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An inexact infeasible arc-search interior-point method for linear programming problems

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Abstract

Inexact interior-point methods (IPMs) are a type of interior-point methods that inexactly solve the linear equation system for obtaining the search direction. On the other hand, arc-search IPMs approximate the central path with an ellipsoidal arc obtained by solving two linear equation systems in each iteration, while conventional line-search IPMs solve one linear system, therefore, the improvement due to the inexact solutions of the linear equation systems can be more beneficial in arc-search IPMs than conventional IPMs. In this paper, we propose an inexact infeasible arc-search interior-point method. We establish that the proposed method is a polynomial-time algorithm through its convergence analysis. The numerical experiments with the conjugate gradient method show that the proposed method can reduce the number of iterations compared to an existing method for benchmark problems; the numbers of iterations are reduced to two-thirds for more than 70% of the problems.

Keywords: interior-point method, arc-search, inexact IPM, infeasible IPM, linear programming.

1 Introduction

Linear programming problems (LPs) have had an important role in both theoretical analysis and practical applications, and many methods have been studied for solving LPs efficiently. Since an interior-point method (IPM) was first proposed by Karmarkar [10], IPMs are extended to larger class of optimization problems, for example, second-order cone programming and semidefinite programming. Many variations of the IPM have been proposed, such as Mehrotra's predictor-corrector method [13].

Inexact IPMs are one of such variations and they inexactly solve a linear equation system (LES) for obtaining the search direction in each iteration. An inexact IPM was first proposed for solving a constrained system of equations by Bellavia [2] and it has been extended for LPs [14, 1]. The inexact IPMs have recently gained much attention due

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to their relevance to quantum computing. Quantum linear system algorithms (QLSAs) have the potential to solve LESs fast; their complexity has a better dependence on the size of variables and the number of constraints but a worse one on other parameters compared to that on classical computers [5]. Recently, inexact IPMs using the QLSA called quantum interior-point methods are proposed in [11, 21].

On the other hand, studies to reduce the number of iterations in IPMs have also contributed to improving the numerical performance. An arc-search IPM was originally proposed by Yang [23]. IPMs numerically trace a trajectory to an optimal solution called the central path. Standard IPMs find the next iterate on a straight line that approximates the central path by computing the search direction; such IPMs are called line-search IPMs in this paper. In contrast, arc-search IPMs employ an ellipsoidal arc for the approximation. Since the central path is generally a smooth curve, the ellipsoidal arc can approximate the central path better than the straight line, and a reduction in the number of iterations can be expected. Several studies [25, 28] found that the arc-search IPMs improve the iteration complexity from the line-search IPM in [20], and the numerical experiments in [24, 28] demonstrated that the number of iterations in solving LP is reduced compared to the existing methods.

Arc-search IPMs solve two LESs in each iteration for computing the search direction while line-search IPMs one LES, thus, the improvement due to solving LESs inexactly is expected to be more beneficial in arc-search IPMs than line-search IPMs. In fact, when the arc-search IPMs are extended to nonlinear programming problems [22] and convex optimization problems [27], the arc-search IPMs can reduce the computation time even if the computation of higher-order derivatives is omitted, i.e., the search direction is obtained inexactly.

In this paper, we propose a novel inexact infeasible arc-search interior-point method (II-arc-IPM) by integrating an inexact IPM and an arc-search IPM. We prove that the II-arc-IPM achieves less iteration complexity than the inexact infeasible line-search IPMs (II-line-IPMs) [14, 15]. Furthermore, the numerical experiments with the conjugate gradient (CG) method as an inexact linear equation solver show that the II-arc-IPM can reduce the number of iterations by a factor of 1.5 compared to II-line-IPM for 70% benchmark problems from Netlib [4].

This paper is organized as follows. Section 2 introduces the standard form of LP problems and the formulas necessary for II-arc-IPM. In Section 3, we describe the proposed method, and in Section 4, we discuss the convergence and the polynomial iteration complexity. Section 5 provides the results of the numerical experiments and the discussion. Finally, Section 6 gives conclusions of this paper and discusses future directions.

1.1 Notations

We use x_i to denote the i-th element of a vector x. The Hadamard product of two vectors u and v is defined by $u \circ v$. The vector of all ones and the identity matrix are denoted by e and I, respectively. We use the capital character $X \in \mathbb{R}^{n\times n}$ as the diagonal matrix whose diagonal elements are taken from the vector $x \in \mathbb{R}^n$. For a set B, we denote the cardinality of the set by $|B|$. Given a matrix $A \in \mathbb{R}^{m\times n}$ and a set

$B \subseteq \{1, \ldots, n\}$, the matrix A_B is the submatrix consisting of the columns $\{A_i : i \in B\}$. Similarly, given a vector $v \in \mathbb{R}^n$ and a set $B \subseteq \{1, \ldots, n\}$ where $|B| = m \leq n$, the matrix $V_B \in \mathbb{R}^{m \times m}$ is the diagonal submatrix consisting of the elements $\{v_i : i \in B\}$. We use $\|x\|_2 = (\sum_i x_i^2)^{1/2}$, $\|x\|_\infty = \max_i |x_i|$ and $\|x\|_1 = \sum_i |x_i|$ for the Euclidean norm, the infinity norm and the ℓ_1 norm of a vector x, respectively. For simplicity, we denote $\|x\| = \|x\|_2$. For a matrix $A \in \mathbb{R}^{m \times n}$, $\|A\|$ denotes the operator norm associated with the Euclidian norm; $\|A\| = \max_{\|z\|=1} \|Az\|$.

2 Preliminaries

In this paper, we consider an LP in the standard form:

$$\min_{x \in \mathbb{R}^n} c^\top x, \quad \text{s.t. } Ax = b, \quad x \geq 0, \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$ with $m \leq n$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ are input data. The associated dual problem of (1) is

$$\max_{y \in \mathbb{R}^m, s \in \mathbb{R}^n} b^\top y, \quad \text{s.t. } A^\top y + s = c, \quad s \geq 0, \tag{2}$$

where y and s are the dual variable vector and the dual slack vector, respectively. Let $\mathcal{S}^*$ be the set of the optimal solutions of (1) and (2). When $(x^*, y^*, s^*) \in \mathcal{S}^*$, it is well-known that (x^*, y^*, s^*) satisfies the KKT conditions:

$$Ax^* = b \tag{3a}$$
$$A^\top y^* + s^* = c \tag{3b}$$
$$(x^*, s^*) \geq 0 \tag{3c}$$
$$x_i^* s_i^* = 0, \quad i = 1, \ldots, n. \tag{3d}$$

We denote the primal and dual residuals in (1) and (2) as

$$r_b(x) = Ax - b \tag{4a}$$
$$r_c(y, s) = A^\top y + s - c, \tag{4b}$$

and define the duality measure as

$$\mu = \frac{x^\top s}{n}. \tag{5}$$

Letting $\zeta \geq 0$, we define the set of ζ-optimal solutions as

$$\mathcal{S}_\zeta^* = \left\{(x, y, s) \in \mathbb{R}^{2n+m} \mid (x, s) \geq 0,\, \mu \leq \zeta,\, \|(r_b(x), r_c(y, s))\| \leq \zeta\right\}. \tag{6}$$

From the KKT conditions (3), we know $\mathcal{S}^* \subset \mathcal{S}_\zeta^*$.

In this paper, we make the following assumptions for the primal-dual pair (1) and (2). These assumptions are common ones in the context of IPMs and are used in many papers (for example, see [20, 26]).

Assumption 2.1. *There exists an interior feasible solution* $(\bar{x}, \bar{y}, \bar{s})$ *such that*

$$A\bar{x} = b,\ A^\top \bar{y} + \bar{s} = c, \text{ and } (\bar{x}, \bar{s}) > 0.$$

Assumption 2.2. *A is a full-row rank matrix, i.e.,* $\operatorname{rank}(A) = m$

Assumption 2.1 guarantees that the optimal set $\mathcal{S}^*$ is nonempty and bounded [20].

IPMs are iterative methods, so we denote the kth iteration by $(x^k, y^k, s^k) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ and the initial point by (x^0, y^0, s^0). Without loss of generality, we assume that the initial point (x^0, y^0, s^0) is bounded. We denote the duality measure of kth iteration as $\mu_k = (x^k)^\top s^k / n$.

Given a strictly positive iteration (x^k, y^k, s^k) such that $(x^k, s^k) > 0$, the strategy of an infeasible IPM is to trace a smooth curve called an approximate central path:

$$\mathcal{C} = \{(x(t), y(t), s(t)) \mid t \in (0, 1]\}, \tag{7}$$

where $(x(t), y(t), s(t))$ is the unique solution of the following system

$$\begin{aligned} Ax(t) - b &= t\ r_b(x^k), && \text{(8a)}\\ A^\top y(t) + s(t) - c &= t\ r_c(y^k, s^k), && \text{(8b)}\\ x(t) \circ s(t) &= t(x^k \circ s^k), && \text{(8c)}\\ (x(t), s(t)) &> 0. && \text{(8d)} \end{aligned}$$

As $t \to 0$, $(x(t), y(t), s(t))$ converges to an optimal solution $(x^*, y^*, s^*) \in \mathcal{S}^*$.

