Tagged particles and size-biased dynamics in mean-field interacting particle systems

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Abstract

We establish a connection between tagged particles and size-biased empirical processes in interacting particle systems, in analogy to classical results on the propagation of chaos. In a mean-field scaling limit, the evolution of the occupation number on the tagged particle site converges to a time-inhomogeneous Markov process with non-linear master equation given by the law of large numbers of size-biased empirical measures. The latter are important in recent efforts to understand the dynamics of condensation in interacting particle systems.

Keywords: interacting particle system ; tagged particle ; size-biased empirical process ; mean-field scaling limit.

MSC2020 subject classifications: NA.

1 Introduction

Based on classical results in [23], propagation of chaos and laws of large numbers for empirical processes have recently attracted significant attention mostly for mean-field interacting diffusion models (see e.g. [18, 8] and references therein). In the context of interacting particle systems (IPS), propagation of chaos has been studied for the evolution of tagged particle locations on regular lattices [21, 20] and for single-site dynamics in mean-field models [11], with recent results also for sparse random graphs [19]. This note is based on results in [11] which provides a law of large numbers for empirical processes with a connection to rate equations studied in the context of cluster aggregation models [6, 22].

We consider the evolution of size-biased empirical measures, which is a useful tool to study the dynamics of condensing IPS with unbounded occupation numbers, such as zero-range [13, 10] or inclusion processes [9]. The dynamics of cluster formation in condensing IPS has attracted significant recent research interest [5, 2], also in the context of metastability (see e.g. [17, 14] and references therein). We show that the occupation number on a tagged particle location in the mean-field limit converges to a time-inhomogeneous Markov process with non-linear master equation given by the law of large numbers for size-biased empirical processes. Our main assumption is a bound on the jump rates by a bi-linear function of departure and target site occupation, which includes the above mentioned examples of condensing systems. In such models, higher order correlation functions diverge with time, so in contrast to recent results with uniform-in-time estimates [15] our results can be only local in time.

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2 Notation and main result

2.1 Mathematical setting

We consider stochastic particle systems $(\eta(t) : t > 0)$ on finite lattices Λ of size $|\Lambda| = L$. Configurations are denoted by $\eta = (\eta_x : x \in \Lambda)$ where $\eta_x \in \mathbb{N}_0$ is the number of particles on site x. We consider systems with a fixed number of particles $N = \sum_{x\in\Lambda} \eta_x$ and the state space of all such configurations is denoted by $E_{L,N} \subset \mathbb{N}_0^\Lambda$. The dynamics of the process is defined by the infinitesimal generator

$$(\mathcal{L}g)(\eta) = \sum_{x,y\in\Lambda} q(x,y)c(\eta_x,\eta_y)(g(\eta^{x\to y}) - g(\eta)) , \quad g \in C_b(E_{L,N}) . \tag{2.1}$$

Here the usual notation $\eta^{x\to y}$ indicates a configuration where one particle has moved from site x to y, i.e. $\eta_z^{x\to y} = \eta_z - \delta_{z,x} + \delta_{z,y}$, and δ is the Kronecker delta. Since $E_{L,N}$ is finite, the generator (2.1) is defined for all bounded, continuous test functions $g \in C_b(E_{L,N})$. For a general discussion and the construction of the dynamics on infinite lattices see e.g. [4, 1].

To ensure that the process is non-degenerate, the jump rates satisfy

$$\begin{cases} c(0,l) = 0 & \text{for all } l \geq 0 \\ c(k,l) > 0 & \text{for all } k > 0 \text{ and } l \geq 0. \end{cases} \tag{2.2}$$

Our main further assumption on the dynamics is that the rates grow sublinearly, in the sense that they are bounded by a bilinear function

$$c(k,l) \leq C_1 k(l + C_2) \quad \text{for constants } C_1, C_2 > 0 . \tag{2.3}$$

We focus on complete graph dynamics, i.e. $q(x,y) = 1/(L-1)$ for all $x \neq y$, and under the above conditions the process is irreducible on $E_{L,N}$ and

$$\sum_{x\in\Lambda} \eta_x(t) \equiv N \quad \text{is the only conserved quantity} . \tag{2.4}$$

To follow the location $(X(t) : t \geq 0)$ of a tagged particle, we extend the state space to $E := E_{L,N} \times \Lambda$ and states $(\eta, x) \in E$ describe the particle configuration $\eta \in E_{L,N}$ and location $x \in \Lambda$ of the tagged particle. In the following, we denote by $\mathbb{P}^L$ and $\mathbb{E}^L$ the law and expectation on the path space $\Omega = D_{[0,\infty)}(E)$ of the joint process $\big((\eta(t), X(t)) : t \geq 0\big)$. As usual, we use the Borel σ-algebra for the discrete product topology on E, and the smallest σ-algebra on Ω such that $\omega \mapsto (\eta_t(\omega), X_t(\omega))$ is measurable for all $t \geq 0$. The joint process is Markov and its evolution is described by the infinitesimal generator

$$\begin{aligned} \tilde{\mathcal{L}}G(\eta,x) = &\sum_{y,z\in\Lambda} \frac{1}{L-1} c(\eta_y,\eta_z)(G(\eta^{y\to z},x) - G(\eta,x))(1-\delta_{xy}) \\ &+ \sum_{z\in\Lambda} \frac{1}{L-1} c(\eta_x,\eta_z) \left[\frac{1}{\eta_x} \left(G(\eta^{x\to z},z) - G(\eta,x)\right) + \frac{\eta_x - 1}{\eta_x} \left(G(\eta^{x\to z},x) - G(\eta,x)\right) \right] \end{aligned} \tag{2.5}$$

for all bounded continuous functions $G \in C_b(E)$. We consider the empirical processes $t \mapsto F_k^L(\eta(t))$ with

$$F_k^L(\eta) := \frac{1}{L} \sum_{x\in\Lambda} \delta_{\eta_x,k} \in [0,1] , \quad k \geq 0 , \tag{2.6}$$

counting the fraction of lattice sites for each occupation number $k \geq 0$.

