# SIMPLICITY OF THE CONTACTOMORPHISM GROUP OF FINITE REGULARITY

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Abstract. For a given coorientable contact manifold $(M^{2n+1}, \xi)$, we consider the group $\text{Cont}_c^{(r,\delta)}(M, \alpha)$ consisting of $C^{r,\delta}$ contactomorphisms with compact support which is equipped with $C^{r,\delta}$-topology of Hölder regularity $(r, \delta)$ for $r \geq 1$ and $0 < \delta \leq 1$. We prove that for all Hölder class exponents with $r > n+2$ or $r = n+1$, $\frac{1}{2} < \delta \leq 1$ (resp. $r < n+1$ or $r = n+1$ and $0 < \delta < \frac{1}{2}$), the group is a perfect (and so a simple) group. In particular, $\text{Cont}_c^r(M, \xi)$ is simple for all integer $r \geq 1$. For the case of $\text{Cont}_c^{(r,\delta)}(M, \alpha)$ of general Hölder regularity, we prove the simplicity for all pairs $(r, \delta)$ leaving *only* the case of $(r, \delta) = (n+1, \frac{1}{2})$ open.

Contents



*Key words and phrases.* $C^r$ contactomorphism group, simplicity, Mather-Rybicki's constructions, contact product, Legendrianization, contact potential, Hölder regularity threshold.

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## 1. Introduction

Let $(M, \xi)$ be a connected smooth contact manifold. The set

$$\{f \in \mathrm{Diff}^r(M) \mid df(\xi) \subset \xi\} =: \mathrm{Cont}^r(M, \xi) \tag{1.1}$$

is a subgroup of $\mathrm{Diff}^r(M)$ for all $r \geq 1$, even for the Hölder regularity $(r, \delta)$ with $r \geq 1$, $0 < \delta \leq 1$. The general topology of this group is not well-behaved. For example, it is not known whether the group is locally contractible to the knowledge of the present author.

On the other hand, when $(M, \xi)$ is coorientable and equipped with a contact form $\alpha$, any smooth contactomorphism $f$ satisfies

$$f^*\alpha = \lambda_f\, \alpha$$

for a nowhere vanishing smooth function $\lambda_f : M \to \mathbb{R}$, which we call the *conformal factor* of $f$. Then we consider the logarithm $\ell_f = \log \lambda_f$ which we call the *conformal exponent* following the practice exercised in [Oh21a, Oh21b, Oh22a].

The following subset of $\mathrm{Cont}^r(M,\xi)$ is a subgroup of $\mathrm{Diff}^r(M)$ which is more suitable e.g., for the simplicity study of the contactomorphism group of finite regularity. Following Tsuboi [T3], we adopt the following definition.

**Definition 1.1** ($C^r$ contact diffeomorphism)**.** An element $C^r$ diffeomorphism $f : M \to M$ is called a $C^r$ contactomorphism with respect to $\alpha$ if $\lambda_f = \lambda_f^\alpha$ is a positive $C^r$ function. We denote by $\mathrm{Cont}^r(M,\alpha)$ the set of $C^r$ contactomorphisms.

It is straightforward to see that this definition does not depend on the choice of *smooth* contact form $\alpha$ and the set of $C^r$ contactomorphisms forms a subgroup of $\mathrm{Diff}^r_c(M)$, which is locally contractible. (See the discussion in [Ly], [Ba2], [T3, Section 2] and Sections 3, 6, especially the identitiy (3.5), of the present paper.)

**Remark 1.2.** One may call an element of $\mathrm{Cont}^r(M,\xi)$ a *weakly-$C^r$ contactomorphism* but we do not concern the group (1.1) in the present paper except when we discuss the case of contact homeomorphisms later in 1.3. Its definition a priori does not involve the choice of a contact form in its definition and so defined even for non-coorientable contact manifold. This is usually denoted by $\mathrm{Cont}(M,\xi)$ in the literature when $r = \infty$. Obviously when $\xi$ is coorientable, we have $\mathrm{Cont}^\infty(M,\xi) = \mathrm{Cont}^\infty(M,\alpha)$ for any choice of smooth contact form $\alpha$, and hence two definitions coincide for the smooth, i.e., for the $C^\infty$ case.

1.1. **Statement of main results.** We denote the set of compactly supported $C^r$ contactomorphisms by

$$\mathrm{Cont}^r_c(M,\alpha)$$

and its identity component by $\mathrm{Cont}^r_c(M,\xi)_0$. We also denote by $B\overline{\mathrm{Cont}}^r_c(M,\alpha)$ Haefliger's classifying space [Ha], [T3] of the group $\mathrm{Cont}^r_c(M,\alpha)$.

The following two results have been previously known concerning the simplicity of contactomorphism groups:

- (Tsuboi [T3]) For $1 \leq r < n + \frac{3}{2}$, $H_1(B\overline{\mathrm{Cont}}^r_c(M,\alpha);\mathbb{Z}) = 0$.
- (Rybicki [Ryb2]) For $r = \infty$, $H_1(B\overline{\mathrm{Cont}}^r_c(M,\alpha);\mathbb{Z}) = 0$.

In particular, $\mathrm{Cont}^r_c(M,\alpha)$ is a perfect (and so simple) group for the corresponding $r$. In their papers, the following contact version of the fragmentation lemma is an important ingredient. The proof follows from the fact that any contactomorphism contact isotopic to the identity is generated by a contact Hamiltonian and so the fragmentation lemma can be proved by the same argument as that of the symplectic case [Ba1]. (See [Ba2].)

**Lemma 1.3** (Fragmentation Lemma)**.** *Let $f \in \mathrm{Cont}_c(M,\alpha)_0$ and let $\{U_i\}_{i=1}^k$ be an open cover of $M$. Then there exists $f_j \in \mathrm{Cont}_c(M,\alpha)_0$, $j = 1,\ldots,\ell$ with $f = f_1 \circ f_2 \cdots \circ f_\ell$ such that $\mathrm{supp}(f_j) \subset U_{i(j)}$ for all $j$. The same holds for contact isotopies of contactomorphisms.*

In the present paper, we prove the following.

**Theorem 1.4.** *For any integer $r \geq n+2$, $H_1(B\overline{\mathrm{Cont}}^r_c(M,\alpha);\mathbb{Z}) = 0$. In particular, $\mathrm{Cont}^r_c(M,\alpha)$ is a simple group.*

Therefore combining the above three results, we obtain the following complete answer to the simplicity for the $C^r$ regularity with integer $r$ (including the case of $r = \infty$ [Ryb2]).

**Corollary 1.5.** *Assume $\dim M = 2n+1$ with $n \geq 1$. Then for any integer $r \geq 1$ including $r = \infty$, $H_1(B\overline{\mathrm{Cont}}^r_c(M,\alpha);\mathbb{Z}) = 0$. In particular, $\mathrm{Cont}^r_c(M,\alpha)$ is a perfect group for all integer $r \geq 1$ and $r = \infty$.*

The above result can be further extended to the Hölder class of regularites $(r,\delta)$ with $r \in \mathbb{N}$ and $0 < \delta \leq 1$. (See also Section 3 for the precise definition of $C^{r,\beta}$ contactomorphisms and the set $\mathrm{Cont}^{(r,\delta)}(M,\alpha)$ consisting thereof.) By definition, this set $\mathrm{Cont}^{(r,\delta)}(M,\alpha)$ forms a subgroup of $\mathrm{Cont}^r(M,\alpha)$ containing $\mathrm{Cont}^{r+1}(M,\alpha)$.

As in [Ma1], we derive Theorem 1.4 as a consequence of the following general result for the case of Hölder regularity class.

**Theorem 1.6.** *Assume* $\dim M = 2n+1$ *with* $n \geq 1$. *Then* $H_1(B\overline{\mathrm{Cont}}_c^{(r,\delta)}(M,\alpha);\mathbb{Z}) = 0$, *and hence* $\mathrm{Cont}_c^{(r,\delta)}(M,\alpha)$ *is a perfect group for all pairs* $(r,\delta)$ *with* $r < n+1$ *or* $r = n+1$ *and* $0 \leq \delta < \frac{1}{2}$.

By reversing the direction of the construction as in [Ma2], we obtain:

**Theorem 1.7.** *Assume* $\dim M = 2n+1$ *with* $n \geq 1$. *Then* $H_1(B\overline{\mathrm{Cont}}_c^{(r,\delta)}(M,\alpha);\mathbb{Z}) = 0$ *and hence* $\mathrm{Cont}_c^{(r,\delta)}(M,\alpha)$ *is a perfect group for all pairs* $(r,\delta)$ *with* $r < n+1$ *or* $r = n+1$ *and* $0 \leq \delta < \frac{1}{2}$.

Appearance of the half integer threshold of $\delta$ has its origin from the asymmetry of the orders of power of $A$ in that when $A \geq 1$, the norm of the contact scaling $(z,q,p) \mapsto (A^2 z, Aq, Ap)$ is $A^2$ but the norm of its inverse $(z,q,p) \mapsto (A^{-2}z, A^{-1}q, A^{-1}p)$ is $A^{-1}$.

These leave the following question open which is the contact analog to the celebrated open question [Ma1, Ma2] on simplicity of the $C^r$ diffeomorphism group with $r = n+1$ for an $n$-manifold. We would like to compare this open problem with that of the diffeomorphism case: Mather proved in [Ma1, Ma2] the corresponding result for the diffeomorphism group $\mathrm{Diff}_c(M^m)^r$ of connected $m$-manifold $M$, *if* $r \neq m+1$. In the mean time, Theorem 1.7 recovers Tsuboi's result [T3] whose proof is in the same spirit as Theorem 1.4 similarly as in [Ma1, Ma2]. On the other hand, Epstein [E1] proved the simplicity of the homeomorphism group $\mathrm{Homeo}_c(M)$.

**Question 1.8.** Suppose $\dim M = 2n+1$. Is $\mathrm{Cont}_c^{(r,\delta)}(M,\xi)$ simple when $r = n+1$ and $\delta = \frac{1}{2}$?

1.2. **Rybicki's contactization of Mather's construction.** The main methodology of the proof is again those introduced by Mather [Ma1], [E2] which also relies on certain fragmentation lemma and the application of Schauder-Tychonoff's fixed point theorem, which has been also applied by Tsuboi [T3] and Rybicki [Ryb2] to the simplicity problem of contactomorphisms. Using the fragmentation lemma and Epstein's reduction [E1], the proof of simplicity (or rather perfectness) is reduced to the case of Euclidean space $\mathbb{R}^m$ ($m = 2n+1$), which in turn crucially relies on the 'linear structure' of the $\mathbb{R}^m$. Some fundamental properties of the Euclidean space (or the torus) used in Mather's proof are the simple facts:

- They carry the abelian group structure induced by the linear addition operator $+$ thereon.
- Any diffeomorphism $C^1$-close to the identity can be written as $f = \mathrm{id} + v$ for $v$ is a $\mathbb{R}^m$-valued function that is $C^1$-close to the zero function.

(See [Ma4] for a detailed analysis of what obstructs the method of [Ma1, Ma2] applied to the case of $r = \dim M + 1$ for the general diffeomorphism case.) *Such a simple linear description of $C^1$ neighborhood fails to hold for the contactomorphisms.* This prevents one from directly borrowing Mather's construction of *rolling-up operators* to the case of contactomorphisms.

New ingredients introduced by Rybicki [Ryb2] in this regard are the following:

(1) A new fragmentation lemma based on this contact potential and the *contact cylinders* $(\mathcal{W}_k^{2n+1}, \alpha_k)$ of the form

$$\begin{aligned} \mathcal{W}_k^{2n+1} &:= S^1 \times T^*(T^k \times \mathbb{R}^{n-k}) \cong S^{k+1} \times \mathbb{R}^{n-k} \times \mathbb{R}^n, \\ \alpha_k &= d\xi_0 - \sum_{i=1} p_i \, d\xi_i \end{aligned} \tag{1.2}$$

where $T^k = (S^1)^k$ and $(\xi_0, \ldots, \xi_k)$ are standard coordinates ($S^1$-valued) of $(S^1)^{k+1}$ and $(\xi_{k+1}, \ldots, \xi_n)$ are those of $\mathbb{R}^{n-k}$ and $p = (p_1, \ldots, p_n)$ the conjugate coordinates of $(\xi_1, \cdots, \xi_n)$.