Arc-search IPMs approximate $\mathcal{C}$ with an ellipsoidal arc. An ellipsoidal approximation of $(x(t), y(t), s(t))$ at (x^k, y^k, s^k) for an angle $\alpha \in [0, \pi/2]$ is obtained by $(x(\alpha), y(\alpha), s(\alpha))$ with the following [26, Theorem 5.1]:

$$\begin{aligned} x(\alpha) &= x - \dot{x}\sin(\alpha) + \ddot{x}(1 - \cos(\alpha)), && \text{(9a)}\\ y(\alpha) &= y - \dot{y}\sin(\alpha) + \ddot{y}(1 - \cos(\alpha)), && \text{(9b)}\\ s(\alpha) &= s - \dot{s}\sin(\alpha) + \ddot{s}(1 - \cos(\alpha)). && \text{(9c)} \end{aligned}$$

Here, $(\dot{x}, \dot{y}, \dot{s})$ and $(\ddot{x}, \ddot{y}, \ddot{s})$ are the first and second derivatives obtained by differentiating both sides of (8) by t, and they are computed as the solutions of the following LESs, respectively:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} r_b(x^k) \\ r_c(y^k, s^k) \\ x^k \circ s^k \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2\dot{x} \circ \dot{s} \end{bmatrix}. \tag{11}$$

Lastly, we define a neighborhood of the approximate central path [20, Chapter 6]:

$$\mathcal{N}(\gamma_1, \gamma_2) := \left\{ (x, y, s) \mid \begin{array}{l} (x, s) > 0,\ x_i s_i \geq \gamma_1 \mu \text{ for } i \in \{1, \dots, n\}, \\ \|(r_b(x), r_c(y, s))\| \leq [\|(r_b(x^0), r_c(y^0, s^0))\| / \mu_0] \gamma_2 \mu \end{array} \right\}, \tag{12}$$

where $\gamma_1 \in (0,1)$ and $\gamma_2 \geq 1$ are given parameters, and $\|(r_b(x), r_c(y,s))\|$ is the norm of the vertical concatenation of $r_b(x)$ and $r_c(y,s)$. This neighborhood will be used in the convergence analysis.

3 The proposed method

In this section, we propose the II-arc-IPM. In the beginning, to guarantee the convergence of the II-arc-IPM, we introduce a perturbation into (10) as follows:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} r_b(x^k) \\ r_c(y^k, s^k) \\ x^k \circ s^k - \sigma\mu_k e \end{bmatrix}, \tag{13}$$

where $\sigma \in (0,1]$ is the constant called centering parameter. In the subsequent discussion, $(\dot{x}, \dot{y}, \dot{s})$ denote the solution of (13). The proposed II-arc-IPM solves (13) and (11) inexactly in each iteration to obtain the ellipsoidal approximation.

Several approaches can be considered for solving the Newton system (13), such as the full Newton system and the Newton equation system (NES) [3]. The NES formula of (13) is

$$M^k \dot{y} = \sigma_1^k, \tag{14}$$

where

$$\begin{aligned} M^k &= A(D^k)^2 A^\top, &\text{(15a)} \\ \sigma_1^k &= A(D^k)^2 r_c(y^k, s^k) + r_b(x^k) - A(S^k)^{-1}(x^k \circ s^k - \sigma\mu_k e) \\ &= A(D^k)^2 A^\top y^k - A(D^k)^2 c + \sigma\mu_k A(S^k)^{-1} e + Ax^k - b, &\text{(15b)} \end{aligned}$$

with $D^k = (X^k)^{\frac{1}{2}}(S^k)^{-\frac{1}{2}}$. When we solve the LES (14) exactly and obtain $\dot{y}$, we can compute the other components $\dot{x}$ and $\dot{s}$ of the solution in (13)

As discussed by Mohammadisiahroudi et al. [15], the iteration complexity of the II-line-IPM can be kept small by the modification to NES (14). This modified NES formula was examined for II-line-IPMs in [1, 16], it is called MNES. Since A is full row rank from Assumption 2.2, we can choose an arbitrary basis $\hat{B} \subset \{1, 2, \ldots, n\}$ where $|\hat{B}| = m$ and $A_{\hat{B}} \in \mathbb{R}^{m \times m}$ is nonsingular. Now we can adapt (14) to

$$\hat{M}^k \dot{z} = \hat{\sigma}_1^k, \tag{16}$$

where

$$\begin{aligned} \hat{M}^k &= (D_{\hat{B}}^k)^{-1} A_{\hat{B}}^{-1} M^k ((D_{\hat{B}}^k)^{-1} A_{\hat{B}}^{-1})^\top, &\text{(17a)} \\ \hat{\sigma}_1^k &= (D_{\hat{B}}^k)^{-1} A_{\hat{B}}^{-1} \sigma_1^k, &\text{(17b)} \end{aligned}$$

with $D_{\hat{B}}^k = (X_{\hat{B}}^k)^{\frac{1}{2}}(S_{\hat{B}}^k)^{-\frac{1}{2}}$. The inexact solution $\tilde{\dot{z}}$ of (16) satisfies

$$\hat{M}^k \tilde{\dot{z}} = \hat{\sigma}_1^k + \hat{r}_1^k, \tag{18}$$

where $\hat{r}_1^k$ is the error of $\tilde{\dot{z}}$ defined as

$$\hat{r}_1^k := \hat{M}^k \tilde{\dot{z}} - \hat{\sigma}_1^k = \hat{M}_k \left(\tilde{\dot{z}} - \dot{z} \right).$$

Then, we can obtain the first derivative $(\tilde{\dot{x}}, \tilde{\dot{y}}, \tilde{\dot{s}})$ from the inexact solution in (18) and the steps below:

$$\tilde{\dot{y}} = \left(\left(D_{\hat{B}}^k \right)^{-1} A_{\hat{B}}^{-1} \right)^\top \tilde{\dot{z}} \tag{19a}$$

$$\tilde{\dot{s}} = r_c(y^k, s^k) - A^T \tilde{\dot{y}} \tag{19b}$$

$$v_1^k = \left(v_{\hat{B}}^k, v_{\hat{N}}^k \right) = \left(D_{\hat{B}}^k \hat{r}_1^k, 0 \right) \tag{19c}$$

$$\tilde{\dot{x}} = x^k - (D^k)^2 \tilde{\dot{s}} - \sigma \mu_k (S^k)^{-1} e - v_1^k. \tag{19d}$$

We also apply the MNES formulation to the second derivative (11). Letting

$$\sigma_2^k = 2A(S^k)^{-1} \tilde{\dot{x}} \circ \tilde{\dot{s}}, \qquad \hat{\sigma}_2^k = (D_{\hat{B}}^k)^{-1} A_{\hat{B}}^{-1} \sigma_2^k,$$

we have

$$\hat{M}^k \ddot{z} = \hat{\sigma}_2^k \tag{20}$$

with the same definition of $\hat{M}^k$ as in (17a). We use $\tilde{\ddot{z}}$ to denote the inexact solution of (20), then we have

$$\hat{M}^k \tilde{\ddot{z}} = \hat{\sigma}_2^k + \hat{r}_2^k, \tag{21}$$

where $\hat{r}_2^k$ is defined as $\hat{r}_2^k := \hat{M}_k \left(\tilde{\ddot{z}} - \ddot{z} \right)$. Similarly to (19), to obtain the inexact second derivative $(\tilde{\ddot{x}}, \tilde{\ddot{y}}, \tilde{\ddot{s}})$ from the inexact solution $\tilde{\ddot{z}}$ in (21), we compute as follows:

$$\begin{aligned}
\tilde{\ddot{y}} &= \left(\left(D_{\hat{B}}^k \right)^{-1} A_{\hat{B}}^{-1} \right)^\top \tilde{\ddot{z}} \\
\tilde{\ddot{s}} &= -A^T \tilde{\ddot{y}} \\
v_2^k &= \left(v_{\hat{B}}^k, v_{\hat{N}}^k \right) = \left(D_{\hat{B}}^k \hat{r}_2^k, 0 \right) \\
\tilde{\ddot{x}} &= -(D^k)^2 \tilde{\ddot{s}} - 2(S^k)^{-1} \tilde{\dot{x}} \circ \tilde{\dot{s}} - v_2^k
\end{aligned}$$

Using the derivatives obtained above, the next iteration will be found on the ellipsoidal arc with the following updated formula:

$$x^k(\alpha) = x^k - \tilde{\dot{x}} \sin(\alpha) + \tilde{\ddot{x}}(1 - \cos(\alpha)), \tag{23a}$$

$$y^k(\alpha) = y^k - \tilde{\dot{y}} \sin(\alpha) + \tilde{\ddot{y}}(1 - \cos(\alpha)), \tag{23b}$$

$$s^k(\alpha) = s^k - \tilde{\dot{s}} \sin(\alpha) + \tilde{\ddot{s}}(1 - \cos(\alpha)). \tag{23c}$$

To give the framework of the proposed method, we prepare some functions below:

$$\begin{aligned}
G_i^k(\alpha) &= x_i^k(\alpha) s_i^k(\alpha) - \gamma_1 \mu_k(\alpha) \text{ for } i \in \{1, \ldots, n\}, \\
g^k(\alpha) &= x^k(\alpha)^\top s^k(\alpha) - (1 - \sin(\alpha))(x^k)^\top s^k, \\
h^k(\alpha) &= (1 - (1 - \beta)\sin(\alpha)) \, (x^k)^\top s^k - x^k(\alpha)^\top s^k(\alpha).
\end{aligned}$$

Here, $h^k(\alpha) \geq 0$ corresponds to the Armijo condition with respect to the duality gap μ. In Section 4, we will show that the proposed algorithm converges to an optimal solution by selecting a step size α that satisfies the following conditions:

$$G_i^k(\alpha) \geq 0 \text{ for } i \in \{1, \dots, n\}, \quad g^k(\alpha) \geq 0, \quad h^k(\alpha) \geq 0. \tag{24}$$

When (24) holds, the next lemma confirm that a next iteration point $(x^k(\alpha), y^k(\alpha), s^k(\alpha))$ is in the neighborhood $\mathcal{N}(\gamma_1, \gamma_2)$. This lemma can be proved in the same approach as Mohammadisiahroudi [15, Lemma 4.5] with Lemma 4.2 below.

