FORMAL DEFORMATIONS, COHOMOLOGY THEORY AND $L_\infty[1]$-STRUCTURES FOR DIFFERENTIAL LIE ALGEBRAS OF ARBITRARY WEIGHT

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ABSTRACT. Generalising a previous work of Jiang and Sheng, a cohomology theory for differential Lie algebras of arbitrary weight is introduced. The underlying $L_\infty[1]$-structure on the cochain complex is also determined via a generalised version of higher derived brackets. The equivalence between $L_\infty[1]$-structures for absolute and relative differential Lie algebras are established. Formal deformations and abelian extensions are interpreted by using lower degree cohomology groups. Also we introduce the homotopy differential Lie algebras. In a forthcoming paper, we will show that the operad of homotopy (relative) differential Lie algebras is the minimal model of the operad of (relative) differential Lie algebras.

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Introduction

This paper studies the cohomology theory, abelian extensions, formal deformations, and L_∞-structures for differential Lie algebras of arbitrary weight.

0.1. **Deformation theory and minimal models.**

An important method to study a mathematical object is to investigate properties of this object under small deformations. Algebraic deformation theory originated from the deformation theory of complex structures due to Kodaira and Spencer and evolved from ideas of Grothendieck, Illusie, Gerstenhaber, Nijenhuis, Richardson, Deligne, Schlessinger, Stasheff, Goldman, Millson, etc. A fundamental idea of deformation theory is that the deformation theory of any given mathematical object can be governed by a certain differential graded Lie algebra or more generally an L_∞-algebra associated to this mathematical object (whose underlying complex is called the deformation complex). This philosophy has been realised as a theorem in characteristic zero by Lurie [15] and Pridham [19]. It is an important problem to determine explicitly this differential graded Lie algebra or L_∞-algebra governing deformation theory of this mathematical object.

Another important problem about algebraic structures is to study their homotopy versions, just like A_∞-algebras for usual associative algebras. Expected result would be providing a minimal model of the operad governing this algebraic structure. When this operad is Koszul, the Koszul duality for operads [7, 16, 13] gives a perfect solution. More precisely, the cobar construction of the Koszul dual cooperad of the operad in question is the minimal model of this operad. However, when the operad is NOT Koszul, essential difficulties arise and few examples of minimal models have been worked out.

These two problems, say, describing controlling L_∞-algebras and constructing minimal models or more generally cofibrant resolutions, are closed related. In fact, given a cofibrant resolution, in particular, a minimal model, of the operad in question, one can deduce from the cofibrant resolution the deformation complex as well as its L_∞-structure as explained by Kontsevich and Soibelman [12].

However, in practice, a minimal model or a small cofibrant resolution is not known a priori. Wang and Zhou [25, 26] recently found a method to solve these two problems for a large class of non-Koszul operads. The method is in fact the original method of Gerstenhaber [4, 5]. We start by studying formal deformations of the algebraic structure in question and construct the deformation complex as well as L_∞-structure on it from deformation equations. Then we consider the homotopy version which corresponds to Maurer-Cartan elements in the L_∞-structure when appropriate spaces are graded. At last we show the differential graded operad governing the homotopy version is the minimal model of the operad of this algebraic structure, and we also found the Koszul dual which, in this case, is usually a homotopy cooperad.

We aim to deal with differential Lie algebras of arbitrary weight in this paper and its sequel by using the method of [25, 26] as well as derived bracket technique.

0.2. **Differential Lie algebras, old and new.**

As is well known, differential operators appeared in Calculus several centuries ago. The study of differential algebras themselves, as the algebraic study of differential equations, only began in the 1930s at the hands of Ritt [20, 21]. From then on, by the work of many mathematicians in the following decades, the subject has been fully developed into a vast area in mathematics.

By definition a **differential algebra** is an associative commutative algebra A endowed with a linear operator d subject to the Leibniz rule:

$$\mathrm{d}(xy) = \mathrm{d}(x)y + x\mathrm{d}(y), x, y \in A.$$

The discretisation of differential operators are **difference operators**. The action of the difference operator Δ on a real function f defined by:

$$(\Delta f)(x) = f(x + \Delta x) - f(x),$$

which satisfies

$$\Delta(fg) = \Delta(f)g + f\Delta(g) + \Delta(f)\Delta(g).$$

In order to unify the study of differential operator and difference operator, Guo and Keigher introduced weighted differential operators [9].

Definition 0.1. ([9]) Let $\lambda \in \mathbf{k}$ be a fixed element. A **differential associative algebra of weight** λ is an associative algebra A together with a linear operator $\mathrm{d} : A \to A$ such that

$$\mathrm{d}(xy) = \mathrm{d}(x)y + x\mathrm{d}(y) + \lambda\mathrm{d}(x)\mathrm{d}(y), x, y \in A.$$

So an usual differential operator is a differential operator of weight 0 and a difference operator is a differential operator of weight 1.

One of the motivations of Guo and his collaborators to introduce the concept of weighted differential operators is to study Rota's program. Rota [22] proposed to classify "interesting" operators on an algebra. In order to promote this program, Guo et al. introduced the Gröbner-Shirshov basis theory for algebras endowed with linear operators [8, 1, 10, 3], and focused on two types of operator algebras: Rota-Baxter type algebras [27] and differential type algebras [10]. Weighted differential algebras fall into the second type.

Inspired by the above research, Guo and Keigher [9] introduced differential Lie algebras of arbitrary weight.

Definition 0.2. ([9]) Let $\lambda \in \mathbf{k}$ be a fixed element. A **differential Lie algebra of weight** λ is a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ together with a linear operator $\mathrm{d}_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$ such that

$$\mathrm{d}_\mathfrak{g}([x, y]) = [\mathrm{d}_\mathfrak{g}(x), y] + [x, \mathrm{d}_\mathfrak{g}(y)] + \lambda[\mathrm{d}_\mathfrak{g}(x), \mathrm{d}_\mathfrak{g}(y)], \quad \forall x, y \in \mathfrak{g}.$$

A related notion is that of relative difference Lie algebras.

Definition 0.3 ([2, 11])**.** A **LieAct triple** is the triple $(\mathfrak{g}, \mathfrak{h}, \rho)$, where $(\mathfrak{g}, [\ ,\ ]_\mathfrak{g})$ and $(\mathfrak{h}, [\ ,\ ]_\mathfrak{h})$ are Lie algebras and $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ is a homomorphism of Lie algebras, where $\mathrm{Der}(\mathfrak{h})$ is the space of derivations on $\mathfrak{h}$.

Let $(\mathfrak{g}, \mathfrak{h}, \rho)$ be a LieAct triple. A linear map $D : \mathfrak{g} \to \mathfrak{h}$ is called a **relative differential operator of weight** λ if the following equality holds:

$$D([x, y]_\mathfrak{g}) = \rho(x)(D(y)) - \rho(y)(D(x)) + \lambda[D(x), D(y)]_\mathfrak{h}, \quad \forall x, y \in \mathfrak{g}.$$

There has been some papers working on differential Lie algebras with nonzero weight before, but they mainly focus on relative differential Lie algebras of weight 1, also called relative difference algebras. When studying non-Abelian extensions of Lie algebras, Lue [14] introduced the concept of cross homomorphisms which are just relative difference operators of weight 1. Caseiro and Costa [2] studied the framework for the existence of cross homomorphisms: LieAct triple $(\mathfrak{g}, \mathfrak{h}, \rho)$, and gave a differential graded Lie algebra where the Maurer-Cartan elements correspond one-to-one to Lie action triples. Pei, Sheng, Tang, and Zhao [18] defined the cohomology theory of cross homomorphisms on Lie algebras, and used the derived bracket technique to find the deformation of a differential graded Lie algebra where the Maurer-Cartan elements correspond one-to-one to the cross homomorphisms. Jiang and Sheng constructed an L_∞-algebra for relative difference Lie algebra using the technique of derived brackets in [11] and they also introduced cohomology theory for relative difference Lie algebras and absolute difference Lie algebras.

This paper is different from papers of Sheng and his collaborators, firstly generalising from weight 1 to arbitrary weight. Moreover, our ultimate goal is to develop systematically the homotopy theory for both absolute and relative differential Lie algebras of arbitrary weight from an operadic viewpoint, including cohomology theory (or deformation complexes), $L_\infty[1]$-structures, homotopy versions, minimal models, and Koszul dual homotopy cooperads. This task will be completed in two papers. This paper is the elementary part of this project, which contains only cohomology theory (or deformation complexes), $L_\infty[1]$-structures and homotopy versions. In a forthcoming paper, we shall prove the operad of homotopy absolute (resp. relative) differential Lie algebras is the minimal model of the operad of absolute (resp. relative) differential Lie algebras, thus justifying the homotopy version we found is the right one.

Apart from the cohomology theory and $L_\infty[1]$-structures, in this paper we introduce a generalised version of derived bracket technique which replace Lie subalgebras by injective maps of Lie algebras. Moreover, this generalised version permits us to establish the equivalence between $L_\infty[1]$-structures for relative and absolute differential Lie algebras. This equivalence is believed by experts and firstly presented explicitly in this paper for differential Lie algebras. In the forthcoming paper, we will establish the equivalence between minimal models for relative and absolute differential Lie algebras by using a coloring and decoloring procedure for (coloured) operads.

0.3. **Layout of the paper.**

The paper is organized as follows.

The first section is of preliminary nature, which contains results which are more or less known. After fixing some notations in Subsection 1.1, we recall basic notions and facts about differential Lie algebra and their representations in Subsection 1.2. In particular, a key descending property for differential representations Lemma 1.7 is presented. A differential Lie algebra is the combination of the underlying Lie algebra and the differential operator. In this light we build the cohomology theory of a differential Lie algebra by combining its components from the Lie algebra and from the differential operator. In Subsection 1.3, we establish the cohomology theory for differential operators of arbitrary weight, which is quite different from the one for the underlying algebra unless the weight is zero. In Subsection 1.4, we combine the Chevalley-Eilenberg cohomology for Lie algebras and the just established cohomology for differential operators of arbitrary weight to define the cohomology of differential Lie algebras of arbitrary weight, with the cochain maps again posing extra challenges when the weight is not zero. The proofs of this section will be given in Subsection 5.1, once the $L_\infty[1]$-structure has been established in Section 4.

As applications and further justification of our cohomology theory for differential Lie algebras, in Section 2, we apply the theory to study abelian extensions of differential Lie algebras of arbitrary weight, and show that abelian extensions are classified by the second cohomology group of the differential Lie algebras.

Further, in Section 3, we apply the above cohomology theory to study formal deformations of differential Lie algebras of arbitrary weight. In particular, we show that if the second cohomology group of a differential Lie algebra with coefficients in the regular representation is trivial, then this differential Lie algebra is rigid.

To deal with the weight case and get $L_\infty[1]$-structure in Section 4, after a reminder on $L_\infty[1]$-structures in Subsection 4.1, a generalised version of the derived bracket technique is introduced in Subsection 4.2, and the weight λ is also incorporated into the statements, which permits us to deal with (absolute) differential Lie algebras of arbitrary weight as well as relative differential Lie algebras of arbitrary weight. With this generalised version at hand, we build the $L_\infty[1]$-structure of relative differential Lie algebras in Subsection 4.3, generalising the result of Jiang and Sheng [11] from weight 1 to arbitrary weight. In Subsection 4.4, we build the $L_\infty[1]$-structure of (absolute) differential Lie algebras. The comparison with the $L_\infty[1]$-structures for relative and absolute differential Lie algebras is provided in Subsection 4.5.

At last in Section 5, based on the $L_\infty[1]$-structure of (absolute) differential Lie algebras found in Section 4, we verify that the cohomoloy theory of Section 1 is the right one in Subsection 5.1, by using a trivial extension construction in order to deal with coefficients. In Subsection 5.2, the structure of homotopy differential Lie algebra with weight is introduced as Maurer-Cartan elements in the $L_\infty[1]$-structure of (absolute) differential Lie algebras when the spaces involved are graded.

1. Differential Lie algebras of arbitrary weight and their cohomology theory

This section recalls some background on differential Lie algebras and their representations and introduce their cohomology theory generalsing the theory introduced by Jiang and Sheng [11] from weight 1 to arbitrary weight.

1.1. Notations.

Throughout this paper, let $\mathbf{k}$ be a field of characteristic 0. Except specially stated, vector spaces are $\mathbf{k}$-vector spaces and all tensor products are taken over $\mathbf{k}$. We use cohomological grading. We will employ Koszul sign rule to determine signs, that is, when exchanging the positions of two graded objects in an expression, we have to multiply the expression by a power of -1 whose exponent is the product of their degrees.

The suspension of a graded space $V = \oplus_{n\in\mathbb{Z}} V^n$ is the graded space sV with $(sV)^n = V^{n+1}$ for any $n \in \mathbb{Z}$. Write $sv \in (sV)^n$ for $v \in V^{n+1}$.

Let $V = \oplus_{n\in\mathbb{Z}} V^n$ be a graded vector space. Recall that the graded symmetric algebra $S(V)$ of V is defined to be the quotient of the tensor algebra $T(V)$ by the two-sided ideal I generated by $x\otimes y - (-1)^{|x||y|} y \otimes x$ for all homogeneous elements $x, y \in V$. For $x_1 \otimes \cdots \otimes x_n \in V^{\otimes n} \subseteq T(V)$, write $x_1 \odot x_2 \odot \cdots \odot x_n$ its image in $S(V)$. For homogeneous elements $x_1, \dots, x_n \in V$ and $\sigma \in S_n$ which is the symmetric group in n variables, the Koszul sign $\varepsilon(\sigma) := \varepsilon(\sigma; x_1, \dots, x_n)$ is defined by

$$x_1 \odot x_2 \odot \cdots \odot x_n = \varepsilon(\sigma) x_{\sigma(1)} \odot x_{\sigma(2)} \odot \cdots \odot x_{\sigma(n)} \in S(V).$$

Denote by $S^n(V)$ the image of $V^{\otimes n}$ in $S(V)$.

Let $V = \oplus_{n\in\mathbb{Z}} V^n$ be a graded vector space. Recall that the graded exterior algebra $\bigwedge(V)$ of V is defined to be the quotient of the tensor algebra $T(V)$ by the two-sided ideal generated by $x\otimes y + (-1)^{|x||y|} y \otimes x$ for all homogeneous elements $x, y \in V$. For $x_1 \otimes \cdots \otimes x_n \in V^{\otimes n} \subseteq T(V)$, write $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ its image in $\bigwedge(V)$. For homogeneous elements $x_1, \dots, x_n \in V$ and $\sigma \in S_n$, the sign $\chi(\sigma) := \chi(\sigma; x_1, \dots, x_n)$ is defined by

$$x_1 \wedge x_2 \wedge \cdots \wedge x_n = \chi(\sigma) x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \cdots \wedge x_{\sigma(n)} \in \bigwedge(V).$$

Denote by $\bigwedge^n(V)$ the image of $V^{\otimes n}$ in $\bigwedge(V)$. Obviously $\chi(\sigma) = \varepsilon(\sigma)\mathrm{sgn}(\sigma)$, where $\mathrm{sgn}(\sigma)$ is the signature of σ.

Recall that for a graded vector space V, $S^n(sV)$ is isomorphic to $s^n \wedge^n V$ via the map

$$sv_1 \odot sv_2 \odot \cdots \odot sv_n \mapsto (-1)^{(n-1)|v_1|+(n-2)|v_2|+\cdots+|v_{n-1}|} s^n(v_1 \wedge v_2 \wedge \cdots \wedge v_n), \tag{1}$$

for $v_1, \ldots, v_n \in V$ homogeneous elements in V.

Let $n \geq 1$. For $0 \leq i_1, \ldots, i_r \leq n$ with $i_1 + \cdots + i_r = n$, $\mathrm{Sh}(i_1, i_2, \ldots, i_r)$ is the set of $(i_1, \ldots, i_r)$-shuffles, i.e., those permutation $\sigma \in S_n$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(i_1),\ \sigma(i_1+1) < \cdots < \sigma(i_1+i_2),\ \ldots,\ \sigma(i_1+\cdots+i_{r-1}+1) < \cdots < \sigma(n).$$

Let $\mathrm{PSh}(i_1, i_2, \ldots, i_r)$ be the subset of $\mathrm{Sh}(i_1, i_2, \ldots, i_r)$ whose elements $(i_1, \ldots, i_r)$-shuffles satisfy:

$$\sigma(1) < \sigma(i_1+1) < \sigma(i_1+i_2+1) < \cdots < \sigma(i_1+\cdot+i_{r-1}+1).$$

1.2. **Differential Lie algebras with weight and their representations.**

In this subsection, we recall basic notions about differential Lie algebras with weight and their representations.

Definition 1.1. ([9]) Let $\lambda \in \mathbf{k}$ be a fixed element. A **differential Lie algebra of weight** λ is a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ together with a linear operator $\mathrm{d}_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$ such that

$$\mathrm{d}_\mathfrak{g}([x,y]) = [\mathrm{d}_\mathfrak{g}(x), y] + [x, \mathrm{d}_\mathfrak{g}(y)] + \lambda[\mathrm{d}_\mathfrak{g}(x), \mathrm{d}_\mathfrak{g}(y)], \quad \forall x, y \in \mathfrak{g}. \tag{2}$$

Such an operator $\mathrm{d}_\mathfrak{g}$ is called a **differential operator of weight** λ or a **derivation of weight** λ.

Given two differential Lie algebras $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$, $(\mathfrak{h}, \mathrm{d}_\mathfrak{h})$ of the same weight λ, a **homomorphism of differential Lie algebras** from $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ to $(\mathfrak{h}, \mathrm{d}_\mathfrak{h})$ is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that $\varphi \circ \mathrm{d}_\mathfrak{g} = \mathrm{d}_\mathfrak{h} \circ \varphi$. We denote by DL_λ the category of differential Lie algebras of weight λ.

A differential operator of weight 1 is also called a crossed homomorphism and a differential Lie algebra of weight 1 is also called a difference Lie algebra [18, 11].

Recall that a representation of a Lie algebra $\mathfrak{g}$ is a pair (V, ρ), where V is a vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V),\ x \mapsto (v \mapsto \rho(x)v)$ is a homomorphism of Lie algebras for all $x, y \in \mathfrak{g}$ and $v \in V$.

Definition 1.2. Let $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ be a differential Lie algebra.

(i) A **representation** over the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ or a **differential representation** is a triple (V, ρ, d_V), where $\mathrm{d}_V \in \mathfrak{gl}(V)$, and (V, ρ) is a representation over the Lie algebra $\mathfrak{g}$, such that for all $x, y \in \mathfrak{g}$, $v \in V$, the following equality holds:

$$\mathrm{d}_V(\rho(x)v) \quad = \quad \rho(\mathrm{d}_\mathfrak{g}(x))v + \rho(x)\mathrm{d}_V(v) + \lambda\rho(\mathrm{d}_\mathfrak{g}(x))\mathrm{d}_V(v).$$

(ii) Given two representations $(U, \rho^U, \mathrm{d}_U)$, $(V, \rho^V, \mathrm{d}_V)$ over $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$, a linear map $f : U \to V$ is called a **homomorphism** of representations, if

$$f \circ \rho^U(x) = \rho^V(x) \circ f, \quad \forall x \in \mathfrak{g}, \quad \text{and} \quad f \circ \mathrm{d}_U = \mathrm{d}_V \circ f.$$

Example 1.3. For a differential Lie algebra $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$, the usual adjoint representation

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),\ x \mapsto (y \mapsto [x, y])$$

over the Lie algebra $\mathfrak{g}$ is also a representation of this differential Lie algebra. It is also called the **adjoint representation** over the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$, denoted by $\mathfrak{g}_{\mathrm{ad}}$.

