{\pm 1}, t)\psi(z)\Gamma_\pm(t) \tag{13}$$

$$\Gamma_\pm(t)\psi^*(z) = \gamma(z^{\pm 1}, t)^{-1}\psi^*(z)\Gamma_\pm(t). \tag{14}$$

We have the Boson-Fermion correspondance allowing to recover the operators ψ_k from the operators α_n:

$$\psi(z) = z^C R\Gamma_-(\{z\})\Gamma_+(-\{z^{-1}\}), \quad \psi^*(z) = R^{-1}z^{-C}\Gamma_-(-\{z\})\Gamma_+(\{z^{-1}\}). \tag{15}$$

3.3. **Intermediate results.** We recall an intermediate result from Okounkov on Schur measures, that is used to prove that the Schur measures are determinantal point processes and expressing the correlation both in terms of the fermionic operators ψ_k.

Proposition 3.1. *Let $\mathbb{P}_{t,t'}$ be the Schur measure with parameters t, t'. Then we have for any finite set $X = \{x_1, \ldots x_m\} \subset \mathbb{Z}'$*

$$\mathbb{P}_{t,t'}(X \subset \mathfrak{S}_0(\lambda)) = Z_{t,t}^{-1}\langle \Gamma_+(t)\psi_{x_1}\psi^*_{x_1}\cdots\psi_{x_m}\psi^*_{x_m}\Gamma_-(t')v_\emptyset, v_\emptyset\rangle.$$

We also recall the following, that is the stepping stone in the proof of our main result. Its proof lies on the Boson-Fermion correspondance.

Proposition 3.2. *Let A be an operator on $\Lambda^{\frac{\infty}{2}}$ such that $A\otimes A$ commutes with the operator*

$$\Psi := \sum_{k\in\mathbb{Z}} \psi_k \otimes \psi_k^*. \tag{16}$$

Set

$$\tilde{A} := \Gamma_+(t)A\Gamma_-(t').$$

Then, the sequence formed by the functions

$$\tilde{\tau}_n(t, t'; A) := \langle \tilde{A}v_n, v_n\rangle$$

are τ-functions for the Toda lattice hierarchy, in the sense that they satisfy the bilinear Hirota equations

$$\begin{aligned}[z^{l-m}]\gamma(z^{-1}, -2s')\tilde{\tau}_{m+1}(t+s, t'+s'+\{z\}; A)\tilde{\tau}_l(t-s, t'-s'-\{z\}; A)\\ = [z^{m-l}]\gamma(z^{-1}, 2s)\tilde{\tau}_m(t+s-\{z\}, t+s'; A)\tilde{\tau}_{l+1}(t-s+\{z\}, t-s'; A),\end{aligned} \tag{17}$$

Proof. We give a proof for completeness. Since the operator commutes with $A\otimes A$ commutes with Ψ, so does the operator $\tilde{A}$. This relation translates into

$$[z^0]\left(\tilde{A}\otimes\tilde{A}\right)(\psi(z)\otimes\psi^*(z)) = [z^0](\psi(z)\otimes\psi^*(z))\left(\tilde{A}\otimes\tilde{A}\right),$$

i.e.

(18)
$$[z^0]\Gamma_+(t)A\Gamma_-(t')\psi(z)\otimes\Gamma_+(t)A\Gamma_-(t')\psi^*(z) = [z^0]\psi(z)\Gamma_+(t)A\Gamma_-(t')\otimes\psi^*(z)\Gamma_+(t)A\Gamma_-(t').$$

Relation (17) now follows by taking the matrix coefficient of eq. (18) for the vector

$$\Gamma_-(s')v_m\otimes\Gamma_-(-s')v_{l+1}$$

against

$$\Gamma_-(s)v_{m+1}\otimes\Gamma_-(-s)v_l.$$

Indeed, from the Boson-Fermion correspondance, we have for $m\in\mathbb{Z}$ and a sequence s'

$$\begin{aligned}\Gamma_+(t)A\Gamma_-(t')\psi(z)\Gamma_-(s')v_m &= \Gamma_+(t)A\Gamma_-(t')z^CR\Gamma_-(\{z\})\Gamma_+(-\{z^{-1}\})\Gamma_-(s')v_m\\ &= z^{m+1}\gamma(z^{-1},-s')\Gamma_+(t)A\Gamma_-(t'+\{z\}+s')v_{m+1}.\end{aligned}$$