For our main result we will consider the thermodynamic limit with density ρ, i.e.

$$L \to \infty,\ N = N_L \to \infty \quad \text{such that} \quad N/L \to \rho \geq 0 . \tag{2.7}$$

For the sequence (in L) of initial conditions $(\eta(0), X(0))$ we first require the minimal condition that there exists a fixed probability distribution $f(0)$ on $\mathbb{N}_0$ with finite moments

$$m_1(0) := \sum_k k f_k(0) = \rho < \infty \quad \text{and} \quad m_2(0) := \sum_{k\geq 1} k^2 f_k(0) < \infty, \tag{2.8}$$

such that we have a weak law of large numbers

$$F_k^L(\eta(0)) \xrightarrow{d} f_k(0) \quad \text{as } L \to \infty, \text{ for all } k \geq 0. \tag{2.9}$$

We need further regularity assumptions on the initial conditions, namely a uniform bound of first, second and third moments,

$$\eta(0) \in E_{L,N}^{\alpha} := \Big\{ \eta : \frac{1}{L}\sum_{x\in\Lambda} \eta_x \leq \alpha_1,\ \frac{1}{L}\sum_{x\in\Lambda} \eta_x^2 \leq \alpha_2,\ \frac{1}{L}\sum_{x\in\Lambda} \eta_x^3 \leq \alpha_3 \Big\} \subset E_{L,N} \quad \text{for all } L \geq 1\ , \tag{2.10}$$

for some fixed $\alpha_1, \alpha_2, \alpha_3 > 0$. Note that (2.10) and conservation of mass (2.4) imply that

$$\frac{1}{L}\sum_{x\in\Lambda} \eta_x(t) = \sum_{k\geq 0} k F_k^L(\eta(t)) \leq \alpha_1\ , \quad \mathbb{P}^L - a.s. \text{ for all } t \geq 0 \text{ and } L \geq 1\ . \tag{2.11}$$

We assume that $N-1$ particles are distributed on the lattice according to some initial conditions satisfying (2.8), (2.9), (2.10) and the N-th particle (the tagged one) is located on position $X(0)$, increasing the value of $\eta_{X(0)}(0)$ by 1 such that

$$\mathbb{E}^L\left[\eta_{X(0)}^2(0)\right] < a_4 \quad \text{holds for some fixed } \alpha_4 > 0 \text{ and all } L \geq 1\ . \tag{2.12}$$

For example, if we distribute $N-1$ particles uniformly, independently on Λ, (2.8) and (2.9) are satisfied with Poisson distribution $f(0)$, and $\mathbb{P}^L\big[\eta(0) \in E_{L,N}^{\alpha}\big] \to 1$ in the limit (2.7), so condition (2.10) is asymptotically no restriction. There are various ways to then choose the initial position of the tagged particle such that (2.12) is satisfied. For example, we could pick a fixed site (e.g. $X(0) = 1$) or select one uniformly at random. On the other hand, selecting for example a site with the maximum occupation number would lead to logarithmic growth with respect to L of $\eta_{X(0)}(0)$, violating (2.12).

2.2 A law of large numbers for empirical processes

Under the above assumptions the following law of large numbers for the empirical process (2.6) was established in [11].

Theorem 2.1. *Consider a process with generator* (2.1) *on the complete graph with sublinear rates (2.3) and initial conditions satisfying* (2.8), (2.9) *and* (2.10). *Then we have in the thermodynamic limit* (2.7) *for any $\rho > 0$ and all functions $h : \mathbb{N}_0 \to \mathbb{R}$ with $|h(k)| \leq c_1(k + c_2)$,*

$$\Big(\sum_{k\geq 0} F_k^L(\eta(t))\, h(k) : t \geq 0 \Big) \to \Big(\sum_{k\geq 0} f_k(t)\, h(k) : t \geq 0 \Big) \quad \textit{weakly on path space as } L \to \infty\ , \tag{2.13}$$

where $t \mapsto f(t) = (f_k(t) : k \in \mathbb{N}_0)$ is the unique global solution of the ***mean-field equation***

$$\begin{aligned} \frac{df_k(t)}{dt} &= \sum_{l\geq 0} c(k+1,l) f_l(t) f_{k+1}(t) + \sum_{l\geq 1} c(l,k-1) f_l(t) f_{k-1}(t) \\ &\quad - \Big(\sum_{l\geq 0} c(k,l) f_l(t) + \sum_{l\geq 0} c(l,k) f_l(t) \Big) f_k(t) \quad \textit{for all } k \geq 0, \end{aligned} \tag{2.14}$$

with initial condition $f(0)$ given by (2.9). Here we use the convention $f_{-1}(t) \equiv 0$ for all $t \geq 0$ and recall that $c(0, l) = 0$ for all $l \geq 0$.

The nonlinear equations (2.14) can be written as

$$\frac{df_k(t)}{dt} = \mu_{k+1}(t)\, f_{k+1}(t) + \beta_{k-1}(t)\, f_{k-1}(t) - \Big(\beta_k(t) + \mu_k(t)\Big) f_k(t)\ , \quad k \geq 0,$$

and thus identified as the master equation of a non-linear birth-death chain on $\mathbb{N}_0$ with time-dependent birth and death rate

$$\beta_k(t) = \sum_{n\geq 1} c(n,k) f_n(t) \quad \text{and} \quad \mu_k(t) = \sum_{n\geq 0} c(k,n) f_n(t)\ , \tag{2.15}$$

respectively. Here we use again the convention $\beta_{-1}(t) \equiv \mu_0(t) \equiv 0$. This corresponds to the limiting dynamics of the occupation number of a fixed site, where any finite set of those evolves as independent birth-death chains according to the propagation of chaos (see [11] and references therein for details).