Example 1.4. Let $\mathfrak{g}$ be a Lie algebra and V be a representation of it. Then $(\mathfrak{g}, \mathrm{Id}_\mathfrak{g})$ is a differential Lie algebra of weight -1 and the pair (V, Id_V) gives a differential representation.

Example 1.5. Let $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ be a differential Lie algebra together with a differential representation (V, d_V). Then for arbitrary nonzero scalar $\kappa \in \mathbf{k}$, $(\mathfrak{g}, \kappa \mathrm{d}_\mathfrak{g})$ is a differential Lie algebra of weight $\frac{\lambda}{\kappa}$ and the pair $(V, \kappa\, \mathrm{d}_V)$ is a differential representation.

It is straightforward to obtain the following result:

Proposition 1.6. *Let (V, ρ, d_V) be a representation of the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$. Then $(\mathfrak{g} \oplus V, \mathrm{d}_\mathfrak{g} + \mathrm{d}_V)$ is a differential Lie algebra, where the Lie algebra structure on $\mathfrak{g} \oplus V$ is given by*

$$\{x + u, y + v\} := [x, y] + \rho(x)v - \rho(y)u, \quad \forall x, y \in \mathfrak{g},\ u, v \in V.$$

*This new differential Lie algebra is denoted by $\mathfrak{g} \ltimes V$, called the **trivial extension** of $\mathfrak{g}$ by V.*

We need the following observation which is a weighted version of [14, Proposition 3.1]:

Lemma 1.7. *Let (V, ρ, d_V) be a representation of the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$. There exists a new representation $(V, \rho_\lambda, \mathrm{d}_V)$ over $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$, where ρ_λ is given by:*

$$\rho_\lambda(x)v = \rho(x + \lambda \mathrm{d}_\mathfrak{g}(x))v, \quad \forall x \in \mathfrak{g}, v \in V.$$

Denote the new presentation by $V_\lambda := (V, \rho_\lambda, \mathrm{d}_V)$.

1.3. **Cohomology of differential operators.**

In this subsection, we define a cohomology theory of differential operators.

Recall that the Chevalley-Eilenberg cochain complex of a Lie algebra $\mathfrak{g}$ with coefficients in a representation (V, ρ) is the cochain complex $(\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}, V), \partial^*_{\mathrm{Lie}})$, where for $n \geq 0$, $\mathrm{C}^n_{\mathrm{Lie}}(\mathfrak{g}, V) = \mathrm{Hom}(\wedge^n \mathfrak{g}, V)$ (in particular, $\mathrm{C}^0_{\mathrm{Lie}}(\mathfrak{g}, V) = V$) and the coboundary operator

$$\partial^n_{\mathrm{Lie}} : \mathrm{C}^n_{\mathrm{Lie}}(\mathfrak{g}, V) \longrightarrow \mathrm{C}^{n+1}_{\mathrm{Lie}}(\mathfrak{g}, V), n \geq 0$$

is given by

$$\begin{aligned}\partial^n_{\mathrm{Lie}}(f)(x_1, \ldots, x_{n+1}) = &\sum_{i=1}^{n+1} (-1)^{i+n} \rho(x_i) f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})\\ &+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+n+1} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}),\end{aligned}$$

for all $f \in \mathrm{C}^n_{\mathrm{Lie}}(\mathfrak{g}, V)$, $x_1, \ldots, x_{n+1} \in \mathfrak{g}$, where $\hat{x}_i$ means deleting this element. The corresponding Chevalley-Eilenberg cohomology is denoted by $\mathrm{H}^*_{\mathrm{Lie}}(\mathfrak{g}, V)$.

When V is the adjoint representation $\mathfrak{g}_{\mathrm{ad}}$, we write $\mathrm{H}^n_{\mathrm{Lie}}(\mathfrak{g}) := \mathrm{H}^n_{\mathrm{Lie}}(\mathfrak{g}, V), n \geq 0$.

Definition 1.8. Let $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ be a differential Lie algebra of weight λ and (V, d_V) be a differential representation over $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$. The **cochain complex of the differential operator** $\mathrm{d}_\mathfrak{g}$ **with coefficients in the differential representation** (V, d_V) is defined to be the Chevalley-Eilenberg cochain complex of the Lie algebra $\mathfrak{g}$ with coefficients in the new differential representation V_λ, denoted by $(\mathrm{C}^*_{\mathsf{DO}_\lambda}(\mathfrak{g}, V), \partial^*_{\mathsf{DO}_\lambda}) := (\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}, V_\lambda), \partial^*_{\mathrm{Lie}})$. The corresponding cohomology, denoted by $\mathrm{H}^*_{\mathsf{DO}_\lambda}(\mathfrak{g}, V)$, is called the **cohomology of the differential operator** $\mathrm{d}_\mathfrak{g}$ **with coefficients in the representation** $(V_\lambda, \mathrm{d}_V)$.

More precisely, for $n \geq 0$, $\mathrm{C}^n_{\mathsf{DO}_\lambda}(\mathfrak{g}, V) = \mathrm{Hom}(\wedge^n \mathfrak{g}, V)$ and the coboundary operator

$\partial^n_{\mathsf{DO}_\lambda}: \mathrm{C}^n_{\mathsf{DO}_\lambda}(\mathfrak{g}, V) \to \mathrm{C}^{n+1}_{\mathsf{DO}_\lambda}(\mathfrak{g}, V)$ is given by

$$\begin{aligned}\partial^n_{\mathsf{DO}_\lambda}(f)(x_1, \dots, x_{n+1}) &= \sum_{i=1}^{n+1}(-1)^{i+n}\rho_\lambda(x_i)f(x_1, \dots, \hat{x}_i, \dots, x_{n+1})\\ &\quad + \sum_{1\le i<j\le n+1}(-1)^{i+j+n+1}f([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1})\\ &= \sum_{i=1}^{n+1}(-1)^{i+n}\rho(x_i + \lambda \mathrm{d}_\mathfrak{g}(x_i))f(x_1, \dots, \hat{x}_i, \dots, x_{n+1})\\ &\quad + \sum_{1\le i<j\le n+1}(-1)^{i+j+n+1}f([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}),\end{aligned}$$

for all $f \in \mathrm{C}^n_{\mathsf{DO}_\lambda}(\mathfrak{g}, V),\ x_1, \dots, x_{n+1} \in \mathfrak{g}$.

1.4. **Cohomology of differential Lie algebras.**

We now combine the classical Chevalley-Eilenberg cohomology of Lie algebras and the newly defined cohomology of differential operators of weight λ to define the cohomology of the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ with coefficients in the representation (V, d_V). In fact, we will define the cochain complex of a differential Lie algebras as, up to shift and signs, the mapping cone of a cochain map from the Chevalley-Eilenberg cochain complex of the Lie algebra to the cochain complex of the differential operators of weight λ.

Notice that [11] has introduced the cohomology theory of a difference Lie algebra with coefficients in a representation. We in fact generalise their construction from weight 1 to arbitrary weight.

Introduce the linear maps

$$\delta^n : \mathrm{C}^n_{\mathsf{Lie}}(\mathfrak{g}, V) \to \mathrm{C}^n_{\mathsf{DO}_\lambda}(\mathfrak{g}, V)$$

by

$$\delta^n f(x_1, \dots, x_n) := \sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\dots<i_k\le n} f(x_1, \dots, \mathrm{d}_\mathfrak{g}(x_{i_1}), \dots, \mathrm{d}_\mathfrak{g}(x_{i_k}), \dots, x_n) - \mathrm{d}_V f(x_1, \dots, x_n),$$

for $f \in \mathrm{C}^n_{\mathsf{Lie}}(\mathfrak{g}, V)$, $n \ge 1$ and

$$\delta^0 v = -\mathrm{d}_V(v), \quad \forall v \in \mathrm{C}^0_{\mathsf{Lie}}(\mathfrak{g}, V) = V.$$

Proposition 1.9. *The linear map δ is a cochain map from the cochain complex* $(\mathrm{C}^*_{\mathsf{Lie}}(\mathfrak{g}, V), \partial^*_{\mathsf{Lie}})$ *to* $(\mathrm{C}^*_{\mathsf{DO}_\lambda}(\mathfrak{g}, V), \partial^*_{\mathsf{DO}_\lambda})$.

Although a direct proof is possible, we shall deduce the above result from the $L_\infty[1]$-structure; see Proposition 5.2.

Remark 1.10. Note that $\mathrm{C}^n_{\mathsf{DO}_\lambda}(\mathfrak{g}, V)$ equals to $\mathrm{C}^n_{\mathsf{Lie}}(\mathfrak{g}, V)$ as linear spaces but they are not equal as cochain complexes unless $\lambda = 0$. When λ is not zero, a new representation structure is needed to define $\partial^*_{\mathsf{DO}_\lambda}$ which eventually leads to the rather long and technical argument in order to establish the cochain map in Proposition 1.9.

Now we can define the cochain complex of a differential Lie algebras $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ with coefficients in the differential representation (V, d_V).

Definition 1.11. Define the **cochain complex** $(\mathrm{C}^*_{\mathsf{DL}_\lambda}(\mathfrak{g},V),\partial^*_{\mathsf{DL}_\lambda})$ **of the differential Lie algebra** $(\mathfrak{g},\mathrm{d}_\mathfrak{g})$ **with coefficients in the differential representation** (V,d_V) to be the negative shift of the mapping cone of the cochain map $\delta^*:(\mathrm{C}^*_{\mathsf{Lie}}(\mathfrak{g},V),\partial^*_{\mathsf{Lie}})\to(\mathrm{C}^*_{\mathsf{DO}_\lambda}(\mathfrak{g},V),\partial^*_{\mathsf{DO}_\lambda})$. More precisely, the space of n-cochains is given by

$$\mathrm{C}^n_{\mathsf{DL}_\lambda}(\mathfrak{g},V):=\begin{cases}\mathrm{C}^n_{\mathsf{Lie}}(\mathfrak{g},V)\oplus\mathrm{C}^{n-1}_{\mathsf{DO}_\lambda}(\mathfrak{g},V), & n\geq 1,\\ \mathrm{C}^0_{\mathsf{Lie}}(\mathfrak{g},V)=V, & n=0.\end{cases}$$

and the differential $\partial^n_{\mathsf{DL}_\lambda}:\mathrm{C}^n_{\mathsf{DL}_\lambda}(g,V)\to\mathrm{C}^{n+1}_{\mathsf{DL}_\lambda}(g,V)$ by

$$\begin{aligned}\partial^n_{\mathsf{DL}_\lambda}(f,g) &:= (\partial^n_{\mathsf{Lie}}f,-\partial^{n-1}_{\mathsf{DO}_\lambda}g-\delta^n f),\quad \forall f\in\mathrm{C}^n_{\mathsf{Lie}}(\mathfrak{g},V),\ g\in\mathrm{C}^{n-1}_{\mathsf{DO}_\lambda}(\mathfrak{g},V),\quad n\geq 1,\\ \partial^0_{\mathsf{DL}_\lambda}v &:= (\partial^0_{\mathsf{Lie}}v,\delta^0 v),\quad \forall v\in\mathrm{C}^0_{\mathsf{Lie}}(\mathfrak{g},V)=V.\end{aligned}$$

Definition 1.12. The cohomology of the cochain complex $(\mathrm{C}^*_{\mathsf{DL}_\lambda}(\mathfrak{g},V),\partial^*_{\mathsf{DL}_\lambda})$, denoted by $\mathrm{H}^*_{\mathsf{DL}_\lambda}(\mathfrak{g},V)$, is called the **cohomology of the differential Lie algebra** $(\mathfrak{g},\mathrm{d}_\mathfrak{g})$ with coefficients in the differential representation (V,d_V).

In the next two sections, we will use the following remarks.

Remark 1.13. We compute 0-cocycles, 1-cocycles and 2-cocycles of the cochain complex $\mathrm{C}^*_{\mathsf{DL}_\lambda}(\mathfrak{g},V)$.

It is obvious that for all $v\in V$, $\partial^0_{\mathsf{DL}_\lambda}v=0$ if and only if

$$\partial^0_{\mathsf{Lie}}v=0,\quad \mathrm{d}_V(v)=0.$$

For all $(f,v)\in\mathrm{Hom}(\mathfrak{g},V)\oplus V$, $\partial^1_{\mathsf{DL}_\lambda}(f,v)=0$ if and only if $\partial^1_{\mathsf{Lie}}f=0$ and

$$\rho_\lambda(x)v=f(\mathrm{d}_\mathfrak{g}(x))-\mathrm{d}_V(f(x)),\quad \forall x\in\mathfrak{g}.$$

For all $(f,g)\in\mathrm{Hom}(\wedge^2\mathfrak{g},V)\oplus\mathrm{Hom}(\mathfrak{g},V)$, $\partial^2_{\mathsf{DL}_\lambda}(f,g)=0$ if and only if $\partial^2_{\mathsf{Lie}}f=0$, and

(3)
$$\rho_\lambda(x)g(y)-\rho_\lambda(y)g(x)-g([x,y])=-\lambda f(\mathrm{d}_\mathfrak{g}(x),\mathrm{d}_\mathfrak{g}(y))-f(\mathrm{d}_\mathfrak{g}(x),y)-f(x,\mathrm{d}_\mathfrak{g}(y))+\mathrm{d}_V(f(x,y)),$$

for all $x,y\in\mathfrak{g}$.

Remark 1.14. we shall need a subcomplex of the cochain complex $\mathrm{C}^*_{\mathsf{DL}_\lambda}(\mathfrak{g},V)$. Let

$$\tilde{\mathrm{C}}^n_{\mathsf{DL}_\lambda}(\mathfrak{g},V):=\begin{cases}\mathrm{C}^n_{\mathsf{Lie}}(\mathfrak{g},V)\oplus\mathrm{C}^{n-1}_{\mathsf{DO}_\lambda}(\mathfrak{g},V), & n\geq 2,\\ \mathrm{C}^1_{\mathsf{Lie}}(\mathfrak{g},V), & n=1,\\ 0, & n=0.\end{cases}$$

Then it is obvious that $(\tilde{\mathrm{C}}^*_{\mathsf{DL}_\lambda}(\mathfrak{g},V)=\oplus_{n=0}^\infty\tilde{\mathrm{C}}^n_{\mathsf{DL}_\lambda}(\mathfrak{g},V),\partial^*_{\mathsf{DL}_\lambda})$ is a subcomplex of the cochain complex $(\mathrm{C}^*_{\mathsf{DL}_\lambda}(\mathfrak{g},V),\partial^*_{\mathsf{DL}_\lambda})$. We denote its cohomology by $\tilde{\mathrm{H}}^*_{\mathsf{DL}_\lambda}(\mathfrak{g},V)$. Obviously, $\tilde{\mathrm{H}}^n_{\mathsf{DL}_\lambda}(\mathfrak{g},V)=\mathrm{H}^n_{\mathsf{DL}_\lambda}(\mathfrak{g},V)$ for $n>2$.

2. Abelian extensions of differential Lie algebras

In this section, we study abelian extensions of differential Lie algebras of weight λ and show that they are classified by the second cohomology, as one would expect of a good cohomology theory.

2.1. **Abelian extensions.**

Notice that a vector space V together with a linear endomorphism d_V can be considered as a **trivial differential Lie algebra of weight** λ endowed with the trivial Lie bracket $[u,v]=0$ for all $u,v\in V$.

Definition 2.1. An **abelian extension** of differential Lie algebras is a short exact sequence of homomorphisms of differential Lie algebras

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "i"] \arrow[d, "\mathrm{d}_V"'] & \hat{\mathfrak{g}} \arrow[r, "p"] \arrow[d, "\mathrm{d}_{\hat{\mathfrak{g}}}"'] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{d}_{\mathfrak{g}}"'] & 0 \\ 0 \arrow[r] & V \arrow[r, "i"] & \hat{\mathfrak{g}} \arrow[r, "p"] & \mathfrak{g} \arrow[r] & 0 \end{tikzcd}$$

where (V,d_V) is a trivial differential Lie algebra. We will call $(\hat{\mathfrak{g}},\mathrm{d}_{\hat{\mathfrak{g}}})$ an **abelian extension** of $(\mathfrak{g},\mathrm{d}_{\mathfrak{g}})$ by (V,d_V).

Definition 2.2. Let $(\hat{\mathfrak{g}}_1,\mathrm{d}_{\hat{\mathfrak{g}}_1})$ and $(\hat{\mathfrak{g}}_2,\mathrm{d}_{\hat{\mathfrak{g}}_2})$ be two abelian extensions of $(\mathfrak{g},\mathrm{d}_{\mathfrak{g}})$ by (V,d_V). They are said to be **isomorphic** if there exists an isomorphism of differential Lie algebras $\zeta:(\hat{\mathfrak{g}}_1,\mathrm{d}_{\hat{\mathfrak{g}}_1})\to(\hat{\mathfrak{g}}_2,\mathrm{d}_{\hat{\mathfrak{g}}_2})$ such that the following commutative diagram holds:

$$\begin{tikzcd} 0 \arrow[r] & (V,\mathrm{d}_V) \arrow[r, "i"] \arrow[d, equal] & (\hat{\mathfrak{g}}_1,\mathrm{d}_{\hat{\mathfrak{g}}_1}) \arrow[r, "p"] \arrow[d, "\zeta"'] & (\mathfrak{g},\mathrm{d}_{\mathfrak{g}}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (V,\mathrm{d}_V) \arrow[r, "i"] & (\hat{\mathfrak{g}}_2,\mathrm{d}_{\hat{\mathfrak{g}}_2}) \arrow[r, "p"] & (\mathfrak{g},\mathrm{d}_{\mathfrak{g}}) \arrow[r] & 0. \end{tikzcd}$$

A **section** of an abelian extension $(\hat{\mathfrak{g}},\mathrm{d}_{\hat{\mathfrak{g}}})$ of $(\mathfrak{g},\mathrm{d}_{\mathfrak{g}})$ by (V,d_V) is a linear map $s:\mathfrak{g}\to\hat{\mathfrak{g}}$ such that $p\circ s=\mathrm{Id}_{\mathfrak{g}}$.

Now for an abelian extension $(\hat{\mathfrak{g}},\mathrm{d}_{\hat{\mathfrak{g}}})$ of $(\mathfrak{g},\mathrm{d}_{\mathfrak{g}})$ by (V,d_V) with a section $s:\mathfrak{g}\to\hat{\mathfrak{g}}$, then there exist a unique linear map $t:\hat{\mathfrak{g}}\to V$ such that:

$$t\circ i=\mathrm{Id}_V,\, t\circ s=0,\, i\circ t+s\circ p=\mathrm{Id}_{\hat{\mathfrak{g}}},$$

then we can define

$$\rho(x)v:=t[s(x),i(v)]_{\hat{\mathfrak{g}}},\quad \forall x\in\mathfrak{g},v\in V.$$

For convenience, we denote it by $\rho(s(x))v$, and observe that $[s(x),i(v)]_{\hat{\mathfrak{g}}}\in i(V)$, and we identify V with $i(V)$.

Now we get a linear map $\rho:\mathfrak{g}\to\mathfrak{gl}(V),\ x\mapsto(v\mapsto\rho(x)v)$.

Proposition 2.3. *With the above notations,* (V,ρ,d_V) *is a representation over the differential Lie algebra* $(\mathfrak{g},\mathrm{d}_{\mathfrak{g}})$.

