Green Functions in Small Characteristic

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Abstract

The values of the ordinary Green functions are known for almost all groups of Lie type, a long term achievement by various authors.

In this note we solve the last open cases, which are for exceptional groups of type $E_8(q)$ where q is a power of 2, 3 or 5.

1 Introduction

Let p be a prime, q some power of p and $\mathbb{F}_q$ the field with q elements. Let $\mathbf{G}$ be a reductive algebraic group over an algebraic closure $\bar{\mathbb{F}}_p$ with a Frobenius endomorphism F defining an $\mathbb{F}_q$-rational structure.

We are interested in class functions of the finite group of fixed points $G(q) := \mathbf{G}^F$. Let $\mathbf{T} \subset \mathbf{G}$ be an F-invariant maximal torus. Deligne and Lusztig [DL76] defined for each irreducible character θ of the abelian group $\mathbf{T}^F$ a generalized character $R^{\mathbf{G}}_{\mathbf{T},\theta}$ of $G(q)$, using certain modules constructed by ℓ-adic cohomology. They show that the values of $R^{\mathbf{G}}_{\mathbf{T},\theta}$ on unipotent elements only depend on the $G(q)$-conjugacy class of $\mathbf{T}$ and not on the character θ, we write $Q^{\mathbf{G}}_{\mathbf{T}}$ for the restriction of $R^{\mathbf{G}}_{\mathbf{T},\theta}$ to unipotent elements, this is called the *Green function* of $\mathbf{T}$.

In [Lus85] Lusztig defined another set of class functions on the unipotent elements using character sheaves, these are called *generalized Green functions.* A subset of them is also associated to $G(q)$-classes of F-invariant maximal tori, and Lusztig showed later [Lus90] that the two types of Green functions coincide.

While from the ℓ-adic cohomology approach it is not so clear how to compute the Green functions explicitly, the definition via character sheaves leads to an algorithm to compute the (generalized) Green functions, see [Lus86].

More precisely, this algorithm determines the Green functions as linear combination of certain functions which are supported on elements C^F for a

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single unipotent class C in $\mathbf{G}$. The values of these functions are clear up to a normalization by some scalar in $\mathbb{C}$ of absolute value 1.

Finding these scalars is a non-trivial task. Computing the (generalized) Green functions for general $\mathbf{G}$ can be reduced to the case of simple simply-connected groups. All cases of groups of classical types were systematically considered by Shoji [Sho06, Sho07, Sho22] (also generalized Green functions), the cases of small rank exceptional groups can be read off from the known character tables, Green functions of groups of type E_6, E_7 and E_8 in good characteristic were considered by Beynon and Spaltenstein [BS85], and various exceptional groups in small characteristic by Malle [Mal93], Porsch [Por94] and more recently Geck [Gec20a]. In this paper we describe a method that enabled us to also handle the only case which was left open so far, that is the groups of type E_8 in bad characteristic.

So, with the results of this paper the Green functions are known in all cases. It only remains to consider the other generalized Green functions in groups of exceptional types.

Our method is a variant of the idea in [Gec20a] where the permutation character of the Borel subgroup in $G(q)$ and computations in a matrix representation where used. Here, we use more general parabolic subgroups and compute with the Steinberg presentation of the considered groups.

2 Notations

For more details on the following basic setup we refer to the introductory sections of the text books [Car93, DM20].

Let $\mathbf{G}$ be a connected reductive group over an algebraic closure $\mathbf{k}$ of a finite field $\mathbb{F}_q$ with q elements in characteristic p. We assume that $\mathbf{G}$ is defined over $\mathbb{F}_q$ and call F the corresponding Frobenius endomorphism of $\mathbf{G}$. We write $G(q) := \mathbf{G}^F$ for the finite group of F-fixed points.

Let $\mathbf{T} \subset \mathbf{G}$ be an F-stable maximal torus of $\mathbf{G}$ that is contained in an F-stable Borel subgroup $\mathbf{B} \subseteq \mathbf{G}$. The group $\mathbf{G}$ is determined up to isomorphism by its root datum with respect to $\mathbf{T}$. The minimal unipotent subgroups X_r normalized by $\mathbf{T}$ are called root subgroups. There is an isomorphism $\mathbf{k}^+ \to X_r$, $a \mapsto x_r(a)$. The root $r : \mathbf{T} \to k^\times$ is an element of the (additive) character group $X(\mathbf{T})$ and decribes the conjugation: $tx_r(a)t^{-1} = x_r(r(t)a)$. We write Φ for the finite set of roots, then we have $\Phi = \Phi^+ \dot\cup \Phi^-$ where a root r lies in Φ^+ when $X_r \subseteq \mathbf{B}$; these subsets are called positive roots and negative roots, respectively, and we have $\Phi^- = -\Phi^+$. The positive roots contain a unique subset Δ such that every positive root is a unique non-negative linear combination of the roots in Δ; the elements of Δ are called simple roots. The height $\mathrm{ht}(r)$ of a positive root $r \in \Phi^+$ is the sum of its coefficients when r is written as linear combination of the roots in Δ.

The group $\mathbf{G}$ is generated by $\mathbf{T}$ and all root subgroups X_r, $r \in \Phi$. The Borel subgroup $\mathbf{B}$ is generated by $\mathbf{T}$ and the X_r with $r \in \Phi^+$. The group $\mathbf{U} = \prod_{r \in \Phi^+} X_r$ is a maximal unipotent subgroup and the unipotent radical of $\mathbf{B}$. We will describe $\mathbf{U}$ in more detail in section 3.

The group $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ is called the Weyl group of $\mathbf{G}$, it is a Coxeter group with Coxeter generators $S := \{s_r \mid r \in \Delta\}$, where s_r is the unique non-trivial coset of $\mathbf{T}$ in $N_{\mathbf{G}}(T) \cap \langle X_r, X_{-r}\rangle$. The length $l(w)$ of an element $w \in W$ is the smallest integer k such that $w = s_1 \cdots s_k$ with $s_i \in S$, the sequence $s_1, \ldots, s_k$ is called a reduced word for w. An element $w \in W$ permutes the roots $r \in \Phi$ where $w(r)$ is defined by $wX_r w^{-1} = X_{w(r)}$. For $r \in \Delta$ we have $s_r(r) = -r$ and $s_r(\Phi^+ \setminus \{r\}) \subset \Phi^+$. So, for $w \in W$ and $r \in \Delta$ we have $l(ws_r) < l(w)$ (and in that case $l(ws_r) = l(w) - 1$) if and only if $w(r) \in \Phi^-$.

The Frobenius endomorphism F restricts to Frobenius endomorphisms of $\mathbf{B}$, $\mathbf{T}$ and $\mathbf{U}$ and induces a natural map on $X(\mathbf{T})$ and an automorphism of W which permutes the set S of generators. We will write F also for the induced maps, and $H(q) = \mathbf{H}^F$ for F-stable subgroups $\mathbf{H}$.