Lemma 3.1. *Assume a step length $\alpha \in (0, \pi/2]$ satisfies $G_i^k(\alpha) \geq 0$ and $g^k(\alpha) \geq 0$. Then, $(x^k(\alpha), y^k(\alpha), s^k(\alpha)) \in \mathcal{N}(\gamma_1, \gamma_2)$.*

Lastly, we discuss the error range such that the inexact solutions still can make the proposed algorithm attain the polynomial iteration complexity. This accuracy will also be used for the convergence proof in Section 4. We assume the following inequality for the error $\hat{r}_1^k$ of (18) and $\hat{r}_2^k$ of (21):

$$\left\| \hat{r}_i^k \right\| \leq \eta \frac{\sqrt{\mu_k}}{\sqrt{n}}, \quad \forall i \in \{1, 2\} \tag{25}$$

where $\eta \in [0, 1)$ is an enforcing parameter.

To prove the polynomial iteration complexity of the proposed algorithm in Lemma 4.6 below, we set the parameters so that

$$(1 - \gamma_1)\sigma - (1 + \gamma_1)\eta > 0, \tag{26a}$$

$$\beta > \sigma + \eta. \tag{26b}$$

We are now ready to give the framework of the proposed method (II-arc-IPM) as Algorithm 1.

Algorithm 1 The inexact infeasible arc-search interior-point method (II-arc-IPM)

Input: $\zeta > 0$, $\gamma_1 \in (0, 1)$, $\gamma_2 \geq 1$, σ, η, β satisfying (26) and an initial point $(x^0, y^0, s^0) \in \mathcal{N}(\gamma_1, \gamma_2)$ such that $x^0 > 0$ and $s^0 > 0$.