Proof. For arbitrary $x,y\in\mathfrak{g},v\in V$, since $s([x,y]_{\mathfrak{g}})-[s(x),s(y)]_{\hat{\mathfrak{g}}}\in V$ implies $s([x,y]_{\mathfrak{g}})v=[s(x),s(y)]_{\hat{\mathfrak{g}}}v$, we have

$$\rho([x,y]_{\mathfrak{g}})(v)=\rho(s([x,y]_{\mathfrak{g}}))v=\rho([s(x),s(y)]_{\hat{\mathfrak{g}}})v\xlongequal{\text{Jacobi identity}}\rho(x)\rho(y)(v)-\rho(y)\rho(x)(v).$$

Hence, ρ is a Lie algebra homomorphism. Moreover, $\mathrm{d}_{\hat{\mathfrak{g}}}(s(x))-s(\mathrm{d}_{\mathfrak{g}}(x))\in V$ means that

$$\rho(\mathrm{d}_{\hat{\mathfrak{g}}}(s(x)))v=\rho(s(\mathrm{d}_{\mathfrak{g}}(x)))v.$$

Thus we have

$$\mathrm{d}_V(\rho(x)v)=\mathrm{d}_V(\rho(s(x))v)=\mathrm{d}_{\hat{\mathfrak{g}}}(\rho(s(x))v)$$

$$
\begin{aligned}
&= \rho(\mathrm{d}_{\hat{\mathfrak{g}}}(s(x)))v + \rho(s(x))\mathrm{d}_{\hat{\mathfrak{g}}}(v) + \lambda\rho(\mathrm{d}_{\hat{\mathfrak{g}}}(s(x)))\mathrm{d}_{\hat{\mathfrak{g}}}(v)\\
&= \rho(s(\mathrm{d}_{\mathfrak{g}}(x)))v + \rho(s(x))\mathrm{d}_V(v) + \lambda\rho(s(\mathrm{d}_{\mathfrak{g}}(x)))\mathrm{d}_V(v)\\
&= \rho(\mathrm{d}_{\mathfrak{g}}(x))v + \rho(x)\mathrm{d}_V(v) + \lambda\rho(\mathrm{d}_{\mathfrak{g}}(x))\mathrm{d}_V(v).
\end{aligned}
$$

Hence, (V, ρ, d_V) is a representation over $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$. □

We further define linear maps $\psi : \mathfrak{g} \otimes \mathfrak{g} \to V$ and $\chi : \mathfrak{g} \to V$ respectively by

$$
\begin{aligned}
\psi(x, y) &= [s(x), s(y)]_{\hat{\mathfrak{g}}} - s([x, y]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g},\\
\chi(x) &= \mathrm{d}_{\hat{\mathfrak{g}}}(s(x)) - s(\mathrm{d}_{\mathfrak{g}}(x)), \quad \forall x \in \mathfrak{g}.
\end{aligned}
$$

We transfer the differential Lie algebra structure on $\hat{\mathfrak{g}}$ to $\mathfrak{g} \oplus V$ by endowing $\mathfrak{g} \oplus V$ with a multiplication $[\cdot, \cdot]_\psi$ and a differential operator d_χ defined by

$$[x + u, y + v]_\psi = [x, y] + \rho(x)v - \rho(y)u + \psi(x, y), \ \forall x, y \in \mathfrak{g},\ u, v \in V, \tag{4}$$

$$\mathrm{d}_\chi(x + v) = \mathrm{d}_{\mathfrak{g}}(x) + \chi(x) + \mathrm{d}_V(v), \ \forall x \in \mathfrak{g},\ v \in V. \tag{5}$$

Proposition 2.4. *The triple $(\mathfrak{g} \oplus V, [\cdot, \cdot]_\psi, \mathrm{d}_\chi)$ is a differential Lie algebra if and only if (ψ, χ) is a 2-cocycle of the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ with the coefficient in (V, d_V).*

Proof. If $(\mathfrak{g} \oplus V, [\cdot, \cdot]_\psi, \mathrm{d}_\chi)$ is a differential Lie algebra, then $[\cdot, \cdot]_\psi$ implies

$$\psi(x, [y, z]) + \rho(x)\psi(y, z) + \psi(y, [z, x]) + \rho(y)\psi(z, x) + \psi(z, [x, y]) + \rho(z)\psi(x, y) = 0, \tag{6}$$

for arbitrary $x, y, z \in \mathfrak{g}$. Since d_χ satisfies (2), we deduce that

$$\chi([x, y]) - \rho_\lambda(x)\chi(y) + \rho_\lambda(y)\chi(x) + \mathrm{d}_V(\psi(x, y)) - \psi(\mathrm{d}_{\mathfrak{g}}(x), y) - \psi(x, \mathrm{d}_{\mathfrak{g}}(y)) - \lambda\psi(\mathrm{d}_{\mathfrak{g}}(x), \mathrm{d}_{\mathfrak{g}}(y)) = 0. \tag{7}$$

Hence, (ψ, χ) is a 2-cocycle, see Eq. (3).

Conversely, if (ψ, χ) satisfies equalities (6) and (7), one can easily check that $(\mathfrak{g} \oplus V, [\cdot, \cdot]_\psi, \mathrm{d}_\chi)$ is a differential Lie algebra. □

2.2. **Classificaiton for Abelian extensions.**

Now we are ready to classify abelian extensions of a differential Lie algebra.

Theorem 2.5. *Let V be a vector space and $\mathrm{d}_V \in \mathrm{End}_{\mathbf{k}}(V)$. Then abelian extensions of a differential Lie algebra $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ by (V, d_V) are classified by the second cohomology group $\tilde{\mathrm{H}}^2_{\mathsf{DL}_\lambda}(\mathfrak{g}, V)$ of $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ with coefficients in the representation (V, d_V).*

Proof. Let $(\hat{\mathfrak{g}}, \mathrm{d}_{\hat{\mathfrak{g}}})$ be an abelian extension of $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ by (V, d_V). We choose a section $s : \mathfrak{g} \to \hat{\mathfrak{g}}$ to obtain a 2-cocycle (ψ, χ) by Proposition 2.4. We first show that the cohomological class of (ψ, χ) does not depend on the choice of sections. Indeed, let s_1 and s_2 be two distinct sections providing 2-cocycles (ψ_1, χ_1) and (ψ_2, χ_2) respectively. We define $\phi : \mathfrak{g} \to V$ by $\phi(x) = s_1(x) - s_2(x)$. Then

$$
\begin{aligned}
\psi_1(x, y) &= [s_1(x), s_1(y)] - s_1([x, y])\\
&= [s_2(x) + \phi(x), s_2(y) + \phi(y)] - (s_2([x, y]) + \phi([x, y]))\\
&= ([s_2(x), s_2(y)] - s_2([x, y])) + [s_2(x), \phi(y)] + [\phi(x), s_2(y)] - \phi([x, y])\\
&= ([s_2(x), s_2(y)] - s_2([x, y])) + [x, \phi(y)] + [\phi(x), y] - \phi([x, y])\\
&= \psi_2(x, y) + \partial^1_{\mathsf{Lie}}\phi(x, y),
\end{aligned}
$$

and

$$
\begin{aligned}
\chi_1(x) &= \mathrm{d}_{\hat{\mathfrak{g}}}(s_1(x)) - s_1(\mathrm{d}_{\mathfrak{g}}(x))\\
&= \mathrm{d}_{\hat{\mathfrak{g}}}(s_2(x) + \phi(x)) - (s_2(\mathrm{d}_{\mathfrak{g}}(x)) + \phi(\mathrm{d}_{\mathfrak{g}}(x)))
\end{aligned}
$$

$$\begin{aligned}&= (\mathrm{d}_{\hat{\mathfrak{g}}}(s_2(x)) - s_2(\mathrm{d}_{\mathfrak{g}}(x))) + \mathrm{d}_{\hat{\mathfrak{g}}}(\phi(x)) - \phi(\mathrm{d}_{\mathfrak{g}}(x))\\ &= \chi_2(x) + \mathrm{d}_V(\phi(x)) - \phi(\mathrm{d}_{\mathfrak{g}}(x))\\ &= \chi_2(x) - \delta^1\phi(x).\end{aligned}$$

That is, $(\psi_1,\chi_1) = (\psi_2,\chi_2) + \partial^1_{\mathrm{DL}_\lambda}(\phi)$. Thus (ψ_1,χ_1) and (ψ_2,χ_2) are in the same cohomological class in $\tilde{\mathrm{H}}^2_{\mathrm{DL}_\lambda}(\mathfrak{g}, V)$.

Next we prove that isomorphic abelian extensions give rise to the same element in $\tilde{\mathrm{H}}^2_{\mathrm{DL}_\lambda}(\mathfrak{g}, V)$. Assume that $(\hat{\mathfrak{g}}_1, \mathrm{d}_{\hat{\mathfrak{g}}_1})$ and $(\hat{\mathfrak{g}}_2, \mathrm{d}_{\hat{\mathfrak{g}}_2})$ are two isomorphic abelian extensions of $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ by (V, d_V) with the associated homomorphism $\zeta : (\hat{\mathfrak{g}}_1, \mathrm{d}_{\hat{\mathfrak{g}}_1}) \to (\hat{\mathfrak{g}}_2, \mathrm{d}_{\hat{\mathfrak{g}}_2})$. Let s_1 be a section of $(\hat{\mathfrak{g}}_1, \mathrm{d}_{\hat{\mathfrak{g}}_1})$. As $p_2 \circ \zeta = p_1$, we have

$$p_2 \circ (\zeta \circ s_1) = p_1 \circ s_1 = \mathrm{Id}_{\mathfrak{g}}.$$

Therefore, $\zeta \circ s_1$ is a section of $(\hat{\mathfrak{g}}_2, \mathrm{d}_{\hat{\mathfrak{g}}_2})$. Denote $s_2 := \zeta \circ s_1$. Since ζ is a homomorphism of differential Lie algebras such that $\zeta|_V = \mathrm{Id}_V$, we have

$$\begin{aligned}\psi_2(x, y) &= [s_2(x), s_2(y)] - s_2([x, y]) = [\zeta(s_1(x)), \zeta(s_1(y))] - \zeta(s_1([x, y]))\\ &= \zeta([s_1(x), s_1(y)] - s_1([x, y])) = \zeta(\psi_1(x, y))\\ &= \psi_1(x, y),\end{aligned}$$

and

$$\begin{aligned}\chi_2(x) &= \mathrm{d}_{\hat{\mathfrak{g}}_2}(s_2(x)) - s_2(\mathrm{d}_{\mathfrak{g}}(x)) = \mathrm{d}_{\hat{\mathfrak{g}}_2}(\zeta(s_1(x))) - \zeta(s_1(\mathrm{d}_{\mathfrak{g}}(x)))\\ &= \zeta(\mathrm{d}_{\hat{\mathfrak{g}}_1}(s_1(x)) - s_1(\mathrm{d}_{\mathfrak{g}}(x))) = \zeta(\chi_1(x))\\ &= \chi_1(x).\end{aligned}$$

Consequently, all isomorphic abelian extensions give rise to the same element in $\tilde{\mathrm{H}}^2_{\mathrm{DL}_\lambda}(\mathfrak{g}, V)$.

Conversely, given two 2-cocycles (ψ_1,χ_1) and (ψ_2,χ_2), we can construct two abelian extensions $(\mathfrak{g} \oplus V, [\cdot, \cdot]_{\psi_1}, \mathrm{d}_{\chi_1})$ and $(\mathfrak{g} \oplus V, [\cdot, \cdot]_{\psi_2}, \mathrm{d}_{\chi_2})$ via equalities (4) and (5). If they represent the same cohomological class in $\tilde{\mathrm{H}}^2_{\mathrm{DL}_\lambda}(\mathfrak{g}, V)$, then there exists a linear map $\phi : \mathfrak{g} \to V$ such that

$$(\psi_1,\chi_1) = (\psi_2,\chi_2) + \partial^1_{\mathrm{DL}_\lambda}(\phi).$$

Define $\zeta : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$ by

$$\zeta(x, v) := x + \phi(x) + v.$$

Then ζ is an isomorphism of these two abelian extensions. □

Remark 2.6. In particular, any vector space V with linear endomorphism d_V can serve as a trivial representation of $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$. In this situation, central extensions of $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ by (V, d_V) are classified by the second cohomology group $\mathrm{H}^2_{\mathrm{DL}_\lambda}(\mathfrak{g}, V)$ of $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ with the coefficient in the trivial representation (V, d_V). Note that for a trivial representation (V, d_V), since $\partial^1_{\mathrm{DO}_\lambda} v = 0$ for all $v \in V$, we have

$$\mathrm{H}^2_{\mathrm{DL}_\lambda}(\mathfrak{g}, V) = \tilde{\mathrm{H}}^2_{\mathrm{DL}_\lambda}(\mathfrak{g}, V).$$

3. Deformations of differential Lie algebras

In this section, we study formal deformations of a differential Lie algebra of weight λ. In particular, we show that if the second cohomology group $\tilde{\mathrm{H}}^2_{\mathrm{DL}_\lambda}(\mathfrak{g}, \mathfrak{g}) = 0$, then the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ is rigid.

Let $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ be a differential Lie algebra. Denote by $\mu_{\mathfrak{g}}$ the multiplication of $\mathfrak{g}$. Consider the 1-parameterized family

$$\mu_t = \sum_{i=0}^{\infty} \mu_i t^i,\ \ \mu_i \in \mathrm{C}^2_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}), \qquad d_t = \sum_{i=0}^{\infty} d_i t^i,\ \ d_i \in \mathrm{C}^1_{\mathsf{DO}_\lambda}(\mathfrak{g}, \mathfrak{g}).$$

Definition 3.1. A **1-parameter formal deformation** of a differential Lie algebra $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ is a pair (μ_t, d_t), which endows the $\mathbf{k}[[t]]$-module $(\mathfrak{g}[[t]], \mu_t, d_t)$ with a differential Lie algebra structure such that $(\mu_0, d_0) = (\mu_{\mathfrak{g}}, \mathrm{d}_{\mathfrak{g}})$.

The pair (μ_t, d_t) generates a 1-parameter formal deformation of the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ if and only if for all $x, y, z \in \mathfrak{g}$, the following equalities hold:

$$0 = \mu_t(x, \mu_t(y, z)) + \mu_t(y, \mu_t(z, x)) + \mu_t(z, \mu_t(x, y)), \tag{8}$$

$$d_t(\mu_t(x, y)) = \mu_t(d_t(x), y) + \mu_t(x, d_t(y)) + \lambda \mu_t(d_t(x), d_t(y)). \tag{9}$$

Expanding these equations and collecting coefficients of t^n, we see that Eqs. (8) and (9) are equivalent to the systems of equations: for each $n \geq 0$,

$$0 = \sum_{i=0}^{n} \mu_i(x, \mu_{n-i}(y, z)) + \mu_i(y, \mu_{n-i}(z, x)) + \mu_i(z, \mu_{n-i}(x, y)), \tag{10}$$

$$\sum_{\substack{k,l\geq 0\\ k+l=n}} d_l \mu_k(x, y) = \sum_{\substack{k,l\geq 0\\ k+l=n}} (\mu_k(d_l(x), y) + \mu_k(x, d_l(y))) + \lambda \sum_{\substack{k,l,m\geq 0\\ k+l+m=n}} \mu_k(d_l(x), d_m(y)). \tag{11}$$

Remark 3.2. For $n = 0$, Eq. (10) is equivalent to the Jabobi identity of $\mu_{\mathfrak{g}}$, and Eq. (11) is equivalent to the fact that $\mathrm{d}_{\mathfrak{g}}$ is a λ-derivation. For $n = 1$, Eq. (10) has the form:

$$\begin{aligned} 0 &= \mu_{\mathfrak{g}}(x, \mu_1(y, z)) + \mu_{\mathfrak{g}}(y, \mu_1(z, x)) + \mu_{\mathfrak{g}}(z, \mu_1(x, y)) \\ &\quad + \mu_1(x, \mu_{\mathfrak{g}}(y, z)) + \mu_1(y, \mu_{\mathfrak{g}}(z, x)) + \mu_1(z, \mu_{\mathfrak{g}}(x, y)). \end{aligned} \tag{12}$$

And for $n = 1$, Eq. (11) has the form:

$$\begin{aligned} d_1\mu_{\mathfrak{g}}(x, y) + \mathrm{d}_{\mathfrak{g}}\mu_1(x, y) &= \mu_1(d_{\mathfrak{g}}(x), y) + \mu_1(x, \mathrm{d}_{\mathfrak{g}}(y)) + \mu_{\mathfrak{g}}(d_1(x), y) + \mu_{\mathfrak{g}}(x, d_1(y)) \\ &\quad + \lambda\mu_1(\mathrm{d}_{\mathfrak{g}}(x), \mathrm{d}_{\mathfrak{g}}(y)) + \lambda\mu_{\mathfrak{g}}(d_1(x), \mathrm{d}_{\mathfrak{g}}(y)) + \lambda\mu_{\mathfrak{g}}(\mathrm{d}_{\mathfrak{g}}(x), d_1(y)). \end{aligned} \tag{13}$$

Proposition 3.3. *Let $(\mathfrak{g}[[t]], \mu_t, d_t)$ be a 1-parameter formal deformation of a differential Lie algebra $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$. Then (μ_1, d_1) is a 2-cocycle of the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ with the coefficient in the adjoint representation $\mathfrak{g}_{\mathrm{ad}}$.*

Proof. For $n = 1$, Eq. (12) is equivalent to $\partial^2_{\mathsf{Lie}}\mu_1 = 0$, and Eq. (13) is equivalent to Eq. (3), that is

$$\partial^1_{\mathsf{DO}_\lambda} d_1 + \delta^2 \mu_1 = 0.$$

Thus (μ_1, d_1) is a 2-cocycle by Remark 1.13. □

If $\mu_t = \mu_{\mathfrak{g}}$ in the above 1-parameter formal deformation of the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$, we obtain a 1-parameter formal deformation of the differential operator $\mathrm{d}_{\mathfrak{g}}$. Consequently, we have

Corollary 3.4. *Let d_t be a 1-parameter formal deformation of the differential operator $\mathrm{d}_{\mathfrak{g}}$. Then d_1 is a 1-cocycle of the differential operator $\mathrm{d}_{\mathfrak{g}}$ with coefficients in the adjoint representation $\mathfrak{g}_{\mathrm{ad}}$.*

Proof. In the special case when $n = 1$, Eq. (11) is equivalent to $\partial^1_{\mathsf{DO}_\lambda} d_1 = 0$, which implies that d_1 is a 1-cocycle of the differential operator $\mathrm{d}_{\mathfrak{g}}$ with coefficients in the adjoint representation $\mathfrak{g}_{\mathrm{ad}}$. □

Definition 3.5. The 2-cocycle (μ_1, d_1) is called the **infinitesimal** of the 1-parameter formal deformation $(\mathfrak{g}[[t]], \mu_t, d_t)$ of $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$.