For $w \in W$ let $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ be a representative. Using the Lang-Steinberg theorem we can find a $g \in \mathbf{G}$ with $gF(g^{-1}) = \dot{w}$. Then $\mathbf{T}_w := \mathbf{T}^g$ is also an F-stable maximal torus of $\mathbf{G}$ (well defined up to $G(q)$-conjugacy). Let $M = \{w_1, \ldots, w_k\}$ be a set of representatives for the F-conjugacy classes of W (where $w, w' \in W$ are F-conjugate if there is a $z \in W$ such that $w' = zwF(z^{-1})$), then $\{\mathbf{T}_{w_1}, \ldots, \mathbf{T}_{w_k}\}$ is a set of representatives for the $G(q)$-conjugacy classes of F-stable maximal tori.

We will consider the Bruhat decomposition of $\mathbf{G}$ as union of disjoint double cosets with respect to the Borel subgroup $\mathbf{B}$:

$$\mathbf{G} = \dot{\bigcup_{w \in W}} \mathbf{B}\dot{w}\mathbf{B},$$

where $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ is again a representative of w. We will use a stronger form of this Bruhat decomposition that says, that an element $g \in \mathbf{B}\dot{w}\mathbf{B}$ can be uniquely written as product $g = ut\dot{w}u'$ with $u \in \mathbf{U}$, $t \in \mathbf{T}$, $u' \in \mathbf{U}_w = \prod_{r \in \Phi^+, w(r) \in \Phi^-} X_r$.

When F acts trivially on Φ and W we can choose the $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})^F$ and have the same decomposition for the corresponding finite groups:

$$G(q) = \dot{\bigcup_{w \in W}} U(q)T(q)\dot{w}U_w(q).$$

We will need one basic property of groups with a BN-pair which have such a Bruhat decomposition: For $w \in W$ and $s \in S$ we have

$$(BsB)(BwB) \begin{cases} = B(sw)B, & \text{if } l(sw) > l(w) \\ \subseteq (BwB) \cup (B(sw)B), & \textit{else.} \end{cases}$$

3 Computing in the unipotent subgroup

We want to do certain computations with elements in $\mathbf{G}$ or $G(q)$ where the elements are represented in the strong form of the Bruhat decomposition. For this one first needs to fix the isomorphisms $k^+ \to X_r$, $t \mapsto x_r(t)$ and the representatives $\dot{w}$ of Weyl group elements. And for the multiplication one needs to know commutator rules for the factors appearing in Bruhat expressions of the form $g = ut\dot{w}u'$. These are provided by the relations in the Steinberg presentation of the group $\mathbf{G}$ or $G(q)$, see for example [Spr98, Ch.9].

It turns out that for our present applications we only need detailed computations in the unipotent subgroup $\mathbf{U}$. This needs the most complicated part of the Steinberg presentation and depends on certain choices (leading to isomorphic groups). We follow the construction of $\mathbf{G}$ as a Chevalley group by Carter in [Car72, Ch.4-8]. This construction starts with a semisimple Lie algebra $\mathcal{L}$ over $\mathbb{C}$ which has the same type of root system as $\mathbf{G}$. It uses a Chevalley basis of $\mathcal{L}$ consisting of elements h_r, $r \in \Delta$, spanning a Cartan subalgebra, and of root vectors e_r, $r \in \Phi$. The e_r are acting as nilpotent linear maps on $\mathcal{L}$ and can be used to construct the elements in the root subgroups:

$$x_r(a) := \exp(a \cdot e_r) \quad \text{for } r \in \Phi,\ a \in \mathbb{C}.$$

The Chevalley bases have the property that all its structure constants (and so the entries of matrices of e_r acting on $\mathcal{L}$ with respect to the Chevalley basis) are in $\mathbb{Z}$. So, for each p (prime or 0) we can reduce the matrix entries modulo p and use the definition of $x_r(a)$ for any element a in a field of characteristic p.

3.1 Different Chevalley bases

We use Carter's description in [Car72, 4.2]. Let $\mathcal{B} = \{h_s, e_r \mid\ s \in \Delta, r \in \Phi\}$ and $\mathcal{B}' = \{h'_s, e'_r \mid\ s \in \Delta, r \in \Phi\}$ be two Chevalley bases. Then there is a unique automorphism of $\mathcal{L}$ with $h_s \mapsto h'_s$ for $s \in \Delta$, $e_r \mapsto \lambda_r e'_r$ for $r \in \Phi$ with $\lambda_r = 1$ for $r \in \Delta$, $\lambda_{-r} = \lambda_r \in \{\pm 1\}$ for all $r \in \Phi$.

The possibilities for the signs can be described as follows, relative to an enumeration $\Delta = \{r_1, \ldots, r_l\}$ of the simple roots. Assign to each root $r \in \Phi^+ \setminus \Delta$ a sign ϵ_r by writing $r = \tilde{r} + r_i$ with $\tilde{r} \in \Phi^+$ and $r_i \in \Delta$ for the largest possible i (Carter calls $(\tilde{r}, r_i)$ an extraspecial pair), then $[e_{\tilde{r}}, e_{r_i}] = N_{\tilde{r},r_i} e_r \neq 0$ and ϵ_r is the sign of $N_{\tilde{r},r_i}$ (this is up to sign the same number for any Chevalley basis). If we also know the signs ϵ'_r for $r \in \Phi^+$, defined in the same way with respect to the basis $\mathcal{B}'$, we can consider all roots $r \in \Phi^+$ by increasing height and determine if the isomorphism above maps e_r to e'_r or $-e'_r$: We have for $r \in \Phi^+ \setminus \Delta$, $r = \tilde{r} + r_i$ as above,

$$\lambda_r = \lambda_{\tilde{r}} \epsilon_r \epsilon'_r.$$

This describes the diagonal transition matrix between the two Chevalley bases and also determines an isomorphism between the groups generated by the root subgroups X_r or X'_r, constructed with respect to the different bases. It maps $x_r(a) \mapsto x'_r(\lambda_r a)$ for all $r \in \Phi$.

For any choice of signs for the extraspecial pairs there exists a corresponding Chevalley basis.

We mention that Geck [Gec17] described an (up to a global sign) canonical choice of a Chevalley basis that does not depend on certain choices of signs.

3.2 Commutator formula

In [Car72, 4.2.2] there is a detailed description of how to compute for a given root system and given signs for extraspecial pairs all (uniquely determined) structure constants of the corresponding Chevalley basis of the Lie algebra $\mathcal{L}$.