Output: ζ-optimal solution (x^k, y^k, s^k)

```
k ← 0
while (x^k, y^k, s^k) ∉ S_ζ do
    μ_k ← (x^k)^⊤ s^k / n
    Calculate (x̃, ỹ, s̃) by solving (16) inexactly satisfying (25).
    Calculate (x̃̃, ỹ̃, s̃̃) by solving (20) inexactly satisfying (25).
    α_k ← max {α ∈ (0, π/2] | α satisfies (24)}
    Set (x^{k+1}, y^{k+1}, s^{k+1}) = (x^k(α_k), y^k(α_k), s^k(α_k)) by (23).
    k ← k + 1
end while
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4 Theoretical proof

In this section, we prove the convergence of Algorithm 1 and its polynomial iteration complexity. Our analysis is close to Mohammadisiahroudi et al. [15], but it also employs properties of arc-search IPMs.

First, we evaluate that the constraint residuals (4). From (18) and (19), the residual appears only in the last equation as a term $S^k v_1^k$, as the following lemma shows.

Lemma 4.1. *For the inexact first derivative* $(\tilde{\dot{x}}, \tilde{\dot{y}}, \tilde{\dot{s}})$ *of* (8) *obtained by the inexact solution of* (16) *and the steps in* (19), *we have*

$$A\tilde{\dot{x}} = r_b(x^k), \tag{27a}$$

$$A^\top \tilde{\dot{y}} + \tilde{\dot{s}} = r_c(y^k, s^k), \tag{27b}$$

$$S^k\tilde{\dot{x}} + X^k\tilde{\dot{s}} = X^k s^k - \sigma\mu_k e - S^k v_1^k. \tag{27c}$$

Lemma 4.1 can be proved from (16) and (19) in the same way as Mohammadisiahroudi [15, Lemma 4.1], thus we omit the proof. As in Lemma 4.1, $(\tilde{\ddot{x}}, \tilde{\ddot{y}}, \tilde{\ddot{s}})$ obtained by (21) and (22) satisfies

$$A\tilde{\ddot{x}} = 0, \tag{28a}$$

$$A^\top \tilde{\ddot{y}} + \tilde{\ddot{s}} = 0, \tag{28b}$$

$$S^k\tilde{\ddot{x}} + X^k\tilde{\ddot{s}} = -2\tilde{\dot{x}} \circ \tilde{\dot{s}} - S^k v_2^k. \tag{28c}$$

Therefore, the following lemma holds from (27a), (27b), (28a) and (28b) due to (23).

Lemma 4.2 ([26, Lemma 7.2])**.** *For each iteration k, the following relations hold.*

$$r_b(x^{k+1}) = r_b(x^k)\left(1 - \sin(\alpha_k)\right),$$
$$r_c(y^{k+1}, s^{k+1}) = r_c(y^k, s^k)\left(1 - \sin(\alpha_k)\right).$$

For the following discussions, we introduce the following notation:

$$\nu_k = \prod_{i=0}^{k-1}(1 - \sin(\alpha_i)).$$

From Lemma 4.2, we can obtain

$$r_b(x^k) = \nu_k r_b(x^0) \tag{29a}$$

$$r_c(y^k, s^k) = \nu_k r_c(y^0, s^0) \tag{29b}$$

In the next proposition, we prove the existence of the lower bound of the step size α_k to guarantee that Algorithm 1 is well defined.

Proposition 4.1. *Let $\{(x^k, y^k, s^k)\}$ be the sequence generated by Algorithm 1. Then, there exists $\hat{\alpha} > 0$ satisfying (24) for any $\alpha_k \in (0, \hat{\alpha}]$ and*

$$\sin(\hat{\alpha}) = \frac{C}{n^{1.5}},$$

where C is a positive constant.

The proof of Proposition 4.1 will be given later. For this proof, we first evaluate x^k and s^k with the ℓ_1 norm.

Lemma 4.3. *There is a positive constant C_1 such that*

$$\nu_k \Big\| (x^k, s^k) \Big\|_1 \le C_1 n \mu_k. \tag{30}$$

The proof below is based on [20, Lemma 6.3].

Proof. From the definition of $\mathcal{N}(\gamma_1, \gamma_2)$ in (12) and $\gamma_2 \ge 1$, we know

$$\frac{\|(r_b(x^k), r_c(y^k, s^k))\|}{\mu_k} \le \gamma_2 \frac{\|(r_b(x^0), r_c(y^0, s^0))\|}{\mu_0} \le \frac{\|(r_b(x^0), r_c(y^0, s^0))\|}{\mu^0},$$

which implies

$$\mu_k \ge \frac{\|(r_b(x^k), r_c(y^k, s^k))\|}{\|(r_b(x^0), r_c(y^0, s^0))\|} \mu_0 = \nu_k \mu_0 \tag{31}$$

from (29). When we set

$$(\bar{x}, \bar{y}, \bar{s}) = \nu_k (x^0, y^0, s^0) + (1 - \nu_k)(x^*, y^*, s^*) - (x^k, y^k, s^k),$$

we have $A\bar{x} = 0$ and $A^\top \bar{y} + \bar{s} = 0$ from (29) and (3), then

$$\begin{aligned}
0 &= \bar{x}^\top \bar{s} \\
&= (\nu_k x^0 + (1 - \nu_k) x^* - x^k)^\top (\nu_k s^0 + (1 - \nu_k) s^* - s^k) \\
&= \nu_k^2 (x^0)^\top s^0 + \nu_k (1 - \nu_k) \Big((x^0)^\top s^* + (x^*)^\top s^0 \Big) + (x^k)^\top s^k + (1 - \nu_k)^2 (x^*)^\top s^* \\
&\quad - \Big(\nu_k ((x^0)^\top s^k + (s^0)^\top x^k) + (1 - \nu_k)((x^k)^\top s^* + (s^k)^\top x^*) \Big)
\end{aligned}$$

is satisfied. Since all the components of x^k, s^k, x^*, s^* are nonnegative, we have $((x^k)^\top s^* + (s^k)^\top x^*) \ge 0$. In addition, we have $(x^*)^\top s^* = 0$ from (3). By using these and rearranging, we obtain

$$\begin{aligned}
\nu_k ((x^0)^\top s^k + (s^0)^\top x^k) &\le \nu_k^2 (x^0)^\top s^0 + \nu_k (1 - \nu_k) \Big((x^0)^\top s^* + (x^*)^\top s^0 \Big) + (x^k)^\top s^k \\
[\because (5)] \quad &= \nu_k^2 n \mu_0 + \nu_k (1 - \nu_k) \Big((x^0)^\top s^* + (x^*)^\top s^0 \Big) + n \mu_k \\
[\because (31)] \quad &\le \nu_k n \mu_k + \frac{\mu_k}{\mu_0} (1 - \nu_k) \Big((x^0)^\top s^* + (x^*)^\top s^0 \Big) + n \mu_k \\
[\because \nu_k \in [0, 1]] \quad &\le 2 n \mu_k + \frac{\mu_k}{\mu_0} \Big((x^0)^\top s^* + (x^*)^\top s^0 \Big).
\end{aligned} \tag{32}$$

Defining a constant ξ by

$$\xi = \min_{i=1,2,\ldots,n} \min(x_i^0, s_i^0) > 0, \tag{33}$$

we have $(x^0)^\top s^k + (s^0)^\top x^k \geq \xi \left\|(x^k, s^k)\right\|_1$. Therefore, from (32), we obtain

$$\nu_k \left\|(x^k, s^k)\right\|_1 \leq \xi^{-1} \left(2 + \frac{(x^0)^\top s^* + (x^*)^\top s^0}{(x^0)^\top s^0}\right) n\mu_k.$$

We complete this proof by setting

$$C_1 = \xi^{-1} \left(2 + \frac{(x^0)^\top s^* + (x^*)^\top s^0}{(x^0)^\top s^0}\right) \tag{34}$$

in (30), where C_1 is independent of n. $\square$

Next, we prove upper bounds of the terms related to $\dot{\tilde{x}}, \dot{\tilde{s}}, \ddot{\tilde{x}}, \ddot{\tilde{s}}$. From (25), the following lemma gives an upper bound of (27c) and (28c):

Lemma 4.4 ([15, Lemma 4.2])**.** *For the derivatives* $(\dot{\tilde{x}}, \dot{\tilde{y}}, \dot{\tilde{s}})$ *and* $(\ddot{\tilde{x}}, \ddot{\tilde{y}}, \ddot{\tilde{s}})$*, when the residuals* $\hat{r}_i^k$ *satisfy* (25)*, it holds that*

$$\left\|S^k v_i^k\right\|_\infty \leq \eta\mu_k. \tag{35}$$

Then, the following lemma holds similarly to [20, Lemma 6.5] and [15, Lemma 4.6].

Lemma 4.5. *There is a positive constant* C_2 *such that*

$$\max\left\{\left\|(D^k)^{-1}\dot{\tilde{x}}\right\|, \left\|D^k\dot{\tilde{s}}\right\|\right\} \leq C_2 n\sqrt{\mu_k}$$

Proof. Let

$$(\bar{x}, \bar{y}, \bar{s}) = (\dot{\tilde{x}}, \dot{\tilde{y}}, \dot{\tilde{s}}) - \nu_k(x^0, y^0, s^0) + \nu_k(x^*, y^*, s^*).$$

From (27a), (27b), (29) and (3), we have $A\bar{x} = 0$ and $A^\top \bar{y} + \bar{s} = 0$, therefore, $\bar{x}^\top \bar{s} = 0$. Thus, we obtain

$$\left\|(D^k)^{-1}\bar{x} + D^k\bar{s}\right\|^2 = \left\|(D^k)^{-1}(\dot{\tilde{x}} - \nu_k(x^0 - x^*))\right\|^2 + \left\|D^k(\dot{\tilde{s}} - \nu_k(s^0 - s^*))\right\|^2. \tag{36}$$

From (27c), it holds that

$$\begin{aligned} S^k\bar{x} + X^k\bar{s} &= (S^k\dot{\tilde{x}} + X^k\dot{\tilde{s}}) - \nu_k S^k(x^0 - x^*) - \nu_k X^k(s^0 - s^*) \\ &= (X^k s^k - \sigma\mu_k e - S^k v_1^k) - \nu_k S^k(x^0 - x^*) - \nu_k X^k(s^0 - s^*). \end{aligned}$$

Consequently, we verify

$$(D^k)^{-1}\bar{x} + D^k\bar{s} = (X^k S^k)^{-\frac{1}{2}}(X^k s^k - \sigma\mu_k e - S^k v_1^k) - \nu_k (D^k)^{-1}(x^0 - x^*) - \nu_k D^k(s^0 - s^*). \tag{37}$$

For any vector $a \in \mathbb{R}^d$,

$$\|a\|_1 \le \sqrt{n}\|a\| \le n\|a\|_\infty \tag{38}$$

holds from [26, Lemma 3.1]. From (36), (37), (38) and Lemma 4.4, we obtain

$$\begin{aligned}
&\left\|(D^k)^{-1}(\tilde{\dot{x}} - \nu_k(x^0 - x^*))\right\|^2 + \left\|D^k(\tilde{\dot{s}} - \nu_k(s^0 - s^*))\right\|^2 \\
&= \left\|(X^kS^k)^{-\frac{1}{2}}(X^ks^k - \sigma\mu_k e - S^kv_1^k) - \nu_k(D^k)^{-1}(x^0 - x^*) - \nu_k D^k(s^0 - s^*)\right\|^2 \\
