GROWTH RATE OF LIQUIDITY PROVIDER'S WEALTH IN G3MS

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Abstract. Geometric mean market makers (G3Ms), such as Uniswap and Balancer, represent a widely used class of automated market makers (AMMs). These G3Ms are characterized by the following rule: the reserves of the AMM must maintain the same (weighted) geometric mean before and after each trade. This paper investigates the effects of trading fees on liquidity providers' (LP) profitability in a G3M, as well as the adverse selection faced by LPs due to arbitrage activities involving a reference market. Our work expands the model described in [MMRZ22, MMR23] for G3Ms, integrating both transaction fees and continuous-time arbitrage into the analysis. Within this context, we analyze G3M dynamics, characterized by stochastic storage processes, and calculate the growth rate of LP wealth. In particular, our results align with and extend the results of [TW20] concerning the constant product market maker, commonly referred to as Uniswap v2.

1. Introduction

Automated Market Makers (**AMMs**) [CJ21, GM23] are innovative algorithms that utilize blockchain technology to automate the process of pricing and order matching on Decentralized Exchanges (**DEXs**). Their foundation on blockchain and the employment of smart contracts enable users to buy and sell crypto assets securely, peer-to-peer, without the dependency on intermediaries or custodians.

A key distinction between AMMs and centralized limit order book models is their price determination mechanisms. An order book model sets prices based on buyers' and sellers' intentions, whereas an AMM's pricing relies on the inventory available in its pool. This "liquidity pool" comprises funds deposited by users, known as liquidity providers (**LPs**), who "lock in" different amounts of tokens in a smart contract, thus providing liquidity for other users' trades. Entry as a liquidity provider is relatively simple, requiring only a self-custody wallet and compatible tokens.

Geometric mean market makers (**G3Ms**), such as Uniswap [AZR20] and Balancer [MM19], are prevalent in decentralized exchanges. In G3Ms, LPs deposit assets into a smart contract's reserves, allowing third parties to trade against these reserves. Trades are executed only if they maintain the constant weighted geometric mean of reserves. LPs are compensated with liquidity pool shares proportionate to their contributions, which can be redeemed anytime for a share of the pool's reserves. G3Ms are known for closely tracking prices in more liquid trading venues [AC20], thanks to arbitrageurs who rebalance reserves to target weights in response to market fluctuations, similar to automated Exchange Traded Fund (**ETF**) rebalancing.

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LPs in AMMs face the challenge of adverse selection, where informed agents, typically arbitrageurs, exploit delayed AMM price updates based on real-time data from centralized exchanges. Previous studies [MMRZ22, MMR23] quantify this adverse selection as 'loss-versus-rebalancing' (LVR).

This paper presents an analysis of AMM dynamics through a stochastic storage model [Har13], driven by arbitrage under the following assumptions:

(a) There exists a reference market characterized by infinite liquidity and no trading costs.
(b) The AMM imposes a fixed trading fee of $(1-\gamma) \times 100\%$, with all pool trades being arbitrages.
(c) Arbitrageurs continuously monitor the market, trading whenever arbitrage opportunities arise.

In such a scenario, the pricing within the AMM would adhere to a non-trading band when compared to the reference market price. We demonstrate that the dynamics of the AMM's pool can be effectively modeled through a stochastic process, referred to as the **mispricing process**, which is influenced by the reference market's price dynamics, as explained in §3. It is important to note that our approach is model-free, distinguishing it from the model-based methodologies employed in [TW20], which utilizes a Markov chain, and [NTYY24], which applies local time for Brownian motion.

In [TW20], Tassy and White focus on the logarithmic return of a Uniswap V2 LP, operating under the assumptions mentioned above. The authors adopt a discretization approach for their model, enabling the application of established results from Markov chain theories on stationary distributions. This methodology allows them to deduce an accurate formula for the expected growth rate of LP wealth. Notably, their formula highlights the impact of volatility drag in a constant rebalanced portfolio (CRP).

In our work, we revisit the Tassy-White formula within the framework of a stochastic storage model, extending their formula beyond Uniswap V2 to the Balancer model. Our principal methodology employs the infinitesimal generator derived from particular reflected diffusion models. Subsequently, we tackle the corresponding parabolic partial differential equation (PDE) characterized by nonzero Neumann boundary conditions. To resolve this, we utilize Sturm-Liouville theory. This method not only aligns with the results of Tassy-White but also extends to a more diverse range of reference market dynamics.

Outline. The paper proceeds as follows: Section 2 introduces G3M mechanisms and setups. Section 3 delves into the arbitrageur's optimization problem and the dynamics of G3Ms driven by discrete and continuous arbitrage. Section 4 presents our main results, connecting LP wealth growth to an associated PDE and solving it via Sturm-Liouville theory.

2. Constant Weight G3Ms

A Geometric Mean Market Maker (G3M) is a type of Automated Market Maker (AMM) whose feasible trade set is defined by the weighted geometric mean of its asset reserves. In this paper, we focus on a G3M that involves only two assets, designated as X and Y, accompanied by a predetermined weight w such that $0 < w < 1$.

2.1. **Market mechanism without transaction costs.** In a G3M with two assets, let's consider $(x, y) \in \mathbb{R}_+^2$ as the reserve pair of the liquidity pool (LP). The G3M is characterized by maintaining a constant weighted geometric mean:

$$x^w y^{1-w} = \ell. \tag{1}$$

Here, ℓ symbolizes the liquidity in the LP. A trade $\Delta = (\Delta_x, \Delta_y) \in \mathbb{R}^2$ is feasible if it adheres to:

$$x^w y^{1-w} = (x + \Delta_x)^w (y + \Delta_y)^{1-w}, \tag{2}$$

where $\Delta_x > 0$ indicates the amount of asset X being added to, and $\Delta_x < 0$ the amount withdrawn from, the pool.

Remark 2.1. G3Ms that operate without transaction costs exhibit path independence [AC20, §2.3], which inherently safeguards them from price manipulation. In other words, traders cannot gain advantages through circular or round-trip trading in such systems.

Differentiating Equation (1) leads to:

$$w x^{w-1} y^{1-w} dx + (1-w) x^w y^{-w} dy = 0$$

yielding:

$$w \frac{dx}{x} + (1-w) \frac{dy}{y} = 0.$$

Denoting P the price of assets X relative to asset Y set by the G3M, we find:

$$P = -\frac{d\Delta_y}{d\Delta_x}\bigg|_{\Delta_x = 0} = -\frac{dy}{dx} = \frac{y/(1-w)}{x/w} = \frac{w}{1-w}\frac{y}{x}. \tag{3}$$

Thus, the G3M's pricing relies solely on the reserve ratios in the LP. Additional observations include:

(a) Equation (3) implies

$$\frac{Px}{w} = \frac{y}{1-w}, \tag{4}$$

indicating the proportion of asset X (and Y) in the pool's total wealth, calculated using the pool price, is always w (and $1-w$, respectively).

(b) By applying logarithms to (1) and (3), we derive expressions for $(\ln x, \ln y)$ in terms of $(\ln \ell, \ln P)$:

$$\begin{aligned} \ln \ell &= w \ln x + (1-w) \ln y \\ \ln P &= -\ln x + \ln y - \ln(1-w) + \ln w. \end{aligned} \tag{5}$$

This gives:

$$\begin{aligned} \ln x &= \ln \ell - (1-w) \ln P - (1-w) \ln(1-w) + (1-w) \ln w \\ \ln y &= \ln \ell + w \ln P + w \ln(1-w) - w \ln w. \end{aligned}$$

This suggests a direct correspondence between (x, y) and (ℓ, P).

(c) Differentiating (5) results in:

$$\frac{dP}{P} = \frac{dy}{y} - \frac{dx}{x} = -\frac{1}{1-w}\frac{dx}{x} = \frac{1}{w}\frac{dy}{y}.$$

This leads to the inference that trading impacts the pool price significantly; specifically, P increases (or decreases) by $\frac{1}{w}\frac{dy}{y}$ percentage when $dy > 0$ (or $dy < 0$, respectively).

2.2. **Market mechanism with transaction costs.** Consider a G3M where $1-\gamma$ represents the proportional transaction cost. The trading process with transaction costs involves two steps:

Step 1: Determining Feasible Trades

- Buying Asset X from the Pool: If the trader purchases asset X, leading to a decrease in x and an increase in y in the pool ($\Delta_y > 0$), the trade Δ must satisfy:

$$(x + \Delta_x)^w (y + \gamma\Delta_y)^{1-w} = \ell. \tag{6}$$

- Selling Asset X to the Pool: Conversely, if the trader sells asset X, resulting in an increase in x and a decrease in y in the pool ($\Delta_x > 0$), the trade Δ must satisfy:

$$(x + \gamma\Delta_x)^w (y + \Delta_y)^{1-w} = \ell. \tag{7}$$

Step 2: Updating Reserves After Transactions

$$\begin{aligned} x &\mapsto x + \Delta_x, \\ y &\mapsto y + \Delta_y, \\ \ell &\mapsto (x + \Delta_x)^w (y + \Delta_y)^{1-w}. \end{aligned}$$

We have the following observations:

- With transaction costs, unlike the scenario without them, liquidity ℓ changes with each trade. Specifically, the liquidity increases for any trade, as:

$$(x + \Delta_x)^w (y + \Delta_y)^{1-w} > \begin{cases} (x + \gamma\Delta_x)^w (y + \Delta_y)^{1-w} & \text{if } \Delta_x > 0, \\ (x + \Delta_x)^w (y + \gamma\Delta_y)^{1-w} & \text{if } \Delta_y > 0, \end{cases} = x^w y^{1-w}.$$

This increment in liquidity is a direct consequence of the trading fees applied in each transaction.

Remark 2.2. (1) In contrast to G3Ms without transaction fees, G3Ms that incorporate transaction costs display path dependence [AC20, §2.3], which indicates that the outcomes of trades are affected by the sequence and size of the transactions.