Furthermore, using the formulae in [Car72, 4.3.1,5.2.2] we can compute for any two different roots $r_1, r_2 \in \Phi^+$ and any $i, j \in \mathbb{Z}_{>0}$ such that $ir_1 + jr_2 \in \Phi^+$ an integer $C_{ijr_1r_2}$. These enable us to compute with elements in the group $U = \langle X_r \mid\ r \in \Phi^+\rangle$ using the following commutator formula [Car72, 5.2.3]:

$$x_{r_2}(a_2)x_{r_1}(a_1) = x_{r_1}(a_1)x_{r_2}(a_2)\prod_{i,j} x_{ir_1+jr_2}(C_{ijr_1r_2}(-a_1)^i a_2^j),$$

where the product is over all $i, j \in \mathbb{Z}_{>0}$ with $ir_1 + jr_2 \in \Phi$ sorted by increasing $i + j$ (factors with the same $i + j$ always commute).

Proposition 3.1. *Let $r_1, r_2, \ldots r_N$ be the positive roots of $\mathbf{G}$ in any fixed order, we write $x_i(a) := x_{r_i}(a)$ for the corresponding root elements.*

(a) Any element of U can be uniquely written in the form

$$x_1(a_1)x_2(a_2)\cdots x_N(a_N).$$

(b) Any product of root elements $x_{t_1}(b_1)\cdots x_{t_k}(b_k)$ for positive roots $r_{t_1}, \ldots, r_{t_k}$ can be rewritten to the form in (a) (where some factors $x_i(0) = 1$ may be omitted) by applying a finite number of the following steps to any pair of consecutive factors $x_{t_i}(b_i)x_{t_{i+1}}(b_{i+1})$ with $t_{i+1} \le t_i$:

- *If $t_i = t_{i+1}$ simplify to one factor $x_{t_i}(b_i + b_{i+1})$.*
- *Otherwise substitute the two factors according to the commutator formula.*

In practice it works best to handle pairs with commuting factors first.

(c) *Let $\prec$ be the partial ordering on the set of positive roots with $r_i \prec r_j$ if and only if $r_j - r_i$ is a non-zero non-negative linear combination of positive roots. Let r_i be a positive root, and*

$$u = x_1(a_1)x_2(a_2)\cdots x_N(a_N)x_i(a).$$

After reordering factors as in (b) we get a product

$$u = x_1(b_1)x_2(b_2)\cdots x_N(b_N).$$

Then we have $b_i = a_i + a$ and $b_j = a_j$ whenever $r_i \not\prec r_j$.

Proof. To show (b) we have to show that the process terminates. For this we count for each positive integer m the number of pairs (t_i, t_j) with $i < j$, $t_i > t_j$ and $\mathrm{ht}(r_{t_1}) + \mathrm{ht}(r_{t_j}) = m$. It is clear that after the first type of substitution in (b) none of these counts will be larger than before. The substitution using the commutator formula will reduce the count for $m = \mathrm{ht}(r_{t_i}) + \mathrm{ht}(r_{t_{i+1}})$ by one and maybe enlarge the counts for some larger m. Since there is an upper bound for the height of all roots, all counts will be zero after a finite number of steps, so the factors are sorted.

Statement (c) follows by applying (b) to the given u and noticing that the commutator formula applied to $x_r(a)x_{r'}(a')$ only introduces new factors $x_{r''}(a'')$ with $r \prec r''$ and $r' \prec r''$.

Part (b) yields a constructive proof of the existence of the form in (a). The uniqueness of the factorization is shown in [Car72, 5.3.3] in the case that the ordering of the roots refines the ordering by height. The general case follows by induction over the lowest height of roots r_i with non-trivial factor $x_i(b_i)$; as in (c) those factors will not change when moved to the left. $\square$

4 Green functions by the Lusztig-Shoji algorithm

Deligne and Lusztig [DL76] defined for each F-stable maximal torus $\mathbf{T}$ and each irreducible character θ of the abelian group $T(q)$ a generalized character $R^{\mathbf{G}}_{\mathbf{T},\theta}$ of $G(q)$. Their values on unipotent elements depend only on the $G(q)$-conjugacy class of $\mathbf{T}$ and not on θ. The restrictions $Q^{\mathbf{G}}_{\mathbf{T}}$ to unipotent elements are called the (ordinary) Green functions of $G(q)$.

As in section 2 let $\{\mathbf{T}_{w_1}, \ldots, \mathbf{T}_{w_k}\}$ be a set of representatives of the $G(q)$-conjugacy classes of maximal tori, where $\{w_1, \ldots, w_k\} \subset W$ are representatives of the F-conjugacy classes of W. We write $Q_{w_i} := Q^{\mathbf{G}}_{\mathbf{T}_{w_i}}$.

Let $\chi \in \mathrm{Irr}(W)$ be an F-stable irreducible character of W. Writing F also for the automorphism of W induced by the Frobenius endomorphism, we consider the semidirect product $W \rtimes \langle F \rangle$; the W-conjugacy classes in the coset WF are in bijection with the F-conjugacy classes of W. The character χ can be extended to an irreducible character $\tilde{\chi}$ of $W \rtimes \langle F \rangle$. For any character $\tilde{\chi}$ of $W \rtimes \langle F \rangle$ we will consider the linear combination

$$Q_{\tilde{\chi}} := \sum_{i=1}^{k} \frac{1}{|C_W(w_i F)|} \tilde{\chi}(w_i F) Q_{w_i}.$$

Let C be an F-stable unipotent class of $\mathbf{G}$ and $u \in C^F$. The Lang-Steinberg theorem shows that the $G(q)$-conjugacy classes in C^F are parameterized by the F-conjugacy classes of the component group $A(u) = C_{\mathbf{G}}(u)/C^0_{\mathbf{G}}(u)$ (the centralizer of u in $\mathbf{G}$ modulo its connected component), or the $A(u)$-conjugacy classes in the coset $A(u)F \subseteq A(u) \rtimes \langle F \rangle$ where now F denotes the automorphism on $A(u)$ induced by the Frobenius endomorphism. We write u_a for an element in the $G(q)$-class of C^F corresponding to $a \in A(u)$. An F-stable irreducible character $\rho \in \mathrm{Irr}(A(u))$ can be extended to an irreducible character $\tilde{\rho}$ of $A(u) \rtimes \langle F \rangle$. We consider the following class function on $G(q)$:

$$Y_{u,\tilde{\rho}}(g) := \begin{cases} \tilde{\rho}(aF), & \text{if } g \text{ is conjugate to } u_a \\ 0, & \text{else.} \end{cases}$$

Using the theory of character sheaves Lusztig [Lus86, 24.] described an algorithm to write another set of class functions, also denoted $Q_{\tilde{\chi}}$, as linear combinations of the functions $\zeta_{u,\tilde{\rho}} Y_{u,\tilde{\rho}}$, where $\zeta_{u,\tilde{\rho}} \in \mathbb{C}$ are scalars of absolute value 1 which are not determined by the algorithm. The main input of the algorithm is the Springer correspondence which is an injective map from $\mathrm{Irr}(W)$ to the set of pairs (u, ρ) modulo $\mathbf{G}$-conjugacy. Further data which are needed are the dimensions of the unipotent classes of $\mathbf{G}$ and the F-character table of the Weyl group.