&\le \left\{\left\|X^kS^k\right\|^{-\frac{1}{2}}\left(\left\|X^ks^k - \sigma\mu_k e\right\| + \left\|S^kv_1^k\right\|\right) + \nu_k\left\|(D^k)^{-1}(x^0 - x^*)\right\| + \nu_k\left\|D^k(s^0 - s^*)\right\|\right\}^2 \\
&\le \left\{\left\|X^kS^k\right\|^{-\frac{1}{2}}\left(\left\|X^ks^k - \sigma\mu_k e\right\| + \sqrt{n}\eta\mu_k\right) + \nu_k\left(\left\|(D^k)^{-1}(x^0 - x^*)\right\| + \left\|D^k(s^0 - s^*)\right\|\right)\right\}^2.
\end{aligned} \tag{39}$$

In addition, $x_i^ks_i^k \ge \gamma\mu_k$ in (12) implies

$$\left\|X^kS^k\right\|^{-\frac{1}{2}} \le \frac{1}{\sqrt{\gamma_1\mu_k}}. \tag{40}$$

From (30) and (40), we have

$$\nu_k\left\|(x^k, s^k)\right\|_1\left\|(XS)^{-1/2}\right\| \le \frac{C_1 n\sqrt{\mu_k}}{\sqrt{\gamma_1}}. \tag{41}$$

According to the derivation in [20, Lemma 6.5], we have

$$\left\|X^ks^k - \sigma\mu_k e\right\| \le n\mu_k, \tag{42}$$

$$\begin{aligned}
&\nu_k\left(\left\|(D^k)^{-1}(x^0 - x^*)\right\| + \left\|D^k(s^0 - s^*)\right\|\right) \\
&\quad \le \nu_k\left\|(x^k, s^k)\right\|_1\left\|(XS)^{-1/2}\right\| \max\left\{\left\|x^0 - x^*\right\|, \left\|s^0 - s^*\right\|\right\}.
\end{aligned} \tag{43}$$

Therefore, from (43) and (41), we obtain

$$\begin{aligned}
&\nu_k\left(\left\|(D^k)^{-1}(x^0 - x^*)\right\| + \left\|D^k(s^0 - s^*)\right\|\right) \\
&\quad \le \frac{C_1}{\sqrt{\gamma_1}} n\sqrt{\mu_k} \max\left\{\left\|x^0 - x^*\right\|, \left\|s^0 - s^*\right\|\right\}.
\end{aligned} \tag{44}$$

Therefore, we have

$$
\begin{aligned}
\left\|(D^k)^{-1}\tilde{\dot{x}}\right\| &\le \left\|(D^k)^{-1}(\tilde{\dot{x}} - \nu_k(x^0 - x^*))\right\| + \nu_k\left\|(D^k)^{-1}(x^0 - x^*)\right\| \\
[\because (39)] \quad &\le \left\|X^k S^k\right\|^{-\frac{1}{2}} \left(\left\|X^k s^k - \sigma\mu_k e\right\| + \sqrt{n}\eta\mu_k\right) \\
&\quad + 2\nu_k \left(\left\|(D^k)^{-1}(x^0 - x^*)\right\| + \left\|D^k(s^0 - s^*)\right\|\right) \\
[\because (40), (42)] \quad &\le \frac{\sqrt{\mu_k}}{\sqrt{\gamma_1}}\left(n + \sqrt{n}\eta\right) + 2\nu_k \left(\left\|(D^k)^{-1}(x^0 - x^*)\right\| + \left\|D^k(s^0 - s^*)\right\|\right) \\
[\because (44)] \quad &\le \frac{\sqrt{\mu_k}}{\sqrt{\gamma_1}}\left(n + \sqrt{n}\eta\right) + \frac{2C_1 n\sqrt{\mu_k}}{\sqrt{\gamma_1}} \max\left\{\left\|x^0 - x^*\right\|, \left\|s^0 - s^*\right\|\right\} \\
&\le \frac{1}{\sqrt{\gamma_1}}\left(1 + \eta + 2C_1 \max\left\{\left\|x^0 - x^*\right\|, \left\|s^0 - s^*\right\|\right\}\right) n\sqrt{\mu_k}.
\end{aligned}
$$

Since the optimal set is bounded from Assumption 2.1 and the initial point is bounded,

$$
C_2 := \gamma_1^{-1/2}\left(1 + \eta + 2C_1 \max\left\{\left\|x^0 - x^*\right\|, \left\|s^0 - s^*\right\|\right\}\right) \tag{45}
$$

is also bounded, and we can prove this lemma by setting this C_2. We can similarly show $\tilde{\dot{s}} \le C_2 n\sqrt{\mu_k}$. ☐

From Lemma 4.5,

$$
\left\|\tilde{\dot{x}} \circ \tilde{\dot{s}}\right\| \le \left\|(D^k)^{-1}\tilde{\dot{x}}\right\|\left\|D^k\tilde{\dot{s}}\right\| \le C_2^2 n^2 \mu_k. \tag{46}
$$

Similarly, we evaluate the terms related to $G_i^k(\alpha)$, $g^k(\alpha)$ and $h^k(\alpha)$.

Lemma 4.6. *There are positive constants C_3 and C_4 such that*

$$
\begin{aligned}
\left\|\tilde{\ddot{x}} \circ \tilde{\ddot{s}}\right\| &\le C_3 n^4 \mu_k, \\
\max\left\{\left\|(D^k)^{-1}\tilde{\ddot{x}}\right\|, \left\|D^k\tilde{\ddot{s}}\right\|\right\} &\le C_4 n^2\sqrt{\mu_k}, \\
\max\left\{\left\|\tilde{\ddot{x}} \circ \tilde{\dot{s}}\right\|, \left\|\tilde{\dot{x}} \circ \tilde{\ddot{s}}\right\|\right\} &\le C_2 C_4 n^3 \mu_k.
\end{aligned}
$$

Proof. When $u^\top v \ge 0$ for any vector pairs of u, v, the inequality

$$
\|u \circ v\| \le 2^{-\frac{3}{2}}\|u + v\|^2
$$

holds from [20, Lemma 5.3], so the following is satisfied:

$$
\left\|\tilde{\ddot{x}} \circ \tilde{\ddot{s}}\right\| = \left\|(D^k)^{-1}\tilde{\ddot{x}} \circ D^k\tilde{\ddot{s}}\right\| \le 2^{-\frac{3}{2}}\left\|(D^k)^{-1}\tilde{\ddot{x}} + D^k\tilde{\ddot{s}}\right\|^2.
$$

From $(D^k)^{-1}\tilde{\ddot{x}} + D^k\tilde{\ddot{s}} = (X^k S^k)^{-1/2}(S^k\tilde{\ddot{x}} + X^k\tilde{\ddot{s}})$,

$$
\begin{aligned}
\left\|(D^k)^{-1}\tilde{\ddot{x}} + D^k\tilde{\ddot{s}}\right\| &\leq \left\|X^k S^k\right\|^{-\frac{1}{2}} \left\|S^k\tilde{\ddot{x}} + X^k\tilde{\ddot{s}}\right\| \\
[\because (28\text{c})] \quad &\leq \left\|X^k S^k\right\|^{-\frac{1}{2}} \left(2\left\|\tilde{\dot{x}} \circ \tilde{\dot{s}}\right\| + \left\|S^k v_2^k\right\|\right) \\
[\because (40), (46), (35), (38)] \quad &\leq \frac{1}{\sqrt{\gamma_1 \mu_k}} \left(2C_2^2 n^2 \mu_k + \sqrt{n}\eta\mu_k\right) \\
&\leq \frac{\sqrt{\mu_k}}{\sqrt{\gamma_1}}(2C_2^2 n^2 + \sqrt{n}\eta). \qquad (47)
\end{aligned}
$$

From the above, we can obtain

$$
\left\|\tilde{\ddot{x}} \circ \tilde{\ddot{s}}\right\| \leq 2^{-\frac{3}{2}} \frac{\mu_k}{\gamma_1}(2C_2^2 n^2 + \sqrt{n}\eta)^2 \leq \frac{(2C_2^2 + \eta)^2}{2^{\frac{3}{2}}\gamma_1} n^4 \mu_k =: C_3 n^4 \mu_k.
$$

From (28a) and (28b), we know

$$
\tilde{\ddot{x}}^\top \tilde{\dot{s}} = 0, \qquad (48)
$$

then (47) leads to

$$
\begin{aligned}
\max\left\{\left\|(D^k)^{-1}\tilde{\ddot{x}}\right\|^2, \left\|D^k\tilde{\ddot{s}}\right\|^2\right\} &\leq \left\|(D^k)^{-1}\tilde{\ddot{x}} + D^k\tilde{\ddot{s}}\right\|^2 \\
&\leq \frac{\mu_k}{\gamma_1}(2C_2^2 n^2 + \sqrt{n}\eta)^2 \\
&\leq \frac{\mu_k}{\gamma_1}(2C_2^2 + \eta)^2 n^4 =: C_4^2 n^4 \mu_k,
\end{aligned}
$$

$$
\left\|\tilde{\ddot{x}} \circ \tilde{\dot{s}}\right\| \leq \left\|(D^k)^{-1}\tilde{\ddot{x}}\right\|\left\|D^k\tilde{\dot{s}}\right\| \leq C_4 n^2 \sqrt{\mu_k} C_2 n \sqrt{\mu_k} = C_2 C_4 n^3 \mu_k.
$$

We can show the boundedness of $\left\|\tilde{\dot{x}} \circ \tilde{\ddot{s}}\right\|$ similarly. $\square$

Using these lemmas, we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. Firstly, we derive the equations necessary for the proofs. We have the following simple identity:

$$
-2(1 - \cos(\alpha)) + \sin^2(\alpha) = -(1 - \cos(\alpha))^2. \qquad (49)
$$

Therefore, we can obtain

$$
\begin{aligned}
x^k(\alpha) \circ s^k(\alpha) =&\ \left(x^k - \dot{\tilde{x}} \sin(\alpha) + \ddot{\tilde{x}}(1 - \cos(\alpha))\right) \circ \left(s^k - \dot{\tilde{s}} \sin(\alpha) + \ddot{\tilde{s}}(1 - \cos(\alpha))\right) \\
=&\ x^k \circ s^k - \left(x^k \circ \dot{\tilde{s}} + \dot{\tilde{x}} \circ s^k\right) \sin(\alpha) + \left(x^k \circ \ddot{\tilde{s}} + \ddot{\tilde{x}} \circ s^k\right) (1 - \cos(\alpha)) \\
&+ \dot{\tilde{x}} \circ \dot{\tilde{s}} \sin^2(\alpha) - \left(\dot{\tilde{x}} \circ \ddot{\tilde{s}} + \ddot{\tilde{x}} \circ \dot{\tilde{s}}\right) \sin(\alpha)(1 - \cos(\alpha)) + \ddot{\tilde{x}} \circ \ddot{\tilde{s}}(1 - \cos(\alpha))^2 \\
[\because (27c), (28c)] \quad =&\ x^k \circ s^k - (x^k \circ s^k - \sigma\mu_k e - S^k v_1^k) \sin(\alpha) + \left(-2\dot{\tilde{x}} \circ \dot{\tilde{s}} - S^k v_2^k\right) (1 - \cos(\alpha)) \\
&+ \dot{\tilde{x}} \circ \dot{\tilde{s}} \sin^2(\alpha) - \left(\dot{\tilde{x}} \circ \ddot{\tilde{s}} + \ddot{\tilde{x}} \circ \dot{\tilde{s}}\right) \sin(\alpha)(1 - \cos(\alpha)) + \ddot{\tilde{x}} \circ \ddot{\tilde{s}}(1 - \cos(\alpha))^2 \\
[\because (49)] \quad =&\ x^k \circ s^k(1 - \sin(\alpha)) + \sigma\mu_k \sin(\alpha) e \\
&+ \left(\ddot{\tilde{x}} \circ \ddot{\tilde{s}} - \dot{\tilde{x}} \circ \dot{\tilde{s}}\right) (1 - \cos(\alpha))^2 - \left(\dot{\tilde{x}} \circ \ddot{\tilde{s}} + \ddot{\tilde{x}} \circ \dot{\tilde{s}}\right) \sin(\alpha)(1 - \cos(\alpha)) \\