(2) When splitting a single trade into multiple smaller transactions in a G3M, the costs incurred are higher. For example, consider a trader selling Δ_x of asset X. The resulting amount Δ_y of asset Y received is determined by the equation:

$$(x + \gamma\Delta_x)^w (y + \Delta_y)^{1-w} = x^w y^{1-w}.$$

However, if the trader divides the trade into two parts, $\Delta_x = \Delta_x^1 + \Delta_x^2$, then the total amount of asset Y received, denoted as $\tilde{\Delta}_y = \Delta_y^1 + \Delta_y^2$, is calculated as follows:

$$(x + \gamma\Delta_x^1)^w (y + \Delta_y^1)^{1-w} = x^w y^{1-w}$$
$$(x + \Delta_x^1 + \gamma\Delta_x^2)^w (y + \Delta_y^1 + \Delta_y^2)^{1-w} = (x + \Delta_x^1)^w (y + \Delta_y^1)^{1-w}.$$

By comparing these equations, it can be inferred that $\Delta_y < \tilde{\Delta}_y < 0$.

2.2.1. *Marginal exchange rate.* The marginal exchange rate of asset X with respect to asset Y represents the price for an infinitesimally small trade. This rate is derived by differentiating the constant geometric mean formula used in the market-making mechanism:

- For a trade where X is purchased, we use the formula:

$$\frac{d}{d\Delta_x}\left((x + \Delta_x)^w (y + \gamma\Delta_y)^{1-w}\right) = 0,$$

leading to:

$$-\frac{d\Delta_y}{d\Delta_x}\bigg|_{\Delta_x=0} = \frac{1}{\gamma}\frac{w}{1-w}\frac{y}{x} = \frac{1}{\gamma}P.$$

In this scenario, the marginal exchange rate exceeds the pool price by a factor of $\frac{1}{\gamma}$.

- In contrast, when selling X, the relevant formula is:

$$\frac{d}{d\Delta_x}\left((x + \gamma\Delta_x)^w (y + \Delta_y)^{1-w}\right) = 0,$$

which yields:

$$-\frac{d\Delta_y}{d\Delta_x}\bigg|_{\Delta_x=0} = \gamma\frac{w}{1-w}\frac{y}{x} = \gamma P.$$

Here, the marginal exchange rate is lower than the pool price by a factor of γ.

Remark 2.3. The transaction cost parameter γ in the Geometric Mean Market Maker framework can be analogously considered as the equivalent of the bid-ask spread typically found in a traditional limit order book.

2.3. **Continuous order flows.** In scenarios where reserve levels and trading activities are continuous, the following dynamics are observed:

- For trades where X is purchased, the order flow adheres to:

$$w\frac{dx}{x} + \gamma(1-w)\frac{dy}{y} = 0. \tag{8}$$

This implies a decrease in x by a percentage $\frac{dx}{x}$ and a corresponding increase in y by $\frac{dy}{y} = -\frac{1}{\gamma}\frac{w}{1-w}\frac{dx}{x}$ (considering $dx < 0$).

- Conversely, when X is sold, the order flow is given by:

$$\gamma w\frac{dx}{x} + (1-w)\frac{dy}{y} = 0. \tag{9}$$

This indicates an increase in x by $\frac{dx}{x}$ and a decrease in y by $\frac{dy}{y} = \gamma\frac{w}{1-w}\frac{dx}{x}$.

- Price P changes via:

$$\frac{dP}{P} = \frac{dy}{y} - \frac{dx}{x} = \begin{cases} (1+\gamma\frac{1-w}{w})\frac{dy}{y} & \text{if } dy > 0, \\ (1+\frac{1}{\gamma}\frac{1-w}{w})\frac{dy}{y} & \text{if } dy < 0. \end{cases} \tag{10}$$

- Liquidity ℓ changes according to:

$$\frac{d\ell}{\ell} = w\frac{dx}{x} + (1-w)\frac{dy}{y} = \begin{cases} (1-\gamma)(1-w)\frac{dy}{y} & \text{if } dy > 0, \\ (1-\frac{1}{\gamma})(1-w)\frac{dy}{y} & \text{if } dy < 0. \end{cases} \tag{11}$$

Thus, liquidity ℓ increases by a specific percentage whether dy is positive or negative.

Remark 2.4. In a continuous trading regime, the G3M mechanism is characterized by maintaining the quantity $x^w y^{\gamma(1-w)}$ (when buying X) and $x^{\gamma w} y^{1-w}$ (when selling X) constant. Consequently, liquidity ℓ consistently rises during such trades.

2.4. **Liquidity provider's wealth profile.** For a liquidity provider (LP), the value of their LP position in terms of the price P, denoted by $V(P)$, is expressed as:

$$V(P) = Px + y.$$

This value can be interpreted either as a function of the reserves (x, y) or of the pair (P, ℓ). Furthermore, we can deduce:

$$V(P) = \frac{Px}{w} = \frac{Px}{w}\left(\frac{\frac{Px}{w}}{\frac{Px}{w}}\right)^w \left(\frac{\frac{y}{1-w}}{\frac{Px}{w}}\right)^{1-w} = \frac{\ell P^w}{w^w(1-w)^{1-w}}, \tag{12}$$

which allows us to express the log growth of the wealth profile as:

$$\ln V(P) = \ln \ell + w \ln P + \mathcal{S}_w, \tag{13}$$

where $\mathcal{S}_w = -w\ln w - (1-w)\ln(1-w)$ represents the entropy component derived from the weight w. This formulation encapsulates the relationship between a liquidity provider's wealth, the pool's price level, and the pool's liquidity, highlighting the impact of these variables on the LP's overall financial position.

3. Arbitrage driven G3M dynamics

In this section, we focus on exploring the dynamics of G3Ms under the premise of arbitrage-driven order flow only. Specifically, we proceed under the following assumption:

Assumption 3.1. The market operates without the interference of noise traders.

This assumption simplifies the market model by excluding random, non-strategic trading activities, allowing us to concentrate on the implications of arbitrage activities within the G3M framework.

3.1. **Trader's arbitrage opportunity.**

Assumption 3.2. There is an external reference market which offers infinite liquidity and incurs no trading cost.

Under Assumption 3.2, the optimal arbitrage problem simplifies to the following:

$$\begin{aligned} &\max_{\Delta\in\mathbb{R}^2} (-1)^{\mathrm{sgn}(-\Delta_x)}\left(S\Delta_x+\Delta_y\right) \\ &\text{u.c. } (x_0+\gamma\Delta_x)^w(y_0+\Delta_y)^{1-w}=x_0^w y_0^{1-w} \quad \text{if } \Delta_x\geq 0 \\ &\qquad (x_0+\Delta_x)^w(y_0+\gamma\Delta_y)^{1-w}=x_0^w y_0^{1-w} \quad \text{if } \Delta_x<0 \end{aligned} \tag{14}$$

with S the reference market price of X w.r.t. Y and (x_0, y_0) the initial reserve. It turns out that maximizing arbitrage profit equates to minimizing the liquidity profile's value (in terms of the reference market price):

$$\begin{aligned} V(S) \triangleq \min_{(x,y)\in\mathbb{R}_+^2} \quad & Sx+y \\ \text{u.c.} \quad & (x_0+\gamma(x-x_0))^w y^{1-w}=x_0^w y_0^{1-w} \quad \text{if } x\geq x_0 \\ & x^w(y_0+\gamma(y-y_0))^{1-w}=x_0^w y_0^{1-w} \quad \text{if } x<x_0, \end{aligned} \tag{15}$$

where (x_0, y_0) is the initial reserve. We refer the reader to [AC20, §2.5] for details.

Given the constraint and initial pool price P_0, the final reserves (x, y) are fully determined by the updated pool price, allowing us to express x and y as a function of P. This leads to the inequality:

$$\frac{d}{dP}\{Sx+y\}\bigg|_{P=P_0}=\left\{S\frac{dx}{dP}+\frac{dy}{dP}\right\}\bigg|_{P=P_0}<0 \Leftrightarrow S<-\frac{dy}{dx}\bigg|_{x=x_0}=\begin{cases}\gamma P & \text{if } P<P_0\\ \gamma^{-1}P & \text{if } P>P_0\end{cases}$$

by the computation of marginal exchange rate in §2.2.1. Therefore, an arbitrage opportunity exists when $S<\gamma P_0$ or $S>\frac{1}{\gamma}P_0$, with the optimal arbitrage given by:

$$P^*=\begin{cases}\gamma^{-1}S & \text{if } P_0>\gamma^{-1}S\\ P_0 & \text{if } \gamma S\leq P_0\leq\gamma^{-1}S\\ \gamma S & \text{if } P_0<\gamma S\end{cases}$$

Note that under Assumption 3.1, the relative pool price P remains constant within the **no-arbitrage interval** $\gamma S\leq P\leq\gamma^{-1}S$.

3.2. **Discrete arbitrage in G3M.** In this subsection, we explore the discrete arbitrage process within a Geometric Mean Market Maker, adapting the approach outlined in [MMR23].

At any time $t\geq 0$, denote S_t as the market price of asset X relative to asset Y, and P_t as the pool-implied price of X in terms of Y, updated according to Section 2.2. We define the log mispricing of the pool as:

$$Z_t \triangleq \ln\frac{S_t}{P_t}. \tag{16}$$

Assumption 3.3. Arbitrageurs arrive at discrete times $0=\tau_0<\tau_1<\tau_2<\cdots\tau_m\leq T$.

Under assumptions 3.1, 3.2, and 3.3, an arbitrageur's objective is to optimize their instantaneous trading profit at each τ_i as formulated in the optimization problem (14). The arbitrage process is outlined as follows:

- When $S_{\tau_i} > \gamma^{-1}P_{\tau_{i-1}}$ (or $Z_{\tau_i^-} > -\ln\gamma$), the arbitrageur buys asset X in the pool and sells it on the external market until $P_{\tau_i} = \gamma S_{\tau_i}$ (resulting in $Z_{\tau_i} = -\ln\gamma$). The profit is then
$$S_{\tau_i}\left\{x(P_{\tau_{i-1}}) - x(\gamma S_{\tau_i})\right\} + \left\{y(P_{\tau_{i-1}}) - y(\gamma S_{\tau_i})\right\},$$
with x and y representing the quantities of assets in the pool as determined by (6).
- In cases where $S_{\tau_i} < \gamma P_{\tau_{i-1}}$ (or $Z_{\tau_i^-} < \ln\gamma$), the arbitrageur sells asset X to the pool and purchases from the external market until $P_{\tau_i} = \gamma^{-1}S_{\tau_i}$ (leading to $Z_{\tau_i} = \ln\gamma$). The profit is then
$$-\left\{S_{\tau_i}\left\{x(P_{\tau_{i-1}}) - x(\gamma^{-1}S_{\tau_i})\right\} + \left\{y(P_{\tau_{i-1}}) - y(\gamma^{-1}S_{\tau_i})\right\}\right\},$$
where x and y are the pool asset amounts as per (7).
- If $\gamma P_{\tau_{i-1}} \leq S_{\tau_i} \leq \gamma^{-1}P_{\tau_{i-1}}$ (or $\ln\gamma \leq Z_{\tau_i^-} \leq -\ln\gamma$), no transaction occurs, leading to $P_{\tau_i} = P_{\tau_{i-1}}$ (and $Z_{\tau_i} = Z_{\tau_i^-}$).