Later, Lusztig showed in [Lus90] that the two types of class functions denoted $Q_{\tilde{\chi}}$ coincide under some conditions on q, and Shoji showed in [Sho95] that the same holds without restrictions on q.

For much more detailed descriptions of this setup we refer to [Lus86, Sho06, Gec20b, Gec20a].

The following proposition reduces the determination of the $\zeta_{u,\rho}$ for untwisted groups to the special case $q = p$, that is the groups $G(p)$ defined over the prime field. It is a special case of [Gec20b, Thm. 3.7].

Proposition 4.1. *Assume that the action of F on Φ is trivial. Let C be an F-stable unipotent class with representative $u \in C^F = C \cap G(q)$. Assume that F acts trivially on $A(u)$ (so that we have $\tilde{\rho} = \rho$ in the discussion above). Let $\rho \in Irr(A(u))$ appear in the Springer correspondence and $\zeta_{u,\rho}$ be the associated scalar. Then we have for any positive integer m that u is also stable under F^m. Write $\zeta^{(m)}_{u,\rho}$ for the scalar associated to (u, ρ) when u is considered as element of $G^{F^m} = G(q^m)$. Then we have $\zeta^{(m)}_{u,\rho} = (\zeta_{u,\rho})^m$.*

What remains to find the Green functions Q_w (or equivalently the $Q_{\tilde{\chi}}$ for F-stable irreducible χ) as class functions is to choose representatives u of unipotent classes and the characters $\tilde{\rho}$ and then to determine the scalars $\zeta_{u,\tilde{\rho}}$. As mentioned in Section 1 this task has been done in almost all cases. We describe a method which enabled us to handle the remaining cases of groups of type $E_8(q)$ in bad characteristic 2, 3 and 5.

For computations we used our own implementation of the Springer correspondence and the Lusztig-Shoji algorithm, but all that is needed is also available in Michel's version of CHEVIE [Mic15].

The following additional information is also useful, see [Lus86, 24.].

Remark 4.2. The Lusztig-Shoji algorithm also returns the matrix Λ whose rows and columns are labeled by the pairs $(u, \tilde{\rho})$ as above and the entry in position $(u, \tilde{\rho})$, $(u', \tilde{\rho}')$ is

$$\sum_{v \in G(q) \text{ unipotent}} \zeta_{u,\tilde{\rho}} Y_{u,\tilde{\rho}}(v) \overline{\zeta_{u',\tilde{\rho}'} Y_{u',\tilde{\rho}'}(v)}.$$

This matrix is block diagonal (one block for each unipotent F-stable class in $\mathbf{G}$), it is symmetric and has values in the rational numbers.

From this we can always determine the sizes of the $G(q)$-classes in C^F for each class: If for a class C there are k classes in C^F and we write Y_C for the $k \times k$ matrix of the values of the $\zeta_{u,\tilde{\rho}} Y_{u,\tilde{\rho}}$, and if Λ_C is the corresponding diagonal block of Λ, then $Y_C \Lambda_C \bar{Y_C}^t$ is a diagonal matrix where the diagonal entries are the class lengths.

These properties sometimes yield restrictions on the values of the scalars $\zeta_{u,\tilde{\rho}}$.

5 Permutation characters of parabolic subgroups

For background about parabolic subgroups we refer to [DM20, Ch.3].

We want to consider standard parabolic subgroups of $G(q)$. These are parameterized by F-stable subsets $J \subseteq \Delta$ of the simple roots. The subgroup $W_J \leq W$ generated by the set $S_J := \{s_r \mid r \in J\}$ is also a Coxeter group with S_J as set of Coxeter generators. The set $\mathbf{P}_J = \bigcup_{w \in W_J} \mathbf{B}\dot{w}\mathbf{B}$ is a subgroup of $\mathbf{G}$ and is called a standard parabolic subgroup of $\mathbf{G}$, and similarly for the finite groups, that is $P_J(q) = \bigcup_{w \in W_J} B(q)\dot{w}U_w(q) \leq G(q)$.

The subset $\Phi_J := W_J(J) \leq \Phi$ is also a root system and the parabolic subgroup has a Levi decomposition $\mathbf{P}_J = \mathbf{L}_J\mathbf{U}_J$, where $\mathbf{U}_J$ is the unipotent radical, generated by the X_r with $r \in \Phi^+ \setminus \Phi_J$, and $\mathbf{L}_J$ is a Levi complement, it is generated by $\mathbf{T}$ and the root subgroups X_r with $r \in \Phi_J$.

5.1 Permutation characters by Deligne-Lusztig characters

We show how to find the values of permutation characters of parabolic subgroups as linear combinations of Deligne-Lusztig characters $R_{\mathbf{T}_w,1}^{\mathbf{G}}$.

Recall that we have a set of representatives $\{w_1, \ldots, w_k\}$ of the F-conjugacy classes of W and that the Q_{w_i} are the corresponding Green functions.

First we mention that the trivial character $1_{G(q)} \in \mathrm{Irr}(G(q))$ is a linear combination of Deligne-Lusztig characters [DM20, 10.2.5]:

$$1_{G(q)} = \sum_{i=1}^{k} \frac{1}{|C_W(w_i F)|} R_{\mathbf{T}_{w_i},1}.$$

So, its restriction to unipotent elements is $Q_{\bar{\chi}}$ for the trivial character $\bar{\chi}$ on WF.

The trivial character on a parabolic subgroup $P_J(q)$ is the inflation of the trivial character of $L_J(q)$ to $P_J(q)$ via the canonical map $P_J(q) \to L_J(q)$. Therefore, the permutation character of $P_J(q) \leq G(q)$ is the Harish-Chandra induction of the trivial character on $L_J(q)$ which is a special case of Lusztig induction $R_{\mathbf{L}_J}^{\mathbf{G}}$, see [DM20, 5.]. Let $\{v_1, \ldots, v_m\}$ be representatives of the F-conjugacy classes in W_J. Then we have $1_{L_J(q)} = \sum_{j=1}^{m} (1/|C_{W_J}(v_j F)|) R_{\mathbf{T}_{v_j},1}^{\mathbf{L}_J}$. Using the transitivity of Lusztig induction [DM20, 9.1.8] we get

$${1_{P_J(q)}}^{G(q)} = R_{\mathbf{L}_J}^{\mathbf{G}}(\sum_{j=1}^{m} \frac{1}{|C_{W_J}(v_j F)|} R_{\mathbf{T}_{v_j},1}^{\mathbf{L}_J}) = \sum_{j=1}^{m} (\frac{1}{|C_{W_J}(v_j F)|} R_{\mathbf{T}_{v_j},1}^{\mathbf{G}}).$$

This shows that the restriction to unipotent elements is $Q_{\bar{\chi}}$ where $\bar{\chi}$ is the permutation character of WF on $W_J F \subseteq WF$.