&+ S^k v_1^k \sin(\alpha) - S^k v_2^k (1 - \cos(\alpha))
\end{aligned} \tag{50}
$$

and

$$
\begin{aligned}
x^k(\alpha)^\top s^k(\alpha) =&\ \left(x^k - \dot{\tilde{x}} \sin(\alpha) + \ddot{\tilde{x}}(1 - \cos(\alpha))\right)^\top \left(s^k - \dot{\tilde{s}} \sin(\alpha) + \ddot{\tilde{s}}(1 - \cos(\alpha))\right) \\
[\because (50), (5), (48)] \quad =&\ (x^k)^\top s^k \left((1 - \sin(\alpha)) + \sigma \sin(\alpha)\right) \\
&- \dot{\tilde{x}}^\top \dot{\tilde{s}}(1 - \cos(\alpha))^2 - \left(\dot{\tilde{x}}^\top \ddot{\tilde{s}} + \ddot{\tilde{x}}^\top \dot{\tilde{s}}\right) \sin(\alpha)(1 - \cos(\alpha)) \\
&+ \sin(\alpha) \sum_{i=1}^n [S^k v_1^k]_i - (1 - \cos(\alpha)) \sum_{i=1}^n [S^k v_2^k]_i.
\end{aligned} \tag{51}
$$

From Lemmas 4.5 and 4.6 and the Cauchy-Schwartz inequality, we know

$$
\left|\dot{\tilde{x}}_i \dot{\tilde{s}}_i\right|, \left|\dot{\tilde{x}}^\top \dot{\tilde{s}}\right| \le \left\|(D^k)^{-1} \dot{\tilde{x}}\right\| \left\|D^k \dot{\tilde{s}}\right\| \le C_2^2 n^2 \mu_k \tag{52a}
$$

$$
\left|\ddot{\tilde{x}}_i \dot{\tilde{s}}_i\right|, \left|\ddot{\tilde{x}}^\top \dot{\tilde{s}}\right| \le \left\|(D^k)^{-1} \ddot{\tilde{x}}\right\| \left\|D^k \dot{\tilde{s}}\right\| \le C_2 C_4 n^3 \mu_k \tag{52b}
$$

$$
\left|\dot{\tilde{x}}_i \ddot{\tilde{s}}_i\right|, \left|\dot{\tilde{x}}^\top \ddot{\tilde{s}}\right| \le \left\|(D^k)^{-1} \dot{\tilde{x}}\right\| \left\|D^k \ddot{\tilde{s}}\right\| \le C_2 C_4 n^3 \mu_k \tag{52c}
$$

$$
\left|\ddot{\tilde{x}}_i \ddot{\tilde{s}}_i\right| \le \left\|(D^k)^{-1} \ddot{\tilde{x}}\right\| \left\|D^k \ddot{\tilde{s}}\right\| \le C_4^2 n^4 \mu_k \tag{52d}
$$

Here, $\left|\ddot{\tilde{x}}^\top \ddot{\tilde{s}}\right| = 0$ holds due to (48). Furthermore, we have

$$
\sin^2(\alpha) = 1 - \cos^2(\alpha) \ge 1 - \cos(\alpha) \tag{53}
$$

from $\alpha \in (0, \pi/2]$.

We prove that the step size α satisfying $g^k(\alpha) \ge 0$ is bounded away from zero. From

(51),

$$\begin{aligned}
x^k(\alpha)^\top s^k(\alpha) &\geq (x^k)^\top s^k \left((1-\sin(\alpha)) + \sigma\sin(\alpha)\right) \\
&\quad - \left|\tilde{\dot{x}}^\top \tilde{\dot{s}}\right|(1-\cos(\alpha))^2 - \left(\left|\tilde{\dot{x}}^\top \tilde{\ddot{s}}\right| + \left|\tilde{\ddot{x}}^\top \tilde{\dot{s}}\right|\right)\sin(\alpha)(1-\cos(\alpha)) \\
&\quad - \left\|S^k v_1^k\right\|_1 \sin(\alpha) - \left\|S^k v_2^k\right\|_1 (1-\cos(\alpha)) \\
[\because (38),(35)] \quad &\geq (x^k)^\top s^k \left((1-\sin(\alpha)) + \sigma\sin(\alpha)\right) \\
&\quad - \left|\tilde{\dot{x}}^\top \tilde{\dot{s}}\right|(1-\cos(\alpha))^2 - \left(\left|\tilde{\dot{x}}^\top \tilde{\ddot{s}}\right| + \left|\tilde{\ddot{x}}^\top \tilde{\dot{s}}\right|\right)\sin(\alpha)(1-\cos(\alpha)) \\
&\quad - \eta n\mu_k(\sin(\alpha) + 1 - \cos(\alpha)).
\end{aligned} \tag{54}$$

Therefore,

$$\begin{aligned}
g^k(\alpha) &= x^k(\alpha)^\top s^k(\alpha) - (1-\sin(\alpha))(x^k)^\top s^k \\
[\because (54)] \quad &\geq \sigma(x^k)^\top s^k \sin(\alpha) - \eta n\mu_k\left(\sin(\alpha) + 1 - \cos(\alpha)\right) \\
&\quad - \left|\tilde{\dot{x}}^\top \tilde{\dot{s}}\right|(1-\cos(\alpha))^2 - \left(\left|\tilde{\dot{x}}^\top \tilde{\ddot{s}}\right| + \left|\tilde{\ddot{x}}^\top \tilde{\dot{s}}\right|\right)\sin(\alpha)(1-\cos(\alpha)) \\
[\because (5),(53)] \quad &\geq \sigma n\mu_k \sin(\alpha) - \eta n\mu_k\left(\sin(\alpha) + \sin^2(\alpha)\right) \\
&\quad - \left|\tilde{\dot{x}}^\top \tilde{\dot{s}}\right|\sin^4(\alpha) - \left(\left|\tilde{\dot{x}}^\top \tilde{\ddot{s}}\right| + \left|\tilde{\ddot{x}}^\top \tilde{\dot{s}}\right|\right)\sin^3(\alpha) \\
[\because (52)] \quad &\geq n\mu_k \sin(\alpha)\left((\sigma-\eta) - \eta\sin(\alpha) - C_2^2 n\sin^3(\alpha) - 2C_2C_4n^2\sin^2(\alpha)\right).
\end{aligned}$$

Since $\left(-\eta\sin(\alpha) - C_2^2 n\sin^3(\alpha) - 2C_2C_4n^2\sin^2(\alpha)\right)$ is monotonically decreasing and $\sigma > \eta$ holds from (26a) and $\gamma_1 \in (0,1)$, there exists the step size $\hat{\alpha}_1 \in (0, \pi/2]$ satisfying the last formula of the right-hand side is no less than 0. When

$$\sin(\hat{\alpha}_1) \leq \frac{\sigma-\eta}{2n}\frac{1}{\max\left\{\eta, C_2^{\frac{2}{3}}, \sqrt{2C_2C_4}\right\}},$$

from $0 < \sigma - \eta < \sigma \leq 1$,

$$\begin{aligned}
&(\sigma-\eta) - \eta\sin(\hat{\alpha}_1) - C_2^2 n\sin^3(\hat{\alpha}_1) - 2C_2C_4n^2\sin^2(\hat{\alpha}_1) \\
&\quad \geq (\sigma-\eta) - \frac{\sigma-\eta}{2n} - \frac{(\sigma-\eta)^3}{8n^2} - \frac{(\sigma-\eta)^2}{4} \\
&\quad \geq (\sigma-\eta)\left(1 - \frac{1}{2} - \frac{1}{8} - \frac{1}{4}\right) \quad \geq 0.
\end{aligned}$$

Therefore, $g^k(\alpha) \geq 0$ is satisfied for any $\alpha \in (0, \hat{\alpha}_1]$.

Next, we consider the range of α such that $G_i^k(\alpha) \geq 0$. From (52),

$$\left|\tilde{\dot{x}}_i\tilde{\dot{s}}_i - \frac{\gamma_1}{n}\tilde{\dot{x}}^\top\tilde{\dot{s}}\right| \leq \left(1 + \frac{\gamma_1}{n}\right)C_2^2n^2\mu_k \leq 2C_2^2n^2\mu_k \tag{55a}$$

$$\left|\tilde{\ddot{x}}_i\tilde{\dot{s}}_i - \frac{\gamma_1}{n}\tilde{\ddot{x}}^\top\tilde{\dot{s}}\right|, \left|\tilde{\dot{x}}_i\tilde{\ddot{s}}_i - \frac{\gamma_1}{n}\tilde{\dot{x}}^\top\tilde{\ddot{s}}\right| \leq 2C_2C_4n^3\mu_k \tag{55b}$$

is satisfied. Therefore, we have

$$
\begin{aligned}
G_i^k(\alpha) &= x_i^k(\alpha)s_i^k(\alpha) - \gamma_1\mu_k(\alpha) \\
[\because (50),(5),(51)] \quad &\geq x_i^k s_i^k(1-\sin(\alpha)) + \sigma\mu_k\sin(\alpha) \\
&\quad + \left(\ddot{\tilde{x}}_i\ddot{\tilde{s}}_i - \dot{\tilde{x}}_i\dot{\tilde{s}}_i\right)(1-\cos(\alpha))^2 - \left(\dot{\tilde{x}}_i\ddot{\tilde{s}}_i + \ddot{\tilde{x}}_i\dot{\tilde{s}}_i\right)\sin(\alpha)(1-\cos(\alpha)) \\
&\quad - \left\|S^k v_1^k\right\|_\infty \sin(\alpha) - \left\|S^k v_2^k\right\|_\infty (1-\cos(\alpha)) \\
&\quad - \frac{\gamma_1}{n}\Bigg(n\mu_k\left((1-\sin(\alpha)) + \sigma\sin(\alpha)\right) \\
&\quad - \ddot{\tilde{x}}^\top\ddot{\tilde{s}}(1-\cos(\alpha))^2 - \left(\dot{\tilde{x}}^\top\ddot{\tilde{s}} + \ddot{\tilde{x}}^\top\dot{\tilde{s}}\right)\sin(\alpha)(1-\cos(\alpha))) \\
&\quad + \left\|S^k v_1^k\right\|_1 \sin(\alpha) + \left\|S^k v_2^k\right\|_1 (1-\cos(\alpha))\Bigg) \\
[\because (12),(35),(38)] \quad &\geq (1-\gamma_1)\sigma\mu_k\sin(\alpha) - (1+\gamma_1)\eta\mu_k(\sin(\alpha) + 1 - \cos(\alpha)) \\
&\quad + \ddot{\tilde{x}}_i\ddot{\tilde{s}}_i(1-\cos(\alpha))^2 - \left(\dot{\tilde{x}}_i\dot{\tilde{s}}_i - \frac{\gamma_1}{n}\ddot{\tilde{x}}^\top\ddot{\tilde{s}}\right)(1-\cos(\alpha))^2 \\
&\quad - \left(\dot{\tilde{x}}_i\ddot{\tilde{s}}_i - \frac{\gamma_1}{n}\dot{\tilde{x}}^\top\ddot{\tilde{s}} + \ddot{\tilde{x}}_i\dot{\tilde{s}}_i - \frac{\gamma_1}{n}\ddot{\tilde{x}}^\top\dot{\tilde{s}}\right)\sin(\alpha)(1-\cos(\alpha)) \\
[\because (53),(52\mathrm{d}),(55)] \quad &\geq \mu_k\sin(\alpha)\Bigg((1-\gamma_1)\sigma - (1+\gamma_1)\eta - (1+\gamma_1)\eta\sin(\alpha) \\
&\quad - (C_4^2n^4 + 2C_2^2n^2)\sin^3(\alpha) - 4C_2C_4n^3\sin^2(\alpha)\Bigg).
\end{aligned}
$$

We can derive the same discussion as $g^k(\alpha)$ using (26a). When

$$
\sin(\hat{\alpha}_2) \leq \frac{(1-\gamma_1)\sigma - (1+\gamma_1)\eta}{2n^{\frac{3}{2}}} \frac{1}{\max\left\{(1+\gamma_1)\eta, (C_4^2 + 2C_2^2)^{\frac{1}{3}}, 2\sqrt{C_2C_4}\right\}},
$$

from $0 < (1-\gamma_1)\sigma - (1+\gamma_1)\eta < \sigma \leq 1$,

$$
\begin{aligned}
&(1-\gamma_1)\sigma - (1+\gamma_1)\eta - (1+\gamma_1)\eta\sin(\hat{\alpha}_2) - (C_4^2n^4 + 2C_2^2n^2)\sin^3(\hat{\alpha}_2) - 4C_2C_4n^3\sin^2(\hat{\alpha}_2) \\