(2) Usage of the local parametrization of $C^1$ neighborhood of the identity via the Legendrianization and the generating functions, which we name the *contact potential*, of the relevant contactomorphisms in his construction of contact version of unfolding-fragmentation operators. This space of *real-valued* functions is the domain of the function space where his application of Schauder-Tychonoff's fixed point theorem is made.

We closely follow the scheme of Rybicki which is used for the $C^\infty$ case. However we need to make both geometric constructions and derivative estimates optimal in all the steps of Rybicki's proof which deals with the $C^\infty$ case by suitably adapting Epstein's simplification of the simpleness proof of $\mathrm{Diff}^\infty(M)$ exercised in [E2] to the contact case (without using the Nash-Moser implicit function theorem originally used in [Th], [Ma4, Appendix]):

(1) We need to package the contact geometry elements employed in [Ryb2] systematically in the framework of contact Hamiltonian geometry and calculus of [Oh21a, Oh21b, Oh22a].
(2) We need to identify the optimal form of *contact homothetic transformations* for the definition of the rolling-up operator and the unfolding-fragmentation operators. (See Section 8.)
(3) Using this optimal geometric package, we derive the *optimal version* of many of the rough estimates carried out in [Ryb2].

For the $C^\infty$ case studied by Rybicki, these optimal choices of homothetic transformation or of the estimates are not needed while only some rough estimates are enough as done in [Ryb2]. But in our study of finite regularity, especially to determine the lower threshold $r = n+2$ and the upper threshold $r = n+1$, we need the optimal version of all the estimates that appear in the course of studying the $C^r$ (or $C^{(r,\beta)}$) norms of various contactomorphisms and of their products. Our estimates also crucially rely on systematic contact Hamiltonian calculus involving the conformal exponents and other basic contact Hamiltonian geometry as exercised in our study of contact instantons in [Oh21b, Oh22a], for example.

**Warning 1.9.** We adopt the notations used in [Ma1]-[Ma3], [E2] and [Ryb2], especially those from [Ryb2] so that the readers can easily compare the details of the present paper with those in [Ryb2] corresponding thereto. However we warn the readers that even though we adopt the same notations for the purpose of comparison, the detailed numerics appearing in the definitions are almost never the same as those from [Ryb2], since we make the optimal choices of various numerical constants and orders of powers. The systematic framework of contact Hamiltonian geometry and calculus developed in [Oh21a, Oh22a] enables us to find these optimal choices for the various constructions which is crucial in our determination of the threshold

$$r = n + 2 \tag{1.3}$$

for the lower threshold, $r = n+1$ for the upper threshold. (Also the latter threshold corresponds to Tusboi's upper threshold $r = n + \frac{3}{2}$ in [T3].) These are the counterparts of Mather's thresholds $r = n+1$ [Ma1] and $r = n$ [Ma2] respectively for the case of diffeomorphisms.

## 1.3. Discussion and open problems.

1.3.1. *Relationship with* [Ryb2]. Once the estimate for $r = n+2$

$$\|u\|_{n+2} \leq \varepsilon_{n+2}$$

is given, we can inductively obtain a sequence $\varepsilon_r$ for $r \geq n+2$, adapting the argument of Epstein [E1, p.121] for the case of $\mathrm{Diff}_c(M)$, such that the map $\vartheta : \mathcal{U} \to \mathcal{U}$ is defined on

$$\mathcal{U} = \{u \in C^\infty_c(\mathbb{R}^{2n+1}) \mid \|u\|_r \leq \varepsilon_r\}$$

which is a convex closed subset of the Frechet space $C^\infty_c(\mathbb{R}^{2n+1})$. Then we can again apply the Schauder-Tychonoff fixed point theorem and conclude that $\mathrm{Cont}^\infty_c(M,\alpha)$ is a perfect group. Rybicki employed the same strategy in [Ryb2] to prove the perfectness of

$\mathrm{Cont}_c^\infty(M,\alpha)$. Even for this case, our proof clarifies the presentation of relevant contact geometry and simplifies the estimates to the optimal level of those given in [Ryb2]. Rybicki also suspected that the perfectness may hold at least for large $r$. Our paper affirmatively confirms this in the optimal way to the level of precisely locating the lower threshold $r = n+2$ and the upper threshold $r = n+1$ for contact manifolds of dimension $2n+1$, similarly as Mather [Ma1, Ma4] did for the diffeomorphism group $\mathrm{Diff}_c(M^n)$ in which case the corresponding thresholds are $n+1$ and $n$. Rybicki also asked whether the contact analogs to the Thurston-Mather type isomorphism from [Ma3], [T2, T3], [Ba2] and [Ryb1] can be proved, and regards as a hard problem. We hope that our systematic study of the background geometry and of the optimal estimates will help making the future researches of such questions easier.

On the other hand, for the case of $1 \leq r < n + \frac{3}{2}$ in the opposite direction, Tsuboi [T3] proved the simplicity of $\mathrm{Cont}_c(M,\xi)$ by utilizing some construction of infinite repetition which has been used in the study of topology of diffeomorphism groups. (We refer readers to [T1] for a detailed exposition on such construction with many illuminating illustrations.) The dual version of the method laid out in the present paper also gives a somewhat different proof of Tsuboi's result for $1 \leq r < n + \frac{3}{2}$. We would like to compare it with the way how Mather's proofs for the case with $r > n+2$ [Ma1] and $r < n+1$ [Ma2] work.

**Remark 1.10.** In our earlier works on contact instantons [Oh21a]-[Oh22a] and others, the Greek letter $\psi$ and $g_\psi$ as the notations of a contact diffeomorphism and of its conformal exponent were used respectively. In the present article, we replace them by the Roman letter $f$ and $\ell_f$respectivley to be in more close contact with the literature related to the study of simplicity problem such as [Ma1, Ma2, Ma3], [E2] and [Ryb2]. We also mention that the conformal factor $\lambda_f$ will be used at all in the present paper neither in our constructions related to the Legendrianization nor in the definitions or the estimates of the norms of contactomorphisms. Only the conformal exponent $\ell_f$ will be used in those matters.

1.3.2. *Towards topological contact dynamics of Müller-Spaeth.* Another interesting direction of research is towards the direction of regularity lower than $C^1$ similarly as in the case of Hamiltonian homeomorphisms (hameomorphisms) as done in [OM], [CHS]. In fact such a study has been carried out by Müller and Spaeth in their series of papers [MS1]–[MS3]. They in particular introduced the notions of *topological contact automorphisms* ([MS1, Definition 6.8]) and and of *contact homeomorphisms* ([MS1, Definitioin 6.7]). They denote them respectively by

$$\mathrm{Homeo}(M,\xi), \quad \mathrm{Aut}(M,\xi). \tag{1.4}$$

By their definition $\mathrm{Homeo}(M,\xi)$ is a subgroup of $\mathrm{Aut}(M,\xi)$. The group $\mathrm{Aut}(M,\xi)$ is the analogue of Eliashberg-Gromov's symplectic homeomorphism group, the $C^0$-closure $\overline{\mathrm{Symp}}(M,\omega)$, which was also denoted by $\mathrm{Sympeo}(M,\omega)$ in [OM].

We prefer to reserve the notation $\mathrm{Homeo}(M,\xi)$ for the set of elements from their $\mathrm{Aut}(M,\xi)$ and to reserve $\mathrm{Hameo}(M,\xi)$ for the set of those from their $\mathrm{Homeo}(M,\xi)$ and call an element therefrom a *contact hameomorphisms* since in the $C^0$-level, there is a priori clear difference between the notions of 'contact' and 'contact Hamiltonian' unlike the smooth case. In this vein we will adopt the notations

$$\mathrm{Hameo}(M,\xi) \subset \mathrm{Homeo}(M,\xi)$$

instead of (1.4) of Müller and Spaeth, to emphasize the fact that the Hamiltonians enter into the definition of the group. One may compare this pair in the contact case with the pair of notations

$$\mathrm{Hameo}(M,\omega) \subset \mathrm{Sympeo}(M,\omega)$$

introduced in [OM] in the symplectic case. (In the same vein, one might prefer to replace the notation $\mathrm{Sympeo}(M,\omega)$ by $\mathrm{Homeo}(M,\omega)$.)

These groups are defined in [MS1] by taking some suitable completions of $\mathrm{Cont}_c^\infty(M,\alpha)$ similarly as in the case of (symplectic) hameomorphisms [OM]. They also showed that $\mathrm{Hameo}(M,\xi)$ is a normal subgroup of $\mathrm{Homeo}(M,\xi)$.

Two natural questions to ask in this regard are the following:

**Question 1.11.** Let $(M, \xi)$ be a coorientable contact manifold.

(1) Is the inclusion $\mathrm{Hameo}(M, \xi) \subset \mathrm{Homeo}(M, \xi)$ proper?
(2) Is $\mathrm{Homeo}(M, \xi)$ a perfect (or a simple) group? How about $\mathrm{Hameo}(M, \xi)$?

One might go further by specializing to the 3 dimensional case of contact manifold as the contact counter part of the area-preserving dynamics on 2 dimensional surface.

Based on the recent developments [CHS] of $C^0$ Hamiltonian dynamics in symplectic geometry, the following problem seems to be a very interesting doable open problem.

**Problem 1.12** (Open problem)**.** Find the answers to the questions asked in Question 1.11 for the 3 dimensional contact manifold $(M, \xi)$, and see if any of the existing Floer-type analytical machinery can be used as in the 2 dimensional area-preserving dynamics.

It is worthwhile to recall the readers that there is one big difference of the contactomorphism group from the symplectomorphism group for the contact case: there is no Calabi-type invariant, at least no apparent one. This seems to prevent one from easily guessing the direction of the answers to the questions, unlike the area-preserving case [OM, Oh10].

The organization of the paper is now in order. In Section 2, we set the notations and various conventions we adopt in contact/symplectic geometry. These are not all the same as those used in [Ryb2]. We also recall various cubical objects and operators defined on $\mathbb{R}^{2n+1}$ appearing in [Ryb2] but none of these cubical objects are the same as the corresponding ones in [Ryb2] in their numerics although the same notations are used. Then in Section 3 - 4 we explain the basic background contact geometry emphasizing our usages of conformal exponent, contact product, the Legendrianization and of an equivariant Darboux-Weinstein theorem. After then we provide the geometric part of various constructions employed in [Ryb2] with some new definitions, refinements, amplifications and corrections, and give the proofs of the main theorems as an application of Schauder-Tychonoff's theorem, *assuming* the existence of the map $\vartheta : \mathcal{L}(\varepsilon, A) \to \mathcal{L}(\varepsilon, A)$ on some closed convex subset $\mathcal{L}(\varepsilon, A) \to \mathcal{L}(\varepsilon, A)$ of the Banach space $C^r_c(\mathbb{R}^{2n+1}, \mathbb{R})$ of real-valued functions. (See Section 22 for the definition of $\vartheta$.)

Then in Part II, we prove all the derivative estimates on the various operators entering in all relevant constructions. All these estimates have their precedents in [Ryb2], some of which in turn have their precedents also in [Ma1] and [E2], but our optimal versions thereof enable us to define the operator $\vartheta : \mathcal{L}(\varepsilon, A) \to \mathcal{L}(\varepsilon, A)$ for a sufficiently small $\varepsilon$ and sufficiently large $A > 1$, *provided* $r + \delta > n + 2$. Then Part III combines the geometry of Part I and the estimates of Part II to complete the proofs of all main theorems. In Appendix A, we derive an implication of the equivariance of the Darboux-Weinstein chart in terms of the independence thereof on the first factor of the contact product.