Definition 3.6. Let $(\mathfrak{g}[[t]], \mu_t, d_t)$ and $(\mathfrak{g}[[t]], \bar{\mu}_t, \bar{d}_t)$ be 1-parameter formal deformations of $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$. A **formal isomorphism** from $(\mathfrak{g}[[t]], \bar{\mu}_t, \bar{d}_t)$ to $(\mathfrak{g}[[t]], \mu_t, d_t)$ is a power series $\Phi_t = \sum_{i\geq 0} \phi_i t^i : \mathfrak{g}[[t]] \to \mathfrak{g}[[t]]$, where $\phi_i : \mathfrak{g} \to \mathfrak{g}$ are linear maps with $\phi_0 = \mathrm{Id}_\mathfrak{g}$, such that

$$\begin{aligned} \Phi_t \circ \bar{\mu}_t &= \mu_t \circ (\Phi_t \times \Phi_t), \\ \Phi_t \circ \bar{d}_t &= d_t \circ \Phi_t. \end{aligned}$$

Two 1-parameter formal deformations $(\mathfrak{g}[[t]], \mu_t, d_t)$ and $(\mathfrak{g}[[t]], \bar{\mu}_t, \bar{d}_t)$ are said to be **equivalent** if there exists a formal isomorphism $\Phi_t : (\mathfrak{g}[[t]], \bar{\mu}_t, \bar{d}_t) \to (\mathfrak{g}[[t]], \mu_t, d_t)$.

Theorem 3.7. *The infinitesimals of two equivalent* 1*-parameter formal deformations of* $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ *are in the same cohomology class in* $\tilde{\mathrm{H}}^2_{\mathrm{DL}_\lambda}(\mathfrak{g}, \mathfrak{g})$.

Proof. Let $\Phi_t : (\mathfrak{g}[[t]], \bar{\mu}_t, \bar{d}_t) \to (\mathfrak{g}[[t]], \mu_t, d_t)$ be a formal isomorphism. For all $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} \Phi_t \circ \bar{\mu}_t(x, y) &= \mu_t \circ (\Phi_t \times \Phi_t)(x, y), \\ \Phi_t \circ \bar{d}_t(x) &= d_t \circ \Phi_t(x). \end{aligned}$$

Expanding the above identities and comparing the coefficients of t, we obtain

$$\begin{aligned} \bar{\mu}_1(x, y) &= \mu_1(x, y) + [\phi_1(x), y] + [x, \phi_1(y)] - \phi_1([x, y]), \\ \bar{d}_1(x) &= d_1(x) + \mathrm{d}_\mathfrak{g}(\phi_1(x)) - \phi_1(\mathrm{d}_\mathfrak{g}(x)). \end{aligned}$$

Thus, we have

$$(\bar{\mu}_1, \bar{d}_1) = (\mu_1, d_1) + \partial^1_{\mathrm{DL}_\lambda}(\phi_1),$$

which implies that $[(\bar{\mu}_1, \bar{d}_1)] = [(\mu_1, d_1)]$ in $\tilde{\mathrm{H}}^2_{\mathrm{DL}_\lambda}(\mathfrak{g}, \mathfrak{g})$. □

Given any differential Lie algebra $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$, interpret $\mu_\mathfrak{g}$ and $\mathrm{d}_\mathfrak{g}$ as the formal power series μ_t and d_t with $\mu_i = \delta_{i,0}\mu_\mathfrak{g}$ and $d_i = \delta_{i,0}\mathrm{d}_\mathfrak{g}$ respectively for all $i \geq 0$, where $\delta_{i,0}$ is the Kronecker sign. Then $(\mathfrak{g}[[t]], \mu_\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ is a 1-parameter formal deformation of $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$.

Definition 3.8. A 1-parameter formal deformation $(\mathfrak{g}[[t]], \mu_t, d_t)$ of $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ is said to be **trivial** if it is equivalent to the deformation $(\mathfrak{g}[[t]], \mu_\mathfrak{g}, \mathrm{d}_\mathfrak{g})$, that is, there exists $\Phi_t = \sum_{i\geq 0} \phi_i t^i : \mathfrak{g}[[t]] \to \mathfrak{g}[[t]]$, where $\phi_i : \mathfrak{g} \to \mathfrak{g}$ are linear maps with $\phi_0 = \mathrm{Id}_\mathfrak{g}$, such that

$$\begin{aligned} \Phi_t \circ \mu_t &= \mu_\mathfrak{g} \circ (\Phi_t \times \Phi_t), \\ \Phi_t \circ d_t &= \mathrm{d}_\mathfrak{g} \circ \Phi_t. \end{aligned}$$

Definition 3.9. A differential Lie algebra $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ is said to be **rigid** if every 1-parameter formal deformation is trivial.

Theorem 3.10. *If* $\tilde{\mathrm{H}}^2_{\mathrm{DL}_\lambda}(\mathfrak{g}, \mathfrak{g}_{\mathrm{ad}}) = 0$*, then the differential Lie algebra* $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ *is rigid.*

Proof. Let $(\mathfrak{g}[[t]], \mu_t, d_t)$ be a 1-parameter formal deformation of $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$. By Proposition 3.3, (μ_1, d_1) is a 2-cocycle. By $\tilde{\mathrm{H}}^2_{\mathrm{DL}_\lambda}(\mathfrak{g}, \mathfrak{g}_{\mathrm{ad}}) = 0$, there exists a 1-cochain $\phi_1 \in \mathrm{C}^1_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g})$ such that

$$(\mu_1, d_1) = -\partial^1_{\mathrm{DL}_\lambda}(\phi_1). \tag{14}$$

Then setting $\Phi_t = \mathrm{Id}_\mathfrak{g} + \phi_1 t$, we have a 1-parameter formal deformation $(\mathfrak{g}[[t]], \bar{\mu}_t, \bar{d}_t)$, where

$$\begin{aligned} \bar{\mu}_t(x, y) &= (\Phi_t^{-1} \circ \mu_t \circ (\Phi_t \times \Phi_t))(x, y), \\ \bar{d}_t(x) &= (\Phi_t^{-1} \circ d_t \circ \Phi_t)(x). \end{aligned}$$

Thus, $(\mathfrak{g}[[t]], \bar{\mu}_t, \bar{d}_t)$ is equivalent to $(\mathfrak{g}[[t]], \mu_t, d_t)$. Moreover, we have

$$\begin{aligned}\bar{\mu}_t(x,y) &= (\mathrm{Id}_{\mathfrak{g}} - \phi_1 t + \phi_1^2 t^2 + \cdots + (-1)^i \phi_1^i t^i + \cdots)(\mu_t(x + \phi_1(x)t, y + \phi_1(y)t)),\\ \bar{d}_t(x) &= (\mathrm{Id}_{\mathfrak{g}} - \phi_1 t + \phi_1^2 t^2 + \cdots + (-1)^i \phi_1^i t^i + \cdots)(d_t(x + \phi_1(x)t)).\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{\mu}_t(x,y) &= [x,y] + (\mu_1(x,y) + [x,\phi_1(y)] + [\phi_1(x),y] - \phi_1([x,y]))t + \bar{\mu}_2(x,y)t^2 + \cdots,\\ \bar{d}_t(x) &= d_A(x) + (d_A(\phi_1(x)) + d_1(x) - \phi_1(\mathrm{d}_{\mathfrak{g}}(x)))t + \bar{d}_2(x)t^2 + \cdots.\end{aligned}$$

By Eq. (14), we have

$$\begin{aligned}\bar{\mu}_t(x,y) &= [x,y] + \bar{\mu}_2(x,y)t^2 + \cdots,\\ \bar{d}_t(x) &= \mathrm{d}_{\mathfrak{g}}(x) + \bar{d}_2(x)t^2 + \cdots.\end{aligned}$$

Then by repeating the argument, we can show that $(\mathfrak{g}[[t]], \mu_t, d_t)$ is equivalent to $(\mathfrak{g}[[t]], \mu_{\mathfrak{g}}, \mathrm{d}_{\mathfrak{g}})$. Thus, $(\mathfrak{g}, \mathrm{d}_{\mathfrak{g}})$ is rigid. □

4. $L_\infty[1]$-STRUCTURE FOR (RELATIVE AND ABSOLUTE) DIFFERENTIAL LIE ALGEBRAS

In this section, we study the $L_\infty[1]$-structure for differential Lie algebras of weight λ. In order to deal with absolute differential Lie algebras, we introduce a generalised version of derived bracket technique; moreover, to consider the weight case, we incorporate the weight λ into the statements. These are the main differences of this paper with [18, 11], since the latter papers consider difference operators, say, relative differential operators of weight 1.

4.1. $L_\infty[1]$-algebras.

In this subsection, we recall some preliminaries on $L_\infty[1]$-algebras.

Definition 4.1. An $L_\infty[1]$**-algebra** is a graded vector space $\mathfrak{L} = \bigoplus_{i\in\mathbb{N}} \mathfrak{L}^i$ endowed with a family of graded linear maps $l_n : \mathfrak{L}^{\otimes n} \to \mathfrak{L}, n \geq 1$ of degree 1 satisfying the following equations: for arbitrary $n \geq 1$, $\sigma \in S_n$ and $x_1, \dots, x_n \in \mathfrak{L}$,

(i) (graded symmetry)

$$l_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \varepsilon(\sigma) l_n(x_1, \dots, x_n),$$

(ii) (generalised Jacobi identity)

$$\sum_{i=1}^{n} \sum_{\sigma\in \mathrm{Sh}(i,n-i)} \varepsilon(\sigma) l_{n-i+1}(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0.$$

Remark 4.2. Let us consider the generalised Jacobi identity for $n \leq 3$ with the assumption of generalised symmetry.

(i) For $n = 1$, then $l_1 \circ l_1 = 0$, that is, l_1 is a differential.
(ii) For $n = 2$, then $l_1 \circ l_2 + l_2 \circ (l_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes l_1) = 0$, that is , l_1 is a derivation with respect to l_2.

(iii) For $n = 3$ and arbitrary homogeneous elements $x_1, x_2, x_3 \in L$, we have

$$\begin{aligned} & l_2(l_2(x_1 \otimes x_2) \otimes x_3) + (-1)^{|x_1|(|x_2|+|x_3|)} l_2(l_2(x_2 \otimes x_3) \otimes x_1) + (-1)^{|x_3|(|x_1|+|x_2|)} l_2(l_2(x_3 \otimes x_1) \otimes x_2) \\ = \ & -\Big(l_1(l_3(x_1 \otimes x_2 \otimes x_3)) + l_3(l_1(x_1) \otimes x_2 \otimes x_3) + (-1)^{|x_1|} l_3(x_1 \otimes l_1(x_2) \otimes x_3) + \\ & (-1)^{|x_1|+|x_2|} l_3(x_1 \otimes x_2 \otimes l_1(x_3)) \Big), \end{aligned}$$

that is, l_2 satisfies the Jacobi identity up to homotopy.

Definition 4.3. A **Maurer-Cartan element** of an $L_\infty[1]$-algebra $(\mathfrak{L}, \{l_n\}_{n\geq 1})$ is an element $\alpha \in \mathfrak{L}^0$ satisfying the Maurer-Cartan equation:

$$\sum_{n=1}^{\infty} \frac{1}{n!} l_n(\alpha^{\otimes n}) = 0,$$

whenever this infinite sum exists. Denote $\mathcal{MC}(\mathfrak{L}) := \{\text{Maurer-Cartan elements of } \mathfrak{L}\}$.

Proposition 4.4. [6, Twisting procedure] *Let α be a Maurer-Cartan element of $L_\infty[1]$-algebra $\mathfrak{L}$, The twisted $L_\infty[1]$-algebra is given by $l_n^\alpha : \mathfrak{L}^{\otimes n} \to \mathfrak{L}$ which is defined as follows:*

$$l_n^\alpha(x_1 \otimes \cdots \otimes x_n) = \sum_{i=0}^{\infty} \frac{1}{i!} l_{n+i}(\alpha^{\otimes i} \otimes x_1 \otimes \cdots \otimes x_n), \ \forall x_1, \dots, x_n \in \mathfrak{L},$$

whenever these infinite sums exist.

The following simple observation with immediate proof will be very useful while considering differential operators of a given weight λ.

Proposition 4.5. *Let $\mathfrak{L} = (\mathfrak{L}, \{l_n\}_{n\geq 1})$ be an $L_\infty[1]$-algebra and $\lambda \in \mathbf{k}$. Consider $l'_n : \mathfrak{L}^{\otimes n} \to \mathfrak{L}, n \geq 1$ given by*

$$l'_n(x_1 \otimes \cdots \otimes x_n) = \lambda^{n-1} l_n(x_1 \otimes \cdots \otimes x_n), \ \forall x_1, \dots, x_n \in \mathfrak{L},$$

then $(\mathfrak{L}, \{l'_n\}_{n\geq 1})$ is also an $L_\infty[1]$-algebra.

If $l_1 = 0$, then imposing $l'_1 = 0$ and $l'_n = \lambda^{n-2} l_n$ for $n \geq 2$ also gives a new $L_\infty[1]$-structure on $\mathfrak{L}$.

4.2. **A generalised version of derived bracket technique.**

Now we introduce a generalised version of the derived bracket technique invented by Voronov [23, 24]. Since we consider differential Lie algebras with weight $\lambda \in \mathbf{k}$, we insert λ in the statements whenever needed by Proposition 4.5, thus modify corresponding results in [2, 11].

Definition 4.6. A **generalised V-data** is a septuple $(\mathfrak{L}, \mathfrak{M}, \iota_\mathfrak{M}, \mathfrak{A}, \iota_\mathfrak{A}, P, \Delta)$, where

(1) $(\mathfrak{L}, [-,-])$ is a graded Lie algebra,
(2) $(\mathfrak{M}, [-,-]_\mathfrak{M})$ is a graded Lie algebra together with an injective linear map $\iota_\mathfrak{M} : \mathfrak{M} \to \mathfrak{L}$ which is a homomorphism of graded Lie algebras,
(3) $\mathfrak{A}$ is an abelian graded Lie algebra endowed with an injective linear map $\iota_\mathfrak{A} : \mathfrak{A} \to \mathfrak{L}$ which is a homomorphism of graded Lie algebras,
(4) $P : \mathfrak{L} \to \mathfrak{A}$ is a linear map such that $P \circ \iota_\mathfrak{A} = \mathrm{Id}_\mathfrak{A}$ and $\mathrm{Ker}(P)$ is a graded Lie subalgebra of $\mathfrak{L}$,
(5) $\Delta \in \mathrm{Ker}(P)^1$ satisfying $[\Delta, \Delta] = 0$ and $[\Delta, \iota_\mathfrak{M}(\mathfrak{M})] \subset \iota_\mathfrak{M}(\mathfrak{M})$.

Theorem 4.7. *Let* $(\mathfrak{L}, \mathfrak{M}, \iota_{\mathfrak{M}}, \mathfrak{A}, \iota_{\mathfrak{A}}, P, \Delta)$ *be a generalised V-datum and* $\lambda \in \mathbf{k}$. *Then the graded vector space* $s\mathfrak{L} \oplus \mathfrak{A}$ *has an* $L_\infty[1]$*-algebra structure which is defined as follows*:

$$\begin{aligned}
l_1(sf) &= (-s[\Delta, f], P(f)),\\
l_1(\xi) &= P[\Delta, \iota_{\mathfrak{A}}(\xi)],\\
l_2(sf, sg) &= (-1)^{|f|}\lambda s[f, g],\\
l_i(sf, \xi_1, \cdots, \xi_{i-1}) &= \lambda^{i-1} P[\cdots[f, \iota_{\mathfrak{A}}(\xi_1)], \cdots, \iota_{\mathfrak{A}}(\xi_{i-1})], i \geq 2,\\
l_i(\xi_1, \cdots, \xi_i) &= \lambda^{i-1} P[\cdots[\Delta, \iota_{\mathfrak{A}}(\xi_1)], \cdots, \iota_{\mathfrak{A}}(\xi_i)], i \geq 2,
\end{aligned}$$

for homogeneous elements $f, g \in \mathfrak{L}$, $\xi, \xi_1, \cdots, \xi_i \in \mathfrak{A}$ *and all other components of* $\{l_i\}_{i=1}^{+\infty}$ *vanish.*

Similarly, the graded vector space $s\mathfrak{M} \oplus \mathfrak{A}$ *has also an* $L_\infty[1]$*-algebra structure which is defined as follows*:

$$\begin{aligned}
l_1(sf) &= (-s\iota_{\mathfrak{M}}^{-1}[\Delta, \iota_{\mathfrak{M}}(f)], P\iota_{\mathfrak{M}}(f)),\\
l_1(\xi) &= P[\Delta, \iota_{\mathfrak{A}}(\xi)],\\
l_2(sf, sg) &= (-1)^{|f|}\lambda s[f, g]_{\mathfrak{M}},\\
l_i(sf, \xi_1, \cdots, \xi_{i-1}) &= \lambda^{i-1} P[\cdots[\iota_{\mathfrak{M}}(f), \iota_{\mathfrak{A}}(\xi_1)], \cdots, \iota_{\mathfrak{A}}(\xi_{i-1})], i \geq 2,\\
l_i(\xi_1, \cdots, \xi_i) &= \lambda^{i-1} P[\cdots[\Delta, \iota_{\mathfrak{A}}(\xi_1)], \cdots, \iota_{\mathfrak{A}}(\xi_i)], i \geq 2,
\end{aligned}$$

for homogeneous elements $f, g \in \mathfrak{M}$, $\xi, \xi_1, \cdots, \xi_i \in \mathfrak{A}$ *and all other components of* $\{l_i\}_{i=1}^{+\infty}$ *vanish.*

Moreover, there exists an injective $L_\infty[1]$*-algebra homomorphism* $\iota : s\mathfrak{M} \oplus \mathfrak{A} \to s\mathfrak{L} \oplus \mathfrak{A}$ *induced by* $\iota_{\mathfrak{M}}$.

Proof. Obviously, the quadruple $(\mathfrak{L}, \iota_{\mathfrak{A}}(\mathfrak{A}), \iota_{\mathfrak{A}} P, \Delta)$ is a V-data in the sense of Voronov [23], by [23, Section 3], there is an $L_\infty[1]$-algebra $\{l_i\}_{i=1}^{+\infty}$ on $s\mathfrak{L} \oplus \iota_{\mathfrak{A}}(\mathfrak{A})$, where for homogeneous elements $f, g \in \mathfrak{L}$, $\xi, \xi_1, \cdots, \xi_i \in \mathfrak{A}$,

$$\begin{aligned}
l_1(sf) &= (-s[\Delta, f], \iota_{\mathfrak{A}} P(f)),\\
l_1(\iota_{\mathfrak{A}}(\xi)) &= \iota_{\mathfrak{A}} P([\Delta, \iota_{\mathfrak{A}}(\xi)]),\\
l_2(sf, sg) &= (-1)^{|f|} s[f, g],\\
l_i(sf, \iota_{\mathfrak{A}}(\xi_1), \cdots, \iota_{\mathfrak{A}}(\xi_{i-1})) &= \iota_{\mathfrak{A}} P[\cdots[f, \iota_{\mathfrak{A}}(\xi_1)], \cdots, \iota_{\mathfrak{A}}(\xi_{i-1})], i \geq 2,\\
l_i(\iota_{\mathfrak{A}}(\xi_1), \cdots, \iota_{\mathfrak{A}}(\xi_i)) &= \iota_{\mathfrak{A}} P[\cdots[\Delta, \iota_{\mathfrak{A}}(\xi_1)], \cdots, \iota_{\mathfrak{A}}(\xi_i)], i \geq 2,
\end{aligned}$$

and all other components of $\{l_i\}_{i=1}^{+\infty}$ vanish. A similar $L_\infty[1]$-algebra structure $\{l_i\}_{i=1}^{+\infty}$ on $s\iota_{\mathfrak{M}}(\mathfrak{M}) \oplus \iota_{\mathfrak{A}}(\mathfrak{A})$ exists, as well as an inclusion $s\iota_{\mathfrak{M}}(\mathfrak{M}) \oplus \iota_{\mathfrak{A}}(\mathfrak{A}) \hookrightarrow s\mathfrak{L} \oplus \iota_{\mathfrak{A}}(\mathfrak{A})$ of $L_\infty[1]$-algebras.