The solutions $t \mapsto (f_k(t) : k \geq 0)$ to this system of equations has been studied in [13, 11] and in detail in [22, 16]. In condensing systems solutions show a bump at occupation numbers increasing with time corresponding to the emergence of cluster sites in the condensed phase. The volume fraction of the latter vanishes in time and corresponds to the integral of the bump. To study the asymptotics of the condensed phase, it is therefore advantageous to consider a size-biased empirical distribution, as has been done for zero-range [13] and inclusion processes [12, 9]. Since (2.14) conserves the total mass $\rho \equiv m_1(t) = \sum_{k\geq 1} k f_k(t)$ for all $t \geq 0$, the corresponding size-biased quantities

$$p_k(t) := \frac{1}{\rho} k f_k(t)\ , \quad k \geq 1 \quad \text{are normalized with} \quad \sum_{k\geq 1} p_k(t) \equiv 1\ , \tag{2.16}$$

and describe the fraction of mass in clusters of size k. From (2.14) and (2.16) it is easy to see that they solve

$$\begin{aligned} \frac{dp_k(t)}{dt} &= \frac{k}{k+1}\mu_{k+1}(t)\, p_{k+1}(t) + \frac{k}{k-1}\beta_{k-1}(t)\, p_{k-1}(t) - \Big(\beta_k(t) + \mu_k(t)\Big) p_k(t)\ , \quad k \geq 1, \\ \frac{dp_1(t)}{dt} &= \frac{1}{2}\mu_2(t)\, p_2(t) + \frac{1}{\rho}\beta_0(t)\, f_0(t) - \Big(\beta_1(t) + \mu_1(t)\Big) p_1(t) \end{aligned} \tag{2.17}$$

with initial condition $p_k(0) = k f_k(0)/\rho$, $k \geq 1$. Here

$$f_0(t) = 1 - \sum_{k\geq 1} f_k(t) = 1 - \rho \sum_{k\geq 1} \frac{p_k(t)}{k}$$

denotes the volume fraction of empty sites, which can also be expressed in terms of $p_k(t)$, $k \geq 1$. Completely analogously to Theorem 2.1 one can show that the empirical mass processes

$$t \mapsto P_k^L(\eta(t)) := \frac{1}{N} \sum_{x\in\Lambda} k \delta_{\eta_x(t),k} \in [0,1]\ , \quad k \geq 1$$

converge to solutions of (2.17). Following our main result, we will see that the latter can be interpreted as the master equation for a process on $\mathbb{N}$, describing the mass on the site of a tagged particle.

2.3 Main result

The evolution of the occupation number on the tagged particle site is denoted by $N^L(t) := \eta_{X(t)}(t)$. To study its dynamics we apply the generator (2.5) to a test function $G(\eta, x) = g(\eta_x)$ and find

$$\hat{\mathcal{L}}^L_\eta g(\eta_x) = \sum_{y\in\Lambda} \frac{1}{L-1} c(\eta_y, \eta_x)(g(\eta_x+1) - g(\eta_x))(1-\delta_{xy})$$
$$+ \sum_{y\in\Lambda} \frac{1}{L-1} c(\eta_x, \eta_y) \left[\frac{1}{\eta_x}\left(g(\eta_y+1) - g(\eta_x)\right) + \frac{\eta_x - 1}{\eta_x}\left(g(\eta_x - 1) - g(\eta_x)\right)\right](1-\delta_{xy})\ . \tag{2.18}$$

Plugging in the process, this can be written for each $n \geq 1$ as

$$\hat{\mathcal{L}}^L_{\eta(t)} g(n) = \frac{L}{L-1} \sum_{k\geq 1} c(k,n) F^L_k(\eta(t))\big(g(n+1) - g(n)\big) +$$
$$\frac{L}{L-1}\left(\frac{1}{n}\sum_{k\geq 0} c(n,k) F^L_k(\eta(t))\left(g(k+1) - g(n)\right) + \frac{n-1}{n}\sum_{k\geq 0} c(n,k) F^L_k(\eta(t))\big(g(n-1) - g(n)\big)\right)$$
$$- \frac{1}{L-1} c(n,n)\left(\frac{n+1}{n}\left(g(n+1) - g(n)\right) + \frac{n-1}{n}\big(g(n-1) - g(n)\big)\right)\ . \tag{2.19}$$

Note that the process $(N^L(t), t \geq 0)$ is itself not a Markov process, since its generator depends also on the state of the configuration $\eta(t)$. Based on Theorem 2.1, we have that for each $n \in \mathbb{N}$ in the limit $L \to \infty$ (2.19) converges to a time-inhomogeneous generator

$$\hat{\mathcal{L}}_t g(n) = \beta_n(t)\big(g(n+1) - g(n)\big) + \frac{n-1}{n}\mu_n(t)\big(g(n-1) - g(n)\big) + \frac{1}{n}\sum_{k\geq 1} c(n, k-1) f_{k-1}(t)\left(g(k) - g(n)\right)\ . \tag{2.20}$$

This generator describes a birth-death process with time-dependent birth and death rates $\beta_n(t)$ and $\mu_n(t)$ as given in (2.15), and with additional long-range jumps when the tagged particle changes position. Here is our main result.