Second equality above is obtained as follows: from (12), we have

$$\Gamma_+(-\{z^{-1}\})\Gamma_-(s') = \gamma(z^{-1},-s')\Gamma_-(s')\Gamma_+(-\{z^{-1}\}).$$

Then, we use (11), the fact that R commutes with the vertex operators and changes v_m to v_{m+1}, and finally that the vertex operators preserve the charge, producing the factor z^{m+1}.

Similarly, we obtain for $l\in\mathbb{Z}$:

$$\Gamma_+(t)A\Gamma_-(t')\psi^*(z)\Gamma_-(-s')v_{l+1} = z^{-l-1}\gamma(z^{-1},-s')\Gamma_+(t)A\Gamma_-(t'-s'-\{z\})v_l.$$

We deduce that

$$\begin{aligned}&[z^0]\left(\tilde{A}\psi(z)\otimes\tilde{A}\psi^*(z)\right)(\Gamma_-(s')v_m\otimes\Gamma_-(-s')v_{l+1})\\ &\qquad= [z^{l-m}]\gamma(z^{-1},-2s')\Gamma_+(t)A\Gamma_-(t'+\{z\}+s')v_{m+1}\otimes\Gamma_+(t)A\Gamma_-(t'-s'-\{z\})v_l,\end{aligned}$$

and thus, using that Γ_+ is the adjoint of Γ_-,

$$\begin{aligned}&[z^0]\left\langle\left(\tilde{A}\psi(z)\otimes\tilde{A}\psi^*(z)\right)(\Gamma_-(s')v_m\otimes\Gamma_-(-s')v_{l+1}),\Gamma_-(s)v_{m+1}\otimes\Gamma_-(-s)v_l\right\rangle\\ &\qquad= [z^{l-m}]\gamma(z^{-1},-2s')\tilde{\tau}_{m+1}(t+s,t'+s'+\{z\};A)\tilde{\tau}_l(t-s,t'-s'-\{z\};A).\end{aligned}$$

Equality

$$\begin{aligned}(19)\quad &[z^0]\left\langle\left(\psi(z)\tilde{A}\otimes\psi^*(z)\tilde{A}\right)(\Gamma_-(s')v_m\otimes\Gamma_-(-s')v_{l+1}),\Gamma_-(s)v_{m+1}\otimes\Gamma_-(-s)v_l\right\rangle\\ &\qquad= [z^{m-l}]\gamma(z^{-1},2s)\tilde{\tau}_m(t+s-\{z\},t+s';A)\tilde{\tau}_{l+1}(t-s+\{z\},t-s';A)\end{aligned}$$

is proved in a similar way, using that z^C is self-adjoint and that R is unitary. ∎

4. Proof of Theorem 1.3

The proof of Theorem 1.3 goes now as follows: we use Proposition 1.2 to then deduce in Lemma 4.1 that the τ-functions $\tau_n(t,t';\sigma)$ fit into the setting of Proposition 3.2. Let us prove first Proposition 1.2.

Proof of Proposition 1.2. Denote by $H_{t,t'}$ and $\check{H}_{t,t'}$ the Hankel operators

$$H_{t,t'}f(x) = \sum_{k\in\mathbb{Z}} J_{k+x}(t,t')f(k),$$
$$\check{H}_{t,t'}f(x) = \sum_{k\in\mathbb{Z}} J_{-k-x}(-t,-t')f(k).$$