By Proposition 4.5, we could insert λ into the construction of the higher Lie brackets. The $L_\infty[1]$-algebra structure on $s\mathfrak{L} \oplus \iota_{\mathfrak{A}}(\mathfrak{A})$ are given by: for homogeneous elements $f, g \in \mathfrak{L}$, $\xi, \xi_1, \cdots, \xi_i \in \mathfrak{A}$,

$$\begin{aligned}
l_1(sf) &= (-s[\Delta, f], \iota_{\mathfrak{A}} P(f)),\\
l_1(\iota_{\mathfrak{A}}(\xi)) &= \iota_{\mathfrak{A}} P([\Delta, \iota_{\mathfrak{A}}(\xi)]),\\
l_2(sf, sg) &= (-1)^{|f|}\lambda s[f, g],\\
l_i(sf, \iota_{\mathfrak{A}}(\xi_1), \cdots, \iota_{\mathfrak{A}}(\xi_{i-1})) &= \lambda^{i-1}\iota_{\mathfrak{A}} P[\cdots[f, \iota_{\mathfrak{A}}(\xi_1)], \cdots, \iota_{\mathfrak{A}}(\xi_{i-1})], i \geq 2,\\
l_i(\iota_{\mathfrak{A}}(\xi_1), \cdots, \iota_{\mathfrak{A}}(\xi_i)) &= \lambda^{i-1}\iota_{\mathfrak{A}} P[\cdots[\Delta, \iota_{\mathfrak{A}}(\xi_1)], \cdots, \iota_{\mathfrak{A}}(\xi_i)], i \geq 2,
\end{aligned}$$

and all other components of higher Lie brackets vanish. The $L_\infty[1]$-algebra structure on $s\iota_{\mathfrak{M}}(\mathfrak{M}) \oplus \iota_{\mathfrak{A}}(\mathfrak{A})$ are given by: for homogeneous elements $f, g \in \mathfrak{M}$, $\xi, \xi_1, \cdots, \xi_i \in \mathfrak{A}$,

$$\begin{aligned}
l_1(s\iota_{\mathfrak{M}}(f)) &= (-s\iota_{\mathfrak{M}}^{-1}([\Delta, \iota_{\mathfrak{M}}(f)]), \iota_{\mathfrak{A}} P\iota_{\mathfrak{M}}(f)),\\
l_1(\iota_{\mathfrak{A}}(\xi)) &= \iota_{\mathfrak{A}} P([\Delta, \iota_{\mathfrak{A}}(\xi)]),\\
l_2(s\iota_{\mathfrak{M}}(f), s\iota_{\mathfrak{M}}(g)) &= (-1)^{|f|}\lambda s\iota_{\mathfrak{M}}([f, g]_{\mathfrak{M}}),\\
l_i(s\iota_{\mathfrak{M}}(f), \iota_{\mathfrak{A}}(\xi_1), \cdots, \iota_{\mathfrak{A}}(\xi_{i-1})) &= \lambda^{i-1}\iota_{\mathfrak{A}} P[\cdots[\iota_{\mathfrak{M}}(f), \iota_{\mathfrak{A}}(\xi_1)], \cdots, \iota_{\mathfrak{A}}(\xi_{i-1})], i \geq 2,\\
l_i(\iota_{\mathfrak{A}}(\xi_1), \cdots, \iota_{\mathfrak{A}}(\xi_i)) &= \lambda^{i-1}\iota_{\mathfrak{A}} P[\cdots[\Delta, \iota_{\mathfrak{A}}(\xi_1)], \cdots, \iota_{\mathfrak{A}}(\xi_i)], i \geq 2,
\end{aligned}$$

and all other components of higher Lie brackets vanish. There exists also an inclusion $s\iota_{\mathfrak{M}}(\mathfrak{M}) \oplus \iota_{\mathfrak{A}}(\mathfrak{A}) \hookrightarrow s\mathfrak{L} \oplus \iota_{\mathfrak{A}}(\mathfrak{A})$ of $L_\infty[1]$-algebras.

Then the theorem holds by the following commutative diagram

$$\begin{tikzcd}
s\mathfrak{L} \oplus \mathfrak{A} \arrow[r, "\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \iota_{\mathfrak{A}} \end{pmatrix}", "\cong"'] & s\mathfrak{L} \oplus \iota_{\mathfrak{A}}(\mathfrak{A}) \\
s\mathfrak{M} \oplus \mathfrak{A} \arrow[u, "\iota"] \arrow[r, "\cong", "\begin{pmatrix} \iota_{\mathfrak{M}} & 0 \\ 0 & \iota_{\mathfrak{A}} \end{pmatrix}"'] & s\iota_{\mathfrak{M}}(\mathfrak{M}) \oplus \iota_{\mathfrak{A}}(\mathfrak{A}). \arrow[u, hook, "\mathrm{inc}"']
\end{tikzcd}$$

□

Definition 4.8. Let $(\mathfrak{L}, \mathfrak{M}, \iota_{\mathfrak{M}}, \mathfrak{A}, \iota_{\mathfrak{A}}, P, \Delta)$ and $(\mathfrak{L}', \mathfrak{M}', \iota_{\mathfrak{M}'}, \mathfrak{A}', \iota_{\mathfrak{A}'}, P', \Delta')$ be two generalised V-data. A **morphism** between them is a triple $f = (f_{\mathfrak{L}}, f_{\mathfrak{M}}, f_{\mathfrak{A}})$, where $f_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{L}'$, $f_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}'$ and $f_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}'$ are three homomorphisms of graded Lie algebras such that $f \circ \iota_{\mathfrak{M}} = \iota_{\mathfrak{M}'} \circ f_{\mathfrak{M}}$, $f \circ \iota_{\mathfrak{A}} = \iota_{\mathfrak{A}'} \circ f_{\mathfrak{A}}$, $f_{\mathfrak{A}} \circ P = P' \circ f_{\mathfrak{L}}$ and $f_{\mathfrak{L}}(\Delta) = \Delta'$.

The following result is obvious.

Proposition 4.9. *Given a morphism of generalised V-data*

$$f : (\mathfrak{L}, \mathfrak{M}, \iota_{\mathfrak{M}}, \mathfrak{A}, \iota_{\mathfrak{A}}, P, \Delta) \to (\mathfrak{L}', \mathfrak{M}', \iota_{\mathfrak{M}'}, \mathfrak{A}', \iota_{\mathfrak{A}'}, P', \Delta'),$$

there exists an $L_\infty[1]$*-algebra homomorphism*

$$\tilde{f} : s\mathfrak{M} \oplus \mathfrak{A} \to s\mathfrak{M}' \oplus \mathfrak{A}'$$

induced by f.

Obviously we have the modified version of Theorem 4.7 as well as a characterisation of their Maurer-Cartan elements.

Proposition 4.10. *Let* $(\mathfrak{L}, \mathfrak{M}, \iota_{\mathfrak{M}}, \mathfrak{A}, \iota_{\mathfrak{A}}, P, \Delta)$ *be a generalised V-datum and* $\lambda \in \mathbf{k}$*. If* $P \circ \iota_{\mathfrak{M}} = 0$ *and* $\Delta = 0$*, then there exists an* $L_\infty[1]$*-algebra structure on* $s\mathfrak{M} \oplus \mathfrak{A}$*, where the higher Lie bracket* $\{l_i\}_{i=1}^{+\infty})$ *are given by*

$$\begin{aligned} l_2(sf, sg) &= (-1)^{|f|} s[f, g]_{\mathfrak{M}}, \\ l_i(sf, \xi_1, \cdots, \xi_{i-1}) &= \lambda^{i-2} P[\cdots[\iota_{\mathfrak{M}}(f), \iota_{\mathfrak{A}}(\xi_1)]_{\mathfrak{L}}, \cdots, \iota_{\mathfrak{A}}(\xi_{i-1})]_{\mathfrak{L}}, i \geq 2, \end{aligned}$$

for homogeneous elements $f, g \in \mathfrak{M}$*,* $\xi_1, \cdots, \xi_i \in \mathfrak{A}$ *and all other components vanish.*

Let $f \in \mathfrak{M}^1, \xi \in \mathfrak{A}^0$*. The pair* (sf, ξ) *is a Maurer-Cartan element in the* $L_\infty[1]$*-algebra* $s\mathfrak{M} \oplus \mathfrak{A}$ *if and only if*

$$f \in \mathcal{MC}(\mathfrak{M}) \text{ and } \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \lambda^{n-2} P[\cdots[\iota_{\mathfrak{M}}(f), \underbrace{\iota_{\mathfrak{A}}(\xi)]_{\mathfrak{L}}, \cdots, \iota_{\mathfrak{A}}(\xi)}_{(n-1) \text{ times}}]_{\mathfrak{L}} = 0.$$

Proof. The first statement follows from Proposition 4.5. For the second statement, (sf, ξ) is a Maurer-Cartan element in the $L_\infty[1]$-algebra $s\mathfrak{M} \oplus \mathfrak{A}$ if and only if

$$\begin{aligned}
0 &= \sum_{n=1}^{\infty} \frac{1}{n!} l_n((sf, \xi)^{\otimes n}) \\
&= \frac{1}{2} l_2(sf, sf) + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} l_n(sf, \underbrace{\xi, \dots, \xi}_{(n-1)\text{ times}}) \\
&= -\frac{1}{2} s[f, f]_{\mathfrak{M}} + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \lambda^{n-2} P[\cdots[\iota_{\mathfrak{M}}(f), \underbrace{\iota_{\mathfrak{A}}(\xi)]_{\mathfrak{L}}, \cdots, \iota_{\mathfrak{A}}(\xi)}_{(n-1)\text{ times}}]_{\mathfrak{L}},
\end{aligned}$$

if and only if

$$[f, f]_{\mathfrak{M}} = 0 \text{ and } \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \lambda^{n-2} P[\cdots[\iota_{\mathfrak{M}}(f), \underbrace{\iota_{\mathfrak{A}}(\xi)]_{\mathfrak{L}}, \cdots, \iota_{\mathfrak{A}}(\xi)}_{(n-1)\text{ times}}]_{\mathfrak{L}} = 0.$$

□

4.3. $L_\infty[1]$-structure for relative differential Lie algebras with weight.

In this subsection, by using the generalised version of derived bracket technique, we found an $L_\infty[1]$-algebra whose Maurer-Cartan elements are in bijection with the set of structures of relative differential Lie algebras of weight λ, thus generalising the result of Jiang and Sheng [11] from weight 1 case to arbitrary weight cases.

We recall the classical Nijenhuis-Richardson brackets and basic facts about LieAct triples; for details, see [2, 11].

Given a vector space V, its exterior algebra is $\bigwedge(V) := \bigoplus_{k=0}^{\infty} \wedge^k V$ and the reduced version is $\bar{\bigwedge}(V) := \bigoplus_{k=1}^{\infty} \wedge^k V$. We consider the graded vector space $\mathrm{Hom}(\bar{\bigwedge}(V), V)$, so for $f \in \mathrm{Hom}(\wedge^{n+1} V, V)$, its degree is n. The graded space $\mathrm{Hom}(\bar{\bigwedge}(V), V)$ is a graded Lie algebra [17] under the Nijenhuis-Richardson bracket $[\ ,\ ]_{\mathrm{NR}}$ which is defined as follows: for arbitrary $f \in \mathrm{Hom}(\wedge^{p+1} V, V)$ and $g \in \mathrm{Hom}(\wedge^{q+1} V, V)$,

$$[f, g]_{\mathrm{NR}} := f \bar{\circ} g - (-1)^{pq} g \bar{\circ} f,$$

where $f \bar{\circ} g \in \mathrm{Hom}(\wedge^{p+q+1} V, V)$ is given by

$$f \bar{\circ} g(x_1, \cdots, x_{p+q+1}) = \sum_{\sigma \in \mathrm{Sh}(q+1, p)} \mathrm{sgn}(\sigma) f(g(x_{\sigma(1)}, \cdots, x_{\sigma(q+1)}), x_{\sigma(q+2)}, \cdots, x_{\sigma(p+q+1)}).$$

Definition 4.11. A **LieAct triple** is a triple $(\mathfrak{g}, \mathfrak{h}, \rho)$, where $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\ ,\ ]_{\mathfrak{h}})$ are Lie algebras and $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ is a homomorphism of Lie algebras.

Remark 4.12. Given a LieAct triple $(\mathfrak{g}, \mathfrak{h}, \rho)$, the Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ means that $\mathfrak{h}$ is a Lie $\mathfrak{g}$-module and there exists an action $\mathfrak{g} \otimes \mathfrak{h} \to \mathfrak{h}$ given by $x \cdot u := \rho(x)(u)$, subject to the Leibniz rule

$$x \cdot [u, v]_{\mathfrak{h}} = [x \cdot u, v]_{\mathfrak{h}} + [u, x \cdot v]_{\mathfrak{h}}, \quad \forall x \in \mathfrak{g} \text{ and } u, v \in \mathfrak{h}.$$

Given two vector spaces V and W, by the isomorphism

$$\begin{aligned}
\varphi : \wedge^n(V \oplus W) &\longrightarrow \bigoplus_{k+l=n, k,l\geq 0} \wedge^k V \otimes \wedge^l W \\
(v_1 + w_1) \wedge \cdots \wedge (v_n + w_n) &\mapsto \sum_{\sigma \in \mathrm{Sh}(k, n-k)} \mathrm{sgn}(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} \otimes w_{\sigma(k+1)} \wedge \cdots \wedge w_{\sigma(n)},
\end{aligned}$$

we have an isomorphism

$$\mathrm{Hom}(\wedge^n(V\oplus W), V\oplus W)\cong (\oplus_{k+l=n}\mathrm{Hom}(\wedge^k V\otimes\wedge^l W, V)\oplus(\oplus_{k+l=n}\mathrm{Hom}(\wedge^k V\otimes \wedge^l W, W))).$$

With this isomorphism in mind, we have the following result.

Proposition 4.13. ([2]) Let $\mathfrak{g}$ and $\mathfrak{h}$ be two vector spaces. Let

$$\mathfrak{L}' = \mathrm{Hom}(\bar{\wedge}(\mathfrak{g}\oplus\mathfrak{h}), \mathfrak{g}\oplus\mathfrak{h})$$

and denote

$$\mathfrak{M}' := \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g})\oplus\mathrm{Hom}(\bar{\wedge}\mathfrak{g}\otimes\bar{\wedge}\mathfrak{h}, \mathfrak{h})\oplus\mathrm{Hom}(\bar{\wedge}\mathfrak{h}, \mathfrak{h}).$$

Let $\iota_{\mathfrak{M}'}:\mathfrak{M}'\to\mathfrak{L}'$ be the natural injection. Then the Nijenhuis-Richardson bracket on $(\mathfrak{L}', [-,-]_{\mathrm{NR}})$ induces a Lie bracket on $\mathfrak{M}$ such that $\iota_{\mathfrak{M}'}:\mathfrak{M}'\to\mathfrak{L}'$ is an injective homomorphism of graded Lie algebras.

Moreover, Maurer-Cartan elements of the graded Lie algebra $\mathfrak{M}'$ are in bijection with the set of LieAct triple structures on $(\mathfrak{g}, \mathfrak{h})$.

Definition 4.14 ([2, 11])**.** Let $(\mathfrak{g}, \mathfrak{h}, \rho)$ be a LieAct triple. A linear map $D:\mathfrak{g}\to\mathfrak{h}$ is called a **relative differential operator of weight** λ if the following equality holds:

$$D([x,y]_{\mathfrak{g}}) = \rho(x)(D(y)) - \rho(y)(D(x)) + \lambda[D(x), D(y)]_{\mathfrak{h}}, \quad \forall x, y\in\mathfrak{g}.$$

The quadruple $(\mathfrak{g}, \mathfrak{h}, \rho, D)$ is called a **relative differential Lie algebra of weight** λ.

Now we exhibit an $L_\infty[1]$-algebra whose Maurer-Cartan elements are in bijection with the set of structures of relative differential Lie algebras of weight λ structures on $(\mathfrak{g}, \mathfrak{h})$, thus generalising the result of Jiang and Sheng [11] from weight 1 to arbitrary weight.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two vector spaces. As above, let

$$\begin{aligned}\mathfrak{L}' &= \mathrm{Hom}(\bar{\wedge}(\mathfrak{g}\oplus\mathfrak{h}), \mathfrak{g}\oplus\mathfrak{h}),\\ \mathfrak{M}' &= \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g})\oplus\mathrm{Hom}(\bar{\wedge}\mathfrak{g}\otimes\bar{\wedge}\mathfrak{h}, \mathfrak{h})\oplus\mathrm{Hom}(\bar{\wedge}\mathfrak{h}, \mathfrak{h}),\end{aligned}$$

and let

$$\mathfrak{A}' = \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{h}).$$

Denote $\iota_{\mathfrak{A}'}:\mathfrak{A}'\to\mathfrak{L}'$ to be the natural injection and $P':\mathfrak{L}'\to\mathfrak{A}'$ to be the natural surjection. Obviously with the induced Lie bracket, $\mathfrak{A}'$ is an abelian graded Lie algebra, $\iota_{\mathfrak{A}'}:\mathfrak{A}'\to\mathfrak{L}'$ is also a homomorphism of graded Lie algebras, $\mathrm{Ker}(P')$ is a graded Lie subalgebra of $\mathfrak{L}'$, $P\circ\iota_{\mathfrak{A}'} = \mathrm{Id}_{\mathfrak{A}'}$. Now let $\Delta' = 0$. Then the following result is easy by direct inspection and by Proposition 4.10.

Proposition 4.15 (Compare with [11, Proposition 3.7])**.** *The data* $(\mathfrak{L}', \mathfrak{M}', \iota_{\mathfrak{M}'}, \mathfrak{A}', \iota_{\mathfrak{A}'}, P', \Delta' = 0)$ *introduced above is a generalised V-datum, and an* $L_\infty[1]$*-algebra structure on* $s\mathfrak{M}'\oplus\mathfrak{A}'$ *is given by*

$$\begin{aligned}l_2(sf, sg) &= (-1)^{|f|}s[f,g]_{\mathrm{NR}},\\ l_i(sf, \xi_1,\cdots,\xi_{i-1}) &= \lambda^{i-2}P[\cdots[f,\xi_1]_{\mathrm{NR}},\cdots,\xi_{i-1}]_{\mathrm{NR}}, i\geq 2,\end{aligned}$$

for homogeneous elements $f, g\in\mathfrak{M}'$, $\xi_1,\cdots,\xi_{i-1}\in\mathfrak{A}'$ *and the other components vanish.*

Theorem 4.16 (Compare with [11, Theorem 3.8])**.** *Let* $\mathfrak{g}, \mathfrak{h}$ *be two vector spaces. Let* $\pi\in\mathrm{Hom}(\wedge^2\mathfrak{g}, \mathfrak{g})$*,* $\rho\in\mathrm{Hom}(\mathfrak{g}\otimes\mathfrak{h}, \mathfrak{h})$*,* $\mu\in\mathrm{Hom}(\wedge^2\mathfrak{h}, \mathfrak{h})$ *and* $D\in\mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$*, then* $(s(\pi+\rho+\mu), D)\in\mathcal{MC}(s\mathfrak{M}'\oplus\mathfrak{A}')$ *if and only if* $(\mathfrak{g}, \mathfrak{h}, \rho)$ *is a LieAct triple and D is a relative differential operator of weight* $\lambda\in\mathbf{k}$.