Given all Q_{w_i}, where the values are written with the so far unknown scalars $\zeta_{u,\tilde{\rho}}$ as independent indeterminates, we can now compute the permutation characters of $G(q)$ on $P_J(q)$ for various J.

Remark 5.1. We get constraints on the possible values of the scalars $\zeta_{u,\tilde{\rho}}$ from general facts about permutation characters. More precisely:

(a) The case of the trivial character of $G(q)$ (the special case $J = \Delta$, $P(q) = G(q)$) yields all $\zeta_{u,\tilde{1}}$.

(b) The values of permutation characters are non-negative rational integers. From this we often see that $\zeta_{u,\tilde{\rho}} \in \mathbb{Q}$, and so $\zeta_{u,\tilde{\rho}} \in \{\pm 1\}$.

(c) If only one of the remaining possibilities for one or several $\zeta_{u,\tilde{\rho}}$ leads to non-negative values then the scalars are determined.

(d) The Green function Q_w for the maximally split torus is the permutation character ${1_{B(q)}}^{G(q)}$. From its values we can find the sizes of the intersections of the unipotent classes in $G(q)$ with $B(q)$ or $U(q)$: For $u \in G(q)$ unipotent we have

$$\begin{aligned} {1_{B(q)}}^{G(q)}(u) &= \tfrac{1}{|B(q)|}\,|\{g \in G(q) \mid \ u^g \in U(q)\}| \\ &= |u^{G(q)} \cap B(q)|\,|C_{G(q)}(u)|\,/\,|B(q)|. \end{aligned}$$

All these values must be positive integers.

We can also use the general facts about Green functions given in [Car93, 7.6]:

(e) The values of the Green function Q_{w_i} are rational integers.

(f) We have $\sum_{u \in G(q) \text{ unipotent}} Q_{w_i}(u) = \frac{G(q)}{T_{w_i}(q)}$.

We could also mention the orthogonality relations for Green functions, but in our applications they never provided useful information.

5.2 Permutation characters by counting fixed points

To find additional equations for the scalars $\zeta_{u,\tilde{\rho}}$ we compute some values of permutation characters by counting fixed points. Note that for any finite group G and subgroup $H \leq G$ the value of the permutation character on the right cosets of H for $g \in G$ is equal to

$${1_H}^G(g) = |\{Hx|\; x \in G, Hx = Hxg\}|.$$

To apply this to parabolic subgroups we need a set of representatives of the right cosets of $P_J(q) \leq G(q)$.

Proposition 5.2. *Let $J \subseteq \Delta$ be a subset of the simple roots of $\mathbf{G}$ and $\mathbf{P}_J$ be the corresponding standard parabolic subgroup and W_J the parabolic subgroup of the Weyl group generated by J. We say that $w \in W$ is J-reduced if $l(s_r w) > l(w)$ for all $r \in J$ (that is w is the shortest possible representative of the coset $W_J w \subseteq W$).*

(a) A set of right coset representatives of $\mathbf{P}_J \setminus \mathbf{G}$ is given by

$$\{\dot{w} \prod_{r \in \Phi^+,\, w(r) \in \Phi^-} x_r(a_r)|\; w \text{ is } J\text{-reduced, } a_r \in k\}.$$

(b) When F acts trivially on Φ and W, then a set of right coset representatives of $P_J(q) \setminus G(q)$ is given by

$$\{\dot{w} \prod_{r \in \Phi^+,\, w(r) \in \Phi^-} x_r(a_r)|\; w \text{ is } J\text{-reduced, } a_r \in \mathbb{F}_q\}.$$

The factors in the products are taken in any fixed order.

Part (b) can be generalized to twisted groups, but the statement becomes more technical. We do not need this in the remainder of this article and omit it.

Proof. Note that for $J = \{\}$, $\mathbf{P}_J = \mathbf{B}$ the Borel subgroup, this is just a reinterpretation of the strong form of the Bruhat decomposition. We consider general J.

Recall that $\mathbf{P}_J$ is generated by the $\mathbf{B}\dot{s}_r\mathbf{B}$ with $r \in J$. Let $g' \in \mathbf{G}$ and $w' \in W$ such that $g' \in \mathbf{B}\dot{w}'\mathbf{B}$. When w' is not J-reduced, there is $r \in J$ with $l(s_r w) < l(w)$ ($\Leftrightarrow w'^{-1}(r) \in \Phi^-$ by [Car72, 2.2.1]). We set $w = s_r w'$ and have $(\mathbf{B}\dot{s}_r\mathbf{B})(\mathbf{B}\dot{w}\mathbf{B}) = \mathbf{B}\dot{w}'\mathbf{B}$. This shows that the coset $\mathbf{P}g' = \mathbf{P}g$ for some $g \in \mathbf{B}\dot{w}\mathbf{B}$. Since the length of w is smaller than the length of w' we will find after a finite number of applications of this step g and w such that w is J-reduced. Since $\mathbf{B} \leq \mathbf{P}_J$ we see from the strong form of the Bruhat decomposition that the elements given in (a) contain representatives of all right cosets in $\mathbf{P}_J \setminus \mathbf{G}$.

The multiplication rule for $(\mathbf{B}\dot{s}_r\mathbf{B})(\mathbf{B}\dot{w}\mathbf{B})$ shows that for $g \in (\mathbf{B}\dot{w}\mathbf{B})$, $g' \in (\mathbf{B}\dot{w}'\mathbf{B})$ with $(\mathbf{P}_J)g = (\mathbf{P}_J)g'$ we have $(W_J)w = (W_J)w'$. Furthermore, for any $w' \in W$ there is exactly one J-reduced $w \in W$ with $(W_J)w = (W_J)w'$ (the J-reduced w maps Φ_J^+ to itself and only the trivial element of W_J has this property). This shows that any $g' \in \mathbf{G}$ determines a unique J-reduced w such that $(\mathbf{P}_J)g = (\mathbf{P}_J)g'$ for some g of the form $g = \dot{w} \prod_{r\in\Phi^+,\, w(r)\in\Phi^-} x_r(a_r)$.

Finally, let $w \in W$ be J-reduced, $u, u' \in U_w$, and $(\mathbf{P}_J)\dot{w}u = (\mathbf{P}_J)\dot{w}u'$. The last condition is equivalent to $\dot{w}uu'^{-1}\dot{w}^{-1} \in \mathbf{P}_J$. We have $uu'^{-1} \in U_w$ and $\dot{w}U_w\dot{w}^{-1} = \prod_{r\in\Phi^+,\, w(r)\in\Phi^-} X_{w(r)}$. Since w is J-reduced, so $w(\Phi_J^+) = \Phi_J^+$, we see $\dot{w}U_w\dot{w}^{-1} \cap \mathbf{P}_J = \{1\}$. Hence $uu'^{-1} = 1$ and $u = u'$ and we conclude that the set of elements in (a) contains a unique representative for each right coset of $\mathbf{P}_J$ in $\mathbf{G}$.