Consequently, at each arbitrage arrival time τ_i, the G3M price updates to:

$$P_{\tau_i} = \begin{cases} \gamma S_{\tau_i} & \text{if } Z_{\tau_i^-} < \ln\gamma, \\ P_{\tau_{i-1}} & \text{if } \ln\gamma \leq Z_{\tau_i^-} \leq -\ln\gamma, \\ \gamma^{-1}S_{\tau_i} & \text{if } Z_{\tau_i^-} > -\ln\gamma. \end{cases} \tag{17}$$

Accordingly, the mispricing process evolves as:

$$Z_{\tau_i} = \max\left\{\min\{Z_{\tau_i^-}, -\ln\gamma\}, \ln\gamma\right\} = \begin{cases} \ln\gamma & \text{if } Z_{\tau_i^-} < \ln\gamma, \\ Z_{\tau_i^-} & \text{if } \ln\gamma \leq Z_{\tau_i^-} \leq -\ln\gamma, \\ -\ln\gamma & \text{if } Z_{\tau_i^-} > -\ln\gamma. \end{cases} \tag{18}$$

Proposition 3.4 (Discrete Mispricing Dynamics)**.** *Given assumptions 3.1, 3.2, and 3.3 and with the initial condition $\gamma P_0 \leq S_0 \leq \gamma^{-1}P_0$, we can define:*

$$J_i = \max\left\{\min\{Z_{\tau_i^-}, -\ln\gamma\}, \ln\gamma\right\} - Z_{\tau_i^-}, \quad L_t = \sum_{i:\tau_i \leq t}\{J_i\}^+, \quad U_t = \sum_{i:\tau_i \leq t}\{J_i\}^-,$$

where $\{a\}^+ = \max\{a, 0\}$ denote the positive part of a, and $\{a\}^- = \max\{-a, 0\}$ denote the negative part. Then for all $t \geq 0$,

$$\begin{aligned} \ln P_t &= \ln P_0 + U_t - L_t, \\ Z_t &= \ln S_t - \ln P_0 + L_t - U_t. \end{aligned}$$

Moreover, L_t and U_t satisfy:

$$L_t = \sup_{i:\tau_i \leq t}\left(-\ln(\gamma P_0) + \ln S_{\tau_i} - U_{\tau_i}\right)^-, \tag{19}$$

$$U_t = \sup_{i:\tau_i \leq t}\left(\ln(\gamma^{-1}P_0) - \ln S_{\tau_i} - L_{\tau_i}\right)^-. \tag{20}$$

Proof. The first assertion is directly inferred from (17) and (18). To establish the second assertion, we note that both sides of (19) and (20) are non-decreasing and piecewise constant, with potential jumps occurring at τ_i for $0 \le i \le m$. Thus, demonstrating the equalities at each τ_i suffices.

The proof employs induction on i. For the base case when $i = 0$, the equalities are valid by the initial condition that $\ln \gamma \le Z_0 \le -\ln \gamma$. Now, assuming the assertion is true for $i < k$, the induction hypothesis yields:

$$L_{\tau_k} = \max\left\{L_{\tau_{k-1}}, \{-\ln \gamma + Z_{\tau_k} - L_{\tau_k}\}^-\right\}.$$

Given that

$$L_{\tau_k} \begin{cases} \ge L_{\tau_{k-1}} & \text{if } Z_{\tau_k} = \ln \gamma, \\ = L_{\tau_{k-1}} & \text{otherwise,} \end{cases}$$

it follows that

$$\{-\ln \gamma + Z_{\tau_k} - L_{\tau_k}\}^- = \min\{L_{\tau_k} + \ln \gamma - Z_{\tau_k}, 0\} \begin{cases} = L_{\tau_k} & \text{if } Z_{\tau_k} = \ln \gamma, \\ < L_{\tau_{k-1}} & \text{otherwise,} \end{cases}$$

thereby validating (19). Applying the same argument confirms (20) as well, thus finishes the proof. □

Remark 3.5. One should think L_t (resp. U_t) as a barrier to keep Z_t lying above $\ln \gamma$ (resp. below $-\ln \gamma$) at each $t = \tau_i$, $1 \le i \le m$.

3.2.1. *Arbitrage profits.* At time τ_i, arbitrageur observes the reference market price S_{τ_i} and adjusts the G3M pool from $(x(P_{\tau_{i-1}}), y(P_{\tau_{i-1}}))$ to $(x(P_{\tau_i}), y(P_{\tau_i}))$. This action involves purchasing $x(P_{\tau_{i-1}}) - x(P_{\tau_i})$ units of asset X from the G3M at average price

$$P_i^{\text{G3M}} \triangleq \frac{y(P_{\tau_i}) - y(P_{\tau_{i-1}})}{x(P_{\tau_i}) - x(P_{\tau_{i-1}})}.$$

The arbitrageur then sells these units in the reference market at price S_{τ_i} accruing profits (in numéraire) from the price difference:

$$\left(S_{\tau_i} - P_i^{\text{G3M}}\right)\left[x(P_{\tau_{i-1}}) - x(P_{\tau_i})\right] = S_{\tau_i}\left[x(P_{\tau_{i-1}}) - x(P_{\tau_i})\right] + \left[y(P_{\tau_{i-1}}) - y(P_{\tau_i})\right].$$

Denoting ARB_T as the cumulative arbitrage profits over time horizon T, then summing over i yields:

$$\begin{aligned} \text{ARB}_T &\triangleq \sum_{i=1}^{m}\left\{S_{\tau_i}\left[x(P_{\tau_{i-1}}) - x(P_{\tau_i})\right] + \left[y(P_{\tau_{i-1}}) - y(P_{\tau_i})\right]\right\} \\ &= \sum_{i=1}^{r} S_{\tau_i}\left[x(P_{\tau_{i-1}}) - x(P_{\tau_i})\right] + y(P_0) - y(P_T) \\ &= S_0 x(P_0) + y(P_0) + \sum_{i=0}^{m} x(P_{\tau_i})\left[S_{\tau_{i+1}} - S_{\tau_i}\right] - S_T x(P_T). \end{aligned} \tag{21}$$

3.3. **Continuous arbitrage dynamics.** In the context of G3Ms, arbitrage opportunities arise when market dynamics cause the market price, S, to deviate significantly from the pool price, P, specifically when $S > \gamma^{-1}P$ or $S < \gamma P$. In such cases, arbitrageurs exploit these discrepancies by trading between the G3M and the reference market.

To understand the continuous arbitrage process in G3Ms, we introduce the following assumption:

Assumption 3.6. Arbitrageurs continuously monitor the market and act immediately when they identify arbitrage opportunities.

Under this assumption, arbitrageurs are in a state of perpetual optimization, dynamically adjusting their trading strategies in response to real-time changes in the reference market price S. Their actions play a key role in guiding the pool price P to remain within the interval $\gamma S < P < \gamma^{-1}S$.

Proposition 3.7 (Continuous Mispricing Dynamics)**.** *Given that the market price S_t is continuous and adheres to the initial condition $\gamma P_0 \le S_0 \le \gamma^{-1}P_0$, and under assumptions 3.1, 3.2, and 3.6, we have:*

a) The mispricing process, denoted as Z_t, can be decomposed into $Z_t = \ln S_t - \ln P_0 + L_t - U_t$ and takes value within the range $[\ln\gamma, -\ln\gamma]$ for all $t \ge 0$.
b) L_t and U_t are both nondecreasing and continuous, with their initial values set at $L_0 = U_0 = 0$.
c) L_t and U_t increase only when $Z_t = -\ln\gamma$ and $Z_t = \ln\gamma$, respectively.

Moreover, L_t and U_t satisfy:

$$L_t = \sup_{0\le s\le t} \left(-\ln(\gamma P_0) + \ln S_s - U_s\right)^{-}, \tag{22}$$

$$U_t = \sup_{0\le s\le t} \left(\ln(\gamma^{-1} P_0) - \ln S_s - L_s\right)^{-}. \tag{23}$$

Proof. Given the assumptions, arbitrageurs are constantly engaged in the optimization problem detailed in (14), adapting in real-time to fluctuations in the reference market price S_t. From this, we can deduce the decomposition $\ln P_t = \ln P_0 + U_t - L_t$, where U_t and L_t represent the adjustments in log price due to buy and sell arbitrage order flows, respectively, and fulfill the conditions outlined in (a)–(c). The second part follows immediately from [Har13, Proposition 2.4]. □

Remark 3.8. The mispricing process Z_t can be viewed as a stochastic storage system with finite buffer capacity, where L_t and U_t function as reflection barriers, akin to the setup described in [Har13, §2.3]. If the market price S_t adheres to a geometric Brownian motion, exemplified by $\ln S_t = \mu t + \sigma B_t$ with B_t being a standard Brownian motion, then Z_t exhibits characteristics of a reflected Brownian motion in Harrison [Har13, §6].

To define the arbitrage process further, we consider the dynamics of the reserve process x_t (or y_t). We say that time t is a point of increase (or decrease) for x_t if there exists a $\delta > 0$ such that $x_{t-\delta_1} < x_{t+\delta_2}$ (or $x_{t-\delta_1} > x_{t+\delta_2}$, resp.) for all $\delta_1, \delta_2 \in (0, \delta]$. The reserve process x_t is said to increase (or decrease) only when $Z_t = a$ if, at every point of increase (or decrease, resp.) for x_t, it holds that $Z_t = a$.