Theorem 2.2. *Consider a tagged particle process with generator* (2.5) *on the complete graph with sublinear rates (2.3) and initial conditions satisfying* (2.8), (2.9), (2.10) *and* (2.12). *In the thermodynamic limit* (2.7) *for any* $\rho > 0$

$$\big(N^L(t), t \geq 0\big) \to \big(\hat{N}(t), t \geq 0\big) \quad \textit{weakly on path space as } L \to \infty\ ,$$

where $\big(\hat{N}(t), t \geq 0\big)$ *is a time-inhomogeneous Markov process on* $\mathbb{N}$ *with generator* $\hat{\mathcal{L}}_t$ (2.20) *and corresponding master equation* (2.17).

Therefore, in a mean-field scaling limit, the evolution of the occupation number on the tagged particle site $\eta_{X(t)}(t)$ converges to a time-inhomogeneous process on $\mathbb{N}$ with (non-linear) master equation (2.17) given by the law of large numbers of size-biased empirical measures. As was demonstrated in [13] for the example of a condensing zero-range process, this can be used to devise efficient numerical schemes to study the coarsening dynamics of the condensed phase emerging from a supercritical homogeneous initial condition. In particular, the expectation

$$\mathbb{E}[\hat{N}(t)] = \sum_{k\geq 1} k p_k(t) = \frac{1}{\rho}\sum_{k\geq 1} k^2 f_k(t)$$

describes the second moment of the particle system which is increasing with t following a coarsening scaling law for condensing systems (see e.g. [13, 10, 22] for details).

3 Proof of the main result

The master equation corresponding to the limiting generator $\hat{\mathcal{L}}_t$ (2.20) is for $n \geq 2$

$$\begin{aligned}\frac{dq_n(t)}{dt} &= -\Big(\beta_n(t) + \frac{n-1}{n}\mu_n(t) + \underbrace{\frac{1}{n}\sum_{k\geq 1} c(n,k-1)f_{k-1}(t)}_{=\mu_n(t)}\Big)\,q_n(t) \\ &\quad + \beta_{n-1}(t)q_{n-1}(t) + \frac{n}{n+1}\mu_{n+1}(t)q_{n+1}(t) + \underbrace{\sum_{k\geq 1}\frac{1}{k}c(k,n-1)f_{n-1}(t)q_k(t)}_{=\frac{1}{n-1}\beta_{n-1}(t)q_{n-1}(t)} \\ &= -\Big(\beta_n(t) + \mu_n(t)\Big)q_n(t) + \frac{n}{n-1}\beta_{n-1}(t)q_{n-1}(t) + \frac{n}{n+1}\mu_{n+1}(t)q_{n+1}(t) \end{aligned} \tag{3.1}$$

and for $n = 1$ we have $\displaystyle \frac{dq_1(t)}{dt} = -\Big(\beta_1(t)+\mu_1(t)\Big)q_1(t) + \frac{1}{2}\mu_2(t)q_2(t) + \underbrace{\sum_{k\geq 1}\frac{c(k,0)}{k}f_0(t)q_k(t)}_{=\beta_0(t)f_0(t)/\rho}$

.

This coincides with (2.17) and the rest of this section is used to prove convergence of the process $t \mapsto N^L(t)$.

3.1 Moment bounds

As a first step we collect some useful results on moments and establish a time-dependent bound on the moments of the processes $N^L(t)$ and $\eta_x(t)$ for $x \in \Lambda$. For any integer $n \geq 0$ denote the n-th moment by

$$m_n^L(t) := \mathbb{E}^L\Big[\frac{1}{L}\sum_{x\in\Lambda}\big(\eta_x(t)\big)^n\Big] = \mathbb{E}^L\Big[\sum_{k\geq 0} k^n F_k^L(\eta(t))\Big]\,. \tag{3.2}$$

We have $m_0^L(t) \equiv 1$ and with (2.9), $m_1^L(0) \to \rho$ and $m_2^L(0) \to m_2(0) < \infty$. The uniform conditions (2.10) on the moments further imply for all $L \geq 1$ that $m_2^L(0) \leq \alpha_2$, and with conservation of mass (2.11) we have $m_1^L(t) \leq \alpha_1$ for all $t \geq 0$, while higher moments typically grow in time for condensing systems (see e.g. [10, 13, 22]). The following result gives a general (but very rough) upper bound.

Proposition 3.1. *Assume that the sequence* $\big(m_n^L(0)\big)_{L\geq 1}$ *is bounded uniformly in* L *for some integer* $n \in \mathbb{N}$. *Then there exists a constant* $C_n > 0$ *independent of* L *such that*

$$m_n^L(t) \leq \big(m_n^L(0) + C_n t\big)e^{C_n t} \quad \textit{for all } t \geq 0 \textit{ and } L \geq 1\,. \tag{3.3}$$

Proof. Applying the generator (2.1) to the function $g(\eta) = \eta_x^k$ for $k \in \mathbb{N}$, we get

$$\mathcal{L}\eta_x^k = \frac{1}{L-1}\sum_{y\neq x}\left(c(\eta_x,\eta_y)\left((\eta_x+1)^k - \eta_x^k\right) + \sum_{y\neq x} c(\eta_y,\eta_x)\left((\eta_x-1)^k - \eta_x^k\right)\right) \tag{3.4}$$

Note that $(x \pm 1)^n - x^n = p_{n-1}^{\pm}(x)$ is a polynomial of degree $n-1$, which implies with