Proof. Let $\chi := \pi+\rho+\mu$. By Proposition 4.10, $(s\chi, D) \in \mathcal{MC}(s\mathfrak{M}'\oplus\mathfrak{A}')$ if and only if $\chi \in \mathcal{MC}(\mathfrak{M}')$ (that is, $(\mathfrak{g}, \mathfrak{h}, \rho)$ is a LieAct triple by Proposition 4.13) and $\sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-1)!} P[\dots, [\chi, \underbrace{D]_{\mathrm{NR}}, \dots, D}_{(k-1)\ \text{times}}]_{\mathrm{NR}} = 0$.

Since $[\chi, D]_{\mathrm{NR}} \in \mathrm{Hom}(\wedge^2\mathfrak{g}, \mathfrak{h}) \oplus \mathrm{Hom}(\mathfrak{g}\otimes\mathfrak{h}, \mathfrak{h})$,

$$[[\chi, D]_{\mathrm{NR}}, D]_{\mathrm{NR}} \in \mathrm{Hom}(\wedge^2\mathfrak{g}, \mathfrak{h}) \text{ and } [[[\chi, D]_{\mathrm{NR}}, D]_{\mathrm{NR}}, D]_{\mathrm{NR}} = 0,$$

then we obtain

$$\begin{aligned} 0 =& \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-1)!} P[\dots, [\chi, \underbrace{D]_{\mathrm{NR}}, \dots, D}_{(k-1)\ \text{times}}]_{\mathrm{NR}} \\ =& P[\chi, D]_{\mathrm{NR}} + \frac{\lambda}{2} P[[\chi, D]_{\mathrm{NR}}, D]_{\mathrm{NR}}. \end{aligned}$$

For arbitrary $(x, y) \in \wedge^2\mathfrak{g}$, we have

$$\begin{aligned} 0 &= P[\chi, D]_{\mathrm{NR}}(x, y) + \frac{\lambda}{2} P[[\chi, D]_{\mathrm{NR}}, D]_{\mathrm{NR}}(x, y) \\ &= \rho\bar{\circ}D(x, y) - D\bar{\circ}\pi(x, y) + \frac{\lambda}{2}(\mu\bar{\circ}D)\bar{\circ}D(x, y) \\ &= \rho(D(x), y) - \rho(D(y), x) - D([x, y]_{\mathfrak{g}}) + \frac{\lambda}{2}(\mu\bar{\circ}D(D(x), y) - \mu\bar{\circ}D(D(y), x)) \\ &= \rho(x)D(y) - \rho(y)D(x) - D([x, y]_{\mathfrak{g}}) + \frac{\lambda}{2}\big(- \mu(D(y), (D(x)) + \mu((D(x), D(y))\big) \\ &= \rho(x)D(y) - \rho(y)D(x) - D([x, y]_{\mathfrak{g}}) + \lambda[D(x), D(y)]_{\mathfrak{h}}, \end{aligned}$$

Hence D is a relative differential operator of weight λ. □

4.4. $L_\infty[1]$-structure for absolute differential Lie algebras.

Let $\mathfrak{g}$ be a vector space. We consider the graded Lie algebra

$$\mathfrak{L} := \mathrm{Hom}(\bar{\wedge}(\mathfrak{g}\oplus\mathfrak{g}), \mathfrak{g}\oplus\mathfrak{g})$$

endowed with the Nijenhuis-Richardson bracket $[\ ,\ ]_{\mathrm{NR}}$. For convenience of presentation, we shall write $\mathfrak{g}'$ for the second $\mathfrak{g}$ in $\mathfrak{g}\oplus\mathfrak{g}$, that is,

$$\mathfrak{L} := \mathrm{Hom}(\bar{\wedge}(\mathfrak{g}\oplus\mathfrak{g}'), \mathfrak{g}\oplus\mathfrak{g}').$$

Let

$$\mathfrak{M} := \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g}) \text{ and } \mathfrak{A} := \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g}).$$

Endow $\mathfrak{M}$ with the Nijenhuis-Richardson bracket and $\mathfrak{A}$ with the trivial bracket. Consider two linear maps $\iota_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{L}$ and $\iota_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{L}$ defined as follows: For given $f : \wedge^{n+1}\mathfrak{g} \to \mathfrak{g} \in \mathfrak{M}$, $\iota_{\mathfrak{M}}(f) = \sum_{i=0}^{n+1} f_i$ where $f_i : \wedge^{n+1-i}\mathfrak{g}\otimes\wedge^i\mathfrak{g}' \to \mathfrak{g}'$, $0 \le i \le n+1$ is given by

$$f_i(x_1 \wedge \cdots \wedge x_{n+1-i} \otimes y_{n+2-i} \wedge \cdots \wedge y_{n+1}) := f(x_1 \wedge \cdots \wedge x_{n+1-i} \wedge y_{n+2-i} \wedge \cdots \wedge y_{n+1}),$$

for $x_1 \wedge \cdots \wedge x_{n+1-i} \otimes y_{n+2-i} \wedge \cdots \wedge y_{n+1} \in \wedge^{n+1-i}\mathfrak{g}\otimes\wedge^i\mathfrak{g}'$; $\iota_{\mathfrak{A}}$ identifies $\mathfrak{A} = \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g})$ with the subspace $\mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g}')$ of $\mathfrak{L}$. Let $P : \mathfrak{L} \to \mathfrak{A}$ be the natural projection identifying the subspace $\mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g}')$ with $\mathfrak{A} = \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g})$.

Proposition 4.17. *Let $\mathfrak{g}$ be a vector space and $\lambda \in \mathbf{k}$. The datum $(\mathfrak{L}, \mathfrak{M}, \iota_{\mathfrak{M}}, \mathfrak{A}, \iota_{\mathfrak{A}}, P, \Delta = 0)$ defined above is a generalised V-datum.*

Proof. Obviously, $\iota_{\mathfrak{M}}:\mathfrak{M}\to\mathfrak{L}$ and $\iota_{\mathfrak{A}}:\mathfrak{A}\to\mathfrak{L}$ are injective. The only unclear statement is that $\iota_{\mathfrak{M}}:\mathfrak{M}\to\mathfrak{L}$ is a homomorphism of graded Lie algebras.

In fact, for arbitrary $f:\wedge^{n+1}\mathfrak{g}\to\mathfrak{g}$ and $g:\wedge^{m+1}\mathfrak{g}\to\mathfrak{g}$ in $\mathfrak{M}$, we have

$$
\begin{aligned}
&[\iota(f),\iota(g)]_{\mathrm{NR}}\\
=&\Big[\sum_{i=0}^{n+1} f_i,\sum_{j=0}^{m+1} g_j\Big]_{\mathrm{NR}}\\
=&[f_0,g_0]_{\mathrm{NR}}+(-1)^{mn+1}\sum_{j=1}^{m} g_j\bar{\circ} f_0+\sum_{i=1}^{n} f_i\bar{\circ} g_0+\sum_{i=1}^{n}\sum_{j=1}^{m}[f_i,g_j]_{\mathrm{NR}}+\sum_{i=1}^{n}[f_i,g_{m+1}]_{\mathrm{NR}}\\
&+\sum_{j=1}^{m}[f_{n+1},g_j]_{\mathrm{NR}}+[f_{n+1},g_{m+1}]_{\mathrm{NR}}\\
=&\left(f_0\bar{\circ} g_0+\sum_{i=1}^{n} f_i\bar{\circ} g_0+\sum_{i=1}^{n}\sum_{j=1}^{m} f_i\bar{\circ} g_j+\sum_{i=1}^{n} f_i\bar{\circ} g_{m+1}+\sum_{j=1}^{m} f_{n+1}\bar{\circ} g_j++f_{n+1}\bar{\circ} g_{m+1}\right)-\\
&(-1)^{mn}\left(g_0\bar{\circ} f_0+\sum_{i=1}^{n} g_0\bar{\circ} f_i+\sum_{j=1}^{m} g_j\bar{\circ} f_0+\sum_{i=1}^{n}\sum_{j=1}^{m} g_j\bar{\circ} f_i+\sum_{j=1}^{m} g_j\bar{\circ} f_{n+1}+\sum_{i=1}^{n} g_{m+1}\bar{\circ} f_i+g_{m+1}\bar{\circ} f_{n+1}\right)\\
=&\iota(f\bar{\circ} g)-(-1)^{mn}\iota(g\bar{\circ} f)\\
=&\iota([f,g]_{\mathrm{NR}}).
\end{aligned}
$$

Thus $\iota_{\mathfrak{M}}$ is a graded Lie algebra homomorphism. □

Lemma 4.18. *Keep the above notations, for arbitrary $f=\sum_{i=0}^{n+1} f_i\in\mathrm{Hom}(\wedge^{n+1}\mathfrak{g},\mathfrak{g})$ and $1\le r\le n+1$, take $\xi_i\in\mathrm{Hom}(\wedge^{m_i+1}\mathfrak{g},\mathfrak{g}')$, $1\le i\le r$. We have*

$$(\cdots((f\bar{\circ}\xi_1)\bar{\circ}\xi_2)\cdots)\bar{\circ}\xi_r=\sum_{\tau\in\mathrm{Sh}(m_r+1,\dots,m_1+1,n+1-r)}(-1)^{\sum_{j=1}^{r}(m_1+\cdots+m_{j-1})m_j} f_r(\xi_r\otimes\cdots\otimes\xi_1\otimes\mathrm{Id}^{\otimes n+1-r})\tau^{-1},$$

where, denote $t=n+\sum_{j=1}^{r} m_j+1$, τ^{-1} acts on $\wedge^t V$ via $\tau^{-1}(x_1,\dots,x_t):=\mathrm{sgn}(\tau)(x_{\tau(1)},\dots,x_{\tau(t)})$.

Proof. In fact, by definition of $\bar{\circ}$, we have

$$
\begin{aligned}
&((\cdots((f\bar{\circ}\xi_1)\bar{\circ}\xi_2)\cdots)\bar{\circ}\xi_r)(x_1,\dots,x_t)\\
=&\sum_{\sigma_1\in\mathrm{Sh}(m_r+1,t-1-m_r)}\mathrm{sgn}(\sigma_1)((\cdots(f_r\bar{\circ}\xi_1)\cdots)\bar{\circ}\xi_{r-1})\Big(\xi_r(x_{\sigma_1(1)},\dots,x_{\sigma_1(m_r+1)}),x_{\sigma_1(m_r+2)},\dots,x_{\sigma_1(t)}\Big)\\
=&\sum_{\substack{\sigma_1\in\mathrm{Sh}(m_r+1,t-1-m_r)\\ \sigma_2\in\mathrm{Sh}(m_{r-1}+1,t-1-m_r-m_{r-1})}}\mathrm{sgn}(\sigma_2)\mathrm{sgn}(\sigma_1)\\
&((\cdots(f_r\bar{\circ}\xi_1)\cdots)\bar{\circ}\xi_{r-2})\Big(\xi_{r-1}(x_{\sigma_1(m_r+\sigma_2(1))},\dots,x_{\sigma_1(m_r+\sigma_2(m_{r-1}+1))}),\xi_r(x_{\sigma_1(1)},\dots,x_{\sigma_1(m_r+1)}),\\
&x_{\sigma_1(m_r+\sigma_2(m_{r-1}+3))},\dots,x_{\sigma_1(m_r+\sigma_2(t-m_r))}\Big)\\
=&\cdots\cdots\cdots
\end{aligned}
$$

$$= \sum_{\substack{\sigma_1\in \mathrm{Sh}(m_r+1,t-1-m_r)\\ \sigma_2\in \mathrm{Sh}(m_{r-1}+1,t-1-m_r-m_{r-1})\\ \cdots\\ \sigma_r\in \mathrm{Sh}(m_1+1,n)}} \mathrm{sgn}(\sigma_r)\cdots\mathrm{sgn}(\sigma_2)\mathrm{sgn}(\sigma_1)$$

$$f_r\Big(\xi_1(x_{\sigma_1(m_r+\sigma_2(m_{r-1}+\cdots+\sigma_{r-1}(m_2+\sigma_r(1))\cdots))},\dots,x_{\sigma_1(m_r+\sigma_2(m_{r-1}+\cdots+\sigma_{r-1}(m_2+\sigma_r(m_1+1))\cdots))}),\dots,$$
$$\xi_{r-1}(x_{\sigma_1(m_r+\sigma_2(1))},\dots,x_{\sigma_1(m_r+\sigma_2(m_{r-1}+1))}),\xi_r(x_{\sigma_1(1)},\dots,x_{\sigma_1(m_r+1)}),$$
$$x_{\sigma_1(m_r+\sigma_2(m_{r-1}+\cdots+\sigma_{r-1}(m_2+\sigma_r(m_1+r+1))\cdots))},\dots,x_{\sigma_1(m_r+\sigma_2(m_{r-1}+\cdots+\sigma_{r-1}(m_2+\sigma_r(m_1+n+1))\cdots))}\Big)$$

$$= \sum_{\substack{\sigma_1\in \mathrm{Sh}(m_r+1,t-1-m_r)\\ \sigma_2\in \mathrm{Sh}(m_{r-1}+1,t-2-m_r-m_{r-1})\\ \cdots\\ \sigma_r\in \mathrm{Sh}(m_1+1,n)}} (-1)^{\frac{r(r-1)}{2}}\mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)\cdots\mathrm{sgn}(\sigma_r)$$

$$f_r\Big(\xi_r(x_{\sigma_1(1)},\dots,x_{\sigma_1(m_r+1)}),\xi_{r-1}(x_{\sigma_1(m_r+\sigma_2(1))},\dots,x_{\sigma_1(m_r+\sigma_2(m_{r-1}+1))}),\dots,$$
$$\xi_1(x_{\sigma_1(m_r+\sigma_2(m_{r-1}+\cdots+\sigma_{r-1}(m_2+\sigma_r(1))\cdots))},\dots,x_{\sigma_1(m_r+\sigma_2(m_{r-1}+\cdots+\sigma_{r-1}(m_2+\sigma_r(m_1+1))\cdots))}),$$
$$x_{\sigma_1(m_r+\sigma_2(m_{r-1}+\cdots+\sigma_{r-1}(m_2+\sigma_r(m_1+r+1))\cdots))},\dots,x_{\sigma_1(m_r+\sigma_2(m_{r-1}+\cdots+\sigma_{r-1}(m_2+\sigma_r(m_1+n+1))\cdots))}\Big)$$

$$= \sum_{\substack{\sigma_1\in \mathrm{Sh}(m_r+1,t-1-m_r)\\ \sigma_2\in \mathrm{Sh}(m_{r-1}+1,t-2-m_r-m_{r-1})\\ \cdots\\ \sigma_r\in \mathrm{Sh}(m_1+1,n)}} (-1)^{\frac{r(r-1)}{2}+\kappa}\mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)\cdots\mathrm{sgn}(\sigma_r)f_r(\xi_r\otimes\cdots\otimes\xi_1\otimes\mathrm{Id}^{\otimes n+1-r})$$

$$\Big(x_{\sigma_1(1)},\dots,x_{\sigma_1(m_r+1)}\big|,x_{\sigma_1(m_r+\sigma_2(1))},\dots,x_{\sigma_1(m_r+\sigma_2(m_{r-1}+1))}\big|,\dots,$$
$$\big|x_{\sigma_1(m_r+\sigma_2(m_{r-1}+\cdots+\sigma_{r-1}(m_2+\sigma_r(1))\cdots))},\dots,x_{\sigma_1(m_r+\sigma_2(m_{r-1}+\cdots+\sigma_{r-1}(m_2+\sigma_r(m_1+1))\cdots))}\big|,$$
$$\big|x_{\sigma_1(m_r+\sigma_2(m_{r-1}+\cdots+\sigma_{r-1}(m_2+\sigma_r(m_1+r+1))\cdots))},\dots,x_{\sigma_1(m_r+\sigma_2(m_{r-1}+\cdots+\sigma_{r-1}(m_2+\sigma_r(m_1+n+1))\cdots))}\Big)$$

(where κ is given by Koszul sign)

$$= \sum_{\tau\in \mathrm{Sh}(m_r+1,\dots,m_1+1,n+1-r)} (-1)^{\sum_{j=1}^{r}(m_1+\cdots+m_{j-1})m_j} f_r(\xi_r\otimes\cdots\otimes\xi_1\otimes\mathrm{Id}^{\otimes n+1-r})\tau^{-1}(x_1,\dots,x_t).$$

□

By Proposition 4.10, we have:

Proposition 4.19. *Keep the above notations, there exists an $L_\infty[1]$-algebra structure on $s\mathfrak{M}\oplus\mathfrak{A}$, where l_i are given by*

$$l_2(sf,sg)=(-1)^{|f|}s[f,g]_{\mathrm{NR}},\quad l_2(sf,\xi)=[f,\xi]_{\mathrm{NR}},$$

and for $3\le i\le n+2$,

$$l_i(sf,\xi_1,\cdots,\xi_{i-1})=\sum_{\tau\in \mathrm{Sh}(m_{i-1}+1,\dots,m_1+1,n+2-i)} (-1)^{\sum_{j=1}^{i-1}(m_1+\cdots+m_{j-1})m_j}\lambda^{i-2}f(\xi_{i-1}\otimes\cdots\otimes\xi_1\otimes\mathrm{Id}^{\otimes n+2-i})\tau^{-1},$$

for homogeneous elements $f\in\mathrm{Hom}(\wedge^{n+1}\mathfrak{g},\mathfrak{g})\subseteq\mathfrak{M}$, $g\in\mathrm{Hom}(\wedge^{m+1}\mathfrak{g},\mathfrak{g})\subseteq\mathfrak{M}$, $\xi\in\mathrm{Hom}(\wedge^{m+1}\mathfrak{g},\mathfrak{g})\subseteq\mathfrak{A}$, and $\xi_j\in\mathrm{Hom}(\wedge^{m_j+1}\mathfrak{g},\mathfrak{g})\subseteq\mathfrak{A}$, $1\le j\le i-1$, and all others components vanish.