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## 2. Notations and conventions

In this section, we gather the notations and conventions that we adopt for the various geometric constructions, and compare them with those used by Mather [Ma1] and by Epstein [E2].

2.1. **Notations and conventions for general contact geometry.** We start with the notations for the general contact geometry.

(1) $(z, q, p) = (z, q_1, \ldots, q_n, p_1, \ldots, p_n)$; The canonical coordinates of the 1-jet bundle $J^1\mathbb{R}^n \cong \mathbb{R} \times T^*\mathbb{R}^n$ *with the $z$-coordinates written first*,
(2) $M_Q := Q \times Q \times \mathbb{R}$; The contact product of contact manifold $(Q, \alpha)$ equipped with contact form $\mathscr{A} = -e^\eta \pi_1^*\alpha + \pi_2^*\alpha$ [Oh22a],
(3) $(x, X, \eta)$: a point in the contact product $Q \times Q \times \mathbb{R}$,
(4) $x = (z, q, p)$ and $X = (Z, Q, P)$ are the 'coordinate system' for the contact product with $Q = J^1\mathbb{R}^n$,
(5) $\Phi_U : U \to V$, $\Phi_{U;A} : U_A \to V_A$; Legendrian Darboux-Weinstein charts. (Note that *the $\mathbb{R}$-factor is written at the last spot* unlike the case of $J^1Q$.)

We also set up our conventions for the definitions of Hamiltonian vector fields both in symplectic and in contact geometry so that they are compatible in some natural sense. We briefly summarize basic calculus of contact Hamiltonian dynamics to set up our conventions on their definitions and signs following [Oh21a].

**Definition 2.1.** Let $\alpha$ be a contact form of $(M, \xi)$. The associated function $H$ defined by

$$H = -\alpha(X) \tag{2.1}$$

is called the *($\alpha$-)contact Hamiltonian* of $X$. We also call $X$ the *($\alpha$-)contact Hamiltonian vector field* associated to $H$.

We alert readers that under our sign convention under which the Reeb vector field $R_\lambda$ as a contact vector field becomes the constant function $H = -1$. We denote by $R_\alpha$ the Reeb vector field of $\alpha$.

Here are more general conventions that we use in the present article.

- The symplectic form on the cotangent bundle $T^*N$ is given by

$$\omega_0 = \sum_{i=1}^n dq_i \wedge dp_i = d(-\theta) \tag{2.2}$$

in the canonical coordinates of $T^*N$ associated to a coordinate system $(x_1, \ldots, x_n)$ of $N$.
- The Hamiltonian vector field associated a real-valued function $H$ on a symplectic manifold $(M, \omega)$ is given by the equation

$$dH = X_H \rfloor \omega. \tag{2.3}$$

- The standard contact form on the 1-jet bundle $J^1N$ is given by

$$\alpha_0 = dz - \sum_{i=1} p_i \, dq_i \tag{2.4}$$

in the canonical coordinates $(z, q_1, \ldots, q_n, p_1, \ldots, p_n)$ of $J^1N$.
- The contact Hamiltonian vector field $X = X_H$ associated to a real-valued function $H$ on general contact manifold $(M, \alpha)$ is uniquely determined by the equation

$$\begin{cases} X \rfloor \alpha = -H, \\ X \rfloor d\alpha = dH - R_\alpha[H]\alpha. \end{cases} \tag{2.5}$$

With these conventions, the contact Hamiltonian vector field $X_H$ has the decomposition

$$X_H = X_H^\parallel \oplus (-H\, R_\alpha) \in \Xi \oplus \mathbb{R}\langle R_\alpha \rangle.$$

In the canonical coordinates $(q, p, z)$ on $\mathbb{R}^{2n+1}$, this expression is reduced to the well-known coordinate formula for the contact Hamiltonian vector field below. (See [Ar, Appendix 4], [OW, Lemma 2.1], [Bhu, Lemma 4.1], [Ryb2, Equation (2.3)], for example but with different sign conventions.)

**Example 2.2.** Let $H : \mathbb{R}^{2n+1} \to \mathbb{R}$ be a smooth function on $\mathbb{R}^{2n+1}$. Then the contact Hamiltonian vector field $X = X_H$ is given by

$$\begin{aligned} X_H &= X_H^{\parallel} - HR_\alpha && (2.6)\\ &= \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i}\frac{D}{\partial q_i} - \frac{DH}{\partial q_i}\frac{\partial}{\partial p_i}\right) - H\frac{\partial}{\partial z} \\ &= \sum_{i=1}^n \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \left(\frac{\partial H}{\partial q_i} + p_i \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(\left\langle p, \frac{\partial H}{\partial p}\right\rangle - H\right)\frac{\partial}{\partial z} && (2.7)\end{aligned}$$

2.2. **Notations of Rybicki in** [Ryb2] **and their variations.** To make it easier for readers to compare various constructions appearing in the present article with those from [Ryb2], we mostly adopt the notations of various objects appearing in Rybicki's constructions in [Ryb2]. We follow the notations of Rybicki as closely as possible but with many changes of numerical constants and orders of power which are necessary to be able to obtain that optimal estimates that give rise to the threshold $r = n + 2$ mentioned in Warning 1.9.

(1) $T^k = (S^1)^k$; The $k$-torus,
(2) [Contact cylinders] $\mathcal{W}_k^m = S^1 \times T^*(T^{k-1} \times \mathbb{R}^{m-k})$ for $m = 2n + 1$ and $1 \leq k \leq n$; the *circular contactization* of symplectic manifold $T^*k \times \mathbb{R}^{m-k}$, equipped with the canonical (partially circular) coordinates $(\xi_0, \xi, p) = (\xi_0, \xi_1, \ldots, \xi_n, p_1, \cdots, p_n)$. Throughout the paper, we will use either $m$ or $2n + 1$ interchangeably as we feel more proper to use.
(3) $\alpha_0 = d\xi_0 - \sum_{i=1} p_i\, d\xi_i$; The *contactization contact form* $\alpha_0 = d\xi_0 - \sum_{i=1} p_i\, d\xi_i$ on $\mathcal{W}^m$,
(4) For $A \geq 1$, $k = 0, \ldots, n$, we define the reference rectangulapid
$$I_A = [-2, 2] \times [-2, 2]^n \times [-2A, 2A]^n.$$
(5) We define
$$\begin{aligned} J_A^{(k)} &= S^1 \times T^{k-1} \times [-2A^2, 2A^2]^{n-k} \times [-2A^3, 2A^3]^n \\ K_A^{(k)} &= S^1 \times T^{k-1} \times [-2, 2] \times [-2A^2, 2A^2]^{n-k+1} \times [-2A^3, 2A^3]^n \end{aligned}$$
for $k = 1, \ldots, n$, and
$$\begin{aligned} J_A^{(0)} &= J_A = [-3A^5, 3A^5] \times [-2A^2, 2A^2]^n \times [-2A^3, 2A^3]^n \\ K_A^{(0)} &= K_A = [-2, 2] \times [-2A^2, 2A^2]^n \times [-2A^3, 2A^3]^n. \end{aligned}$$
(Compare these definitions with those appearing in [Ryb2, p.3312] with the same notations.)
(6) $E_A = [-A, A]^m$, $E_A^k = S^1 \times T^k \times [-A, A]^{m-k}$,
(7) [Subinterval] A closed subset $E \subset E_A$ of type $E_A$ is called a *subinterval*,
(8) [Contact scaling] $\chi_A$: the map defined by $\chi_A(z, q, p) = (A^2 z, Aq, Ap)$,
(9) [twisted $p$-translations] $\sigma_i^t$: the contact cut-off of the map $S_i^t$ defined by $S_i^t(z, q, p) = (z + tq_i, q, p + t\frac{t}{\partial p_i})$,
(10) [Contact scaling preceded by $p$-translations]
$$\rho_{A;\mathbf{t}} = \chi_A^2 \circ \sigma^{\mathbf{t}}, \quad \sigma^{\mathbf{t}} = \sigma_1^{t_1} \circ \cdots \circ \sigma_n^{t_n}$$
for $\mathbf{t} = (t_1, \ldots, t_n)$. (This map is different from the one given in [Ryb2, p.3307] with the same notation.)
(11) [Mather-Rybicki's rolling-up operators]
$$\Theta_A^{(k)} : \mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}_{K_A^{(k)}}(\mathcal{W}_{k+1}^{2n+1}, \alpha_0)_0,$$
(12) [Mather-Rybicki's unfolding-fragmentation operators]
$$\Xi_{A;N}^{(k)} : \mathrm{Cont}_{J_A^{(k)}}(\mathcal{W}_{k+1}^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0,$$

(13) [Rybicki's rolling-up operators]
$$\Psi_A^{(k)} : \mathrm{Cont}_{J_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}_{J_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0$$
where the rolling occurs in the $q_k$-coordinate direction for $k = 1, \ldots, n$ and for $z$ for $k = 0$.

We now organize the domains and the codomains entering into the definitions of these operators in the following diagram: For the study of the lower threshold, we consider the diagram

$$\begin{tikzcd}
& & {[-2,2]^{2n+1}} \arrow[dll, "p_0\rho_{A,\sigma_{t_0}}"'] \arrow[dl, "p_1\rho_{A,\sigma_{t_1}}"] \arrow[d, "p_*\rho_{A,\sigma_{t_*}}"] \arrow[dr, "p_n\rho_{A,\sigma_{t_n}}"] & \\
J_A^{(0)} \arrow[r] \arrow[d, "\iota_0"'] & J_A^{(1)} \arrow[r] \arrow[d, "\iota_1"'] & \cdots\cdots \arrow[r] \arrow[d, "\iota_*"'] & J_A^{(n)} \arrow[d, "\iota_n"'] \\
\mathcal{W}_0^{2n+1} \arrow[r, "\pi_0"'] & \mathcal{W}_1^{2n+1} \arrow[r, "\pi_1"'] & \cdots\cdots \arrow[r, "\pi_n"'] & \mathcal{W}_n^{2n+1}.
\end{tikzcd}$$

Here $\pi_k$ are the covering projections induced by $\mathbb{R}^{k+1} \to T^{k+1}$ for $k = 0, \ldots, n$, the map induced by the identity map of $T^{k+1}$ and the inclusion map
$$[-2A^2, 2A^2]^{n-k} \times [-2A^3, 2A^3]^n \hookrightarrow \mathbb{R}^{2n-k}.$$
The map $p_i : \mathbb{R}^{2n+1} \to T^{i+1} \times \mathbb{R}^{2n-i}$ are the obvious covering projections. Finally, the maps $\rho_{A,t_i} : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ are the translation map followed by the contact rescaling map given by
$$\rho_{A,t_i} := \chi_A^2 \circ \sigma_i^{t_i}.$$
(See (8.2) for the precise definition thereof.) Compare this sequence
$$\mathbb{R}^{2n+1} \supset J_A^{(0)} \to J_A^{(1)} \cdots \to J_A^{(n)} \subset J^1\mathbb{R}^n$$
with the sequence
$$[-2,2]^n = D_n \subset D_{n-1} \subset \cdots \subset D_0 = [-2A, 2A]^n$$
from [Ma1, Section 3], [E2, Section 2], and observe that *both sequences have $n$ terms in them*, which is essential for the determination of the lower threshold $r = n+2$ for the contact case and $r = n+1$ for the diffeomorphism case, respectively here and in [Ma1, Ma2].

Similarly as in [Ma2], we reverse the horizontal arrows for the study of the upper threshold.