(3.4) and sublinear rates (2.3) that

$$\begin{aligned}
\frac{d}{dt}m_n^L(t) &= \frac{1}{L}\sum_{x\in\Lambda}\mathbb{E}^L\left[\mathcal{L}\eta_x^n(t)\right] = \frac{1}{L-1}\mathbb{E}^L\bigg[\sum_{k,l\geq 0}c(k,l)p_{n-1}^+(k)\Big(F_k(\eta(t))L-\delta_{k,l})F_l(\eta(t))\Big) \\
&\qquad\qquad + \sum_{k,l\geq 0}c(l,k)p_{n-1}^-(k)\Big(F_k(\eta(t))L-\delta_{k,l})\Big)F_l(\eta(t))\bigg] \\
&= \frac{L}{L-1}\mathbb{E}^L\bigg[\sum_{k,l\geq 0}\Big(c(k,l)p_{n-1}^+(k)+c(l,k)p_{n-1}^-(k)\Big)F_k(\eta(t))F_l(\eta(t))\bigg] \\
&\qquad\qquad - \frac{1}{L-1}\mathbb{E}^L\bigg[\sum_{k\geq 1}c(k,k)\Big(p_{n-1}^+(k)+p_{n-1}^-(k)\Big)F_k(\eta(t))\bigg] \\
&\leq \mathbb{E}^L\bigg[\sum_{k,l\geq 0}\Big(2C_1kl+C_1C_2(k+l)\Big)p_{n-1}^+(k)F_k(\eta(t))F_l(\eta(t))\bigg] \\
&\leq C\mathbb{E}^L\bigg[\sum_{k\geq 0}p_n(k)F_k(\eta(t))\bigg]
\end{aligned} \tag{3.5}$$

for some constant $C>0$ which does not depend on L. Here we used that $p_{n-1}^+(k)\geq p_{n-1}^-(k)$ and $p_{n-1}^+(k)+p_{n-1}^-(k)\geq 0$ in the first inequality, and conservation of mass (2.11) in the second inequality with a polynomial $p_n(k)$ of degree n. Since $m_n^L(t)\leq m_{n+1}^L(t)$ for all $n\geq 1$, this implies for some constant $C_n>0$

$$\frac{d}{dt}m_n^L(t)\leq C_n\big(1+m_n^L(t)\big)\ .$$

The result then follows by Gronwall's Lemma. ☐

In the following, we denote the n-th moment of the process $N^L(t)$ by

$$\hat{m}_n^L(t) := \mathbb{E}^L\big[(N^L(t))^n\big] = \mathbb{E}^L[\eta_{X(t)}^n(t)] \tag{3.6}$$

and we have the following exponential bound:

Proposition 3.2. *Assume that the sequence $\big(m_{n+1}^L(0)\big)_{L\geq 1}$ is bounded for some integer $n\in\mathbb{N}_+$. Then, there exist constants $A_n,B_n,C_n>0$ independent of L such that*

$$\hat{m}_n^L(t)\leq\big(A_n+B_nt+C_n\hat{m}_n^L(0)\big)e^{C_nt}\quad\textit{for all } t\geq 0 \textit{ and } L\geq 1\ . \tag{3.7}$$

Proof. Applying the generator (2.19) to the function $g(x)=x^n$, we get (writing $N(t)=N^L(t)$ for simplicity of notation)

$$\begin{aligned}
\frac{d\hat{m}_n^L(t)}{dt} &= \mathbb{E}^L\left[\hat{\mathcal{L}}_{\eta(t)}^L N^n(t)\right] = \mathbb{E}^L\left[\frac{L}{L-1}\sum_{k\geq 1}c(k,N(t))F_k^L(\eta(t))p_{n-1}^+(N(t))\right] \\
&\quad + \frac{L}{L-1}\mathbb{E}^L\left[\frac{1}{N(t)}\sum_{k\geq 0}c(N(t),k)F_k^L(\eta(t))\Big((k+1)^n-N^n(t)\Big)\right] \\
&\quad - \frac{L}{L-1}\mathbb{E}^L\left[\frac{N(t)-1}{N(t)}\sum_{k\geq 0}c(N(t),k)F_k^L(\eta(t))p_{n-1}^+(N(t)-1))\right] \\
&- \mathbb{E}^L\left(\frac{1}{L-1}c(N(t),N(t))\frac{1}{N(t)}\big[(N(t)+1)p_{n-1}^+(N(t))-(N(t)-1)p_{n-1}^+(N(t)-1)\big]\right)
\end{aligned} \tag{3.8}$$

where we used $p_{n-1}^-(k) = -p_{n-1}^+(k-1)$. Since the functions $x \mapsto (x+1)p_{n-1}^+(x)$ for all $n \in \mathbb{N}$, $n \mapsto m_n(t)$ for all $t \geq 0$ and $n \mapsto \hat{m}_n(t)$ for all $t \geq 0$ are non-decreasing, we have that for a certain polynomial p_n of degree n:

$$\begin{aligned}\frac{d\hat{m}_n^L(t)}{dt} &\leq \frac{L}{L-1} C_1 a_1 \mathbb{E}^L \left[p_n(N(t))\right] + \frac{L}{L-1} C_1 \mathbb{E}^L \left[\sum_{k \geq 0} p_{n+1}(k) F_k^L(\eta(t))\right] \\ &\leq D_n(\hat{m}_n^L(t) + m_{n+1}^L(t)) \leq \hat{C}_n \Big(\hat{m}_n^L(t) + \big(m_{n+1}^L(0) + \hat{C}_n t\big) e^{\hat{C}_n t}\Big) ,\end{aligned}$$

where in the last line we used relation (3.7) and $D_n, \hat{C}_n$ are positive constants. The result then follows by Gronwall's Lemma. □

Based on Proposition 3.2 and assumptions (2.10)-(2.12), we have the following corollary:

Corollary 3.3. *There exist constants $A_2, B_2, C_2 > 0$ independent of L such that*

$$\hat{m}_2^L(t) \leq \big(A_2 + B_2 t + C_2 a_4\big) e^{C_2 t} \quad \textit{for all } t \geq 0\,,\ L \geq 1\,. \tag{3.9}$$

3.2 Existence of limit processes

Lemma 3.4. *Consider the process with generator* (2.19) *and conditions as in Theorem 2.2. Denote by $\mathbb{Q}^L$ the measure of the process $t \mapsto N^L(t)$ on path space $D_{[0,\infty)}(\mathbb{N})$, which is the image measure of $\mathbb{P}^L$ under the mapping $(\eta, x) \mapsto \eta_x$. Then $\mathbb{Q}^L$ is tight as $L \to \infty$.*

Proof. Using a version of Aldous' criterion to establish tightness for $\mathbb{Q}^L$ (cf. Theorem 16.10 in [7]), it suffices to show that for all $t \geq 0$

$$\lim_{a\to\infty} \limsup_{L\to\infty} \sup_{(\zeta,x)\in E_{L,N}^\alpha \times \Lambda} \mathbb{P}_{(\zeta,x)}^L \big[N(t) \geq a\big] = 0, \tag{3.10}$$

(writing again $N(t) = N^L(t)$) and that for any $\epsilon > 0$

$$\lim_{\delta\to 0+} \limsup_{L\to\infty} \sup_{t<\delta} \sup_{(\zeta,x)\in E_{L,N}^\alpha \times \Lambda} \mathbb{P}_{(\zeta,x)}^L \big[|N(t) - \zeta_x| > \epsilon\big] = 0. \tag{3.11}$$

Here $(\zeta, x) \in E_{L,N}^\alpha \times \Lambda$ denotes a fixed initial condition of the full process (2.5) with ζ satisfying (2.10), and $\mathbb{P}_{(\zeta,x)}^L$ the corresponding path measure of the process.

Since by Proposition 3.2 and assumption (2.12) the first moment of $N^L(t)$ is bounded (uniformly in L), (3.10) follows directly from Markov's inequality.

$$\mathbb{P}_{(\zeta,x)}^L \big[N(t) \geq a\big] \leq \frac{(A_1 + B_1 t + C_1 a_4) e^{C_1 t}}{a} \quad \text{for all } L \geq 1 \text{ and } (\zeta, x) \in E_{L,N}^\alpha \times \Lambda\,.$$

Now fix $\delta > 0$ and consider $t < \delta$. By Itô's formula, we have

$$N(t) - \zeta_x = \int_0^t \hat{\mathcal{L}}_{\eta(s)}^L N(s)\, ds + M(t)\,, \tag{3.12}$$

where $(M(t) : t > 0)$ is a martingale with predictable quadratic variation given by integrating the 'carré du champ' operator

$$[M](t) = \int_0^t \Big(\hat{\mathcal{L}}_{\eta(s)}^L N^2(s) - 2N(s) \hat{\mathcal{L}}_{\eta(s)}^L N(s)\Big) ds\,. \tag{3.13}$$

Using again Markov's inequality in (3.11) we have to bound

$$\mathbb{E}^L_{(\zeta,x)}\Big[\big|N(t)-\zeta_x\big|\Big] \leq \int_0^t \mathbb{E}^L_{(\zeta,x)}\big[|\hat{\mathcal{L}}^L_{\eta(s)}N(s)|\big]\,ds + \mathbb{E}^L_{(\zeta,x)}\big[[M](t)\big]^{1/2}\,, \tag{3.14}$$

where we used Hölder's inequality and $\mathbb{E}^L_{(\zeta,x)}\big[M^2(t)\big] = \mathbb{E}^L_{(\zeta,x)}\big[[M](t)\big]$ for the martingale.

Regarding the first term on the right of (3.14), we have

$$\begin{aligned}\mathbb{E}^L_{(\zeta,x)}\left[\big|\hat{\mathcal{L}}^L_{\eta(t)}N(t)\big|\right] \leq \frac{L}{L-1}\mathbb{E}^L_{(\zeta,x)}\left[\sum_{k\geq 1} c(k,N(t))F^L_k(\eta(t))\right] + \\ \frac{L}{L-1}\mathbb{E}^L_{(\zeta,x)}\left[\frac{1}{N(t)}\sum_{k\geq 0} c(N(t),k)F^L_k(\eta(t))\big|k+1-N(t)\big| + \frac{N(t)-1}{N(t)}\sum_{k\geq 0} c(N(t),k)F^L_k(\eta(t))\right] \\ + \mathbb{E}^L_{(\zeta,x)}\left[\frac{1}{L-1}c(N(t),N(t))\frac{2}{N(t)}\right]\end{aligned} \tag{3.15}$$

and therefore

$$\begin{aligned}\mathbb{E}^L_{(\zeta,x)}\left[\big|\hat{\mathcal{L}}^L_{\eta(t)}N(t)\big|\right] &\leq C_1 a_1\left(\hat{m}^L_1(t)+C_2\right)\left(1+\tfrac{1}{L}\right) \\ &+ C_1\Big(m^L_2(t)+(a_1+C_2)\hat{m}^L_1(t)) + C_2 a_1\Big)\left(1+\tfrac{1}{L}\right) + 2C_1(a_1+C_2)\,,\end{aligned} \tag{3.16}$$

where we used that $\frac{N(t)}{L}\leq a_1$ and the factor $(1+1/L)$ results from replacing $1/(L-1)$ by $1/L$. So, based on Propositions 3.1 and 3.2, we conclude that

$$\begin{aligned}\int_0^t \mathbb{E}^L_{(\zeta,x)}\left[\big|\hat{\mathcal{L}}^L_{\eta(s)}g(N(s))\big|\right]ds \leq \delta\Bigg(& 2C_1 a_1\left((A_1+B_1\delta+C_1 a_4)e^{C_1\delta}+C_2\right) \\ &+ C_1((\alpha_2+C\delta)e^{C\delta}+(a_1+C_2)\big(A_1+B_1\delta+C_1 a_4\big)e^{C_1\delta}) + 2C_2 a_1 + 2C_1(a_1+C_2)\Bigg) \to 0\end{aligned} \tag{3.17}$$

as $\delta\to 0$, which holds uniformly in $(\zeta,x)\in E^\alpha_{L,N}\times\Lambda$ and $L\geq 1$.