&\quad \geq ((1-\gamma_1)\sigma - (1+\gamma_1)\eta)\left(1 - \frac{1}{2n^{\frac{3}{2}}} - \frac{1}{2^3n^{\frac{1}{2}}} - \frac{1}{2^2}\right) \\
&\quad \geq ((1-\gamma_1)\sigma - (1+\gamma_1)\eta)\left(1 - \frac{1}{2} - \frac{1}{8} - \frac{1}{4}\right) \\
&\quad \geq 0.
\end{aligned}
$$

Therefore, $G_i^k(\alpha) \geq 0$ is satisfied for $\alpha \in (0, \hat{\alpha}_2]$.

Lastly, we consider $h^k(\alpha) \geq 0$. Similarly to the derivation of (54), we can obtain the following:

$$
\begin{aligned}
x^k(\alpha)^\top s^k(\alpha) \leq (x^k)^\top s^k &\left((1-\sin(\alpha)) + \sigma\sin(\alpha)\right) \\
&+ \left|\ddot{\tilde{x}}^\top\ddot{\tilde{s}}\right|(1-\cos(\alpha))^2 + \left(\left|\dot{\tilde{x}}^\top\ddot{\tilde{s}}\right| + \left|\ddot{\tilde{x}}^\top\dot{\tilde{s}}\right|\right)\sin(\alpha)(1-\cos(\alpha)) \\
&+ \eta n\mu_k(\sin(\alpha) + 1 - \cos(\alpha)),
\end{aligned} \tag{56}
$$

Therefore,

$$\begin{aligned}
h^k(\alpha) =&\ (1-(1-\beta)\sin(\alpha))\,(x^k)^\top s^k - x^k(\alpha)^\top s^k(\alpha)\\
[\because (56)] \quad \geq&\ (x^k)^\top s^k\,(\beta\sin(\alpha)-\sigma\sin(\alpha)) - \eta n\mu_k(\sin(\alpha)+1-\cos(\alpha))\\
&- \left|\tilde{\dot{x}}^\top\tilde{\dot{s}}\right|(1-\cos(\alpha))^2 - \left(\left|\tilde{\dot{x}}^\top\tilde{\ddot{s}}\right| + \left|\tilde{\ddot{x}}^\top\tilde{\dot{s}}\right|\right)\sin(\alpha)(1-\cos(\alpha))\\
[\because (5)] \quad =&\ n\mu_k\,(\beta\sin(\alpha)-\sigma\sin(\alpha)-\eta(\sin(\alpha)+1-\cos(\alpha)))\\
&- \left|\tilde{\dot{x}}^\top\tilde{\dot{s}}\right|(1-\cos(\alpha))^2 - \left(\left|\tilde{\dot{x}}^\top\tilde{\ddot{s}}\right| + \left|\tilde{\ddot{x}}^\top\tilde{\dot{s}}\right|\right)\sin(\alpha)(1-\cos(\alpha))\\
[\because (52)] \quad \geq&\ n\mu_k\,((\beta-\sigma-\eta)\sin(\alpha)-\eta(1-\cos(\alpha)))\\
&- C_2^2 n^2\mu_k(1-\cos(\alpha))^2 - 2C_2C_4n^3\mu_k\sin(\alpha)(1-\cos(\alpha))\\
[\because (53)] \quad \geq&\ n\mu_k\sin(\alpha)\left((\beta-\sigma-\eta)-\eta\sin(\alpha)-C_2^2 n\sin^3(\alpha)-2C_2C_4n^2\sin^2(\alpha)\right).
\end{aligned}$$

The last coefficient on the right-hand side is cubic for $\sin(\alpha)$ and monotonically decreasing for α. Therefore, it is possible to take a step size $\hat{\alpha}_3$ satisfying $h^k(\hat{\alpha}_3) \geq 0$ from (26b). When

$$\sin(\hat{\alpha}_3) \leq \frac{\beta-\sigma-\eta}{2n}\frac{1}{\max\left\{\eta, C_2^{\frac{2}{3}}, \sqrt{2C_2C_4}\right\}},$$

from $0 < \beta-\sigma-\eta < \beta < 1$, we know

$$\begin{aligned}
&(\beta-\sigma-\eta)-\eta\sin(\hat{\alpha}_3)-C_2^2 n\sin^3(\hat{\alpha}_3)-2C_2C_4n^2\sin^2(\hat{\alpha}_3)\\
&\quad \geq (\beta-\sigma-\eta) - \frac{\beta-\sigma-\eta}{2n} - \frac{(\beta-\sigma-\eta)^3}{8n^2} - \frac{(\beta-\sigma-\eta)^2}{4}\\
&\quad > (\beta-\sigma-\eta)\left(1-\frac{1}{2}-\frac{1}{8}-\frac{1}{4}\right)\\
&\quad = \frac{\beta-\sigma-\eta}{8} > 0.
\end{aligned}$$

Therefore, $g^k(\alpha) \geq 0$ is satisfied for $\alpha \in (0, \hat{\alpha}_3]$.

From the above discussions, when $\hat{\alpha}$ is taken such that

$$\sin(\hat{\alpha}) = \frac{1}{n^{\frac{3}{2}}}\frac{\min\{(1-\gamma_1)\sigma-(1+\gamma_1)\eta, \beta-\sigma-\eta\}}{2\max\left\{(1+\gamma_1)\eta, (C_4^2+2C_2^2)^{\frac{1}{3}}, 2\sqrt{C_2C_4}\right\}}, \tag{57}$$

$g^k(\alpha), G_i^k(\alpha), h^k(\alpha) \geq 0$ are satisfied for all k and $\alpha \in (0, \hat{\alpha}]$. □

Since $\hat{\alpha}$ defined in (57) can satisfy the conditions in line 6 of Algorithm 1, we can find the step length $\alpha_k \geq \hat{\alpha} > 0$. Therefore, Algorithm 1 is well-defined. From $h^k(\alpha_k) \geq 0$ for all k,

$$\begin{aligned}
h^k(\alpha_k) \geq 0 \Rightarrow x^k(\alpha_k)^\top s^k(\alpha_k) &\leq (1-(1-\beta)\sin(\alpha_k))(x^k)^\top s^k\\
&\leq (1-(1-\beta)\sin(\hat{\alpha}))(x^k)^\top s^k\\
&\leq (1-(1-\beta)\sin(\hat{\alpha}))^k(x^0)^\top s^0.
\end{aligned} \tag{58}$$

Due to (29), it also holds that

$$\|(r_b(x^k), r_c(y^k, s^k))\| \le (1-\sin(\hat{\alpha}))^k \|(r_b(x^0), r_c(y^k, s^0))\|. \quad (59)$$

We can prove the polynomial complexity of the proposed method based on the following theorem.

Theorem 4.1 ([26, Theorem 1.4])**.** *Suppose that an algorithm for solving* (3) *generates a sequence of iterations that satisfies*

$$\mu_{k+1} \le \left(1 - \frac{\delta}{n^\omega}\right)\mu_k, \quad k = 0, 1, 2, \ldots,$$

for some positive constants δ and ω. Then there exists an index K with

$$K = \mathcal{O}(n^\omega \log(\mu_0/\zeta))$$

such that

$$\mu_k \le \zeta \text{ for } \forall k \ge K.$$

Applying (58), (12), $(x^k, y^k, s^k) \in \mathcal{N}(\gamma_1, \gamma_2)$, (59) and a result that $\sin(\hat{\alpha})$ is propositional to $n^{-1.5}$ in (57) to this theorem, we can obtain the following theorem.

Theorem 4.2. *Algorithm 1 generates a ζ-optimal solution in at most*

$$\mathcal{O}\left(n^{1.5} \log\left(\frac{\max\{\mu_0, \|r_b(x^0), r_c(y^0, s^0)\|\}}{\zeta}\right)\right)$$

iterations.

Al-Jeiroudi et al. [1] and Mohammadisiahroudi et al. [15] analyzed that the iteration complexity of II-line-IPM is $\mathcal{O}(n^2 L)$. Theorem 4.2 indicates that II-arc-IPM can reduce the iteration complexity from n^2 to $n^{1.5}$, by a factor of $\sqrt{n}$. This reduction is mainly brought by the ellipsoidal approximation in the arc-search method.

5 Numerical experiments

In this section, we describe the implementation and the numerical experiments of the proposed method. The experiments were conducted on a Linux server with Opteron 4386 (3.10GHz), 16 cores, and 128GB RAM, and the methods were implemented with Python 3.10.9.

In the following, we use "II-line-IPM" [15, Algorithm 1] as the existing method for comparisons with the proposed method (II-arc-IPM, Algorithm 1).

5.1 Implementation details

5.1.1 Parameter settings

In these numerical experiments, we set

$$\sigma = 0.4, \quad \eta = 0.3, \quad \gamma_1 = 0.1, \quad \gamma_2 = 1, \quad \beta = 0.9.$$

These parameters satisfy (26), and we use the same parameters for II-line-IPM as well.

5.1.2 Initial points

To determine the initial point, we employed the same approach as in [24, Section 4.1]. However, the point generated by this approach does not always satisfy $(x^0, y^0, s^0) \in \mathcal{N}(\gamma_1, \gamma_2)$. In such a case, we set the initial point as $(x^0, y^0, s^0) = 10^4(e, 0, e)$.

5.1.3 Solving LESs

To solve the LESs inexactly, we employ the conjugate gradient (CG) method in Scipy package. Although we examined other iterative solvers than CG, the preliminary experiments showed that CG was the fastest inexact solver in the II-arc-IPM.

The proposed method uses the MNES formulation in Section 3, but preliminary experiments showed that MNES lacks numerical stability. Specifically, CG did not converge to a certain accuracy even when a preconditioner was employed, and the search direction did not satisfy (24). A possible cause is that the condition number of the coefficient matrix $\hat{M}^k$ for MNES is extremely worse than that for NES; it is known that the condition number of MNES can grow up to the square of that of NES [17].

Therefore, in the numerical experiments, we choose the NES formulations (14) and

$$M^k \ddot{y} = \sigma_2^k, \tag{60}$$

instead of the MNES (16) and (20), respectively. The inexact solution of (14) satisfies

$$M^k \tilde{\dot{y}} = \sigma_1^k + r_1^k, \tag{61}$$