## 3. Conformal exponents, contact product and Legendrianization

Now let $(M, \xi)$ be a contact manifold of dimension $m = 2n+1$, which is coorientable. We denote by $\mathrm{Cont}_+(M, \xi)$ the set of orientation preserving contactomorphisms and by $\mathrm{Cont}_0(M, \xi)$ its identity component. Equip $M$ with a contact form $\alpha$ with $\ker \alpha = \xi$.

### 3.1. **Conformal exponents of contactomorphisms.**

For any coorientation-preserving contactomorphism $g$ we have
$$g^*\alpha = \lambda_g^\alpha\, \alpha.$$
We adopt the convention of systematically calling the function $\lambda_g^\alpha$ the conformal factor, and
$$\ell_g^\alpha := \log(\lambda_g^\alpha) \tag{3.1}$$
the *conformal exponent* of $g$, following the practice of [Oh21a, Oh21b, Oh22a]. Since the contact form $\alpha$ will be fixed throughout the paper, we omit $\alpha$ from notations by writing $\lambda_f$ and $\ell_f$ respectively from now on. The following lemma is well-known and straightforward to check.

**Lemma 3.1.** *For any two contactomorphisms $g$, $f$, we have*
$$\ell_{gf}^\alpha = \ell_g^\alpha \circ f + \ell_f^\alpha. \tag{3.2}$$

For each contact form $\alpha$ of the given contact manifold $(M, \xi)$, we consider the function $\varphi_\alpha : G := \mathrm{Cont}(M, \xi) \to C^\infty(M)$, defined by

$$\varphi_\alpha(f) = \ell_f^\alpha. \tag{3.3}$$

Regard it as a one-cochain in the group cohomology complex $(C^1(G, C^\infty(M)), \delta)$ with the coboundary map $\delta : C^1(G, C^\infty(M)) \to C^2(G, C^\infty(M))$ on the right $\mathbb{Z}[G]$-module $C^1(G, C^\infty(M))$ with the right composition

$$(f, \ell) \mapsto \ell \circ f$$

as the action of $G$ on $C^1(G, C^\infty(M))$. Then the above lemma can be interpreted as

$$\ell_{gf} = (\delta\varphi_\alpha)(g, f). \tag{3.4}$$

Furthermore for a different choice of contact form $\alpha'$ of the same contact structure $\xi$ in the same orientation class, we make $\alpha$-dependence on $\ell_f$ explicit by writing $\ell_f^\alpha$ and $\ell_f^{\alpha'}$. Furthermore we have

$$\alpha' = e^{h_{(\alpha'\alpha)}}\alpha \tag{3.5}$$

for some smooth function $h_{(\alpha'\alpha)}$ depending on $\alpha$, $\alpha'$. Then we have

**Proposition 3.2.** *Consider the two zero-cochains $\varphi_{\alpha'}$, $\varphi_\alpha$ in the group cohomology complex $C^*(G, C^\infty(M))$. Then $\varphi_{\alpha'}$, $\varphi_\alpha$ are cohomologus to each other.*

*Proof.* A straightforward calculation leads to

$$\ell_f^{\alpha'} = h_{\alpha'\alpha} \circ f + \ell_f^\alpha. \tag{3.6}$$

This itself can be written as

$$\varphi_{\alpha'} - \varphi_\alpha = \delta h_{(\alpha'\alpha)}$$

where we regard $h_{(\alpha'\alpha)}$ as a zero-cochain. This finishes the proof. □

An iteration of (3.2) gives rise to the following suggestive form of the identity

$$\ell_{g_m \circ \cdots \circ g_1} = \sum_{k=0}^{m-1} \ell_{g_k} \circ (g_{k-1} \circ \cdots \circ g_1) \tag{3.7}$$

for all $g_i \in \mathrm{Cont}(M, \xi)$ with $i = 1, \cdots, m$.

3.2. **Defintion of $C^{r,\delta}$ contactomorphisms.** Now, we give the precise definition of $C^r$ (resp. $C^{r,\beta}$) contactomorphisms.

**Definition 3.3** ($C^r$ contact diffeomorphism)**.** A $C^r$ diffeomorphism $f : M \to M$ is called a $C^r$ contactomorphism if $\lambda_f$, or equivalently $\ell_F$, is a positive $C^r$ function.

This definition can be extended to the Hölder regularity classes. More precisely, let $\beta$ be a modulus of continuity of the form $\beta : [0, \infty) \to [0, 1]$ and Denote by $\mathrm{Diff}^{(r,\beta)}(M, \alpha)$ be the set of $C^{r,\beta}$ diffeomorphisms.

We specialize to the Hölder regularity $\beta(x) = x^\delta$ and define the set of $C^{r,\delta}$ contactomorphism as follows.

**Definition 3.4** ($\mathrm{Cont}^{(r,\delta)}(M, \alpha)$)**.** We define the set of $C^{r,\delta}$ contactomorphisms to be the intersection

$$\mathrm{Cont}^{(r,\delta)}(M, \alpha) := \mathrm{Cont}^r(M, \alpha) \cap \mathrm{Diff}^{(r,\delta)}(M) \subset \mathrm{Diff}^{(r,\delta)}(M). \tag{3.8}$$

An immediate corollary of (3.2) is the following.

**Corollary 3.5.** *For any $1 \le r \le \infty$, the set $\mathrm{Cont}_c^r(M, \alpha)$ (resp. $\mathrm{Cont}_c^{(r,\delta)}(M, \alpha)$) is a topological subgroup of $\mathrm{Diff}_c^r(M)$.*

*Proof.* We have only to prove that for any contact diffeomorphisms $f$, $g$ such that they are of class $C^r$ as well as $\ell_f$, $\ell_g$ are $C^r$, $\ell_{gf}$ are $C^r$. But this is apparent by the formula (3.2).

For the case of $\mathrm{Cont}_c^{(r,\delta)}(M, \alpha)$, it is well-known [Ma1, Section 2] that $\mathrm{Diff}^{(r,\delta)}(M)$ is a subgroup of $\mathrm{Diff}^r(M)$, which finishes the proof. □

It is straightforward to derive from the identity (3.5) that the definition of $\mathrm{Cont}^{(r,\delta)}(M,\alpha)$ and its topology do not depend on the choice of the contact form $\alpha$.

**Remark 3.6.** However, we alert the readers that the product operation is not closed in the intersection

$$\mathrm{Cont}_c^{r+1}(M,\alpha) \cap \mathrm{Diff}_c^r(M),$$

according to the definition given in Definition 1.1, and hence the intersection does not form a subgroup of $\mathrm{Diff}_c^r(M)$ because the conformal exponent of an element may not be $C^r$. Therefore $\mathrm{Cont}_c^r(M,\xi)$ is *not* a closed subgroup of $\mathrm{Diff}_c^r(M)$, because a priori one loses a regularity by 1 to define the conformal exponent. As a result, the $C^r$-convergence of contactomorphisms $f_i$ does not guarantee convergence $\ell_{f_i}$. The following inclusion is a proper inclusion

$$\overline{\mathrm{Cont}_c^{r+1}(M,\xi)} \subset \mathrm{Cont}_c^r(M,\xi)$$

where the closure is the one of $\mathrm{Cont}_c^{r+1}(M,\xi) \subset \mathrm{Diff}_c^r(M)$ taken in $\mathrm{Diff}_c^r(M)$. The case $r < 1$ is particularly interesting which may be regarded as a contact counterpart of the group $\mathrm{Hameo}(M,\omega)$ of *hameomorphisms* in symplectic geometry [OM]. (It was conjectured in [OM] that $\mathrm{Hameo}(M,\omega)$ is a *proper* subgroup of the area-preserving homeomorphism group in the 2 dimensional case such as $M = D^2$, $S^2$, and the conjecture has been recently proved by Cristofaro-Gardiner, Humilière and Seyfaddini [CHS].) Some related researches in this direction have been carried out by Müller and Spaeth in their series of works [MS1, MS2, MS3] and others.

3.3. **Contact product and Legendrianization.** We now consider the product $(M_Q, \Xi)$

$$M_Q := Q \times Q \times \mathbb{R}, \quad \Xi := \ker \mathscr{A} \tag{3.9}$$

with contact distribution $\Xi = \ker \mathscr{A}$ for a specifically chosen contact form

$$\mathscr{A} := -e^{\eta}\, \pi_1^*\alpha + \pi_2^*\alpha. \tag{3.10}$$

Here we follow the sign convention of [Oh22a]. We then consider the operation of *Legendrianization* of contactomorphisms, which is the contact analog to the *Lagrangianization* of canonically associating to each symplectomorphism the Lagrangian submanifold in the product which is nothing but its graph.

We summarize basic properties of this contact product and the Legendrianization, some of which are well-known, e.g., can be found from [Ly, Bhu].

**Proposition 3.7.** *Let $(Q,\xi)$ be any contact manifold and a contact form $\alpha$ be given. We define $(M_Q,\mathscr{A})$ as above. Denote by $\pi_i : M_Q \to Q$ the projection to the i-th factor of the product for $i = 1, 2$, and $\eta : M_Q \to \mathbb{R}$ the projection to $\mathbb{R}$. Then the following hold:*

1. *The fibers of the projection maps $(\pi_i, \eta)$, $i = 1, 2$ are Legendrian in $M_Q$.*
2. *The Reeb vector field $R_{\mathscr{A}}$ is given by $(0, R_\alpha, 0)$.*
3. *For any contactomorphism $g$ of $(Q,\xi)$ with $g^*\alpha = e^{\ell_g}\alpha$, the map*

$$j_g(y) = (y, g(y), \ell_g(y)) \tag{3.11}$$

*is a Legendrian embedding of $Q$ into $(M_Q,\mathscr{A})$.*

*We call the image of the map $j_g$ the* Legendrianization *of $g$* [Oh22a].

One can utilize this to give a local parametrization of a neighborhood of the identity in $\mathrm{Cont}(M,\xi)$, which also shows local contractibility of the $\mathrm{Cont}_c^r(M,\xi)$. (See [Ly], [Ba2].)

We will apply this construction to the case when $Q = J^1\mathbb{R}^n \cong \mathbb{R}^{2n+1}$ equipped with the standard contact form $\alpha_0 = dz - pdq$ in the canonical coordinates

$$(z, q_1, \cdots, q_n, p_1, \cdots, p_n).$$

In terms of the coordinate system $(x, X, \eta)$, which we call the contact product coordinate system, of

$$M_{\mathbb{R}^{2n+1}} = \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R}$$

with $x = (z, q, p)$ and $X = (Z, Q, P)$, we have

$$\pi_1^*\alpha_0 = dz - p\, dq, \quad \pi_2^*\alpha_0 = dZ - P\, dQ.$$

Then the form $\mathscr{A}$ is given by

$$\mathscr{A}_{\mathbb{R}^{2n+1}} = -e^\eta(dz - p\, dq) + (dZ - P\, dQ) \tag{3.12}$$

in the given coordinate system. We highlight the location of the $\mathbb{R}$-coordinate $\eta$ that is written *at the last spot*, while the $\mathbb{R}$-coordinate in the 1-jet coordinate system of $J^1\mathbb{R}^{2n+1}$ is written *at the first spot. This practice will be consistently used throughout the present paper.*

**Remark 3.8** (Locations of the $\mathbb{R}$ coordinates)**.** Here is the reason why we exercise the aforementioned practice. There are quite a few constructions carried out on

$$\mathbb{R}^{2(2n+1)+1} \cong J^1\mathbb{R}^{2n+1} \cong M_{\mathbb{R}^{2n+1}}.$$

Some of them are more natural to consider in $J^1\mathbb{R}^{2n+1}$ but others are in $M_{\mathbb{R}^{2n+1}}$. Some of the constructions may be the most natural even in $\mathbb{R}^{2(2n+1)+1}$. We would like to make it clear on which space we perform the constructions by distinguishing the ways of representing points in $M_{\mathbb{R}^{2n+1}}$ and in $J^1\mathbb{R}^{2n+1}$ at least by putting the $\mathbb{R}$ coordinates differently. Luckily, the articles [Bhu] and the present author's preprints [Oh21b], [Oh22a] have already used the same convention putting the $\mathbb{R}$-coordinate in the last spot as $Q \times Q \times \mathbb{R} = M_Q$ which makes the current practice consistent with them. Furthermore we take the same practice as [Ryb2] put the $\mathbb{R}$ coordinate in the 1-jet bundle $J^1\mathbb{R}^{2n+1}$ in the first spot so that comparing the notations and the details from [Ryb2] and from the current paper is hoped to become easier.