Observe that we have $K_{t,t',\sigma}\sigma = H_{t,t'}\sigma\check{H}_{t,t'}$. Making use of the identity

$$\det(1+AB) = \det(1+BA),$$

we have

$$\begin{aligned}\det\left(1-K_\sigma\right)_{l^2\{n+1/2,n+3/2,\dots\}} &= \det\left(1-\mathbf{1}_{\{n+1/2,n+3/2,\dots\}}H_{t,t'}\sigma\check{H}_{t,t'}\mathbf{1}_{\{n+1/2,n+3/2,\dots\}}\right)_{l^2(\mathbb{Z})}\\ &= \det\left(1-\sqrt{\sigma}\check{H}_{t,t'}\mathbf{1}_{n+1/2,n+3/2,\dots}H_{t,t'}\sqrt{\sigma}\right)_{l^2(\mathbb{Z})}\\ &= \det\left(1-\sqrt{\sigma(\cdot-n)}\check{H}_{t,t'}\mathbf{1}_{\mathbb{Z}'_{>0}}H_{t,t'}\sqrt{\sigma(\cdot-n)}\right)_{l^2(\mathbb{Z})}\\ &= \det\left(1-\sigma(\cdot-n)\check{H}_{t,t'}\mathbf{1}_{\mathbb{Z}'_{>0}}H_{t,t'}\right)_{l^2(\mathbb{Z})}.\end{aligned}$$

The proof now follows from the fact that $\mathbb{P}_{t,t'}$ is a determinantal point process with kernel

$$K_{t,t',\mathbf{1}_{\mathbb{Z}'_{>0}}} = \check{H}_{t,t'}\mathbf{1}_{\mathbb{Z}_{>0}}H_{t,t'},$$

see Theorem 2.1 and equation (4). $\square$

Lemma 4.1. *Consider the operator*

$$A_\sigma := \prod_{k\in\mathbb{Z}}(1-\sigma(k))\psi_k\psi_k^*.$$

Then we have

$$\langle\Gamma_+(-t)A_\sigma\Gamma_-(-t')v_n, v_n\rangle = \tau_n(t,t';\sigma).$$

Moreover, the operator $A\otimes A$ commutes with the operator Ψ, defined in (16).

Proof. The fact that $A\otimes A$ commutes with Ψ is easy. For the first point, we expand

(20) $$\langle\Gamma_+(-t)A_\sigma\Gamma_-(-t')v_n, v_n\rangle = \langle\Gamma_+(-t)\prod_{k\in\mathbb{Z}}(1-\sigma(k-n))\psi_{k-n}\psi^*_{k-n}\Gamma_-(-t')v_\emptyset, v_\emptyset\rangle$$

$$= \sum_{m\geq 0}(-1)^m\sum_{\{x_1,\dots,x_m\}\subset\mathbb{Z}}\sigma(x_1-n)\cdots\sigma(x_m-n)\langle\Gamma_+(-t)\psi_{x_1}\psi^*_{x_1}\cdots\psi_{x_m}\psi^*_{x_m}\Gamma_-(-t')v_\emptyset, v_\emptyset\rangle.$$

From Proposition 3.1 and Theorem 2.1, we have

$$\langle\Gamma_+(-t)\psi_{x_1}\psi^*_{x_1}\cdots\psi_{x_m}\psi^*_{x_m}\Gamma_-(-t')v_\emptyset, v_\emptyset\rangle = Z_{t,t'}\det\left(K_{t,t',\mathbf{1}_{\mathbb{Z}_{\geq 0}}}(x_i,x_j)\right)_{i,j=1}^m.$$

Thus, one recognizes on the last line of equation (20) the Fredholm determinant expansion of

$$Z_{t,t'}\det\left(1-\sigma(\cdot-n)K_{t,t',\mathbf{1}_{\mathbb{Z}_{\geq 0}}}\right)_{l^2(\mathbb{Z})} = Z_{t,t'}\mathbb{E}_{\mathbb{P}_{-t',-t}}\left[\prod_{x\in X}(1-\sigma(x-n))\right].$$

We conclude by Lemma 1.2. $\square$

We now conclude the proof of Theorem 1.3 by applying Proposition 3.2 to $A = A_\sigma$.

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