Proof. The assertion follows from Proposition 4.17, Proposition 4.10, and Lemma 4.18. Indeed, by Propositions 4.17 and 4.10,

$$\begin{aligned} l_2(sf,sg) &= (-1)^{|f|}s[f,g]_{\mathrm{NR}},\\ l_i(sf,\xi_1,\cdots,\xi_{i-1}) &= \lambda^{i-2}P[\cdots[\iota_{\mathfrak{M}}(f),\iota_{\mathfrak{A}}(\xi_1)]_{\mathrm{NR}},\cdots,\iota_{\mathfrak{A}}(\xi_{i-1})]_{\mathrm{NR}}, i\ge 2.\end{aligned}$$

On one hand, for $i = 2$, we have

$$\begin{aligned}
l_2(sf, \xi) &= P[\iota_{\mathfrak{M}}(f), \iota_{\mathfrak{A}}(\xi)]_{\mathrm{NR}} \\
&= P\Big(\sum_{i=0}^{n+1} f_i \bar{\circ} \iota_{\mathfrak{A}}(\xi) - (-1)^{|f||\xi|} \iota_{\mathfrak{A}}(\xi) \bar{\circ} \sum_{i=0}^{n+1} f_i\Big) \\
&= P(\sum_{i=1}^{n+1} f_i \bar{\circ} \iota_{\mathfrak{A}}(\xi) - (-1)^{|f||\xi|} \iota_{\mathfrak{A}}(\xi) \bar{\circ} f_0\Big) \\
&= f \bar{\circ} \xi - (-1)^{|f||\xi|} \xi \bar{\circ} f \\
&= [f, \xi]_{\mathrm{NR}}.
\end{aligned}$$

On the other hand, for $i \geq 3$, we have

$$\begin{aligned}
& P[\cdots[\iota_{\mathfrak{M}}(f), \iota_{\mathfrak{A}}(\xi_1)]_{\mathrm{NR}}, \ldots, \iota_{\mathfrak{A}}(\xi_{i-1})]_{\mathrm{NR}} \\
&= P[\cdots[\sum_{j=1}^{n+1} f_j \bar{\circ} \iota_{\mathfrak{A}}(\xi_1) - (-1)^{|f||\xi_1|} \iota_{\mathfrak{A}}(\xi_1) \bar{\circ} f_0, \iota_{\mathfrak{A}}(\xi_2)]_{\mathrm{NR}}, \ldots, \iota_{\mathfrak{A}}(\xi_{i-1})]_{\mathrm{NR}} \\
&= P[\cdots[\sum_{j=2}^{n+1} (f_j \bar{\circ} \iota_{\mathfrak{A}}(\xi_1)) \bar{\circ} \iota_{\mathfrak{A}}(\xi_2), \iota_{\mathfrak{A}}(\xi_3)]_{\mathrm{NR}}, \ldots, \iota_{\mathfrak{A}}(\xi_{i-1})]_{\mathrm{NR}} \\
&= P((\cdots \sum_{j=i-1}^{n+1} (f_j \bar{\circ} \iota_{\mathfrak{A}}(\xi_1)) \cdots) \bar{\circ} \iota_{\mathfrak{A}}(\xi_{i-1})) \\
&= P((\cdots (f_{i-1} \bar{\circ} \iota_{\mathfrak{A}}(\xi_1)) \bar{\circ} \cdots) \bar{\circ} \iota_{\mathfrak{A}}(\xi_{i-1})) \\
&= (\cdots (f \bar{\circ} \xi_1) \bar{\circ} \cdots) \bar{\circ} \xi_{i-1} \\
&= \sum_{\tau \in \mathrm{Sh}(m_{i-1}+1, \ldots, m_1+1, n+2-i)} (-1)^{\sum_{j=1}^{i-1} (m_1 + \cdots + m_{j-1}) m_j} \lambda^{i-2} f(\xi_{i-1} \otimes \cdots \otimes \xi_1 \otimes \mathrm{Id}^{\otimes n+2-i}) \tau^{-1},
\end{aligned}$$

where the last equality follows from Lemma 4.18. □

Example 4.20. Consider a special case where $n = 1, i = 3, t = 2, m_1 = m_2 = 0$. For homogeneous elements $\pi \in \mathrm{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$, $\xi_1 = \xi_2 = D \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$, we have

$$\begin{aligned}
l_3(s\pi, D, D)(x, y) &= \sum_{\tau \in \mathrm{Sh}(1,1)} \lambda \pi(D \otimes D)\, \tau^{-1}(x, y) \\
&= \lambda \pi(Dx, Dy) - \lambda \pi(Dy, Dx) \\
&= 2\lambda \pi(Dx, Dy).
\end{aligned}$$

Theorem 4.21. *Let $\mathfrak{g}$ be a vector space. Let $\pi \in \mathrm{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$, $D \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$, then $(s\pi, D) \in \mathcal{MC}(s\mathfrak{M} \oplus \mathfrak{A})$ if and only if $(\mathfrak{g}, \pi, D)$ is a differential Lie algebra of weight $\lambda \in \mathbf{k}$.*

Proof. By Proposition 4.10, $(s\pi, D) \in \mathcal{MC}(s\mathfrak{M} \oplus \mathfrak{A})$ if and only if $[\pi,\pi]_{\mathrm{NR}} = 0$ (that is, π is a Lie bracket) and

$$\begin{aligned}
0 &= \sum_{k=2}^{\infty} \frac{1}{(k-1)!} l_k(s\pi, \underbrace{D, \dots, D}_{(k-1)\text{ times}}) \\
&= l_2(s\pi, D) + \frac{\lambda}{2} l_3(s\pi, D, D) \\
&= \pi \bar{\circ} D - D \bar{\circ} \pi + \frac{\lambda}{2} l_3(s\pi, D, D).
\end{aligned}$$

For arbitrary $(x, y) \in \wedge^2 \mathfrak{g}$, we have

$$\begin{aligned}
0 &= (\pi \bar{\circ} D - D \bar{\circ} \pi + \frac{\lambda}{2} l_3(s\pi, D, D))(x, y) \\
&= \pi \bar{\circ} D(x, y) - D \bar{\circ} \pi(x, y) + \lambda \pi(Dx, Dy) \quad \text{(by Example 4.20)} \\
&= \pi(D(x), y) - \pi(D(y), x) - D([x, y]) + \lambda [D(x), D(y)] \\
&= [D(x), y] - [D(y), x] - D([x, y]) + \lambda [D(x), D(y)] \\
&= [D(x), y] + [x, D(y)] - D([x, y]) + \lambda [D(x), D(y)].
\end{aligned}$$

Hence D is a differential operator of weight λ. □

4.5. **Equivalences between $L_\infty[1]$-structures for absolute and relative differential Lie algebras with weight.**

4.5.1. *From relative to absolute.*

Let $(\mathfrak{g}, \mu, \mathrm{d})$ be a differential Lie algebra of weight λ.

Let $\mathfrak{h} = \mathfrak{g}$, $\rho : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ be the adjoint representation. Consider d as a map $\mathrm{d} : \mathfrak{g} \to \mathfrak{h}$. Then $(\mathfrak{g}, \mathfrak{h}, \rho, \mathrm{d})$ is a relative differential Lie algebra of weight λ. As seen in Subsection 4.3, let

$$\begin{aligned}
\mathfrak{L}' &= \mathrm{Hom}(\bar{\wedge}(\mathfrak{g} \oplus \mathfrak{h}), \mathfrak{g} \oplus \mathfrak{h}), \\
\mathfrak{M}' &= \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}(\bar{\wedge}\mathfrak{g} \otimes \bar{\wedge}\mathfrak{h}, \mathfrak{h}) \oplus \mathrm{Hom}(\bar{\wedge}\mathfrak{h}, \mathfrak{h}), \\
\mathfrak{A}' &= \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{h}).
\end{aligned}$$

Denote $\iota_{\mathfrak{A}'} : \mathfrak{A}' \to \mathfrak{L}'$ and $\iota_{\mathfrak{M}'} : \mathfrak{M}' \to \mathfrak{L}'$ to be the natural injection and $P : \mathfrak{L}' \to \mathfrak{A}'$ to be the natural surjection. Then $(\mathfrak{L}', \mathfrak{M}', \iota_{\mathfrak{M}'}, \mathfrak{A}', \iota_{\mathfrak{A}'}, P, \Delta' = 0)$ is a generalised V-datum.

Recall that in Subsection 4.4, let $\mathfrak{g}' = \mathfrak{g}$ and let

$$\begin{aligned}
\mathfrak{L} &= \mathrm{Hom}(\bar{\wedge}(\mathfrak{g} \oplus \mathfrak{g}'), \mathfrak{g} \oplus \mathfrak{g}'), \\
\mathfrak{M} &= \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g}), \\
\mathfrak{A} &= \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g}).
\end{aligned}$$

Let two linear maps $\iota_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{L}$ and $\iota_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{L}$ be defined as follows: For given $f : \wedge^{n+1}\mathfrak{g} \to \mathfrak{g} \in \mathfrak{M}$, $\iota_{\mathfrak{M}}(f) = \sum_{i=0}^{n+1} f_i$ where $f_i : \wedge^{n+1-i}\mathfrak{g} \otimes \wedge^i \mathfrak{g}' \to \mathfrak{g}'$, $0 \le i \le n+1$ is given by

$$f_i(x_1 \wedge \cdots \wedge x_{n+1-i} \otimes y_{n+2-i} \wedge \cdots \wedge y_{n+1}) := f(x_1 \wedge \cdots \wedge x_{n+1-i} \wedge y_{n+2-i} \wedge \cdots \wedge y_{n+1}),$$

for $x_1 \wedge \cdots \wedge x_{n+1-i} \otimes y_{n+2-i} \wedge \cdots \wedge y_{n+1} \in \wedge^{n+1-i}\mathfrak{g} \otimes \wedge^i \mathfrak{g}'$; $\iota_{\mathfrak{A}}$ identifies $\mathfrak{A} = \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g})$ with the subspace $\mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g}')$ of $\mathfrak{L}$. Let $P : \mathfrak{L} \to \mathfrak{A}$ be the natural projection identifying the subspace $\mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g}')$ with $\mathfrak{A} = \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g})$. By Proposition 4.17, $(\mathfrak{L}, \mathfrak{M}, \iota_{\mathfrak{M}}, \mathfrak{A}, \iota_{\mathfrak{A}}, P, \Delta = 0)$ is a generalised V-datum.

By identifying $\mathfrak{g}'$ with $\mathfrak{h}$, let $f_{\mathfrak{L}} : \mathfrak{L} \to \mathfrak{L}'$ and $f_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}'$ be the identity maps, let $f_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}'$ be exactly defined as $\iota_{\mathfrak{M}}$.

The following result is clear.

Proposition 4.22. *The triple $f = (f_{\mathfrak{L}}, f_{\mathfrak{M}}, f_{\mathfrak{A}})$ is a morphism of generalised V-data from $(\mathfrak{L}, \mathfrak{M}, \iota_{\mathfrak{M}}, \mathfrak{A}, \iota_{\mathfrak{A}}, P, \Delta = 0)$ to $(\mathfrak{L}', \mathfrak{M}', \iota_{\mathfrak{M}'}, \mathfrak{A}', \iota_{\mathfrak{A}'}, P, \Delta' = 0)$. It induces an injective homomorphism of $L_\infty[1]$-algebras from $s\mathfrak{M} \oplus \mathfrak{A}$ introduced in Proposition 4.19 to $s\mathfrak{M}' \oplus \mathfrak{A}'$ introduced in Proposition 4.15.*

The above result means that one can deduce the $L_\infty[1]$-structure of absolute differential Lie algebras from that of relative differential Lie algebras.

4.5.2. *From absolute to relative.*

Let $(\mathfrak{g}, \mathfrak{h}, \rho)$ be a LieAct triple and $D : \mathfrak{g} \to \mathfrak{h}$ be a linear map.

By [18], there exists a Lie algebra $\mathfrak{g} \ltimes_\rho \mathfrak{h}$, where the Lie bracket is given by

$$[x + u, y + v]_\ltimes = [x, y]_\mathfrak{g} + \rho(x)v - \rho(y)u + \lambda[u, v]_\mathfrak{h}, \forall x, y \in \mathfrak{g}, u, v \in \mathfrak{h}.$$

The map D extends to $\tilde{D} : \mathfrak{g} \ltimes_\rho \mathfrak{h} \to \mathfrak{g} \ltimes_\rho \mathfrak{h}$ by $\tilde{D}(x + u) = D(x) - u, \forall x \in \mathfrak{g}, u \in \mathfrak{h}$. Furthermore, we have the following observation which seems to be new.

Proposition 4.23. *$\tilde{D}$ is a differential operator of weight λ if and only if D is a relative differential operator of weight λ.*

Proof. According to the definition of $\tilde{D}$, we have

$$\begin{aligned} & \tilde{D}([x + u, y + v]_\ltimes) \\ =& \tilde{D}([x, y]_\mathfrak{g} + \rho(x)v - \rho(y)u + \lambda[u, v]_\mathfrak{h}) \\ =& D([x, y]_\mathfrak{g}) - \rho(x)v + \rho(y)u - \lambda[u, v]_\mathfrak{h}. \end{aligned} \tag{15}$$

On the other hand,

$$\begin{aligned} & [x + u, \tilde{D}(y + v)]_\ltimes + [\tilde{D}(x + u), y + v]_\ltimes + \lambda[\tilde{D}(x + u), \tilde{D}(y + v)]_\ltimes \\ =& [x + u, D(y) - v]_\ltimes + [D(x) - u, y + v]_\ltimes + \lambda[D(x) - u, D(y) - v]_\mathfrak{h} \\ =& \rho(x)(D(y) - v) + \lambda[u, D(y) - v]_\mathfrak{h} - \rho(y)(D(x) - u) - \lambda[v, D(x) - u]_\mathfrak{h} \\ & + \lambda[D(x) - u, D(y) - v]_\mathfrak{h} \\ =& \rho(x)(D(y)) - \rho(x)(v) + \lambda[u, D(y)]_\mathfrak{h} - \lambda[u, v]_\mathfrak{h} - \rho(y)(D(x)) + \rho(y)(u) + \lambda[D(x), v]_\mathfrak{h} \\ & - \lambda[u, v]_\mathfrak{h} + \lambda[D(x), D(y)]_\mathfrak{h} - \lambda[D(x), v]_\mathfrak{h} - \lambda[u, D(y)]_\mathfrak{h} + \lambda[u, v]_\mathfrak{h}. \end{aligned} \tag{16}$$

Compare the Eq. (15) with Eq. (16), it is easy to see $\tilde{D}$ is a differential operator of weight λ if and only if D is a relative differential operator of weight λ. □

Now let $(\mathfrak{g}, \mathfrak{h}, \rho, D)$ be a relative differential Lie algebra of weight λ, then $(\mathfrak{g} \ltimes_\rho \mathfrak{h}, [\ ,\ ]_\ltimes, \tilde{D})$ is a differential Lie algebra of weight λ by Proposition 4.23. As seen in Subsection 4.4, let $(\mathfrak{g} \ltimes_\rho \mathfrak{h})' = \mathfrak{g} \ltimes_\rho \mathfrak{h}$ and

$$\begin{aligned} \mathfrak{L} &= \mathrm{Hom}(\bar{\wedge}(\mathfrak{g} \ltimes_\rho \mathfrak{h} \oplus (\mathfrak{g} \ltimes_\rho \mathfrak{h})'), \mathfrak{g} \ltimes_\rho \mathfrak{h} \oplus (\mathfrak{g} \ltimes_\rho \mathfrak{h})'), \\ \mathfrak{M} &= \mathrm{Hom}(\bar{\wedge}(\mathfrak{g} \ltimes_\rho \mathfrak{h}), \mathfrak{g} \ltimes_\rho \mathfrak{h}), \\ \mathfrak{A} &= \mathrm{Hom}(\bar{\wedge}(\mathfrak{g} \ltimes_\rho \mathfrak{h}), \mathfrak{g} \ltimes_\rho \mathfrak{h}). \end{aligned}$$

The two linear maps $\iota_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{L}$ and $\iota_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{L}$ defined as that in Subsection 4.4. Then $(\mathfrak{L}, \mathfrak{M}, \iota_{\mathfrak{M}}, \mathfrak{A}, \iota_{\mathfrak{A}}, P, \Delta = 0)$ is a generalised V-datum.

Recall in Subsection 4.3, let

$$\begin{aligned}\mathfrak{L}' &= \mathrm{Hom}(\bar{\wedge}(\mathfrak{g}\oplus\mathfrak{h}), \mathfrak{g}\oplus\mathfrak{h}),\\ \mathfrak{M}' &= \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g})\oplus\mathrm{Hom}(\bar{\wedge}\mathfrak{g}\otimes\bar{\wedge}\mathfrak{h}, \mathfrak{h})\oplus\mathrm{Hom}(\bar{\wedge}\mathfrak{h}, \mathfrak{h}),\\ \mathfrak{A}' &= \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{h}).\end{aligned}$$

Denote $\iota_{\mathfrak{A}'}:\mathfrak{A}'\to\mathfrak{L}'$ to be the natural injection and $P:\mathfrak{L}'\to\mathfrak{A}'$ to be the natural surjection. Then $(\mathfrak{L}', \mathfrak{M}', \iota_{\mathfrak{M}'}, \mathfrak{A}', \iota_{\mathfrak{A}'}, P, \Delta'=0)$ is a generalised V-datum.

Let $f_{\mathfrak{L}'}:\mathfrak{L}'\to\mathfrak{L}$, $f_{\mathfrak{M}'}:\mathfrak{M}'\to\mathfrak{M}$ and $f_{\mathfrak{A}'}:\mathfrak{A}'\to\mathfrak{A}$ be natural injection maps.

The following result is clear.

Proposition 4.24. *The triple $f=(f_{\mathfrak{L}'}, f_{\mathfrak{M}'}, f_{\mathfrak{A}'})$ is a morphism of generalised V-data from $(\mathfrak{L}', \mathfrak{M}', \iota_{\mathfrak{M}'}, \mathfrak{A}', \iota_{\mathfrak{A}'}, P, \Delta'=0)$ to $(\mathfrak{L}, \mathfrak{M}, \iota_{\mathfrak{M}}, \mathfrak{A}, \iota_{\mathfrak{A}}, P, \Delta=0)$. It induces an injective homomorphism of $L_\infty[1]$-algebras from $s\mathfrak{M}'\oplus\mathfrak{A}'$ introduced in Proposition 4.15 to $s\mathfrak{M}\oplus\mathfrak{A}$ introduced in Proposition 4.19.*

The above result means that one can deduce the $L_\infty[1]$-structure of relative differential Lie algebras from that of absolute differential Lie algebras.

5. Application of $L_\infty[1]$-structure for differential Lie algebras

In this section, we will derive from the $L_\infty[1]$-structure the cohomology theory of differential Lie algebras of arbitrary weight and the notion of homotopy differential Lie algebras of arbitrary weight.

5.1. Cohomology of differential Lie algebras from $L_\infty[1]$-structure.

Let $\mathfrak{g}$ be a vector space. Let

$$\mathfrak{M} := \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g}) \text{ and } \mathfrak{A} := \mathrm{Hom}(\bar{\wedge}\mathfrak{g}, \mathfrak{g}).$$

Recall that we have constructed an $L_\infty[1]$-structure on $s\mathfrak{M}\oplus\mathfrak{A}$ in Proposition 4.19.

Let $(\mathfrak{g}, \mu, \mathrm{d})$ be a differential Lie algebra. By Theorem 4.21, $(s\mu, \mathrm{d})$ is a Maurer-Cartan element in the $L_\infty[1]$-algebra $s\mathfrak{M}\oplus\mathfrak{A}$. By Proposition 4.4, twisting $s\mathfrak{M}\oplus\mathfrak{A}$ by $(s\mu, \mathrm{d})$ gives a new $L_\infty[1]$-algebra, whose new differential is denoted by $l_1^{(s\mu,\mathrm{d})}$.

Consider the cochain complex $\mathrm{C}^*_{\mathsf{DL}_\lambda}(\mathfrak{g}, \mathfrak{g}_{\mathrm{ad}})$ of the differential Lie algebra $(\mathfrak{g}, \mu, \mathrm{d})$ with coefficients in the adjoint representation $\mathfrak{g}_{\mathrm{ad}}$. Note that for each $n\geq 1$,

$$\mathrm{C}^n_{\mathsf{DL}_\lambda}(\mathfrak{g}, \mathfrak{g}_{\mathrm{ad}}) = \mathrm{C}^n_{\mathsf{Lie}}(\mathfrak{g}, \mathfrak{g}_{\mathrm{ad}})\oplus\mathrm{C}^{n-1}_{\mathsf{DO}_\lambda}(\mathfrak{g}, \mathfrak{g}_{\mathrm{ad}}) = \mathrm{Hom}(\wedge^n\mathfrak{g}, \mathfrak{g})\oplus\mathrm{Hom}(\wedge^{n-1}\mathfrak{g}, \mathfrak{g}) = (s\mathfrak{M}\oplus\mathfrak{A})^{n-2}$$

is exactly the degree $n-2$ part of $s\mathfrak{M}\oplus\mathfrak{A}$.