Corollary 3.9 (Inventory Dynamics in Arbitrage)**.** *Under the same assumptions as in Proposition 3.7, the following hold:*

(a) x_t *and* y_t *are predictable processes.*
(b) x_t *(or* y_t*) increases only when* $Z_t = \ln\gamma$ *(or* $Z_t = -\ln\gamma$*, resp.), and decreases only when* $Z_t = -\ln\gamma$ *(or* $Z_t = \ln\gamma$*, resp.).*
(c) $\ln x_t$ *and* $\ln y_t$ *are continuous and of bounded variation on any bounded interval in* $[0, \infty)$*.*
(d) The arbitrage inventory process can be characterized by:

$$\begin{aligned} d\ln x_t &= \frac{1-w}{1-w+\gamma w} dL_t - \frac{\gamma(1-w)}{\gamma(1-w)+w} dU_t, \\ d\ln y_t &= \frac{w}{\gamma(1-w)+w} dU_t - \frac{\gamma w}{1-w+\gamma w} dL_t. \end{aligned} \tag{24}$$

Proof. Given Proposition 3.7 (a)–(c), we infer that U_t, L_t and $\ln P_t$ are continuous predictable processes of bounded variation. The assertions then follow directly from equation (10). □

Incorporating the equations from (24) into (11), we derive an equation to describe the dynamics of liquidity growth based on the mispricing process.

Corollary 3.10 (Liquidity Dynamics in Arbitrage)**.** *Under the same assumptions as in Proposition 3.7, the liquidity process* ℓ_t *is a nondecreasing, predictable process. Its rate of change is given by:*

$$d\ln \ell_t = \frac{(1-\gamma)w(1-w)}{1-w+\gamma w} dL_t + \frac{(1-\gamma)w(1-w)}{\gamma(1-w)+w} dU_t. \tag{25}$$

3.4. **LP wealth growth from arbitrage.** Let S_t^X and S_t^Y denote the market prices of assets X and Y, respectively, in the reference market. The wealth process V_t of a liquidity provider in a G3M is defined by:

$$V_t = S_t^X x_t + S_t^Y y_t = S_t^Y \left(S_t x_t + y_t\right). \tag{26}$$

Utilizing equation (4), we derive the following inequalities:

$$\begin{aligned} S_t x_t + y_t &\geq \gamma P_t x_t + y_t = \gamma\left(P_t x_t + y_t\right) + (1-\gamma) y_t = \left(1 - w(1-\gamma)\right)\left(P_t x_t + y_t\right); \\ S_t x_t + y_t &\leq \frac{1}{\gamma} P_t x_t + y_t = \frac{1}{\gamma}\left(P_t x_t + y_t\right) + \left(1 - \frac{1}{\gamma}\right) y_t = \left(1 + w\left(\frac{1}{\gamma} - 1\right)\right)\left(P_t x_t + y_t\right). \end{aligned}$$

Consequently, the logarithm of the LP's wealth can be expressed as:

$$\begin{aligned} \ln V_t &= \ln S_t^Y + \ln\left(P_t x_t + y_t\right) + \ln \frac{S_t x_t + y_t}{P_t x_t + y_t} \qquad (27) \\ &= \ln S_t^Y + \ln \ell_t + w \ln P_t + O(1) \\ &= \ln \ell_t + w \ln S_t^X + (1-w)\ln S_t^Y + O(1) \\ &= \frac{(1-\gamma)w(1-w)}{1-w+\gamma w} L_t + \frac{(1-\gamma)w(1-w)}{\gamma(1-w)+w} U_t + w\ln S_t^X + (1-w)\ln S_t^Y + O(1) \end{aligned}$$

Here, the first equality is derived from (26), the second estimation is based on equation (13), the third equality comes from the relationship that $\ln P_t = \ln S_t + O(1) = \ln S_t^X - \ln S_t^Y + O(1)$, and the last equality is a consequence of (25).

Remark 3.11. The term $\ln \ell_t$ can be viewed as analogous to the concept of excess growth rate in Stochastic Portfolio Theory [Fer02, §1.1]. Specifically, in a scenario focused only on arbitrage, $\ln \ell_t$ tends toward the excess growth rate as the fee tier γ approaches 1. This alignment underscores the relationship between liquidity wealth dynamics in G3Ms and the principles of SPT, especially under the frictionless market assumption.

4. LP wealth growth analysis

In this section, we outline a methodology for calculating the ergodic growth rates of L_t, U_t, and consequently, the wealth process V_t. We assume that for any $t > 0$, the limits

$$\mu_X := \lim_{T\to\infty} \frac{\mathbb{E}_t\left[\ln S_T^X\right]}{T-t} \quad \text{and} \quad \mu_Y := \lim_{T\to\infty} \frac{\mathbb{E}_t\left[\ln S_T^Y\right]}{T-t} \quad \text{exist.} \tag{28}$$

For example, this is the case if S_t^X and S_t^Y satisfy

$$\begin{aligned} d\ln S_t^X &= \mu_X dt + \sigma_X dB_t^X, \\ d\ln S_t^Y &= \mu_Y dt + \sigma_Y dB_t^Y, \end{aligned}$$

where μ_X, μ_Y, σ_X, σ_Y are constants and B_t^X and B_t^Y are Brownian motions.

4.1. Reflected diffusion.

Assume that the log price $s_t := \ln S_t$ follows the diffusion process

$$ds_t = \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t,$$

where $\tilde{\sigma}_t$ and $\tilde{\mu}_t$ may be stochastic. It follows from Proposition 3.7 that the mispricing process $Z_t = \ln S_t - \ln P_t$ is governed by the reflected diffusion in the bounded interval $[-c, c]$, $c = -\ln\gamma$, satisfying

$$dZ_t = ds_t + dL_t - dU_t = \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t + dL_t - dU_t,$$

where recall that L_t and U_t are both nondecreasing, continuous with $L_0 = U_0 = 0$ and increase only when $Z_t = -c$ and $Z_t = c$, respectively. Define

$$\mu(t, Z_t) = \mathbb{E}\left[\tilde{\mu}_t | Z_t\right], \qquad \sigma(t, Z_t) = \sqrt{\mathbb{E}\left[\tilde{\sigma}_t^2 | Z_t\right]}. \tag{29}$$

We remark that the notation $\mathbb{E}_t[\cdot]$ is applied for conditional expectation conditioned on corresponding filtration up to time t hereafter.

The following lemma shows a Feynman-Kac style formula for reflected diffusions.

Lemma 4.1. *For any given deterministic functions α_t and β_t of t, the solution to the following parabolic PDE*

$$u_t + \frac{\sigma^2(t,z)}{2} u_{zz} + \mu(t,z) u_z + f(t,z) = 0 \tag{30}$$

with boundary condition

$$u_z(t, -c) = \alpha_t, \quad u_z(t, c) = \beta_t$$

and terminal condition $u(T, z) = g(z)$ has the following stochastic representation

$$u(t, z) = \mathbb{E}_t \left[g(Z_T) + \int_t^T f(\tau, Z_\tau) d\tau - \int_t^T \alpha_\tau dL_\tau + \int_t^T \beta_\tau dU_\tau \right], \tag{31}$$

where Z_t is the reflected diffusion driven by

$$dZ_t = \mu(t, Z_t)dt + \sigma(t, Z_t)dW_t + dL_t - dU_t. \tag{32}$$

Proof. Itô's formula implies that

$$\begin{aligned} & u(T, Z_T) - u(t, Z_t) \\ = & \int_t^T \left(\partial_t u + \frac{1}{2} \tilde{\sigma}_\tau^2 u_{zz} + \tilde{\mu}_\tau u_z \right) d\tau + \int_t^T \tilde{\sigma}_\tau u_z dW_\tau + \int_t^T u_z dL_\tau - \int_t^T u_z dU_\tau. \end{aligned}$$

By taking into account the boundary and terminal conditions and taking conditional expectation $\mathbb{E}_t [\cdot]$ on both sides of the last equation we obtain

$$\begin{aligned} & \mathbb{E}_t [g(Z_T)] - u(t, z) \\ = & \mathbb{E}_t \left[\int_t^T \left(u_t + \frac{1}{2} \mathbb{E}_t \left[\tilde{\sigma}_\tau^2 | Z_\tau \right] u_{zz} + \mathbb{E}_t \left[\tilde{\mu}_\tau | Z_\tau \right] u_z \right) d\tau + \int_t^T \alpha_\tau dL_\tau - \int_t^T \beta_\tau dU_\tau \right] \\ = & \mathbb{E}_t \left[\int_t^T \left(u_t(\tau, Z_\tau) + \frac{1}{2} \sigma^2(\tau, Z_\tau) u_{zz} + \mu(\tau, Z_\tau) u_z \right) d\tau + \int_t^T \alpha_\tau dL_\tau - \int_t^T \beta_\tau dU_\tau \right] \\ = & \mathbb{E}_t \left[- \int_t^T f(\tau, Z_\tau) d\tau + \int_t^T \alpha_\tau dL_\tau - \int_t^T \beta_\tau dU_\tau \right] \end{aligned}$$

since u satisfies the PDE (30). It follows, by rearranging terms, that the stochastic representation (31) holds. □

The solution to the terminal-boundary value problem in Lemma 4.1 in general admits no simple analytical expression. We shall present two special cases that expressions in terms of the eigensystem associated with the infinitesimal generator of Z_t with Neumann boundary conditions are readily accessible. To that end, the following lemma is required for dealing parabolic PDEs with nonvanishing Neumann boundary conditions.

Lemma 4.2. *The solution u to the parabolic PDE*

$$u_t + \frac{\sigma^2(t, x)}{2} u_{xx} + \mu(t, x) u_x = 0 \tag{33}$$

with boundary conditions

$$u_x(t, -c) = a, \quad u_x(t, c) = b$$

for some constants a, b, and terminal condition $u(T, x) = 0$ is given by

$$u(t, x) = v(t, x) + \frac{b - a}{4c} x^2 + \frac{b + a}{2} x, \tag{34}$$

where v is the solution to the following inhomogeneous parabolic PDE

$$v_t + \frac{\sigma^2(t, x)}{2} v_{xx} + \mu(t, x) v_x + \frac{\sigma^2(x)}{2} \frac{b - a}{2c} + \mu(x) \left(\frac{b - a}{2c} x + \frac{b + a}{2} \right) = 0$$

with Neumann boundary conditions $v_x(t,-c) = v_x(t,c) = 0$ *and terminal condition*

$$v(T,x) = -\frac{b-a}{4c}x^2 - \frac{b+a}{2}x.$$

Proof. Straightforward calculations. □

4.2. **Time-homogeneous reflected diffusion.** Assume that the reflected diffusion (32) is time-homogeneous, in other words, the functions μ and σ in (32) are independent of t. We show in this case that the conditional expectation in (31) can expressed in terms of the eigensystem associated with the infinitesimal generator of the reflected diffusion (32).