To compute $[M](t)$, we notice that (suppressing the time dependence of N)

$$\begin{aligned}&\hat{\mathcal{L}}^L_\eta N^2 - 2N\hat{\mathcal{L}}^L_\eta N = \frac{L}{L-1}\sum_{k\geq 1}c(k,N)F^L_k(\eta) \\ &+\frac{L}{L-1}\left(\frac{1}{N}\sum_{k\geq 0}c(N,k)F^L_k(\eta)\,(k+1-N)^2 + \frac{N-1}{N}\sum_{k\geq 0}c(N,k)F^L_k(\eta)\right) - \frac{4}{L-1}c(N,N)\,.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\mathbb{E}^L_{(\zeta,x)}\left[\hat{\mathcal{L}}^L_{\eta(s)}N^2(s) - 2N(s)\hat{\mathcal{L}}^L_{\eta(s)}N(s)\right] &\leq C_1 a_1\left(\hat{m}^L_1(s)+C_2\right)\left(1+\tfrac{1}{L}\right) \\ &+\Big(m^L_3(s)+(a_1+C_2)\hat{m}^L_2(t)+C_2 m^L_2(s)+(a_1+C_2)\hat{m}^L_1(t)\Big)\left(1+\tfrac{1}{L}\right)\,.\end{aligned}$$

Based on Assumptions (2.8)-(2.12) and Propositions 3.1 and 3.2, we conclude that

$$\sup_{t<\delta}\mathbb{E}^L_{(\zeta,x)}\big[[M](t)\big]\to 0 \text{ as } \delta\to 0$$

and this holds again uniformly in $(\zeta,x)\in E^\alpha_{L,N}\times\Lambda$ and $L\geq 1$, and finishes the proof. □

By Prokhorov's theorem, the tightness result in Lemma 3.4 implies the existence of limit points of the sequence $(N^L(t) : t \geq 0)$ in the usual topology of weak convergence on path space. More specifically, we have existence of sub-sequential weak limits of $\mathbb{Q}^L$ in the Skorohod topology, and we denote any such limit by $\mathbb{Q}$.

3.3 Generator of the limit process

In order to identify the limit $\mathbb{Q}$ we need to show that for all $t \geq 0$ and $g \in C_b$

$$g(\omega(t)) - g(\omega(0)) - \int_0^t \hat{\mathcal{L}}_s g(\omega(s))ds \;\text{ is a martingale w.r.t. } \mathbb{Q}\;, \tag{3.18}$$

where $\omega \in D_{[0,\infty)}(\mathbb{N})$ denotes an element in path space. Together with the uniqueness of the martingale problem associated with $\hat{\mathcal{L}}_t$, this implies convergence of $\mathbb{Q}^L$ and characterizes the limit $\mathbb{Q}$ as the law of the Markov process $(\hat{N}(t) : \; t \geq 0)$ with generator $\hat{\mathcal{L}}_t$ (2.20). More specifically, following [3], Section 8, we need to show that

$$\mathbb{E}^{\mathbb{Q}}\left[f\left((\omega(u) : 0 \leq u \leq s)\right) \left(g(\omega(t)) - g(\omega(s)) - \int_s^t \hat{\mathcal{L}}_s g(\omega(u))du \right) \right] = 0 \tag{3.19}$$

for all $0 \leq s \leq t$ and continuous bounded functions $f : D_{[0,\infty)}(\mathbb{N}) \to \mathbb{R}$. Since $\hat{\mathcal{L}}_t$ (2.20) corresponds to a birth-death process on $\mathbb{N}$ with additional long-range jumps which however happen at uniformly bounded rates, we have that $\mathbb{Q}[\omega(t) \neq \omega(t-)] = 0$. Then Lemma 8.1 in [3] implies that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^L}&\left[f\left((\omega(u) : 0 \leq u \leq s)\right) \left(g(\omega(t)) - g(\omega(s)) - \int_s^t \hat{\mathcal{L}}_s g(\omega(u))du \right) \right] \to \\ &\mathbb{E}^{\mathbb{Q}}\left[f\left((\omega(u) : 0 \leq u \leq s)\right) \left(g(\omega(t)) - g(\omega(s)) - \int_s^t \hat{\mathcal{L}}_s g(\omega(u))du \right) \right] \;.\end{aligned} \tag{3.20}$$

Therefore, in order to prove (3.19), it suffices to prove that

$$\mathbb{E}^L\left[\left| g(N^L(t)) - g(N^L(s)) - \int_s^t \hat{\mathcal{L}}_s g(N^L(u))du \right| \right] \to 0\;. \tag{3.21}$$

Since $\Big((\eta(t), X(t)) \,:\, t \geq 0\Big)$ is a Markov process, we know that with initial condition (ζ, x)

$$\begin{aligned}g(N^L(t)) - g(\zeta_x) &- \int_0^t \hat{\mathcal{L}}^L_{\eta(s)} g(N^L(s))\,ds = \\ &= g(N^L(t)) - g(\zeta_x) - \int_0^t \hat{\mathcal{L}}_s g(N^L(s)) + \int_0^t \Big(\hat{\mathcal{L}}_s g(N^L(s)) - \hat{\mathcal{L}}_{\eta(s)} g(N^L(s)) \Big)\,ds\end{aligned}$$

is a martingale for all $t \geq 0$ and $L \in \mathbb{N}$. Thus, we only need to prove that for all $T > 0$

$$\mathbb{E}^L\left[\left| \int_0^T \Big(\hat{\mathcal{L}}_t g(N^L(t)) - \hat{\mathcal{L}}_{\eta(t)} g(N^L(t)) \Big)\,dt \right| \right] \to 0 \tag{3.22}$$

as $L \to \infty$.