Proposition 5.1. *The underlying complex of the twisted $L_\infty[1]$-algebra $s\mathfrak{M}\oplus\mathfrak{A}$ is exactly the double shift of the cochain complex $\mathrm{C}^*_{\mathsf{DL}_\lambda}(\mathfrak{g}, \mathfrak{g}_{\mathrm{ad}})$, up to signs.*

Proof. It suffices to make explicit the differential $l_1^{(s\mu,\mathrm{d})}$.

For $n\geq 1$, $f\in\mathrm{Hom}(\wedge^n\mathfrak{g}, \mathfrak{g})$, $g\in\mathrm{Hom}(\wedge^{n-1}\mathfrak{g}, \mathfrak{g})$,

$$\begin{aligned}l_1^{(s\mu,\mathrm{d})}(sf, g) &= \sum_{k=0}^{\infty}\frac{1}{k!}l_{k+1}(\underbrace{(s\mu,\mathrm{d}),\cdots,(s\mu,\mathrm{d})}_{k\text{ times}}, (sf, g))\\ &= l_2((s\mu,\mathrm{d}), (sf, g)) + \sum_{k=2}^{\infty}\frac{1}{k!}l_{k+1}(\underbrace{(s\mu,\mathrm{d}),\cdots,(s\mu,\mathrm{d})}_{k\text{ times}}, (sf, g))\end{aligned}$$

$$= \quad (l_2(s\mu, sf), l_2(s\mu, g) + l_3(s\mu, \mathrm{d}, g) + l_2(sf, \mathrm{d}) + \sum_{k=2}^{n} \frac{1}{k!} l_{k+1}(sf, \underbrace{\mathrm{d}, \cdots, \mathrm{d}}_{k \text{ times}})).$$

It is easy to see that $l_2(s\mu, sf) = -s[\mu, f]_{\mathrm{NR}}$ is exactly $-s\partial^n_{\mathrm{Lie}}(f)$.

Let us compute $l_2(s\mu, g) + l_3(s\mu, \mathrm{d}, g) + l_2(sf, \mathrm{d}) + \sum_{k=2}^{n} \frac{1}{k!} l_{k+1}(sf, \underbrace{\mathrm{d}, \cdots, \mathrm{d}}_{k \text{ times}})$.

For $x_1, \dots, x_n \in \mathfrak{g}$, we have

$$\begin{aligned}
& (l_2(s\mu, g) + l_3(s\mu, \mathrm{d}, g))(x_1, \dots, x_n) \\
= \ & [\mu, g]_{\mathrm{NR}}(x_1, \dots, x_n) + \lambda \sum_{\tau \in \mathrm{Sh}(1, n-1)} \mu(g \otimes \mathrm{d})\tau^{-1}(x_1, \dots, x_n) \\
= \ & \sum_{i=1}^{n} (-1)^{i+n-1} \rho(x_i) g(x_1, \dots, \hat{x}_i, \dots, x_n) \\
& + \sum_{1 \le i < j \le n}^{n-1} (-1)^{i+j+n} g([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n) \\
& + \lambda \sum_{i=1}^{n} (-1)^{i+n-1} \rho(\mathrm{d}(x_i)) g(x_1, \dots, \hat{x}_i, \dots, x_n) \\
= \ & \partial^{n-1}_{\mathsf{DO}_\lambda}(g)(x_1, \dots, x_n).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& l_2(sf, \mathrm{d}) + \sum_{k=2}^{n} \frac{1}{k!} l_{k+1}(sf, \underbrace{\mathrm{d}, \cdots, \mathrm{d}}_{k \text{ times}})(x_1, \dots, x_n) \\
= \ & (-\mathrm{d} \circ f + \sum_{k=1}^{n} \frac{1}{k!} \lambda^{k-1} f \bar{\circ} \{\underbrace{\mathrm{d}, \cdots, \mathrm{d}}_{k \text{ times}}\})(x_1, \dots, x_n) \\
= \ & -\mathrm{d}(f(x_1, \cdots, x_n)) + \sum_{k=1}^{n} \frac{1}{k!} \sum_{\tau \in \mathrm{Sh}(\underbrace{1, \cdots, 1}_{k \text{ times}}, n-k)} \lambda^{k-1} f(\underbrace{\mathrm{d} \otimes \cdots \otimes \mathrm{d}}_{k \text{ times}} \otimes \mathrm{Id}^{\otimes n-k})(x_1, \dots, x_n) \\
= \ & \sum_{\sigma \in \mathrm{Sh}(k, n-k)} \mathrm{sgn}(\sigma) f_k(x_1, \cdots, \hat{x_{i_1}}, \cdots, \hat{x_{i_k}}, \cdots, x_n, \mathrm{d}(x_{i_1}), \cdots, \mathrm{d}(x_{i_k})) - \mathrm{d}(f(x_1, \cdots, x_n)) \\
= \ & \delta^n(f).
\end{aligned}$$

That is, we have

$$l_1^{(s\mu, \mathrm{d})}(sf, g) = (-s\partial^n_{\mathrm{Lie}}(f), \partial^{n-1}_{\mathsf{DO}_\lambda}(g) + \delta^n(f)) = -\partial^n_{\mathsf{DL}_\lambda}(f, g).$$

□

Let (V, d_V) be a representation of the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ with weight λ. Now we justify the cochain complex $(\mathrm{C}^*_{\mathsf{DL}_\lambda}(\mathfrak{g}, V), \partial^*_{\mathsf{DL}_\lambda})$ of the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ with coefficients in the representation (V, d_V) introduced in Definition 1.11 and Proposition 1.9.

To this end, we consider the trivial extension $\mathfrak{g} \ltimes V$ of the differential Lie algebra $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ by the representation (V, d_V) in Proposition 1.6, and we get the complex $(\mathrm{C}^*_{\mathsf{DL}_\lambda}(\mathfrak{g} \ltimes V, (\mathfrak{g} \ltimes V)_{\mathrm{ad}}), \partial^*_{\mathsf{DL}_\lambda})$, by Proposition 5.1. The following result can be proved by direct inspection.

Proposition 5.2. *The cochain complex* $(\mathrm{C}^*_{\mathsf{DL}_\lambda}(\mathfrak{g}, V), \partial^*_{\mathsf{DL}_\lambda})$ *of the differential Lie algebra* $(\mathfrak{g}, \mathrm{d}_\mathfrak{g})$ *with coefficients in the representation* (V, d_V) *introduced in Definition 1.11 is a subcomplex of* $(\mathrm{C}^*_{\mathsf{DL}_\lambda}(\mathfrak{g} \ltimes V, (\mathfrak{g} \ltimes V)_{\mathrm{ad}}), \partial^*_{\mathsf{DL}_\lambda})$.

Proof. For vector spaces U and W, we have an isomorphism

$$\wedge^p(U \oplus W) \simeq \wedge^p U \oplus (\oplus_{i=1}^{p-1} \wedge^i U \otimes \wedge^{p-i} W) \oplus \wedge^p W.$$

Hence, there exists a natural injection from

$$\mathrm{C}^n_{\mathsf{DL}_\lambda}(\mathfrak{g}, V) = \mathrm{Hom}(\wedge^n \mathfrak{g}, V) \oplus \mathrm{Hom}(\wedge^{n-1}\mathfrak{g}, V)$$

to

$$\mathrm{C}^n_{\mathsf{DL}_\lambda}(\mathfrak{g} \ltimes V, (\mathfrak{g} \ltimes V)_{\mathrm{ad}}) = \mathrm{Hom}(\wedge^n(\mathfrak{g} \oplus V), \mathfrak{g} \oplus V) \oplus \mathrm{Hom}(\wedge^{n-1}(\mathfrak{g} \oplus V), \mathfrak{g} \oplus V).$$

Now it suffices to show that this map commutes with differentials.

□

5.2. **Homotopy differential Lie algebras with weight.**

In this section, we need to use graded Nijenhuis-Richardson brackets.

Let $\mathfrak{g}$ be a graded vector space and denote $\mathfrak{l} = s\mathfrak{g}$. Consider the graded vector space $\mathfrak{C}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}) := \mathrm{Hom}(S(\mathfrak{l}), \mathfrak{l})$.

Recall that by Eq. (1), $S^n(s\mathfrak{g})$ is isomorphic to $s^n \wedge^n \mathfrak{g}$. When $\mathfrak{g} = \mathfrak{g}^0 = V$ is ungraded, then $\mathrm{Hom}(S^n(sV), sV)$, which has degree $n-1$, is isomorphic to $\mathrm{Hom}(\wedge^n V, V)$ as vector spaces. This justifies why we have imposed degree $n-1$ on the latter in Subsection 4.3.

The graded Nijenhuis-Richardson bracket $[-,-]_{\mathrm{NR}}$ on the graded vector space $\mathfrak{C}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g})$ can be defined as follows: for $f \in \mathrm{Hom}(S^n(\mathfrak{l}), \mathfrak{l})$ of degree p and $g \in \mathrm{Hom}(S^m(\mathfrak{l}), \mathfrak{l})$ of degree q,

$$[f, g]_{\mathrm{NR}} := f \bar{\circ} g - (-1)^{pq} g \bar{\circ} f,$$

where $f \bar{\circ} g \in \mathrm{Hom}(S^{m+n-1}(\mathfrak{l}), \mathfrak{l})$ of degree $p+q$ is defined by

$$f \bar{\circ} g(v_1, \ldots, v_{m+n-1}) = \sum_{\sigma \in \mathrm{Sh}(m, n-1)} \varepsilon(\sigma) f(g(v_{\sigma(1)}, \ldots, v_{\sigma(m)}), v_{\sigma(m+1)}, \ldots, v_{\sigma(m+n-1)}),$$

for $v_1, \ldots, v_{m+n-1} \in \mathfrak{l}$. It is well known that $(\mathfrak{C}_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}), [\ ,\ ]_{\mathrm{NR}})$ is a graded Lie algebra whose Maurer-Cartan elements correspond bijectively to $L_\infty[1]$-algebra structures on $\mathfrak{l}$.

We consider the graded Lie algebra

$$\mathfrak{L} := \mathfrak{C}_{\mathrm{Lie}}(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g} \oplus \mathfrak{g})$$

endowed with the graded Nijenhuis-Richardson bracket $[\ ,\ ]_{\mathrm{NR}}$.

Let

$$\mathfrak{M} := \mathrm{Hom}(\bar{S}(\mathfrak{l}), \mathfrak{l}) \text{ and } \mathfrak{A} := \mathrm{Hom}(\bar{S}(\mathfrak{l}), \mathfrak{l}).$$

Endow $\mathfrak{M}$ with the graded Nijenhuis-Richardson bracket and $\mathfrak{A}$ with the trivial bracket. With the same analysis in Subsection 4.4, we have:

Proposition 5.3. *Let* $\mathfrak{g}$ *be a graded vector space. Then we get a generalised V-datum* $(\mathfrak{L}, \mathfrak{M}, \iota_\mathfrak{M}, \mathfrak{A}, \iota_\mathfrak{A}, P, \Delta = 0)$ *and there is an* L_∞*[1]-algebra* $s\mathfrak{M} \oplus \mathfrak{A}$ *given below:*

$$l_2(sf, sg) = (-1)^{|f|} s[f, g]_{\mathrm{NR}}, \quad l_2(sf, \xi) = [f, \xi]_{\mathrm{NR}},$$

and for $3 \leq i \leq n+2$,

$$l_i(sf, \xi_1, \cdots, \xi_{i-1}) = \sum_{\tau \in \mathrm{Sh}(m_{i-1}+1, \ldots, m_1+1, n+2-i)} (-1)^{\sum_{j=1}^{i-1}(|\xi_1| + \cdots + |\xi_{j-1}|)|\xi_j|} \lambda^{i-2} f(\xi_{i-1} \otimes \cdots \otimes \xi_1 \otimes \mathrm{Id}^{\otimes n+2-i}) \tau^{-1},$$

for homogeneous elements $f \in \mathrm{Hom}(S^{n+1}(\mathfrak{l}), \mathfrak{l}) \subseteq \mathfrak{M}$*,* $g \in \mathrm{Hom}(S^{m+1}(\mathfrak{l}), \mathfrak{l}) \subseteq \mathfrak{M}$*,* $\xi \in \mathrm{Hom}(S^{m+1}(\mathfrak{l}), \mathfrak{l}) \subseteq \mathfrak{A}$*, and* $\xi_j \in \mathrm{Hom}(S^{m_j+1}(\mathfrak{l}), \mathfrak{l}) \subseteq \mathfrak{A}$*,* $1 \leq j \leq i-1$*, and all others components vanish.*

Definition 5.4. Let $\mathfrak{g}$ be a graded vector space. A structure of **homotopy differential Lie algebra with weight** λ on $\mathfrak{g}$ is defined to be a Maurer-Cartan element of the $L_\infty[1]$-algebra $s\mathfrak{M} \oplus \mathfrak{A}$ introduced in Proposition 5.3.

Theorem 5.5. *Let* $\mathfrak{g}$ *be a graded vector space and denote* $\mathfrak{l} = s\mathfrak{g}$*. A homotopy differential Lie algebra with weight* $\lambda \in \mathbf{k}$ *on* $\mathfrak{g}$ *is equivalent to the pair* $(\mu = \{\mu_i\}_{i\geq 1}, D = \{D_i\}_{i\geq 1})$*, where for each* $i \geq 1$*,* $\mu_i : S^i(\mathfrak{l}) \to \mathfrak{l}$ *is a degree* 1 *map such that* $(\mathfrak{g}, \mu)$ *is an* $L_\infty[1]$*-algebra and for each* $i \geq 1$*,* $D_i : S^i(\mathfrak{l}) \to \mathfrak{l}$ *is of degree* 0 *which form a homotopy differential operator of weight* λ*. More precisely, for* $n \geq 1$ *and* $x_1, \dots, x_n \in \mathfrak{l}$*, we have*

$$\sum_{i=1}^{n} \sum_{\sigma\in \mathrm{Sh}(i,n-i)} \varepsilon(\sigma)\mu_{n-i+1}(\mu_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}) = 0, \tag{17}$$

and

$$\begin{aligned}&\sum_{p\geq 2}\sum_{t=p-1}^{n}\sum_{m_1+\cdots+m_{p-1}=t}\sum_{\substack{\sigma\in\mathrm{Sh}(m_{p-1},\cdots,m_1,n-t)\\ \sigma|_t\in \mathrm{PSh}(m_{p-1},\cdots,m_1)}} \varepsilon(\sigma)\lambda^{p-2}\mu_{n-t+p-1}(D_{m_{p-1}}(x_{\sigma(1)},\cdots,x_{\sigma(m_{p-1})}),\cdots,\\ &\qquad \cdots, D_{m_1}(x_{\sigma(m_2+\cdots+m_{p-1}+1)},\cdots,x_{\sigma(t)}), x_{\sigma(t+1)}\cdots,x_{\sigma(n)})\\ &-\sum_{j=1}^{n}\sum_{\sigma\in\mathrm{Sh}(j,n-j)}\varepsilon(\sigma)D_{n-j+1}(\mu_j(x_{\sigma(1)},\cdots,x_{\sigma(j)}),x_{\sigma(j+1)},\cdots,x_{\sigma(n)}) = 0.\end{aligned} \tag{18}$$

Proof. Let $\mu = \sum_{i=1}^{\infty}\mu_i : \bar{S}(\mathfrak{l}) \to \mathfrak{l}$, with $\mu_i : S^i(\mathfrak{l}) \to \mathfrak{l}$ and $D = \sum_{i=1}^{\infty} D_i : \bar{S}(\mathfrak{l}) \to \mathfrak{l}$, with $D_i : S^i(\mathfrak{l}) \to \mathfrak{l}$. Then by Proposition 4.10, $(s\mu, D) \in \mathcal{MC}(s\mathfrak{M} \oplus \mathfrak{A})$ is equivalent to

$$[\mu,\mu]_{\mathrm{NR}} = 0 \text{ and } \sum_{p=2}^{\infty}\frac{1}{(p-1)!}l_p(s\mu, \underbrace{D,\dots,D}_{(p-1)\text{ times}}) = 0.$$

It is well known that, see for instance, [2, Theorem 4.2], that $[\mu,\mu]_{\mathrm{NR}} = 0$ if and only if Eq. (17) holds if and only if $\mathfrak{g}$ is an $L_\infty[1]$-algebra.

On the other hand, we have

$$\begin{aligned} 0 &= \sum_{p=2}^{\infty}\frac{1}{(p-1)!}l_p(s\mu, \underbrace{D,\dots,D}_{(p-1)\text{ times}})\\ &= \sum_{p=2}^{\infty}\frac{1}{(p-1)!}l_p(s\sum_{m=1}^{\infty}\mu_m, \sum_{m_1=1}^{\infty}D_{m_1},\dots,\sum_{m_{p-1}=1}^{\infty}D_{m_{p-1}})\\ &= \sum_{p=2}^{\infty}\frac{1}{(p-1)!}\sum_{m,m_1,\dots,m_{p-1}=1}^{\infty} l_p(s\mu_m, D_{m_1},\dots,D_{m_{p-1}})\\ &= \sum_{m=1}^{\infty}\sum_{j=1}^{\infty}[\mu_m, D_j]_{\mathrm{NR}} + \sum_{p=3}^{\infty}\frac{1}{(p-1)!}\sum_{m,m_1,\dots,m_{p-1}=1}^{\infty}\sum_{\tau\in\mathrm{Sh}(m_{p-1},\dots,m_1,m+1-p)}(-1)^{\sum_{j=1}^{p-1}(|D_{m_1}|+\cdots+|D_{m_{j-1}}|)|D_{m_j}|}\\ &\qquad \lambda^{p-2}\mu_m(D_{m_{p-1}}\otimes\cdots\otimes D_{m_1}\otimes \mathrm{Id}^{\otimes m+1-p})\tau^{-1}\end{aligned}$$

$$= \sum_{m=1}^{\infty}\sum_{j=1}^{\infty}[\mu_m, D_j]_{\mathrm{NR}}$$
$$+ \sum_{p=3}^{\infty}\frac{1}{(p-1)!}\sum_{m,m_1,\ldots,m_{p-1}=1}^{\infty}\sum_{\tau\in \mathrm{Sh}(m_{p-1},\ldots,m_1,m+1-p)}\lambda^{p-2}\mu_m(D_{m_{p-1}}\otimes\cdots\otimes D_{m_1}\otimes \mathrm{Id}^{\otimes m+1-p})\tau^{-1},$$

which is equivalent to Eq. (18). □

Example 5.6. Since Eq. (17) is the well known $L_\infty[1]$-structure, we only need to focus on homotopy differential operators.

When $n = 1$, Eq. (18) gives

$$\mu_1 D_1 = D_1\mu_1.$$

which means that D_1 is a cochain map.

When $n = 2$, Eq. (18) gives

$$\begin{aligned} D_1(\mu_2(x,y)) - \mu_2(D_1(x),y) - \mu_2(x,D_1(y)) - \lambda\mu_2(D_1(x),D_1(y)) \\ = \mu_1(D_2(x,y)) - D_2(\mu_1(x),y) - D_2(x,\mu_1(y)). \end{aligned}$$

which means that D_1 is a differential operator (with respect to the multiplication μ_2), but only up to the homotopy given by D_2.

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