Denote by $\mathcal{L}$ the differential operator $\frac{\sigma^2(x)}{2}\partial_x^2 + \mu(x)\partial_x$. Note that $\mathcal{L}$ can be written in the Sturm-Liouville form (47) as

$$\mathcal{L} = \frac{1}{\omega(x)}\frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}\right),$$

where

$$p(x) := e^{\int \frac{2\mu(x)}{\sigma^2(x)}dx}, \qquad \omega(x) := \frac{2}{\sigma^2(x)}e^{\int \frac{2\mu(x)}{\sigma^2(x)}dx}.$$

We remark that ω is indeed the *speed measure* in classical diffusion theory. Let $\{(\lambda_n, e_n(x))\}_{n=0}^{\infty}$ be the normalized eigensystem associated with $\mathcal{L}$ with Neumann boundary condition in the interval $[-c,c]$. The following theorem provides an eigensystem expansion for conditional expectations of reflected diffusions.

Theorem 4.3. *Let* X_t *be the reflected diffusion in the interval* $[-c,c]$ *governed by*

$$dX_t = \sigma(X_t)dW_t + \mu(X_t)dt + dL_t - dU_t.$$

For given constants a *and* b*, the conditional expectation*

$$\mathbb{E}_t\left[-\int_t^T a dL_\tau + \int_t^T b dU_\tau\right] \tag{35}$$

can be written in terms of eigensystem associated with $\mathcal{L}$ *with Neumann boundary condition as*

$$\begin{aligned} &\mathbb{E}_t\left[-\int_t^T a dL_\tau + \int_t^T b dU_\tau\right] \qquad (36)\\ &= \frac{b-a}{4c}x^2 + \frac{b+a}{2}x + \xi_0(T-t) - \eta_0 + \sum_{n=1}^{\infty}\left\{\frac{\xi_n}{\lambda_n}\left[1 - e^{-\lambda_n(T-t)}\right] - \eta_n e^{-\lambda_n(T-t)}\right\}e_n(x). \end{aligned}$$

The coefficients ξ_n *and* η_n*, for* $n \geq 0$*, are given by*

$$\xi_n = \int_{-c}^{c} h(x)e_n(x)\omega(x)dx, \qquad \eta_n = \int_{-c}^{c} k(x)e_n(x)\omega(x)dx \tag{37}$$

where

$$h(x) = \frac{\sigma^2(x)}{2}\frac{b-a}{2c} + \mu(x)\left(\frac{b-a}{2c}x + \frac{b+a}{2}\right), \qquad k(x) = -\frac{b-a}{4c}x^2 - \frac{b+a}{2}x.$$

Proof. Let

$$u(t,x) = \mathbb{E}_t\left[-\int_t^T a dL_\tau + \int_t^T b dU_\tau\right].$$

Then, by Lemma 4.1, u satisfies the PDE

$$u_t + \frac{\sigma^2(x)}{2}u_{xx} + \mu(x)u_x = 0$$

with boundary condition $u_x(t,-c) = a$, $u_x(t,c) = b$, for all $t < T$, and terminal condition $u(T,x) = 0$. Thus, by Lemma 4.2, the solution u can be written as

$$u(t,x) = v(t,x) + \frac{b-a}{4c}x^2 + \frac{b+a}{2}x,$$

where v is the solution to the following inhomogeneous parabolic PDE

$$v_t + \frac{\sigma^2(x)}{2}v_{xx} + \mu(x)v_x + \frac{\sigma^2(x)}{2}\frac{b-a}{2c} + \mu(x)\left(\frac{b-a}{2c}x + \frac{b+a}{2}\right) = 0$$

with Neumann boundary conditions $v_x(t,-c) = v_x(t,c) = 0$ and terminal condition

$$v(T,x) = -\frac{b-a}{4c}x^2 - \frac{b+a}{2}x.$$

Hence, the eigensystem expansion for v as in (36) in Section 6 is given by

$$v(t,x) = \xi_0(T-t) - \eta_0 + \sum_{n=1}^{\infty}\left\{\frac{\xi_n}{\lambda_n}\left[1 - e^{-\lambda_n(T-t)}\right] - \eta_n e^{-\lambda_n(T-t)}\right\}e_n(x),$$

where the coefficients ξ_n's and η_n's are defined in (37). Finally, since

$$\mathbb{E}_t\left[-\int_t^T a dL_\tau + \int_t^T b dU_\tau\right] = u(t,x) = \frac{b-a}{4c}x^2 + \frac{b+a}{2}x + v(t,x),$$

it follows that the eigensystem expansion (36) holds. $\square$

Given the expression in (36), the long-term time-averaged limit of the conditional expectation (35) is immediate, which we summarize in the following corollary without proof.

Corollary 4.4. *As $T \to \infty$, the limit of time-average of the conditional expectation (35) is given by*

$$\begin{aligned} & \lim_{T\to\infty}\frac{1}{T-t}\mathbb{E}_t\left[-a(L_T - L_t) + b(U_T - U_t)\right] \\ = \quad & \xi_0 = \frac{\int_{-c}^{c}\left\{\frac{\sigma^2(x)}{2}\frac{b-a}{2c} + \mu(x)\left(\frac{b-a}{2c}x + \frac{a+b}{2}\right)\right\}\omega(x)dx}{\int_{-c}^{c}\omega(x)dx}. \end{aligned}$$

We thus obtain the long-term expected logarithm growth rate of LP's wealth as given in the following theorem.

Theorem 4.5 (Log Growth Rate of LP Wealth in G3M Time-homogeneous)**.** *Assume that the mispricing process Z_t follows the time-homogeneous reflected diffusion process in the interval $[-c,c]$ governed by*

$$dZ_t = \sigma(Z_t)dW_t + \mu(Z_t)dt + dL_t - dU_t, \tag{38}$$

where recall that the coefficients $\sigma(z)$ and $\mu(z)$ are given in (29). The long-term expected logarithmic growth rate of a LP's wealth in a G3M can be expressed as

$$\lim_{T\to\infty}\frac{\mathbb{E}_t[\ln V_T]}{T-t} = w\mu_X + (1-w)\mu_Y + \frac{(1-\gamma)w(1-w)}{1-w+\gamma w}\alpha + \frac{(1-\gamma)w(1-w)}{\gamma(1-w)+w}\beta,$$

where μ_X and μ_Y are given by (28), and

$$\alpha = \frac{\int_{\ln\gamma}^{-\ln\gamma} -\left\{\frac{\sigma^2(x)}{4\ln\gamma} + \mu(x)\left(\frac{1}{2\ln\gamma}x - \frac{1}{2}\right)\right\}\omega(x)dx}{\int_{\ln\gamma}^{-\ln\gamma}\omega(x)dx},$$

$$\beta = \frac{\int_{\ln\gamma}^{-\ln\gamma} -\left\{\frac{\sigma^2(x)}{4\ln\gamma} + \mu(x)\left(\frac{1}{2\ln\gamma}x + \frac{1}{2}\right)\right\}\omega(x)dx}{\int_{\ln\gamma}^{-\ln\gamma}\omega(x)dx},$$

$$\omega(x) = \frac{2}{\sigma^2(x)}e^{\int\frac{2\mu(x)}{\sigma^2(x)}dx}.$$

Proof. The theorem follows readily from (27) together with (28) and Corollary 4.4. □

As an example, when the market price S_t adheres to a Geometric Brownian Motion (GBM), the terms involved in the expression for long-term expected logarithmic growth rate can be calculated explicitly.

Corollary 4.6 (Log Growth Rate of LP under GBM)**.** *The log growth rate of a LP's wealth in a G3M under a GBM market model is given by*

$$\lim_{T\to\infty}\frac{\mathbb{E}_t[\ln V_T]}{T-t} = w\mu_X + (1-w)\mu_Y + \frac{(1-\gamma)w(1-w)}{1-w+\gamma w}\alpha + \frac{(1-\gamma)w(1-w)}{\gamma(1-w)+w}\beta,$$

where α and β are defined as follows:

- *If $\mu = 0$, then $\alpha = \beta = -\frac{\sigma^2}{4\ln\gamma}$.*
- *If $\mu \neq 0$, then $\alpha = \frac{\mu}{\gamma^{-2\theta}-1}$ and $\beta = \frac{\mu}{1-\gamma^{2\theta}}$, with $\theta = \frac{2\mu}{\sigma^2}$.*

Proof. In this case, since $\mu(x) \equiv \mu$ and $\sigma(x) \equiv \sigma$ are constants, we have

$$\omega(x) = \begin{cases} \frac{2}{\sigma^2} & \text{if } \mu = 0, \\ \frac{2}{\sigma^2}e^{\theta x} & \text{if } \mu \neq 0, \end{cases}$$

$$\int_{\ln\gamma}^{-\ln\gamma}\omega(x)dx = \begin{cases} \frac{-2\ln\gamma}{\sigma^2} & \text{if } \mu = 0, \\ \frac{1}{\mu}\left[\gamma^{-\theta} - \gamma^{\theta}\right] & \text{if } \mu \neq 0, \end{cases}$$

$$\int_{\ln\gamma}^{-\ln\gamma}x\omega(x)dx = \begin{cases} 0 & \text{if } \mu = 0, \\ \frac{-\ln\gamma}{\mu}\left[\gamma^{-\theta} + \gamma^{\theta}\right] + \frac{\sigma^2}{2\mu^2}\left[\gamma^{-\theta} - \gamma^{\theta}\right] & \text{if } \mu \neq 0. \end{cases}$$

Thus, the corollary follows immediately from Theorem 4.5. □

Remark 4.7. The corollary aligns with the computation in [Har13, Proposition 6.6] through the application of the stationary distribution for Z.

We conclude the section by presenting the steady-state distribution of the mispricing process Z_t, see Theorem 6.2 for its proof.