Proof of (3.22). Since the process $t \mapsto \hat{\mathcal{L}}_t g(N^L(t))$ is uniformly bounded with respect to L on compact time intervals, it suffices to prove that for all $T > 0$

$$\int_0^T \mathbb{E}^L\left[\left| \hat{\mathcal{L}}_t g(N(t)) - \frac{L-1}{L} \hat{\mathcal{L}}_{\eta(t)} g(N(t)) \right| \right] dt \;\to 0 \tag{3.23}$$

as $L \to \infty$. Since $g \in C_b(\mathbb{N}_+)$ and because of condition (2.3), we find

$$\left|\hat{\mathcal{L}}_t g(N(t)) - \frac{L-1}{L}\hat{\mathcal{L}}_{\eta(t)} g(N(t))\right| \leq 2||g||_\infty \Bigg(\sum_{k\geq 1} c(k,N(t)) \left|F_k^L(\eta(t)) - f_k(t)\right|$$

$$+C_1 \sum_{k\geq 0}(k+C_2)\left|F_k^L(\eta(t)) - f_k(t)\right| + \sum_{k\geq 0} c(N(t),k)\left|F_k^L(\eta(t)) - f_k(t)\right| + \frac{2C_1(C_2+1)N^2(t)}{L}\Bigg)$$

$$\leq 2||g||_\infty \Bigg(C_1(2+C_2)N(t)\sum_{k\geq 1} k\left|F_k^L(\eta(t)) - f_k(t)\right|$$

$$+C_1 \sum_{k\geq 0}(C_2+(1+C_2)k)\left|F_k^L(\eta(t)) - f_k(t)\right| + \frac{2C_1(C_2+1)N^2(t)}{L}\Bigg)$$

Notice that for all $M > 0$, we have

$$\mathbb{E}^L\Bigg[N(t)\sum_{k\geq 1} k\left|F_k^L(\eta(t)) - f_k(t)\right|\Bigg] = \mathbb{E}^L\Bigg[N(t)\sum_{k\geq 1} k\left|F_k^L(\eta(t)) - f_k(t)\right|\mathbb{1}\{N(t)\leq M\}\Bigg] +$$

$$\mathbb{E}^L\Bigg[N(t)\sum_{k\geq 1} k\left|F_k^L(\eta(t)) - f_k(t)\right|\mathbb{1}\{N(t)> M\}\Bigg]$$

$$\leq M\mathbb{E}^L\Bigg[\sum_{k\geq 1} k\left|F_k^L(\eta(t)) - f_k(t)\right|\Bigg] + 2a_1 \sup_{L\in\mathbb{N},t\leq T}\mathbb{E}^L\big[N(t)\mathbb{1}\{N(t)>M\}\big]\ .$$

Therefore, for all $M > 0$ and because of relation (3.9), we find:

$$\int_0^T \mathbb{E}^L\left[\left|\hat{\mathcal{L}} g(N(s)) - \frac{L-1}{L}\hat{\mathcal{L}}_{\eta(s)} g(N(s))\right|\right] ds \leq$$

$$2M||g||_\infty C_1(2+C_2)\int_0^T \mathbb{E}^L\Bigg[\sum_{k\geq 1} k\left|F_k^L(\eta(t)) - f_k(t)\right|\Bigg]dt +$$

$$4a_1 T||g||_\infty C_1(2+C_2)\sup_{L\in\mathbb{N},t\leq T}\mathbb{E}^L\big[N(t)\mathbb{1}\{N(t)>M\}\big] +$$

$$+C_1\int_0^T \mathbb{E}^L\Bigg[\sum_{k\geq 0}(C_2+(1+C_2)k)\left|F_k^L(\eta(t)) - f_k(t)\right|\Bigg]dt + \frac{2C_1(C_2+1)\big(A_2+B_2T+C_2\alpha_3\big)e^{C_2T}}{L}T$$

In the limit $L \to \infty$, by bounded convergence and Theorem 2.1, we have that

$$\int_0^T \mathbb{E}^L\Bigg[\sum_{k\geq 1} k\left|F_k^L(\eta(t)) - f_k(t)\right|\Bigg]dt \to 0$$

Therefore, for all $M > 0$,

$$\int_0^T \mathbb{E}^L\left[\left|\hat{\mathcal{L}} g(N^L(s)) - \frac{L-1}{L}\hat{\mathcal{L}}_{\eta(s)} g(N^L(s))\right|\right] ds \leq$$

$$4a_1 T||g||_\infty C_1(2+C_2)\sup_{L\in\mathbb{N},t\leq T}\mathbb{E}^L\big[N^L(t)\mathbb{1}\{N^L(t)>M\}\big]\ .$$

In the limit $M \to \infty$, the uniform integrability of $\{N^L(t)\}_{L,t\leq T}$ due to relation (3.9), gives (3.23). $\square$

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