## 4. Basic contact vector fields and contactomorphisms of $\mathbb{R}^{2n+1}$

We take the (global) frame

$$\left\{\frac{\partial}{\partial z}, \frac{D}{\partial q_i}, \frac{\partial}{\partial p_i}\right\},$$

on $\mathbb{R}^{2n+1} = J^1\mathbb{R}^n$ which is equipped with the standard contact form

$$\alpha_0 = dz - \sum_{i=1}^n p_i\, dq_i.$$

Here we write

$$\frac{\partial}{\partial q_i} + q_i\frac{\partial}{\partial z} =: \frac{D}{\partial q_i}$$

following the notation from [LOTV]. We mention that $\{\frac{D}{\partial q_i}, \frac{\partial}{\partial p_i}\}$ is a Darboux frame of the contact distribution $\xi$. More specifically, they are tangent to the distribution $\xi$ and satisfies

$$d\alpha\left(\frac{D}{\partial q_i}, \frac{\partial}{\partial p_j}\right) = \delta_{ij}.$$

Note that the collection $\{\frac{\partial}{\partial q_i} + q_i\frac{\partial}{\partial z}, \frac{\partial}{\partial p_i}\}$ provides a global Darboux frame of the contact distribution $\xi$ of $J^1\mathbb{R}^n$. We mention that except the Reeb vector field $\frac{\partial}{\partial z}$ (whose Hamiltonian is the constant function $-1$) none of these vector fields are contact.

### 4.1. Basic contact Hamiltonian vector fields.

Now we consider the flows of various basic Hamiltonian vector fields:

(1) Consider the Hamiltonian $H = -q_i$ whose Hamiltonian vector field is given by

$$X_{-q_i} = \frac{\partial}{\partial p_i} + q_i\frac{\partial}{\partial z} = \vec{f}_i + q_i\vec{e}_z.$$

It generates the translation flow in the $p_i$-direction

$$\psi^t_{-q_i}(z, q, p) = (z + tq_i, q_1, \cdots, q_n, p_1, \cdots, p_{i-1}, p_i + t, p_{i+1}, \cdots, p_n) =: T_i^t.$$

In particular, we write

$$T_i := \psi^1_{-q_i}$$

the translation by 1 in the $p_i$-direction.

(2) Next, we consider $H = p_i$ and its Hamiltonian vector field
$$X_{p_i} = \frac{\partial}{\partial q_i} = \vec{e}_i$$
whose flow is given by
$$\psi^t_{p_i} = (z, q_1, \cdots, q_i + t, \cdots, q_n, p_1, \cdots, p_n) =: S^t_i$$
In particular, we have
$$\psi^1_{p_i}(z, q, p) = (z, q + \vec{e}_i, p) =: S_i.$$

(3) We also consider the *contact Euler vector field* $E^c$ defined by
$$E^c = 2z\frac{\partial}{\partial z} + \sum_{i=1}^n \left(q_i \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial p_i}\right) \tag{4.1}$$
which is a contact vector field associated to the Hamiltonian $H^c := 2z - \sum_{i=1}^n q_i p_i$, and generates the flows the *contact rescaling* $\eta_t$ given by
$$\psi^t_{H^c}(z, q, p) = (e^{2t} z,\ e^t\, q, e^t\, p). \tag{4.2}$$

(4) The *front Euler vector field*
$$E^f = \sum_{i=1}^n q_i \frac{\partial}{\partial q_i} + z\frac{\partial}{\partial z} \tag{4.3}$$
is associated to the Hamiltonian $H^f := z - \sum_{i=1}^n q_i p_i$ and generate and the *front rescaling* $\chi_t$ given by
$$\psi_{H^f}(q, p, t) = (e^t,\ e^t q, p). \tag{4.4}$$

We denote the time-one maps of $E^c$ and $E^f$ by $R^c$ and $R^f$ respectively.

**Remark 4.1.** We would like to warn the readers that our definition is different from that of [Ryb2] for the signs in that the Hamiltonian associated to a contact vector field $X$ is given by $H := -\alpha(X)$, while [Ryb2] adopts its negative $H = \alpha(X)$.

4.2. **Contact cut-off of basic Hamiltonian vector fields.** In summary, we will consider the following collection of contact diffeomorphisms
$$\{T_i,\ S_i\}_{i=1}^n \cup \{R^c,\ R^f\} \tag{4.5}$$
throughout the paper, which we call the *basic contact transformations* on $\mathbb{R}^{2n+1}$. Obviously, none of these contactomorphisms are compactly supported on $\mathbb{R}^{2n+1}$ while our concern in on the compactly supported contactomorphisms. However all of them are generated by contact Hamiltonian vector fields and so we can cut down their supports by multiplying a bump function *to the associated Hamiltonian functions* as done in [Ba1] for the symplectic case and in [Ryb2] for the contact case, when there is given a compact subset $K \subset \mathbb{R}^{2n+1}$.

It deserves some explanations to explain this contact cut-off of basic Hamiltonian vector fields and their induced contactomorphisms, which we feel is somewhat counter intuitive. We will focus on the cut-off of the case of transformation $T_j = \psi^1_{p_i}$ which is generated by the Hamiltonian vector field
$$\frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial z}$$
on $\mathbb{R}^{2n+1} \cong J^1\mathbb{R}^n$. It generates the flow
$$T^t(z, q, p) = (z + q_i t, q, p + t\vec{f}_i).$$
If we multiply a cut-off function of the form
$$\kappa = \kappa_0 \times \left(\prod_{i=1}^n \kappa_i\right) \times \left(\prod_{j=1}^n \kappa_{n+j}\right)$$

supported in the interior of a compact set

$$E = \{(z,q,p) \mid |z| \leq 2a_0,\ |q_i| \leq 2a_i,\ |p_j| \leq 2b_j\}$$

that also satisfy $\kappa_i \equiv 1$ on $[-a_i, a_i]$ and $\kappa_{n+j} \equiv b_j$ on $[-b_j, b_j]$ for some positive numbers $a_i$, $b_j > 0$ for $i = 0, \ldots, n$ and $j = 1, \ldots, n$

We write $\rho_i := (\kappa/\kappa_i)$ which does not depend on $q_i$ by definition. Then we compute the Hamiltonian vector field of $H = q_j \cdot \kappa = (q_j\chi_j) \cdot \rho_j$. We substitute $H$ into (2.6) and obtain

$$X_H = -\left((q_i\kappa_i)'\frac{\partial}{\partial p_i} + q_i(q_i\kappa_i)'\frac{\partial}{\partial z}\right) + X'$$

where $X'$ is the vector field that does not contain components of $\frac{\partial}{\partial q_i}$ and $\frac{\partial}{\partial z}$ which is still supported in $E$. We not examine the components in the two directions of $q_i$ and $z$, which is

$$(q_i\kappa_i)'\frac{\partial}{\partial p_i} + q_i(q_i)'\frac{\partial}{\partial z} = (\kappa_i + q_i')\frac{\partial}{\partial p_i} + q_i(\kappa_i + q_i\kappa_i' - q_i\kappa_i\rho_i)\frac{\partial}{\partial z}.$$

Therefore the $p_i$ component of its flow is given by

$$t \mapsto p_i + t((\kappa_i(q_i) + q_i\kappa_i'(q_i)) =: p_i(t)$$

and the $z$ component is given by

$$t \mapsto z + q_i(\kappa_i + q_i\kappa_i' - q_i\kappa_i\rho_i)t =: z(t).$$

The upshot of the above calculation is to show that *the coordinate function $-q_i$ does not generate a pure $p_i$-translation but only the one coupled with a $z$-translation, while $p_i$ generates the pure $q+i$-translation.*

**Lemma 4.2.** *The cut-off flow $\tau^t$ maps $E$ into $E$ and is a translation with constant speed $q_i$ in the direction of $p_i$ on $(-b_i, b_i)$.*

*Proof.* The first statement immediately follows from the observation that the vector field $X_{p_i}$ vanishes outside $E$, and the second follows from the property $\equiv 1$ on $\frac{1}{2}E \subset E$. □

These preparations on the cut-off being made and mentioned, we will omit the process throughout the paper, without further mentioning of converting (4.5) to the relevant compactly supported counterparts see [Ma1, Construction in p. 519] for the precise description of the process in the case of $\mathrm{Diff}^r_c(M)$): We denote the compactly supported contactomorphisms obtained this way by

$$\{\tau_i, \sigma_i\} \cup \{\chi, \eta\}. \tag{4.6}$$

We will also consistently utilize the following notations for the corresponding collection

$$\chi_A := \psi_{H^c}^{\log A}, \quad \eta_A := \psi_{H^f}^{\log A}, \quad A > 1$$

respectively, where we also denote by $\chi_A$, $\eta_A$ the associated cut-off version thereof by an abuse of notations.

## Part 1. Mather-Rybicki's constructions for $\mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)_0$

### 5. An equivariant Darboux-Weinstein chart

An immediate corollary of the standard Darboux-Weinstein theorem is that any $C^1$-small perturbation $R'$ of the given Legendrian submanifold $R$ can be uniquely written as

$$R' = \Phi_U^{-1}(j^1u(R))$$

for some smooth function $u : R \to \mathbb{R}$ where $j^1u$ is the 1-jet graph of $f$ via the chart $\Phi_U : U \to V$. Conversely for any $C^1$-small (resp. $C^k$-small) smooth function $u$, its 1-jet graph corresponds to is a $C^0$-small (resp. $C^{k-1}$-small) smooth perturbation of $R$. However we want some additional equivariant property for the chart with respect to some actions by the group

$$G = \mathbb{R}^{n+1} \quad \text{or} \quad T^k \times \mathbb{R}^{n+1-k}.$$

with respect to some actions for our purpose of the present paper. This leads us to involve a variation of the *contact product* [Ly, Ba2, Bhu, Oh22a, Ryb2], when we apply it to the standard 'linear' contact manifold $\mathbb{R}^{2n+1}$.