The proof of (b) is the same using the corresponding finite groups instead of $\mathbf{G}$, $\mathbf{B}$, $\mathbf{P}_J$ and X_r. □

5.3 Computing a character value

Let

$$v = x_{j_1}(c_1)\cdots x_{j_k}(c_k) \in U(q),$$

with $c_1, \ldots, c_k \in \mathbb{F}_q \setminus \{0\}$ be a unipotent element for which we want to find the value of the permutation character ${1_{P_J(q)}}^{G(q)}(v)$.

For each J-reduced $w \in W$ we fix an ordering $r_1, \ldots, r_N$ of the positive roots such that for some l we have $w(r_i) \in \Phi^+$ for $1 \leq i \leq l$ and $w(r_i) \in \Phi^-$ for $l+1 \leq i \leq N$. We consider all cosets represented by

$$\dot{w}x_{l+1}(a_{l+1})\cdots x_N(a_N), \text{ with } a_i \in \mathbb{F}_q$$

at once. For the computation we use independent indeterminates $y_{l+1}, \dots y_N$ over $\mathbb{F}_q$ (or even over $\mathbb{Z}$) instead of the a_i in the expression above. We have to count for how many specializations $a_i \in \mathbb{F}_q$ of the y_i the elements

$$g_1 = \dot{w} x_{l+1}(a_{l+1}) \cdots x_N(a_N) \text{ and } g_2 = g_1 v$$

are in the same right coset of $P_J(q)$. We apply the reordering algorithm in Proposition 3.1(b) to rewrite

$$\dot{w}\ x_{l+1}(y_{l+1}) \cdots x_N(y_N) x_{j_1}(c_1) \cdots x_{j_k}(c_k)$$

in the form

$$\dot{w}\ x_1(b_1) \cdots x_l(b_l) x_{l+1}(b_{l+1}) \cdots x_N(b_N),$$

where we get $b_1, \dots, b_N$ as polynomials in the indeterminates $y_{l+1}, \dots, y_N$.

The chosen ordering of the positive roots yields that any specialization of the indeterminates in

$$\dot{w}\ x_1(b_1) \cdots x_l(b_l) \dot{w}^{-1}$$

in $\mathbb{F}_q$ yields an element in $U(q) \subset P_J(q)$.

So, we want to count the tuples $(a_{l+1}, \dots, a_N) \in \mathbb{F}_q^{N-l}$ such that specializing the y_i by a_i in the expressions

$$\dot{w}\ x_{l+1}(y_{l+1}) \cdots x_N(y_N) \text{ and } \dot{w}\ x_{l+1}(b_{l+1}) \cdots x_N(b_N)$$

yields representatives of the same right coset of $P_J(q)$.

Using Proposition 5.2(b) this translates to counting the solutions in $\mathbb{F}_q^{N-l}$ of the system of polynomial equations

$$y_{l+1} - b_{l+1} = 0, \dots, y_N - b_N = 0.$$

5.4 Counting solutions

We have reduced the computation of values of permutation characters to counting solutions over $\mathbb{F}_q$ of systems of multivariate polynomial equations over $\mathbb{F}_q$. In general, such counting is a difficult task. In principle one could use Gröbner bases with respect to some elimination monomial ordering. But computing Gröbner bases is itself a difficult problem. We mention here some heuristics which often solve the problem for systems of equations that occur in the context of the previous subsection.

- For small q we can reduce the degrees of the polynomials by the substitution $y = y^q$ for any indeterminate y (since $a^q - a = 0$ for all $a \in \mathbb{F}_q$).

- It happens quite often that we have an equation of the form $y = f$ with f a polynomial in indeterminates different from y. We use this to substitute y accordingly in all equations, and hence eliminate y from the system. We prefer cases where f has few terms, and we fix an upper bound for the amount of memory to use and stop substitutions when the limit is reached.

- There are many examples where the previous steps lead to an equation $c = 0$ for a non-zero constant c (so, there are no solutions) or that we end up with no equation (so, the number of solutions is q^k when there are k independent variables left).

- Otherwise, we use a straight forward backtrack search: Specialize one variable to all possible values and solve for each value recursively the resulting system of equations in the remaining variables.

5.5 Detecting cases with no solutions

In the setup of Section 5.3 we can often detect quickly that the system of polynomial equations we get for a J-reduced $w \in W$ has no solutions.

We consider the roots $\Psi' = \{r_{j_1}, \ldots, r_{j_k}\}$ corresponding to the non-trivial root element factors of the element v for which we want to compute the value of a permutation character. We determine the subset $\Psi \subseteq \Psi'$ consisting of r which correspond to only one factor in the given product v and such that $r' \not\prec r$ for all $r' \in \Psi' \setminus \{r\}$.

Now let $w \in W$ be J-reduced and assume that $w(r) \in \Phi^-$ for some $r \in \Psi$. Let $r_i = r$ for some $l + 1 \leq i \leq N$ and $x_{r_i}(c)$ (with $c \neq 0$) be the corresponding factor in v. Then we see from Proposition 3.1(c) that our method leads to an equation $y_i = y_i + c$ causing that there are no solutions.

So, we can skip our computation for the double coset of w whenever there is at least one $r \in \Psi$ such that $w(r)$ is negative.

6 Application to $\mathbf{E_8(q)}$

As an application we want to determine the (ordinary) Green functions for the exceptional groups of type $E_8(q)$ in bad characteristic $p = 2, 3, 5$. For good characteristic $p > 5$ this problem was solved by Beynon and Spaltenstein in [BS85]. In Section 4 we have explained that we need to find appropriate class representatives of the unipotent classes in $\mathbf{G}$ and certain associated complex scalars $\zeta_{u,\bar{\rho}}$ of absolute value 1. The following theorem summarizes the result.

Theorem 6.1. *With one exception we can find in each unipotent conjugacy class C of $E_8(\bar{\mathbb{F}}_q)$ an element $u \in C^F$ such that F acts trivially on $A(u)$ and such that we have for all associated scalars $\zeta_{u,\rho} = 1$.*

The exception is the class C labeled $D_8(a_3)$ when $q \equiv -1(3)$ (so $p \neq 3$). In this case there is a unique G^F-class of $u \in C^F$ such that F acts trivially on $A(u) \cong S_3$. Then $\zeta_{u,\rho} = 1$ except when $\rho = -1$ is the sign character of S_3 where $\zeta_{u,-1} = -1$.