where the error r_1^k is defined as $r_1^k := M^k \tilde{\dot{y}} - \sigma_1^k = M^k \left(\tilde{\dot{y}} - \dot{y}\right)$, and that of (60) satisfies

$$M^k \tilde{\ddot{y}} = \sigma_2^k + r_2^k, \tag{62}$$

where the error r_2^k is defined similar to r_1^k. As for the solution accuracy, we set the following threshold as in (25):

$$\left\| r_i^k \right\| \leq \eta \frac{\sqrt{\mu_k}}{\sqrt{n}} \quad \forall i \in \{1, 2\}. \tag{63}$$

We set the maximum number of iterations of CG to $100m$ (100 times the dimension of the coefficient matrix M^k).

The coefficient matrix M^k has to be a symmetric positive definite matrix when solving (14) and (60) in the II-arc-IPM and [15, (NES)] in the II-line-IPM by CG of Scipy. Though this condition should hold theoretically from Assumption 2.2 and $x^k, s^k > 0$, M^k may not be positive definite due to numerical errors. Therefore, when the CG method fails to satisfy (63), we replace M^k with $M^k + 10^{-3}I$, as indicated in [12].

5.1.4 The modification of $(\tilde{\tilde{x}}, \tilde{\tilde{y}}, \tilde{\tilde{s}})$

If $\left\|-2\tilde{x} \circ \tilde{s}\right\|_\infty \le \eta\mu_k$ is satisfied, (28) and (35) can hold with $(\tilde{\tilde{x}}, \tilde{\tilde{y}}, \tilde{\tilde{s}}) = (0, 0, 0)$. Therefore, to shorten the computation time, we skip solving (60) and set $(\tilde{\tilde{x}}, \tilde{\tilde{y}}, \tilde{\tilde{s}}) = (0, 0, 0)$. In this case, (23) can be interpreted as a line-search method.

Furthermore, when the inexact solution of (60) satisfies $\left\|M_2^k \tilde{\tilde{y}} - \sigma_2^k\right\| > \left\|\sigma_2^k\right\|$, $\tilde{\tilde{y}}$ is replaced with a zero vector as in [14] to avoid a large error.

5.1.5 Step size

In line 6 of Algorithm 1 and [15, Algorithm 1, Line 9], since it is difficult to obtain the solution of (24) analytically, Armijo's rule [20] is employed to determine an actual step size α_k.

5.1.6 Stopping criteria

The algorithms are designed to terminate when $(x^k, y^k, s^k) \in \mathcal{S}_\zeta^*$ is satisfied. The condition $\mu_k \le \zeta$, however, does not consider the magnitude of the data, thus it is not practical especially when the magnitude of the optimal values is relatively large.

Therefore, as in [22], we terminate the algorithms when the following condition is met:

$$\max\left\{\frac{\left\|r_b(x^k)\right\|}{\max\{1, \|b\|\}}, \frac{\left\|r_c(y^k, s^k)\right\|}{\max\{1, \|c\|\}}, \frac{\mu_k}{\max\left\{1, \|c^\top x^k\|, \|b^\top y^k\|\right\}}\right\} < \epsilon, \tag{64}$$

where we set the threshold $\epsilon = 10^{-7}$.

In addition, we stop the algorithm immaturely when one of two conditions is detected; (i) the iteration number k reaches 100 or (ii) the step size α_k diminishes as $\alpha_k < 10^{-7}$.

5.2 Numerical Results

We compare the II-arc-IPM with the II-line-IPM by solving the benchmark problems using CG in this section.

We use the Netlib test repository [4] and choose 85 test problems by excluding the largest instances. We applied the same preprocessing as in [9, Section 5.1] to the problem, e.g., removing redundant rows of the constraint matrix A.

Firstly, Figure 1 shows a performance profile [6, 7] on the numbers of iterations of the II-arc-IPM and the II-line-IPM. To output the performance profile, we used a Julia package [19].

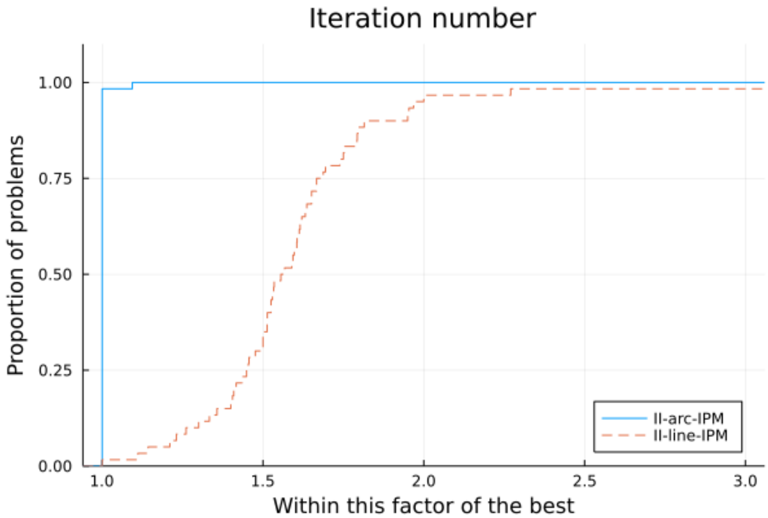

Figure 1: Performance profile of the number of iterations with the II-arc-IPM and II-line-IPM

We observe from Figure 1 that the II-arc-IPM demands fewer iterations than the II-line-IPM in almost all the problems. For nearly 70% of the test problems, the numbers of iterations in the II-line-IPM are 1.5 times of those in the II-arc-IPM or more. Therefore, these results verify that the number of iterations can be reduced by approximating the central path with the ellipsoidal arc, even if the LESs for the search direction are solved inexactly.

Next, Figure 2 shows a performance profile on the computation time. Since the computation times on small problems are too short, we compare the methods with 25 larger problems for which both methods spent 10 seconds or longer. This figure indicates that the II-arc-IPM can solve more than 25% of problems faster than the II-line-IPM, and the computation times are comparable by a factor of around 1.2.

The reason that the computational times did not decrease as much as the number of iterations is that the number of the time-consuming CG executions in the II-arc-IPM is larger than that in the II-line-IPM. For example, the total time to solve a test instance PILOT ($n = 5348$, $m = 2173$) with the II-arc-IPM is 2,014 seconds, and the computation time of CG for 114 executions is 1,721 seconds. The number of iterations is 57, which means that both first- and second-order derivatives are obtained by CG in all iterations. On the other hand, in the case of the II-line-IPM, the total time is 1,758 seconds, and the computation time of CG for 93 executions is 1,489 seconds. The number of iterations is 93, equal to that of CG executions.

When more iterations of IPMs are necessary for the convergence, the CG part tends to take longer times, since the condition number of the the coefficient matrix M^k would be larger [18] and the higher accuracy in (63) would be required.

Therefore, it can be expected that the effect of switching from the inexact solver to an exact solver after a certain number is large in the II-arc-IPM, since it solves LESs twice

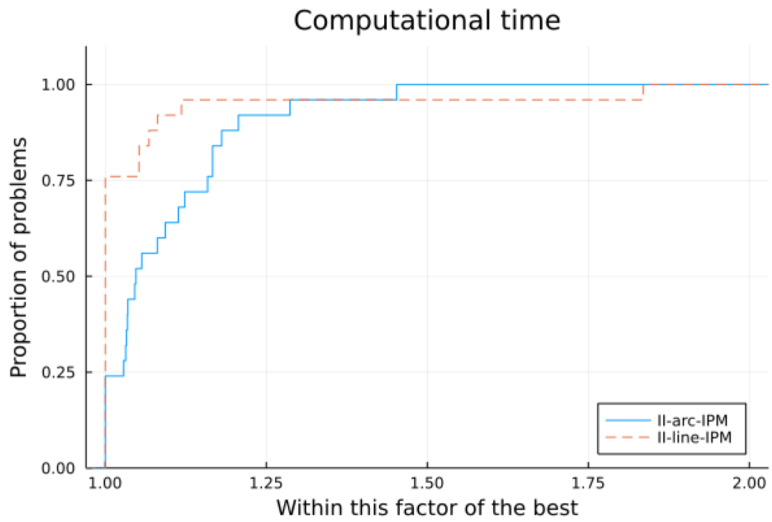

Figure 2: Performance profile of the computation time with Algorithm 1 and II-line-IPM

in each iteration. As an additional investigation on the proposed method, we examined an experiment that utilized an exact solver based on the Cholesky factorization from the middle of the algorithm. Figure 3 compares the computation time of the II-arc-IPM with that of the II-line-IPM when the LESs are solved exactly in the subsequent iterations after $\mu_k < 1/n$ is reached. We compare the methods with 28 large problems for which both methods spent 10 seconds or longer. Figure 3 indicates that the improvement in the computation time per iteration is larger in the II-arc-IPM than that in the II-line-IPM.

6 Conclusion

In this work, we proposed an inexact infeasible arc-search interior-point method (II-arc-IPM) for solving LPs. In particular, by formulating MNES and setting the parameters appropriately, we showed that the proposed method achieves polynomial iteration complexity that is smaller than the II-line-IPM by a factor of $n^{0.5}$. In the numerical experiments with CG as the solver for the LESs, the II-arc-IPM reduced the number of iterations compared to the II-line-IPM. Additionally, if the exact search direction is employed in subsequent iterations after $\mu_k < 1/n$ is satisfied, the improvement of the computational time in the II-arc-IPM is better than that in the II-line-IPM.

As a future direction, we can consider an II-arc-IPM that utilizes QLSA, for solving the LESs inexactly, such as the HHL [8] algorithm. In addition, the reduction of the LESs should be investigated further to shorten the entire computation time, for example, by combining Nesterov's restarting strategy into the framework of arc-search IPMs as in [9].

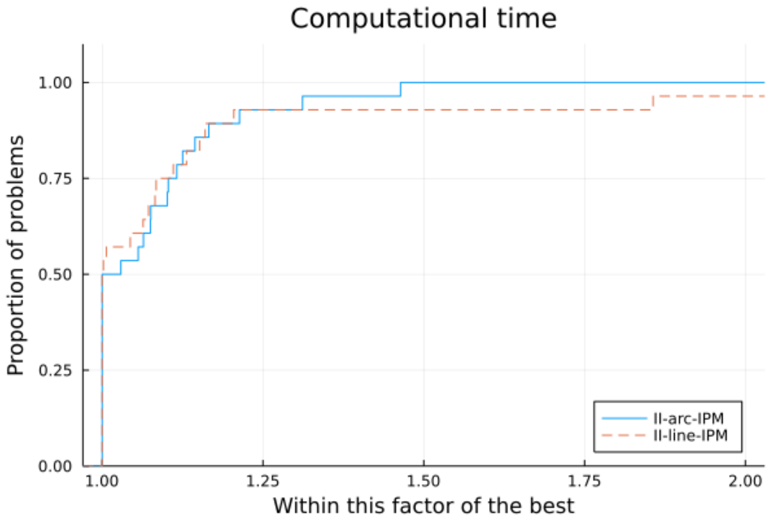

Figure 3: Performance profile of the computation time with the II-arc-IPM and II-line-IPM

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