5.1. **Coordinate transforms on $\mathbb{R}^{2(2n+1)+1}$.** Following Rybicki [Ryb2], we consider the contact cylinders denoted by

$$\mathcal{W}_k^{2n+1} = T^k \times \mathbb{R}^{2n+1-k} \cong (T^k \times \mathbb{R}^{n-k}) \times \mathbb{R}^n.$$

We first consider its 1-jet bundle

$$J^1\mathcal{W}_k^{2n+1} \cong \mathbb{R} \times T^*\mathcal{W}_k^{2n+1} \cong \mathbb{R} \times \mathcal{W}_k^{2n+1} \times \mathbb{R}^{2n+1}, \quad \alpha_0 = d\xi_0 - \sum_{i=1}^n p_i d\xi_i.$$

(For the simplicity of notations, we sometime write $2n+1 =: m$.) We write the associated canonical coordinates as $(t, x, \mathbf{p}_x)$ with $x = (z, q, p)$ where $\mathbf{p}_x = (z_x, q_x, p_x)$ is the conjugate coordinates of $x$. Here we regard $q_i$ as $S^1$-valued for $i = 1, \ldots, k$ and real-valued for $i = k+1, \ldots, m$ when we consider $\mathcal{W}_k^m$ instead of $\mathbb{R}^{2n+1}$. Consider the Legendrian submanifold

$$R := \mathcal{Z}_{\mathcal{W}_k^m} \cong \{0\}_{\mathbb{R}} \times \mathcal{W}_k^m \times \{0\}_{(\mathbb{R}^n)^*} \subset J^1\mathcal{W}_k^m \cong \mathbb{R} \times T^*\mathcal{W}_k^m, \tag{5.1}$$

the zero section of $J^1\mathcal{W}_k^m$. By the definition of $C^r$-topology of $\mathrm{Cont}(M, \alpha)$ (Definition 3.3) in general, it follows that any contactomorphism of $\mathcal{W}_k^m$ $C^1$-close the identity can be uniquely lifted to a contactomorphism $\widetilde{f}$ $C^1$-close to the identity on $\mathbb{R}^{2n+1}$:

$$\begin{tikzcd} \mathbb{R}^{2n+1} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi"] & \mathbb{R}^{2n+1} \arrow[d, "\pi"] \\ \mathcal{W}_k^m \arrow[r, "f"] & \mathcal{W}_k^m. \end{tikzcd}$$

**Remark 5.1** (Notational abuse).

(1) We can uniquely lift the chart $\Phi_U$ to $\widetilde{\Phi}_U : \widetilde{U} \to \widetilde{V}$ under the covering projection $\mathbb{R}^{2n+1} \to \mathcal{W}_k^{2n+1}$ so that the following diagram commutes:

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\widetilde{\Phi}_U"] \arrow[d, "\pi"'] & \widetilde{V} \arrow[d, "\pi"'] \\ U \arrow[r, "\Phi_U"'] & V \end{tikzcd}$$

where $\widetilde{U} = \pi^{-1}(U)$ and $\widetilde{V} = \pi^{-1}(V)$.

(2) Since we will be interested in $C^1$-small diffeomorphisms, we may and will always assume that this unique lifting is fixed. Once this is being said, we will drop the tilde from notations and abuse the notation $\Phi_U : U \to V$ also to denote the latter. Since the covering projections of $\Pi$ are linear isometries with respect to the standard linear structure and the metric of $\mathbb{R}^{2(2n+1)+1}$ and $\delta$ is a linear map, all the estimates we will perform will be done on $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2(2n+1)+1}$. This enables us to freely use the global coordinates of $\mathbb{R}^{2(2n+1)+1}$. We will adopt this approach in the rest of the paper, especially when we do the estimates in Part 2 of the present paper, unless there is a danger of confusion. (See [Ma1, p.525] for the similar practice laid out for the same purpose of study of derivative estimates in the similar setting of covering projections $\mathbb{R}^n \to \mathscr{C}_i$ with $\mathscr{C}_i \cong S^1 \times \mathbb{R}^{n-1}$.

5.2. **Contact product of $\mathcal{W}_k^m$ and equivariant Darboux-Weinstein chart.** Recall the general definition of the *contact product* $(M_Q, \mathscr{A})$ of a contact manifold $(Q, \alpha)$, $M_Q := Q \times Q \times \mathbb{R}$ equipped with the contact form

$$\mathscr{A} = -e^{\eta}\pi_1^*\alpha_0 + \pi_2^*\alpha_0.$$

Then the *contact diagonal* is given by

$$\{(x, x, 0) \in M_Q \mid x \in Q\} =: \Gamma_{\text{id}}. \tag{5.2}$$

By construction, it follows that $\Gamma_{\text{id}}$. is Legendrian with respect to $\mathscr{A}$, which is diffeomorphic to $R$ given in (5.1). More generally, recall that the *graph*

$$\Gamma_f : Q \to M_Q; \quad \Gamma_f(x) := (x, f(x), \ell_f(x)) \tag{5.3}$$

is a Legendrian embedding of $(M_Q, \mathscr{A})$ which is called the *Legendrianization* of $f$ in [Oh22a]. With a slight abuse of notations, we will denote by $\Gamma_f$ both the map and its image.

For the purpose of applying a Mather-type construction on $Q = \mathbb{R}^{2n+1}$, we would like to achieve parameterize each contactomorphism $f$ of $\mathcal{W}_k^m$ *sufficiently $C^1$-close to the identity map* by a *real-valued function $u_f$* on $\mathbb{R}^{2n+1}$. For this purpose, we want to associate to the map

$$\widetilde{f} - id : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1},$$

not $f$ itself, a Legendrian submanifold in $(J^1\mathbb{R}^{2n+1}, \alpha_0)$ so that it becomes the graph of 1-jet map $j^1 u_f : \mathbb{R}^{2n+1} \to J^1\mathbb{R}^{2n+1}$ for some real-valued function $u_f$. To achieve this goal, we consider the map $\delta : M_{\mathbb{R}^{2n+1}} \to M_{\mathbb{R}^{2n+1}}$ defined by

$$\delta(x, X, \eta) = (x, X + x, \eta). \tag{5.4}$$

Obviously its inverse is given by

$$\delta^{-1}(x, X, \eta) = (x, X - x, \eta). \tag{5.5}$$

(See [Ryb2, p. 3300] for a similar consideration.)

**Remark 5.2.** This operation of taking the difference $\widetilde{f} - id$ is natural on $M_{\mathbb{R}^{2n+1}}$ while it is very unnatural as one made on $J^1\mathbb{R}^{2n+1}$. Therefore the operation $\delta$ is defined as one on $M_{\mathbb{R}^{2n+1}}$. However it is not a contact diffeomorphism of $(M_{\mathbb{R}^{2n+1}}, \mathscr{A})$ and hence the graph $\widetilde{f} - \text{id}$ is not Legendrian for contactomorphism $f$ unlike $\Gamma_f$.

Here we use the collective coordinate system $(x, X, \eta)$ on $\mathbb{R}^{2(2n+1)+1}$ with

$$x = (z, q, p), \quad X = (Z, Q, P)$$

as the coordinate system on the contact product $M_{\mathbb{R}^{2n+1}}$ by identifying $R^{2(2n+1)+1}$ with $\mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R}$.

**Remark 5.3.** We emphasize the fact that this kind of linear contactomorphism is a construction that exists only for the linear contact manifold $\mathbb{R}^{2n+1}$, not for for general contact manifold $Q$, while the contact product is a general functorial construction applied to an arbitrary contact manifold $Q$.

We pull-back $\mathscr{A}$ by the diffeomorphism $\delta$ and set a new contact form

$$\widehat{\mathscr{A}}|_{(t,x,X)} := \delta_* \mathscr{A}|_{(t,x,X)} = -e^t(dz - pdq) + d(z + Z) - \sum_{i=1}^n (p + P) d(q + Q) \tag{5.6}$$

on $M_{\mathbb{R}^{2n+1}}$. We also consider the tautological map

$$\Pi : J^1\mathbb{R}^{2n+1} \to M_{\mathbb{R}^{2n+1}} = \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R} \tag{5.7}$$

defined by $\Pi(t, x, X) = (x, X, t)$ where we identify $X$ with the conjugate $p_x$ of $x$ on the domain of the map $\Pi$, and have the diagram

$$\begin{tikzcd} M_{\mathbb{R}^{2n+1}} \arrow[r, "\delta"] & M_{\mathbb{R}^{2n+1}} \arrow[d, "\mathrm{pr}_2"'] \\ J^1\mathbb{R}^{2n+1} \arrow[u, "\Pi"] \arrow[r, "\pi"'] & \mathbb{R}^{2n+1}. \end{tikzcd} \tag{5.8}$$

Obviously the map

$$\Delta_f := (\delta\Pi)^{-1} \circ \Gamma_f : \mathbb{R}^{2n+1} \to J^1\mathbb{R}^{2n+1}$$

is not Legendrian with respect to $\alpha_0$ but becomes *Legendrian* with respect to the contact form $\widehat{\mathscr{A}}$ for all contactomorphism $f$ of $\mathbb{R}^{2n+1}$. Similarly as we did for $\Gamma_f$, we denote by $\Delta_f$ both the map and its image.

The following is by now obvious and reflects the 'linear structure', which plays a significant role in Mather's construction, on the contact manifold $\mathbb{R}^{2n+1}$, and is utilized in Rybicki's contact version [Ryb2] of the rolling-up and homothetic transformations of Mather [Ma1].

**Lemma 5.4.** *Let $f \in \mathrm{Cont}(\mathbb{R}^{2n+1}, \alpha_0)$ and consider the graph of the map $\Delta_f : \mathbb{R}^{2n+1} \to J^1\mathbb{R}^{2n+1}$ given by*

$$\Delta_f := (\delta\Pi)^{-1} \circ \Gamma_f = \left\{(\ell_f(x), x, f(x) - x) \in J^1\mathbb{R}^{2n+1} \mid x \in R^{2n+1}\right\}. \tag{5.9}$$

*Then for any $f$ $C^1$-close to the identity map, $\Delta_f$ is a section of the projection $\pi : J^1\mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$, which is also Legendrian with respect to the contact form $\widehat{\mathscr{A}}$. For $f = id$, we have*

$$\Delta_{\mathrm{id}} = \mathcal{Z}_{\mathcal{W}_k^{2n+1}}. \tag{5.10}$$

*Proof.* By definition, we have $\pi \circ \Delta_f$ is bijective on $\mathbb{R}^{2n+1}$ which shows that $\Delta_f$ is associated to a section of the projection $\pi$. By definition, we have

$$(\delta\Pi) \circ \Delta_f = \Gamma_f \tag{5.11}$$

and hence

$$\Delta_f^*(\widehat{\mathscr{A}}) = (\delta^{-1} \circ \Gamma_f)^* \widehat{\mathscr{A}} = \Gamma_f^* \mathscr{A}$$

This shows that $\Delta_f$ is Legendrian with respect to $\Pi^*\widehat{\mathscr{A}}$ if and only if $\Gamma_f$ is Legendrian with respect to $\mathscr{A}$. This finishes the proof. □

**Remark 5.5.** In [Ryb2], the notation $\Gamma_f$ is associated to $f - \mathrm{id}$, not to $f$, unlike the present paper. In the present paper, we reserve the notation $\Gamma_f$ the Legendrian graph in $M_{\mathbb{Q}}$ in general to be compatible with the notation from [Bhu, Oh22a]. The notation $\Delta_f$ for the Legendrian map is reserved for a Legendrian submanifold associated to $f - \mathrm{id}$ applied *only for the 1-jet bundle $J^1\mathcal{W}_k^{2n+1}$ with respect to the contact form* $\Pi^*\widehat{\mathscr{A}}$ which is closer to $\Gamma_f$ adopted in [Ryb2].

Then we would like to associate to each such $f$ a real-valued function on $\mathcal{W}_k^{2n+1}$. Knowing that (5.10) holds, we can apply the standard Darboux-Weinstein theorem (contact version). On the other hand, we will also want some fiber preserving property for the chart. (See [Ryb2, p.3300].) To naturally construct such a chart, we will apply some *equivariance* for the Darboux-Weinstein chart exploiting the linear structure of $\mathbb{R}^m$ (resp. $\mathcal{W}_k^m$) whose description is now in order.

We consider the abelian groups

$$G := \mathbb{R}^{2n+1} \quad \text{or} \quad T^k \times \mathbb{R}^{n+1-k} \tag{5.12}$$

and consider its actions on $M_{\mathcal{W}_k^{2n+1}}$ and on $J^1\mathbb{R}^{2n+1}$ respectively. We consider the contact $G$-actions on $(M_{\mathbb{R}^{2n+1}}, \widehat{\mathscr{A}})$

$$\mathcal{G}_1 : (g, (t, x, X)) \mapsto (t, x + (g, 0), X + (g, 0))$$

and on $(J^1\mathbb{R}^{2n+1}, \alpha_0)$ the one given by

$$\mathcal{G}_2 : (g, (t, x, X) \mapsto (t, x + (g, 0), X).$$

Here $(g, 0) \in \left(S^1 \times (T^{k-1} \times R^{n-k+1})\right) \times \mathbb{R}^n \cong \mathcal{W}_k^{2n+1}$ and we identify the momentum coordinates$p_x$ with $X$. which is possible by the linearity of $Q = \mathbb{R}^{2n+1}$.