The starting point of the proof is Spaltenstein's table in [Spa85] which describes the generalized Springer correspondence for groups of type E_8 (the description is slightly different in the cases $p = 2, 3, 5, > 5$). The table has labels for the unipotent classes in the algebraic group and also gives the dimensions of the classes and the isomorphism types of the component groups $A(u)$.

The unipotent conjugacy classes were determined by Mizuno [Miz80] and Spaltenstein used the labeling of Mizuno. We will use some explicit class representatives found by Mizuno. Some care is needed when using Mizuno's tables because they contain (very few) errors. An important correction of two $A(u)$ in characteristic 2 is mentioned in [Spa85, 5.5]. For some explicit computations in the groups we want to use elements given in Mizuno's paper. Therefore, we use the same structure constants as Mizuno and read off the extraspecial signs he used (there are some errors in the table of structure constants in the paper, but we recompute everything from the signs for extraspecial pairs.)

We can conclude from Mizuno's results that all unipotent classes contain an F-stable element u such that F acts trivially on $A(u)$: Otherwise there would be a class C such that F acts on $A(u)$ as a non-inner automorphism for any $u \in C^F$. All groups occuring as $A(u)$ have the easy to check property that the number of ϕ-conjugacy classes for any non-inner automorphism ϕ is smaller than the number of conjugacy classes. So, for some power F^m of F such that F^m acts trivially on all $A(u)$ the group G^{F^m} would have more unipotent classes than G^F. But Mizuno showed that the number of unipotent classes only depends on the characteristic p.

From now on we assume that $p = 2$. The cases $p = 3, 5$ must be considered separately, but the arguments to obtain our result are the same (and sometimes a bit easier because there are fewer unipotent classes).

Step (1). So, assuming that we choose representatives $u \in C^F$ with trivial action of F on $A(u)$ for each unipotent class C of $\mathbf{G}$, we can use the Lusztig-Shoji algorithm to compute the (generalized) Green functions as linear combinations of the functions $\zeta_{u,\rho} Y_{u,\rho}$, see Section 4. For this we only need the information in Spaltenstein's table and the character tables of (relative) Weyl groups, which are, e.g., available in GAP [GAP22] or CHEVIE [GHL+96]. For the computations in this step we use independent indeterminates instead of the so far unknown scalars $\zeta_{u,\rho}$.

Up to this point we do not need to be more specific on the choice of class representatives $u \in C^F$. (But, of course, the $\zeta_{u,\rho}$ depend on the choice of u.)

We have 74 unipotent classes in $\mathbf{G}$ and 146 unipotent classes in $G(q) =$

$E_8(q)$ (where q is any power of $p = 2$). We need to determine 112 scalars $\zeta_{u,\rho}$ which so far are represented by 112 indeterminates in our table of Green functions.

Step (2). We consider the trivial character of $G(q)$ (the permutation character on $P_J(q)$ for $J = \Delta$) and use Remark 5.1(a). This shows that the 74 scalars $\zeta_{u,1}$ all must be 1 (independent of the choice of representatives u). We specialize their corresponding indeterminates in our table of Green functions such that only 38 unknown $\zeta_{u,\rho}$ remain.

From the rationality of Green functions, see Remark 5.1(e) or (b), we see that all of these $\zeta_{u,\rho} \in \{\pm 1\}$.

Step (3). We determine the sizes of the unipotent classes in $G(q)$ with the information from Remark 4.2. In many cases they are uniquely determined from the matrix Λ, and they are the same for all possible values of the not yet known $\zeta_{u,\rho}$. For example this is the case for the class $D_8(a_3)$ (where $A(u) \cong S_3$ for u in this class).

In other cases certain possibilities for $\zeta_{u,\rho}$ do not lead to a diagonal matrix of class length as described in 4.2. For example for the class $D_4(a_1)$, also with $A(u) \cong S_3$, we get that the two not yet known scalars can only be 1, and this determines the class lengths.

An interesting case are classes C where for $u \in C$ we have $A(u)$ of order 2 and the two classes in C^F have different length. There both choices in $\{\pm 1\}$ of the scalar for the non-trivial character of $A(u)$ lead to a diagonal matrix with both class lengths, but in different order. So, here we can just assume that the scalar is $\zeta_{u,-1} = 1$ and we get the class length of u.

Another case is the class $D_6(a_1)$ with $A(u)$ elementary abelian of order 4, where only one unknown scalar appears in the ordinary Green functions. We get the class lengths but not the scalar. Also for the class $2A_4$ with $A(u) \cong S_5$ we get the class lengths but not the scalars.

Now we use the property of Green functions from Remark 5.1(f) for the split torus of order $(q-1)^8$. We can evaluate this sum using the lengths of the unipotent classes.

We get an equation of the form

$$z_1 f_1(q) + \ldots + z_{18} f_18(q) = \frac{|G(q)|}{(q-1)^8}.$$

where $z_1, \ldots, z_{18}$ are 18 out of our 38 indeterminates remaining after step (2) and the $f_1(q), \ldots, f_{18}(q)$ are polynomial expressions in q.

For all $f_i(q)$ it is easy to see that they evaluate to positive integers for q any power of 2. Also, the sum of the $f_i(q)$ equals the right hand side of the equation. So, the only way to satisfy the equation by substituting all z_i by complex numbers of absolute value 1 is to set all z_i to 1.

Step (4). Let C be one of the classes $(A_5 + A_1)''$, $A_5 + 2A_1$, $D_6 + A_1$, D_8 where for $u \in C$ we have $|A(u)| = 2$. In these cases the two classes in C^F have the same order and the two possibilities for $\zeta_{u,-1} \in \{\pm 1\}$ exchange all values of Green functions on these two classes. So, there is a choice of $u \in C^F$ such that $\zeta_{u,-1} = 1$.

A similar argument works for the class $C = E_7 + A_1$ where for $u \in C$ the group $A(u)$ is elementary abelian of order 4. Setting the unknown scalar to $\{\pm 1\}$ leads to the same permutation of the values on the four classes in C^F for all Green functions. So, there is a choice of u such that the scalar is 1.

We have 15 remaining indeterminates.

Step (5). Now we use Remark 5.1(d) and compute the intersections of unipotent classes with the Borel subgroup $B(q)$. Some of these expressions are rational polynomials in q and one or several of the unknown scalars.

The expressions are big, but now we also use Proposition 4.1 and specialize q to $q = 2$ to get some rational linear combinations of unknown scalars which must be positive integers. For example, the expressions for the class $A_5 + A_2$ with component group S_3 contain two unknown scalars. Setting both scalars to 1 yields positive integers, but the other three possibilities either yield a negative or a non-integer rational number. (Note that in the case of the non-abelian $A(u) \cong S_3$ the $G(q)$-class of u is already fixed by the condition that F acts trivially on $A(u)$.)

With this method we find another 11 of the scalars to be 1.