Theorem 4.8 (Steady-State Distribution)**.** *The reflected diffusion Z_t defined in (38) has a steady-state distribution π as $T \to \infty$ given by*

$$\pi(dz) = \frac{\omega(z)}{\int_{-c}^{c} \omega(\zeta) d\zeta} dz, \quad z \in [-c, c].$$

In other words, the steady-state distribution π is equal to the normalized speed measure, assuming the integrability of w within the interval $[-c, c]$.

The following corollary is a direct consequence of Theorem 4.8 and Corollary 4.6.

Corollary 4.9 (Steady-State Distribution under GBM)**.** *Assuming μ and σ are constants, let π be the steady-state distribution of the mispricing process Z_t. If $\mu = 0$, then π is the uniform distribution on $[\ln \gamma, -\ln \gamma]$. Otherwise, π is represented by the truncated exponential distribution*

$$\pi(dz) = \frac{\theta e^{\theta z}}{\gamma^{-\theta} - \gamma^{\theta}} dz,$$

where $\theta = \frac{2\mu}{\sigma^2}$.

4.3. **Time-inhomogeneous reflected diffusion.** The eigensystem representation for the solution in Section 4.2 does not apply when the coefficients are time dependent since in this case there exist no universal eigenfunctions that are associated with the time dependent eigenvalues. However, since we are only concerned with the long-term expected logarithmic growth rate of the wealth, it suffices to derive an asymptotic behavior for the time-averaged expectation in the long term. We show in this section how to determine the limit of time-averaged expectation in long term for time-inhomogeneous reflected diffusion.

Let $p(T, y|t, x)$ be the transition density for the time-inhomogeneous reflected diffusion

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t + dL_t - dU_t$$

within the interval $[-c, c]$. Assume that

$$\lim_{t\to\infty} \sigma(t, x) = \bar{\sigma}(x), \qquad \lim_{t\to\infty} \mu(t, x) = \bar{\mu}(x)$$

in L^2 and that the limits are smooth and bounded. Let

$$q(y) = \frac{\bar{w}(y)}{\int_{-c}^{c} \bar{w}(\eta) d\eta}, \qquad \text{where } \bar{w}(y) := \frac{2}{\bar{\sigma}^2(y)} e^{\int \frac{2\bar{\mu}(y)}{\bar{\sigma}^2(y)} dy}.$$

In other words, q is the speed measure, as well as the stationary distribution, associated with the (time-homogeneous) reflected diffusion with drift coefficient $\bar{\mu}$ and diffusion $\bar{\sigma}$.

Theorem 4.10 in the following characterizes the value of long-term time-averaged expectation for time-inhomogeneous reflect diffusion, whose proof is postponed till Appendix II in Section 7.

Theorem 4.10. *Let $u = u(t, x)$ be the solution to the parabolic PDE*

$$u_t + \frac{\sigma^2(t, x)}{2} u_{xx} + \mu(t, x) u_x + f(t, x) = 0$$

with Neumann boundary condition $u_x(t,-c) = u_x(t,c) = 0$ and terminal condition $u(T,x) = g(x)$. Further assume that $g \in L^2$ and there exists a function $\bar{f}$ so that

$$\lim_{t\to\infty} f(t,x) = \bar{f}(x)$$

in L^2. We have the following asympotic for u as T approaches infinity. For any t and $x \in (-c,c)$,

$$\lim_{T\to\infty} \frac{u(t,x)}{T-t} = \int_{-c}^{c} \bar{f}(y)q(y)dy.$$

As a corollary of Theorem 4.10, the following result is immediate.

Corollary 4.11 (Log Growth Rate of LP Wealth in G3M Time-inhomogeneous)**.** *Assume that the mispricing process Z_t follows the time-inhomogeneous reflected diffusion process in the interval $[-c,c]$ governed by*

$$dZ_t = \sigma(t,Z_t)dW_t + \mu(t,Z_t)dt + dL_t - dU_t, \tag{39}$$

where recall that the coefficients $\sigma(t,z)$ and $\mu(t,z)$ are given in (29). Further assume that the L^2 limits of the coefficients σ and μ as t approaches infinity exist. Specifically,

$$\lim_{t\to\infty} \sigma(t,z) = \bar{\sigma}(z), \qquad \lim_{t\to\infty} \mu(t,z) = \bar{\mu}(z) \tag{40}$$

for smooth and bounded functions $\bar{\mu}$ and $\bar{\sigma}$. The long-term expected logarithmic growth rate of a LP's wealth in a G3M can be expressed as

$$\lim_{T\to\infty} \frac{\mathbb{E}_t[\ln V_T]}{T-t} = w\mu_X + (1-w)\mu_Y + \frac{(1-\gamma)w(1-w)}{1-w+\gamma w}\bar{\alpha} + \frac{(1-\gamma)w(1-w)}{\gamma(1-w)+w}\bar{\beta},$$

where

$$\bar{\alpha} = \frac{\int_{\ln\gamma}^{-\ln\gamma} -\left\{\frac{\bar{\sigma}^2(x)}{4\ln\gamma} + \bar{\mu}(x)\left(\frac{1}{2\ln\gamma}x - \frac{1}{2}\right)\right\}\bar{\omega}(x)dx}{\int_{\ln\gamma}^{-\ln\gamma}\bar{\omega}(x)dx},$$

$$\bar{\beta} = \frac{\int_{\ln\gamma}^{-\ln\gamma} -\left\{\frac{\bar{\sigma}^2(x)}{4\ln\gamma} + \bar{\mu}(x)\left(\frac{1}{2\ln\gamma}x + \frac{1}{2}\right)\right\}\bar{\omega}(x)dx}{\int_{\ln\gamma}^{-\ln\gamma}\bar{\omega}(x)dx},$$

$$\bar{\omega}(x) = \frac{2}{\bar{\sigma}^2(x)} e^{\int \frac{2\bar{\mu}(x)}{\bar{\sigma}^2(x)}dx}.$$

4.4. **Independent stochastic volatility and drift.** In this section, we assume that the log price $s_t = \ln S_t$ follows the diffusion process

$$ds_t = \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t,$$

where $\tilde{\sigma}_t$ and $\tilde{\mu}_t$ are stochastic but independent of the driving Brownian motion W_t. The mispricing process $Z_t = \ln S_t - \ln P_t$ is then governed by the reflected diffusion in the bounded interval $[-c,c]$, $c = -\ln\gamma$, satisfying

$$dZ_t = ds_t + dL_t - dU_t = \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t + dL_t - dU_t.$$

Further assume the strong ellipticity of $\tilde{\sigma}_t$, i.e., $\tilde{\sigma}_t \geq \epsilon > 0$, for some ϵ, almost surely for all t, we derive the long-term limit for the time-averaged logarithmic growth of LP's wealth in the following. We remark that, as for the time-inhomogeneous reflected diffusion in Section

4.3, the eigensystem approach employed in Section 4.2 does not apply when $\tilde{\mu}_t \neq 0$ since there does not exist a set of eigenfunctions that are independent of time.

4.4.1. *Time-dependent reflected diffusion.* We first consider the case where $\tilde{\sigma}_t$ is a deterministic function of t and $\tilde{\mu}_t = 0$ for all t. In this case the (time-dependent) infinitesimal generator is given by $\mathcal{L}_t = \frac{\sigma^2(t)}{2}\partial_x^2$. We have in this case that the eigenvalues and a set of eigenfunctions associated with the operator $\mathcal{L}_t$, defined by

$$\mathcal{L}_t u = \frac{\sigma^2(t)}{2} u_{xx} = -\lambda u$$

with Neumann boundary conditions $u_x(-c) = u_x(c) = 0$, are given by

$$\lambda_n(t) = \frac{\sigma^2(t)}{2}\left(\frac{n\pi}{c}\right)^2, \qquad e_0(x) = \sqrt{\frac{1}{2c}}, \qquad e_n(x) = \sqrt{\frac{1}{c}}\cos\left(\frac{n\pi}{c}x\right) \text{ for } n \geq 1. \tag{41}$$

Notice that, though the eigenvalues λ_n depend on time, the eigenfunctions e_n are independent of t. Then by applying the same argument as in Section 4.2, the conditional expectation

$$\mathbb{E}_t[-a(L_T - L_t) + b(U_T - U_t)]$$

can be expressed in terms of the eigensystem (41). We summarize the result in the following theorem without proof.

Theorem 4.12. *Let X_t be the reflected diffusion in the interval $[-c, c]$ governed by*

$$dX_t = \sigma(t)dW_t + dL_t - dU_t.$$

For given constants a and b, the conditional expectation

$$\mathbb{E}_t\left[-\int_t^T a dL_\tau + \int_t^T b dU_\tau\right] \tag{42}$$

can be written in terms of the eigensystem (41) associated with $\mathcal{L}_t$ with Neumann boundary condition as

$$\mathbb{E}_t\left[-\int_t^T a dL_\tau + \int_t^T b dU_\tau\right] = \frac{b-a}{4c}x^2 + \frac{b+a}{2}x + \sum_{n=0}^{\infty} v_n(t)e_n(x), \tag{43}$$

where the time dependent coefficients v_n is given by

$$v_n(t) = e^{-\frac{1}{2}\left(\frac{n\pi}{c}\right)^2 \int_t^T \sigma^2(s)ds} k_n + \int_t^T h_n(s) e^{-\frac{1}{2}\left(\frac{n\pi}{c}\right)^2 \int_t^s \sigma^2(\tau)d\tau} ds. \tag{44}$$

and

$$h_n(t) = \frac{\sigma^2(t)}{4c}\int_{-c}^{c}(b-a)e_n(x)dx, \qquad k_n = -\int_{-c}^{c}\left(\frac{b-a}{4c}x^2 + \frac{b+a}{2}x\right)e_n(x)dx.$$

Observe from (43) that we have

$$\lim_{T\to\infty}\frac{1}{T-t}v_n(t)=0\quad \forall n\geq 1,$$
$$\lim_{T\to\infty}\frac{1}{T-t}v_0(t)=\lim_{T\to\infty}\frac{1}{T-t}k_0+\lim_{T\to\infty}\frac{1}{T-t}\int_t^T h_0(s)ds$$
$$=\frac{b-a}{4c}\sqrt{2c}\lim_{T\to\infty}\frac{1}{T-t}\int_t^T\sigma^2(s)ds.$$

Consequently,

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T-t}\mathbb{E}_t[-a(L_T-L_t)+b(U_T-U_t)]\\
=&\lim_{T\to\infty}\frac{1}{T-t}\left\{\sum_{n=0}^{\infty}v_n(t)e_n(x)+\frac{b-a}{4c}x^2+\frac{b+a}{2}x\right\}\\
=&\lim_{T\to\infty}\frac{1}{T-t}v_0(t)e_0(x)\\
=&\frac{b-a}{4c}\lim_{T\to\infty}\frac{1}{T-t}\int_t^T\sigma^2(s)ds.\end{aligned}$$

This leads to the following theorem.