Then the map $\delta\Pi$ is a $(\mathcal{G}_1, \mathcal{G}_2)$-equivariant contact diffeomorphism which makes the following commuting diagram

$$\begin{tikzcd}
(J^1\mathbb{R}^{2n+1}, \Pi^*\widehat{\mathscr{A}}) \arrow[rr, "\delta\Pi"] \arrow[dr, "\pi"'] & & (M_{\mathbb{R}^{2n+1}}, \mathscr{A}) \arrow[dl, "\mathrm{pr}_2"] \\
& \mathbb{R}^{2n+1}. &
\end{tikzcd}$$

The following is a special form of an equivariant Darboux-Weinstein theorem implicitly utilized by Rybicki [Ryb2]. Recall $\Delta_{\mathrm{id}} = \mathcal{Z}_{\mathcal{W}_k^m}$.

**Proposition 5.6.** *There exists a $\mathcal{G}_2$-equivariant open neighborhoods $U$ and $V$ of $\mathcal{Z}_{\mathcal{W}_k^m} = R$ in $J^1\mathbb{R}^{2n+1}$ and $\mathcal{G}_2$-equivariant diffeomorphism $\Phi_U : U \to V$ such that*

*(1) $\Phi_U^*\alpha_0 = \widehat{\mathscr{A}}$ and $\Phi_U \circ \Delta_{\mathrm{id}} = \mathcal{Z}_{\mathcal{W}_k^m}$.*
*(2) It satisfies*

$$T_{(0,x,0)}\Phi_U(\xi_x, \xi_x + \xi_X, \xi_t) = (\xi_t, \xi_x, \xi_X) \in T_{(0,x,0)}J^1\mathbb{R}^{2n+1}$$

*for each element $(\xi_x, \xi_x + \xi_X, \xi_t) \in T_{(0,x,0)}M_{\mathbb{R}^{2n+1}}$.*

**Remark 5.7.** The geometric meaning of Statement (2) is that the derivative $d\Phi_U$ maps the linear polarization

$$\{(a, \xi_x, v + \xi_x) \mid v \in \mathbb{R}^{2n+1}, a \in \mathbb{R}\}$$

to

$$\{(a, \xi_x, v) \mid v \in \mathbb{R}^{2n+1}, a \in \mathbb{R}\}$$

for each given vector $v_x \in \mathbb{R}^{2n+1} = T_x\mathcal{W}_k^{n+1}$. For $\xi_x = 0$, the two subspaces coincide while for $\xi_x \neq 0$, the first subspace is moved by a translation of the second by $\xi_x$ in the $X$ direction (i.e., the fiber direction).

An immediate corollary of the $G$-equivariance is the following. For readers' convenience, we give its proof in Appendix A.

**Corollary 5.8.** *We have the expression*

$$\Pi\Phi_U^{-1}(t, x, X) = (x + h_x(X,t), x + h_X(X,t), (t + h_t(X,t)) \in M_{\mathbb{R}^{2n+1}}$$

*such that $h_t(0,x,0) = 0$, $h_x(0,x,0) = 0$ and $h_X(0,x,0) = 0$. If we define the map $\mathsf{H}$ to be*

$$\mathsf{H}(x, X, t) = (h_t(X,t), h_x(X,t), h_X(X,t)) \tag{5.13}$$

*as a map $M_{\mathbb{R}^{2n+1}} \to J^1\mathbb{R}^{2n+1}$, it does not depend on $x$.*

Obviously from this corollary, the map

$$(\Pi\Phi_U^{-1})^{-1}\delta^{-1} : (M_{\mathbb{R}^{2n+1}}, \mathscr{A}) \to (J^1\mathbb{R}^{2n+1}, \alpha_0)$$

is a contact diffeomorphism which has the form

$$(\Pi\Phi_U^{-1})^{-1}\delta^{-1} = \Pi^{-1} + \mathsf{H} \tag{5.14}$$

on $U'$. (Recall that $\Pi$ is just the $\mathbb{R}$ coordinate swapping map and the identity map if we identify $J^1\mathbb{R}^{2n+1}$ and $M_{\mathbb{R}^{2n+1}}$ with $\mathbb{R}^{2(2n+1)+1}$.) We summarize the above discussion into the diagram

$$\begin{tikzcd}
& (M_{\mathbb{R}^{2n+1}}, \widehat{\mathscr{A}}) \arrow[r, "\delta"] & (M_{\mathbb{R}^{2n+1}}, \mathscr{A}) \arrow[d, "\mathrm{pr}_2"] \arrow[dl, "(\Pi\Phi_U^{-1})^{-1}\delta^{-1}"'] \\
(J^1\mathbb{R}^{2n+1}, \Pi^*\widehat{\mathscr{A}}) \arrow[ur, "\Pi"] \arrow[r, "\Phi_U"'] & (J^1\mathbb{R}^{2n+1}, \alpha_0) \arrow[u, "\Pi\circ\Phi_U^{-1}"] \arrow[r, "\pi"'] & \mathbb{R}^{2n+1}
\end{tikzcd}$$

where all maps in the left triangle and $\delta$ are contact diffeomorphisms by definition. In particular, the map

$$(\Pi\Phi_U^{-1})^{-1}\delta^{-1} \circ \Gamma_f = \Phi_U(\delta\Pi)^{-1} \circ \Gamma_f = \Phi_U \circ \Delta_f \tag{5.15}$$

is a Legendrian embedding of $\mathbb{R}^{2n+1}$ into $(J^1\mathbb{R}^{2n+1}, \alpha_0)$ that is also graph-like for any contactomorphism $f$ sufficiently $C^1$-close to the identity. (See (5.9).)

We remark that the map $\mathsf{H}$ depends only on the Darboux-Weinstein chart $\Phi_U$.

## 6. Parametrization of $C^1$-small contactomorphisms by 1-jet potentials

We recall that any Legendrian submanifold $R$ $C^1$-close to the zero section $\mathcal{Z}_N \subset J^1N$ can be written as the 1-jet graph

$$R_u := \operatorname{Image} j^1u = \{(z,q,p) \mid p = du(q),\, z = u(q)\}$$

for some smooth function $u : R \to \mathbb{R}$. We call $u$ the *(strict) generating function* of $R$. This correspondence is one-to-one in a $C^1$-small perturbations of the zero section.

**Remark 6.1.** We remark that if $f$ is a contactomorphism $C^1$-close to the identity (say, $M_1^*(f) = \max\{\|f\|_1, \|\ell_f\|_1\} < \frac{1}{4}$), its lift to $\mathbb{R}^{2n+1}$ can be written as $\widetilde{f} = \mathrm{id} + \widetilde{v}$ for a map that is $C^1$ close to the zero map. This also implies that the map $v$ is $C^1$-close to the zero section map of the *front projection* $J^1\mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ and $v \equiv 0$ when $f = \mathrm{id}_{\mathcal{W}_k^m}$. However unlike the diffeomorphism case of [Ma1], *there is no obvious such* linear perturbation result *of contactomorphisms*. We suspect that this phenomenon leads to the contact counterpart of the discussion about the failure of Mather's scheme in [Ma1, Ma2] on the nose for the critical case $r = n+1$ as explained in [Ma2]. We will investigate this phenomenon elsewhere.

By a suitable contact conformal rescaling of $\Phi_U$, we may assume

$$\pi(\Phi_U(U)) \supset [-1,1]^{2n+1} \tag{6.1}$$

for the chart $\Phi_U$ where $\pi : J^1\mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is the canonical projection. We set

$$K_{\Phi_U,r} := \sup_{0 \le s \le r+1} \max\left\{ \left\| D^s \Phi_{U \cap \widetilde{E}_1^{(0)}} \right\|, \left\| D^s (\Phi_{U \cap \widetilde{E}_1^{(0)}})^{-1} \right\| \right\}, \tag{6.2}$$

This constant is a universal constant depending only on the Darboux-Weinstein chart $\Phi_U$ and $r$. We remark that the set $U \cap \widetilde{E}_1^{(0)}$ is relatively compact and so $K_{\Phi_U,r} < \infty$ for all $r \geq 1$.

*We will fix the chart $\Phi_U$ as the reference in the rest of the paper* for other charts that will be obtained by further conformal rescalings. The latter will be denoted by

$$\Phi_{U;A}, \quad A \geq 1$$

depending on the constant $A$, whose precise definition is now in order.

With slight abuse of terminology, we call such a contactomorphism $C^1$ close to the identity a $C^1$*-small* contactomorphism.

### 6.1. Rescaled Darboux-Weinstein chart $\Phi_{U;A}$.

We recall the maps $\chi_A$ and $\eta_A$ given in (4.4) and (4.2) respectively with $t = A$. Both satisfy that

$$\chi_A^* \alpha_0 = A^2 \alpha_0 = \eta_A^* \alpha_0 \tag{6.3}$$

where $\alpha_0$ is the standard contact form on $\mathbb{R}^{2n+1} \cong J^1\mathbb{R}^n$. Furthermore, we have

$$\chi_A\left([-1,1]^{2n+1}\right) = [-2A^2, 2A^2] \times [-A,A]^{2n} \tag{6.4}$$

$$\eta_A\left([-1,1]^{2n+1}\right) = [-A,A]^{n+1} \times [-1,1]^n. \tag{6.5}$$

By a suitable conjugating process by the maps $\chi_A$ and $\nu_A$, we lift the map $\chi_A$ on $\mathcal{W}_k^m$ to the maps

$$\begin{aligned} \mu_A &= \chi_A \times \chi_A \times \mathrm{id}_{\mathbb{R}} \quad \text{on } \mathcal{W}_k^m \times \mathcal{W}_k^m \times \mathbb{R} = M_{\mathcal{W}_k^m} \\ \nu_A &= (A^2\, \mathrm{id}_{\mathbb{R}}) \times \chi_A \times \eta_A \quad \text{on } \mathbb{R} \times T^*\mathcal{W}_k^m = J^1\mathcal{W}_k^m \end{aligned}$$

as contact automorphisms, respectively. In fact they can be lifted to $\mathbb{R}^{2(2n+1)+1}$ explicitly expressed as

$$\mu_A(z,q,p,Z,Q,P,\eta) = (A^2z, Aq, Ap, A^2Z, AQ, AP, \eta), \tag{6.6}$$

$$\nu_A(t,z,q,p,Z,Q,P) = (A^2t, A^2z, Aq, Ap, AZ, AQ, P) \tag{6.7}$$

in terms of the standard coordinates of $M_{\mathbb{R}^{2n+1}}$ and $J^1\mathbb{R}^{2n+1}$ respectively. It is easy to check from this that they indeed satisfy

$$\mu_A^* \widehat{\mathscr{A}} = A^2 \widehat{\mathscr{A}}, \quad \nu_A^* \alpha_0 = A^2 \alpha_0$$

respectively, i.e., $\mu_A$ and $\nu_A$ define contactomorphisms of $(J^1\mathbb{R}^{2n+1}, \widehat{\mathscr{A}})$ and $(J^1\mathbb{R}^{2n+1}, \alpha_0)$, respectively. This being said, we will also denote by the same notation $\mu_A$ for the obvious action on $J^1\mathbb{R}^{2n+1}$ conjugate by the map $\Pi$, recalling that $\Pi$ is essentially the identity map as a map defined on $\mathbb{R}^{2(2n+1)+1}$.

**Definition 6.2** ($\Phi_{U;A}$)**.** Let $A \geq 1$ be given and consider the expression

$$\Phi_{U;A} := \nu_A \circ \Phi_U \circ \mu_A^{-1} : U_A \to V_A \tag{6.8}$$

and write

$$U_A := \mu_A(U),\ V_A := \nu_A(V)$$

where we regard the subsets as ones either on $J^1\mathbb{R}^{2n+1}$ or on $J^1\mathcal{W}_k^m$.