Step (6). Two of the remaining four unknown scalars belong to the class $C = D_8(a_3)$. For $u \in C$ we have $A(u) \cong S_3$, so that the $G(q)$-class of u is fixed by the condition that F acts trivially on $A(u)$. We still need to determine $\zeta_{u,-1}$ and $\zeta_{u,\rho}$ where we write -1 for the sign character and ρ for the character of degree 2 of S_3.

Computing values of permutation characters on unipotent elements for various parabolic subgroups $P_J(q)$ as in Section 5.1 we find that, e.g., for J of type D_4 the mentioned scalars appear in the values on the classes in C^F.

Using Proposition 4.1 we specialize $q = 2$. Then the four possibilities for the scalars would lead to different character values of ${1_{P_J(q)}}^{G(q)}$ on the classes in C^F, namely the tuples: $(92897, 177889, 89825)$, $(179937, 90849, 176865)$, $(88801, 177889, 91873)$, $(175841, 90849, 178913)$.

Now we use Section 5.3 and compute the values of this permutation character on the elements z_{52}, z_{53}, z_{54} given in Mizuno [Miz80] and find the first of the tuples above which corresponds to $\zeta_{u,-1} = -1$ and $\zeta_{u,\rho} = 1$. This is the exception stated in our Theorem 6.1. Note that by Proposition 4.1 this exception only occurs when q is an odd power of 2 (or $q \equiv -1(3)$).

Remarks: The computation of each value took about 5 seconds, there are 3628800 J-reduced elements w, but only about 500 of them do not fulfill the criterion in Section 5.5. Note that with our computation we can actually show that the elements z_{52}, z_{53}, z_{54} in Mizuno's list are indeed in the class $C = D_8(a_3)$ because the same values do not occur on other classes. In fact, computing the value of z_{44} in Mizuno's list also yields the value 177889 and this shows that this element is conjugate to z_{53}.

Step (7). The last two unknown scalars correspond to the class $C = D_8(a_1)$. For $u \in C$ the group $A(u)$ is a dihedral group of order 8 which has 5 conjugacy classes and two non-trivial linear characters ϵ', ϵ'' which appear in the Springer correspondence. All values of (ordinary) Green functions are the same on the classes of $u = u_1$ and u_a where a is the non-trivial element in the center of $A(u)$.

As in step (6) we can use the parabolic subgroup for J of type D_4. The four possibilities for the two unknown scalars always lead to the same four character values, but they differ in which of the four values appears twice. For $q = 2$ the four values are $\{6785, 2625, 6401, 2241\}$. Using the five representative z_i, $27 \leq i \leq 31$, in Mizuno's list and computing the character values for these elements we get the value 6785 twice, namely for z_{27} and z_{31}. This shows that also the scalars $\zeta_{u,\epsilon'} = \zeta_{u,\epsilon''} = 1$.

Remarks. In our first proof of the Theorem 6.1 we used arguments as in steps (6) and (7), that is computations as explained in Section 5.2, for many more of the unknown scalars. Even large J for which the computations are pretty fast provide a lot of useful information. We were a bit surprised that in the revised version presented here we could avoid almost all of these computations. Nevertheless, as hinted in the remarks to step (6), the computational method can be useful to obtain more detailed information as in Theorem 6.1. For example, we have seen above that we can identify the $G(q)$-class of some unipotent elements. The values of some permutation characters are useful class invariants which often distinguish classes from all others.

References

[BS85] W. M. Beynon. and N. Spaltenstein. Green functions of finite Chevalley groups of type E_n $(n = 6, 7, 8)$. *J. Algebra*, 88:584–614, 1985.

[Car72] R.W. Carter. *Simple Groups of Lie Type.* A Wiley-Interscience publication, London, 1972.

[Car93] R. W. Carter. *Finite groups of Lie type.* Wiley Classics Library. John Wiley & Sons, Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[DL76] P. Deligne and G. Lusztig. Representations of reductive groups over finite fields. *Annals of Math.*, 103:103–161, 1976.

[DM20] F. Digne and J. Michel. *Representations of finite groups of Lie type (2nd edition)*, volume 95 of *London Mathematical Society Student Texts.* Cambridge University Press, 2020.

[GAP22] GAP – Groups, Algorithms, and Programming, Version 4.12.2. `http://www.gap-system.org`, Nov 2022.

[Gec17] M. Geck. On the construction of semisimple Lie algebras and Chevalley groups. *Proc. Amer. Math. Soc.*, 145:3233–3247, 2017.

[Gec20a] M. Geck. Computing Green functions in small characteristic. *J. Algebra*, 561:163–199, 2020.

[Gec20b] M. Geck. Green functions and Glauberman degree-divisibility. *Annals of Math.*, 192:229–249, 2020.

[GHL+96] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer. CHEVIE—a system for computing and processing generic character tables. *Appl. Algebra Engrg. Comm. Comput.*, 7(3):175–210, 1996. Computational methods in Lie theory (Essen, 1994).

[Lus85] G. Lusztig. Character sheaves II. *Adv. in Math.*, 57:226–265, 1985.

[Lus86] G. Lusztig. Character sheaves V. *Adv. in Math.*, 61:103–155, 1986.

[Lus90] G. Lusztig. Green functions and character sheaves. *Ann. of Math. (2)*, 131(2):355–408, 1990.

[Mal93] G. Malle. Green functions for groups of types E_6 and F_4 in characteristic 2. *Comm. Algebra*, 21:747–798, 1993.

[Mic15] J. Michel. The development version of the `CHEVIE` package of `GAP3`. *J. Algebra*, 435:308–336, 2015.

[Miz80] K. Mizuno. The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8. *Tokyo J. Math.*, 3(2):391–461, 1980.

[Por94] U. Porsch. *Die Greenfunktionen der endlichen Gruppen $E_6(q)$, $q = 3^n$*. Diploma thesis, Universität Heidelberg, 1994.

[Sho95] T. Shoji. Character sheaves and almost characters of reductive groups. I, II. *Adv. Math.*, 111(2):244–313, 314–354, 1995.

[Sho06] T. Shoji. Generalized Green functions and unipotent classes of finite reductive groups, I. *Nagoya J. Math*, 184:155–198, 2006.

[Sho07] T. Shoji. Generalized Green functions and unipotent classes of finite reductive groups, II. *Nagoya J. Math*, 188:133–170, 2007.

[Sho22] T. Shoji. Generalized Green functions and unipotent classes of finite reductive groups, III. *Nagoya J. Math*, 247:552–573, 2022.

[Spa85] N. Spaltenstein. On the generalized Springer correspondence of exceptional groups. *Advanced Studies in Pure Mathematics*, 6, 1985.

[Spr98] T. A. Springer. *Linear Algebraic Groups, Second Edition*, volume 9 of *Progress in Mathematics*. Birkhäuser, Boston, 1998.