Theorem 4.13 (Log Growth Rate of LP Wealth under Time-Dependent Volatility)**.** *Assuming $\mu_X=\mu_Y=\mu$, the logarithmic growth rate of an LP's wealth in a G3M can be expressed as*

$$\lim_{T\to\infty}\frac{\mathbb{E}_t[\ln V_T]}{T-t}=\mu-\left[\frac{(1-\gamma)w(1-w)}{1-w+\gamma w}+\frac{(1-\gamma)w(1-w)}{\gamma(1-w)+w}\right]\frac{1}{4\ln\gamma}\lim_{T\to\infty}\frac{1}{T-t}\int_t^T\sigma^2(s)ds.$$

Furthermore, since the volatility process $\tilde{\sigma}_t$ is independent of the mispricing process Z_t, by conditioning on the σ-algebra generated by $\tilde{\sigma}_t$ then applying the tower property for conditional expectation, we obtain the logarithmic growth rate of LP's wealth under driftless, independent stochastic volatility as follows.

Theorem 4.14 (Log Growth Rate under Independent Stochastic Volatility)**.** *Assuming $\mu_X=\mu_Y=\mu$, the logarithmic growth rate of an LP's wealth in a G3M can be expressed as*

$$\begin{aligned}&\lim_{T\to\infty}\frac{\mathbb{E}_t[\ln V_T]}{T-t}\\
=&\ \mu-\left[\frac{(1-\gamma)w(1-w)}{1-w+\gamma w}+\frac{(1-\gamma)w(1-w)}{\gamma(1-w)+w}\right]\frac{1}{4\ln\gamma}\lim_{T\to\infty}\frac{1}{T-t}\int_t^T\mathbb{E}_t\left[\tilde{\sigma}_s^2\right]ds.\end{aligned}$$

We conclude the section by showing that if the stochastic, but independent, drift and volatility converge to their corresponding L^2 limits, similar asymptotics in expectation as in Corollary 4.11 can be also be obtained.

Theorem 4.15 (Log Growth Rate of LP Wealth in G3M Independent Volatility and Drift)**.** *Assume that the mispricing process Z_t follows the reflected diffusion process in the interval*

$[-c, c]$ governed by

$$dZ_t = \tilde{\sigma}_t dW_t + \tilde{\mu}_t dt + dL_t - dU_t, \tag{45}$$

where the coefficients $\tilde{\sigma}_t$ and $\bar{\mu}_t$ are stochastic but independent of the driving Brownian motion W_t. Further assume that the limits of the coefficients $\tilde{\sigma}_t$ and $\tilde{\mu}_t$ as t approaches infinity exist almost surely and in L^2. Specifically, there exists an $\epsilon > 0$ such that

$$\lim_{t\to\infty} \sigma_t^2 = \bar{\sigma}^2 \geq \epsilon, \qquad \lim_{t\to\infty} \mu_t = \bar{\mu} \tag{46}$$

almost surely and in L^2, where $\bar{\mu}$ and $\bar{\sigma}^2$ are square integrable random variables. The long-term expected logarithmic growth rate of a LP's wealth in a G3M can be expressed as

$$\lim_{T\to\infty} \frac{\mathbb{E}_t[\ln V_T]}{T-t} = w\mu_X + (1-w)\mu_Y + \frac{(1-\gamma)w(1-w)}{1-w+\gamma w}\bar{\alpha} + \frac{(1-\gamma)w(1-w)}{\gamma(1-w)+w}\bar{\beta},$$

where

$$\bar{\alpha} = \bar{\beta} = -\frac{1}{4\ln\gamma}\mathbb{E}\left[\bar{\sigma}^2\right], \qquad \text{if } \bar{\mu} = 0 \text{ almost surely;}$$

$$\bar{\alpha} = \mathbb{E}\left[\frac{\bar{\mu}}{\gamma^{-2\theta}-1}\right], \quad \bar{\beta} = \mathbb{E}\left[\frac{\bar{\mu}}{1-\gamma^{2\theta}}\right], \quad \text{if } \bar{\mu} \neq 0,$$

where $\theta = \frac{2\bar{\mu}}{\bar{\sigma}^2}$, should the expectations exist.

Proof. The proof essentially is based on conditioning on the realizations of $\tilde{\mu}_t$ and $\tilde{\sigma}_t$ followed by applying the tower property since $\tilde{\mu}$ and $\tilde{\sigma}$ are independent of the Brownian motion W_t. □

We remark that the long term expected logarithmic growth rate considered in Theorem 4.15 can also be obtained differently by first calculating the condition expectations as in (29), then apply the asymptotic result given in Corollary 4.11. This route is applicable even when $\tilde{\mu}_t$ and $\tilde{\sigma}_t$ are not independent of W_t, it is however subject to the determination of the conditional expectations in (29), which in general do not admit easy to access analytical expressions. The expression obtained in Theorem 4.15 is more tractable in the sense that it is subject to the determination of the limiting distributions for $\bar{\mu}$ and $\bar{\sigma}^2$ as well as the corresponding expectations. However, it applies only if $\tilde{\mu}_t$ and $\tilde{\sigma}_t$ are independent of W_t.

5. Conclusion

In this paper, we explored the growth of liquidity providers' (LPs) wealth in Geometric Mean Market Makers (G3Ms), with a focus on the impacts of continuous-time arbitrage and transaction fees. By extending the analysis beyond constant product models, such as those introduced by Tassy and White [TW20], to a broader spectrum of G3Ms, we delved into a broader range of G3Ms. Our findings illustrate the significant role of arbitrage in influencing LP profitability and detail the intricate dynamics between G3Ms and reference markets. This study advances the domain of decentralized finance by presenting a model-free analysis of G3Ms under the influence of arbitrage and fees, enhancing existing models [MMRZ22, MMR23]. It lays the groundwork for future research into AMMs' complex dynamics and their market implications, marking a step forward in understanding decentralized finance's evolving landscape.

A key insight from our analysis is the depiction of LP wealth growth via a stochastic model driven by the mispricing process between G3Ms and reference markets. This process elucidates the crucial role of trading fees in affecting LPs' returns and outlines the conditions under which arbitrage can lead to a non-trading band relative to the reference market price. Our findings also indicate that the adverse selection posed by arbitrage activities necessitates a nuanced understanding of LP wealth dynamics within these decentralized financial platforms.

Looking to the future, our research suggests numerous promising avenues for exploration. These include the incorporation of order flows from noise trades, the investigation of interplay between different AMM liquidity pools, and the refinement of LP wealth growth models. Moreover, relating stochastic portfolio theory (SPT) [Fer02, KF09] to G3Ms with dynamic weights, as discussed in Evans [Eva21], holds great potential for further study. Pursuing these lines of inquiry will advance our understanding of AMMs' functionality and their wider impact on liquidity providers and the DeFi ecosystem.

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6. Appendix I - Sturm-Liouville theory

For reader's convenience, we review in this section the classical Sturm-Liouville theory that is relevant to the problem considered in the article.

6.1. **Eigensystem.** For a second-order differential operator $\mathcal{L}$ given by

$$\mathcal{L}u = \frac{1}{\omega(x)}\frac{d}{dx}\left(p(x)\frac{du}{dx}\right), \tag{47}$$

where ω and p are smooth functions dependent on x, let (λ_n, e_n) represent the eigensystem of the operator $\mathcal{L}$, such that $\mathcal{L}e_n = -\lambda_n e_n$ for $n \geq 0$ with Neumann boundary conditions $e_n'(-c) = e_n'(c) = 0$ within the interval $[-c, c]$. Note that since a constant function is invariably a solution, we have $\lambda_0 = 0$, and the corresponding normalized eigenfunction is $e_0 \equiv \frac{1}{K}$, where $K = (\int_{-c}^{c} \omega(x)dx)^{\frac{1}{2}}$. The following properties hold:

- $\lambda_n > 0$ for all $n \in \mathbb{N}_{>0}$ and each eigenvalue is of multiplicity one.
- The normalized eigenfunctions e_n form an orthonormal basis for $L^2[-c, c]$ functions, weighted by ω. Specifically,

$$\int_{-c}^{c} e_n(x)e_m(x)\omega(x)dx = \delta_{nm}$$

and, for any function $f \in L^2[-c, c]$, we have

$$f(x) = \sum_{n=0}^{\infty} \xi_n e_n(x),$$

where the coefficient ξ_n is given by

$$\xi_n = \int_{-c}^{c} f(x) e_n(x) \omega(x) dx$$

and the convergence is in the L^2 sense, weighted by ω.

6.2. **Solution to inhomogeneous PDE with general terminal condition.** For the inhomogeneous parabolic PDE

$$u_t + \mathcal{L}u + f(x) = 0 \tag{48}$$

with terminal condition $u(T, x) = g(x)$ and Neumann boundary conditions $u_x(t, -c) = u_x(t, c) = 0$ for $t < T$, we show how to formulate its solution using the eigensystem given in Section 6.1. Let the eigenfunction expansions for functions f and g be represented as

$$f(x) = \sum_{n=0}^{\infty} \xi_n e_n(x), \qquad g(x) = \sum_{n=0}^{\infty} \eta_n e_n(x)$$

where the coefficients ξ_n and η_n are defined as:

$$\xi_n = \int_{-c}^{c} f(x) e_n(x) \omega(x) dx, \qquad \eta_n = \int_{-c}^{c} g(x) e_n(x) \omega(x) dx.$$

In particular, the term ξ_0, expressed as

$$\xi_0 = \frac{\int_{-c}^{c} f(x) \omega(x) dx}{\int_{-c}^{c} \omega(x) dx}, \tag{49}$$

is the weighted average of f over the interval $[-c, c]$, weighted by ω.

The solution to the terminal-boundary value problem (48) can be expressed in terms of eigenvalues and eigenfunctions for $\mathcal{L}$ as

$$u(t, x) = \xi_0(T - t) + \eta_0 + \sum_{n=1}^{\infty} \left\{ \frac{\xi_n}{\lambda_n} \left[1 - e^{-\lambda_n (T-t)} \right] + \eta_n e^{-\lambda_n (T-t)} \right\} e_n(x). \tag{50}$$

Consequently, the following long-term time-averaged limit of u exists

$$\lim_{T \to \infty} \frac{u(t, x)}{T - t} = \xi_0,$$

where recall that ξ_0, given in (49), is the zeroth Fourier coefficient of the inhomogeneous term f. We note that this long-term time-averaged limit depends only on the zeroth coefficient of the inhomogeneous term, no other higher order coefficients are involved. Furthermore, we have the following long-term limit of u as $T \to \infty$

$$\lim_{T \to \infty} \{ u(t, x) - \xi_0(T - t) \} = \eta_0 + \sum_{n=1}^{\infty} \frac{\xi_n}{\lambda_n} e_n(x).$$

6.3. **Transition density in terms of eigensystem.** The following proposition shows that the transition density of a reflected diffusion within a bounded interval can be expressed in terms of the eigensystem of its infinitesimal generator with Neumann boundary condition.

Proposition 6.1. *The transition density p of a reflected diffusion in the interval $[-c.c]$ with infinitesimal generator $\mathcal{L}$ given in (47) can be expressed in terms of the eigensystem for $\mathcal{L}$ as*

$$p(T,y|t,x)=\sum_{n=0}^{\infty}e^{-\lambda_n(T-t)}e_n(x)e_n(y)\omega(y).$$

We conclude that the steady-state distribution for a reflected diffusion can be obtained as follows.

Theorem 6.2 (Steady-State Distribution)**.** *The reflected diffusion within the interval $[-c,c]$ with infinitesimal generator (47) has a steady-state distribution π given by*

$$\pi(dx)=\frac{\omega(x)}{\int_{-c}^{c}\omega(\xi)d\xi}dx,\quad x\in[-c,c].$$

Proof. By Proposition 6.1, as $T\to\infty$, the steady-state distribution is given by

$$\begin{aligned}\lim_{T\to\infty}p(T,y|t,x)&=\lim_{T\to\infty}\sum_{n=0}^{\infty}e^{-\lambda_m(T-t)}e_n(x)e_n(y)\omega(y)\\&=e_0(x)e_0(y)\omega(y)=\frac{\omega(y)}{\int_{-c}^{c}\omega(x)dx}\end{aligned}$$

since the zeroth eigenfunction $e_0(x)$ is a constant $e_0(x)=\left(\int_{-c}^{c}\omega(\xi)d\xi\right)^{-\frac{1}{2}}$. ☐

7. Appendix II - Time-inhomogeneous reflected diffusion

In this appendix, we provide the proof of the long-term time averaged expectation of a time-inhomogeneous reflected diffusion as stated in Theorem 4.10. For fixed t,x, we shall sometimes suppress the reference to t, x in the transition density p and simply denote $p(s,y|t,x)$ by $p(s,y)$ for simplicity. For any function φ defined in $[-c,c]$, $\|\varphi\|_2$ denotes the L^2 norm of φ in $[-c,c]$. We start with stating an estimate of the L^2-norm between the transition density p and the stationary density q in the following lemma, whose proof is omitted (for interested reader we refer it to for instance [Kah83], see (3.21) on P.276), is classical and crucial to the proof that follows.

Lemma 7.1. *Assume that the infinitesimal generator operator $\mathcal{L}_t:=\frac{\sigma^2(t,x)}{2}\partial_x^2+\mu(t,x)\partial_x$ is strongly elliptic, i.e., there exists an $\epsilon>0$ such that $\sigma(t,x)\geq\epsilon$ for all t, x, and that the coefficients σ and μ are smooth and bounded, the following estimates holds. For any $T>t$, we have*

$$\|p(T,\cdot|t,x)-q\|_2\leq\frac{C}{\sqrt{T-t}}\tag{51}$$

for some constant C depending only on the interval $[-c, c]$. As a result, we note that the L^2 norm of $p(s, y)$ is bounded above by

$$\|p(s,\cdot)\|_2 \leq \|p(s,\cdot) - q\|_2 + \|q\|_2 \leq \frac{C}{\sqrt{s-t}} + \|q\|_2 \tag{52}$$

for $s > t$.

With Lemma 7.1 in hand, we provide the proof of Theorem 4.10 as follows. Note that we have

$$\begin{aligned} u(t,x) &= \mathbb{E}_t\left[g(X_T) + \int_t^T f(s, X_s)ds\right] \\ &= \int_{-c}^{c} g(y)p(T,y|t,x)dy + \int_t^T \int_{-c}^{c} f(s,y)p(s,y|t,x)dyds \end{aligned}$$

since p is the transition density. Consider

$$\frac{u(t,x)}{T-t} = \frac{1}{T-t}\int_{-c}^{c} g(y)p(T,y|t,x)dy + \frac{1}{T-t}\int_t^T \int_{-c}^{c} f(s,y)p(s,y|t,x)dyds, \tag{53}$$

we determine the limits of the two terms on the right hand side of (53) separately.

For the first term in (53), by applying the Cauchy-Schwarz inequality we obtain that, for $T \geq t$,

$$\left|\int_{-c}^{c} g(y)p(T,y|t,x)dy\right| \leq \|g\|_2 \|p(T,\cdot|t,x)\|_2 \leq \|g\|_2 \left\{\frac{C}{\sqrt{T-t}} + \|q\|_2\right\}$$

where in the second inequality we applied the upper bound for $p(T,\cdot)$ given in (52). It follows that

$$\lim_{T\to\infty} \frac{1}{T-t}\left|\int_{-c}^{c} g(y)p(T,y|t,x)dy\right| \leq \lim_{T\to\infty} \frac{\|g\|_2}{T-t}\left\{\frac{C}{\sqrt{T-t}} + \|q\|_2\right\} = 0. \tag{54}$$

For the second term in (53), we claim that, as $\lim_{t\to\infty} f(t,x) = \bar{f}(x)$ in L^2, we have

$$\lim_{T\to\infty} \frac{1}{T-t}\int_t^T \int_{-c}^{c} f(s,y)p(s,y|t,x)dyds = \int_{-c}^{c} \bar{f}(y)q(y)dy \tag{55}$$

for every t and x. Note that, by applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left|\int_{-c}^{c} \left\{f(s,y)p(s,y) - \bar{f}(y)q(y)\right\} dy\right| \\ \leq &\int_{-c}^{c} \left|f(s,y) - \bar{f}(y)\right| p(s,y)dy + \int_{-c}^{c} |\bar{f}(y)||p(s,y) - q(y)|dy \\ \leq &\|f(s,\cdot) - \bar{f}\|_2 \|p(s,\cdot)\|_2 + \|\bar{f}\|_2 \|p(s,\cdot) - q\|_2. \end{aligned} \tag{56}$$

We shall deal with the two pieces in (56) separately. For the first piece, since $f(s,y) \to \bar{f}(y)$ as $s \to \infty$ in L^2, for any $\epsilon > 0$, there exists a $t_1 \geq t$ such that

$$\|f(s,\cdot) - \bar{f}\|_2 < \epsilon$$

for all $s > t_1$. Hence, for given $T > t_1$ we have

$$\begin{aligned}
&\int_t^T \|f(s,\cdot) - \bar{f}\|_2 \|p(s,\cdot)\|_2 ds \\
= \;& \int_t^{t_1} \|f(s,\cdot) - \bar{f}\|_2 \|p(s,\cdot)\|_2 ds + \int_{t_1}^T \|f(s,\cdot) - \bar{f}\|_2 \|p(s,\cdot)\|_2 ds \\
\leq \;& M \int_t^{t_1} \left\{ \frac{C}{\sqrt{s-t}} + \|q\|_2 \right\} ds + \epsilon \int_{t_1}^T \left\{ \frac{C}{\sqrt{s-t}} + \|q\|_2 \right\} ds \\
= \;& M \left\{ 2C\sqrt{t_1 - t} + \|q\|_2 (t_1 - t) \right\} + \epsilon \left\{ 2C\sqrt{T - t_1} + \|q\|_2 (T - t_1) \right\},
\end{aligned}$$

where in the inequality we applied the upper bound for p given in (52) and the constant $M > 0$ is defined as

$$\infty > M := \max_{t \leq s \leq t_1} \|f(s,\cdot)\|_2 + \|\bar{f}\|_2 \geq \|f(s,\cdot)\|_2 + \|\bar{f}\|_2 \geq \|f(s,\cdot) - \bar{f}\|_2.$$

It follows that

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T-t} \int_t^T \|f(s,\cdot) - \bar{f}\|_2 \|p(s,\cdot)\|_2 ds \\
\leq \;& \lim_{T\to\infty} \frac{M}{T-t} \left\{ 2C\sqrt{t_1 - t} + \|q\|_2 (t_1 - t) \right\} + \lim_{T\to\infty} \frac{\epsilon}{T-t} \left\{ 2C\sqrt{T - t_1} + \|q\|_2 (T - t_1) \right\} \\
= \;& \epsilon \|q\|_2.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain the limit of time-average of the first piece in (56) as T approaches infinity as

$$\lim_{T\to\infty} \frac{1}{T-t} \int_t^T \|f(s,\cdot) - \bar{f}\|_2 \|p(s,\cdot)\|_2 ds = 0.$$

Next, for the second piece in (56), note that from (51) we have

$$\int_t^T \|p(s,\cdot) - q\|_2 ds \leq \int_t^T \frac{C}{\sqrt{s-t}} ds = 2C\sqrt{T-t}.$$

It follows immediately that

$$\lim_{T\to\infty} \frac{1}{T-t} \int_t^T \|\bar{f}\|_2 \|p(s,\cdot) - q\|_2 ds \leq 2C \lim_{T\to\infty} \frac{\|\bar{f}\|_2}{T-t} \sqrt{T-t} = 0.$$

Finally, by combing (54) and (55) we conclude that

$$\lim_{T\to\infty} \frac{1}{T-t} u(t,x) = \int_{-c}^{c} \bar{f}(y) q(y) dy.$$

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