Then the rescaled chart map

$$\Phi_{U;A} : (U_A, \Pi^*\widehat{\mathscr{A}}) \to (V_A, \alpha_))$$

is a well-defined strict contactomorphism for all $A > 1$.

**Proposition 6.3** (Compare Proposition 4.2 (2) [Ryb2])**.** *Let $r \geq 2$ be given. For any given $A_0 > 1$, let $1 \leq A \leq A_0$ and consider a subinterval $E \subset E_A^{(k)}$.*

*(1) Then the map*

$$\Phi_{U;A} : U_A \to V_A$$

*is defined and satisfies $\Phi_{U;A}|_R = id_R$, $\Phi_{U;A}^*\widehat{\mathscr{A}} = \alpha_0$.*

*(2) Define the constants*

$$K_{\Phi_U,r,A} := \sup_{0 \leq s \leq r+1} \max\left\{ \left\| D^s \Phi_{U;A}|_{U_A \cap \widetilde{E}_A^{(0)}} \right\|, \left\| D^s (\Phi_{U;A}|_{U_A \cap \widetilde{E}_A^{(0)}})^{-1} \right\| \right\}. \tag{6.9}$$

*Then $K_{\Phi_U,r,A} \leq A^2 K_{\Phi_{U,r}}$ for all $0 \leq s \leq r$.*

An upshot of this proposition is that when $A_0 > 0$ and $r$ are given, the constants cam be uniformly controlled depending only on the original chart $\Phi_U$ and on $A_0$ which however does not depend on individual $A$ from $1 \leq A \leq A_0$.

Following [Ryb2], we consider the cylinders

$$\mathcal{W}_k^{2n+1} := (S^1)^k \times \mathbb{R}^{2n+1-k} \tag{6.10}$$

for $k = 0, \ldots, n$ where we write the $z$ coordinates *first* and set $q_0 := z$. For $k = 0$, we have $\mathcal{W}_0^{2n+1} = \mathbb{R}^{2n+1}$ and for $k = 1, \ldots, n$, and for $k \geq 1$ we can write

$$\mathcal{W}_k^{2n+1} \cong S^1 \times T^*(T^{k-1} \times \mathbb{R}^{n-k+1})$$

which is manifestly a contact manifold as the (circular) contactization of the symplectic manifold. $T^*(T^{k-1} \times \mathbb{R}^{n-k+1})$.

For $k = 1, \ldots, n$, we consider

$$E_A^{(k)} := (S^1)^k \times [-A, A]^{2n+1-k} \subset \mathcal{W}_k^{2n+1} \tag{6.11}$$

and for $k = 0$

$$E_A^{(0)} := [-A, A]^{2n+1} \subset \mathbb{R}^{2n+1}.$$

We write the associated coordinates by $(\xi_0, \cdots, \xi_n, p_1, \cdots, p_n)$ with

$$\begin{aligned} \xi_j &\equiv q_j \mod 1 && \text{for } j = 0, \cdots, k, \\ \xi_j &= q_j && \text{for } j = k+1, \cdots, n. \end{aligned}$$

We then consider the family of subsets

$$\begin{aligned} \widetilde{E}_A^{(0)} : &= [-2A^2, 2A^2] \times [-A, A]^{m-1} \times \mathbb{R}^{m+1} \left(= \chi_A([-1,1]^m \times \mathbb{R}^{m+1}\right), \\ \widetilde{E}_A^{(k)} : &= T^k \times [-A, A]^{m-k} \times \mathbb{R}^{m+1} \left(= \pi\left(\mathbb{R}^k \times [-A, A]^{m-k} \times \mathbb{R}^{m+1}\right)\right) \end{aligned}$$

and equip the contact forms induced from $\widehat{\mathscr{A}}$ on the contact product $M_{\mathbb{R}^{2n+1}}$.

6.2. **Representation of contactomorphisms by their 1-jet potentials.** Let $C^\infty_E(\mathcal{W}^{2n+1}_k,\mathbb{R})$ be the set of $\mathbb{R}$-valued functions compactly supported in a closed subset $E \subset \mathcal{W}^{2n+1}_k$.

Composing the 1-jet map $j^1u$ with the Darboux-Weinstein chart $\Phi_{U;A}$, we obtain the following parametrization of $C^1$-small neighborhood of the identity map in $\mathrm{Cont}(\mathcal{W}^m_k,\xi)$.

**Proposition 6.4.** *Let $A_0 > 1$ be given. Then there exist $\mathcal{U}_A$ is a $C^1$-small neighborhood of the identity and $\mathcal{V}_A$ a $C^2$-small neighborhood of zero in $C^\infty_E(\mathcal{W}^{2n+1}_k,\mathbb{R})$ and a one-to-one correspondence*

$$\mathscr{G}_A : \mathrm{Cont}_E(\mathcal{W}^{2n+1}_k,\alpha_0) \supset \mathcal{U}_A \to \mathcal{V}_A \subset C^\infty_E(\mathcal{W}^{2n+1}_k,\mathbb{R})$$

*that satisfies $\mathscr{G}_A(id) = 0$ and is continuous over $1 < A \leq A_0$.*

*Proof.* Let $\pi : J^1\mathcal{W}^m_k \to \mathcal{W}^m_k$ be the canonical projection. Consider submanifold

$$R_{g,A} := \mathrm{Image}\,\Phi_{U;A} \circ \Delta_g$$

which is Legendrian by construction. Moreover the projection $\pi$ restricted to $R_{g,A}$ becomes one-to-one provided the $C^1$-norm of $g$ is sufficiently small. Therefore there exists a unique real-valued function $u$ such that we can express $R_{g,A} = \mathrm{Image}\,j^1u$ for the 1-jet map of $u$.

We define $\mathscr{G}_A(g)$ to be the unique function $u : \mathcal{W}^{2n+1}_k \to \mathbb{R}$ satisfying

$$\mathrm{Image}(\Phi_{U;A} \circ \Delta_g) = \mathrm{Image}(j^1u) \tag{6.12}$$

where $u = u_{g,A}$ depends not only on $g$, $A$ but also on the chart $\Phi_{U;A}$. (See Diagram 5.8.) We alert readers that *while their images coincide they are different as a map.*

**Remark 6.5.** The fact that the two maps are not the same complicates the relationship between the contactomorphism $g$ and the function $u_{g,A}$ as shown below. This will give rise to some difficulty later in Section 15 when we try to compare the $C^r$ estimates of $g$ and that of the function $u_{g,A}$.

Then we put

$$\mathscr{G}_A(g) := u_{g;A}. \tag{6.13}$$

For the statement on the properties, we further examine the definition. By the definition of the Legendrianization parametrization map $\mathscr{G}_A$. By (6.12), there exists $y = y(x)$ such that

$$\Phi_{U;A} \circ \Delta_g(x) = (u_{g,A}(y), y, Du_{g,A}(y))$$

for each $x \in \mathcal{W}^m_k$, and that such $y$ is unique, provided $g$ is sufficiently $C^1$-small and the neighborhood $U$ is sufficiently small. We can express

$$\begin{aligned} y &= \pi_2\Phi_{U;A} \circ \Delta_g(x) \\ u_{g,A}(y) &= \pi_1\Phi_{U;A} \circ \Delta_g(x) \\ Du_{g,A}(y) &= \pi_3\Phi_{U;A} \circ \Delta_g(x). \end{aligned}$$

Furthermore the map

$$\pi_2\Phi_{U;A} \circ \Delta_g =: \Upsilon_{g,A}$$

is a self diffeomorphism of $\mathbb{R}^{2n+1}$ map if $g$ is sufficiently $C^1$-small. We can write the first equation as

$$x = \Upsilon^{-1}_{g,A}(y). \tag{6.14}$$

Then we can express

$$\begin{aligned} u_{g,A} &= \pi_1\Phi_{U;A} \circ \Delta_g \circ \Upsilon^{-1}_{g;A} \\ Du_{g,A} &= \pi_3\Phi_{U;A} \circ \Delta_g \circ \Upsilon^{-1}_{g;A}. \end{aligned}$$

This expression already clearly shows the continuity of the map

$$\mathscr{G}_A : (g, A) \mapsto u_{g,A}$$

in the $C^r$ topology of $g$ and in the $C^{r+1}$ topology of $u = u_{g,A}$, respectively.

The last statement immediately follows from this presentation. ∎

**Definition 6.6** (1-jet potential)**.** Let

$$\mathcal{U}_1 \subset \mathrm{Cont}_c(\mathcal{W}_k^{2n+1}, \alpha_0) \tag{6.15}$$

be a $C^1$-small neighborhood of the identity of $\mathrm{Cont}_c(\mathcal{W}_k^{2n+1}, \alpha_0)$. For any $C^1$-small contactomorphism $g \in \mathcal{U}_1$, we call the function $u = u_g$ satisfying (6.13) (for $A = 1$) the *1-jet potential* of contactomorphism$g \in \mathcal{U}_1$ with respect to $\alpha$ and $\Phi_U$.

We will fix a cut-off function $\psi : \mathcal{W}_k^{2n+1} \to [0,1]$ whose precise defining properties will be given later.

**Proposition 6.7** (Compare with Proposition 5.4 [Ryb2])**.** *Let $E \subset E_A^{(k)}$ be a sub-interval of $E_A^{(k)}$. There exists a $C^1$-neighborhood $\mathcal{U}_{\chi,A} \subset \mathcal{U}_1$ of the identity in $\mathrm{Cont}_{E_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)$ such that for any $g \in \mathcal{U}_{\chi,A}$ with support $E$ the contactomorphism*

$$g^\psi := \mathscr{G}_A^{-1}(\psi \mathscr{G}_A(g)) = \mathscr{G}_A^{-1}(\psi u_{g,A})$$

*is well-defined and $\mathrm{supp}(g^\psi) \subset E$. More precisely, we have $\mathrm{supp}(g^\psi) \subset \mathrm{supp}(\chi)$ and $g^\psi = g$ on any open subset $U \subset \mathcal{W}_k^{2n+1}$ with $g = 1$ on $U$.*

We will just write $u_g = u_{g;A}$ as in [Ryb2] for the simplicity of notation, whenever there is no danger of confusion.

**Remark 6.8.** Observe that the identity (6.12) relates the two maps $\Gamma_g$ and $j^1 u_g$ for $u_g = \mathscr{G}_A(f)$ explicitly via the chart $\Phi_{U;A}$ which depends only on the fixed chart $\Phi_U$ and the rescaling constant $A > 1$. In particular, the identity shows the equivalence of the two norms

$$M_0^*(g) = \max\{\|g - \mathrm{id}\|_{C^0}, \|\ell_g\|_{C^0}\}$$

and

$$\|j^1 u_g\|_{C^0} = \max\{\|Du_g\|_{C^0}, \|u_g\|_{C^0}\}$$

when $A$ varies $1 < A \le A_0$ for any fixed constant $A_0 > 1$.

## 7. Correcting contactomorphisms via the Legendrianization

The construction in the present section, which was introduced and utilized by Rybicki [Ryb2], presents its feature applicable only to the case of contactomorphisms in which the parametrization of a $C^1$ neighborhood of the identity of $\mathrm{Cont}_c(\mathcal{W}_k^{2n+1}, \alpha_0)_0$ is achieved by a $C^2$-neighborhood of 0 in $C^\infty(\mathcal{W}_k^{2n+1}, \mathbb{R})$ through taking the 1-jet potentials. Such a construction was not needed for the case of general diffeomorphisms or even for the case of symplectic diffeomorphisms [Ba1].

One important aspect of the Euclidean space $\mathbb{R}^n$ in the study of its diffeomorphism groups, although not manifest enough at the time of the advent of [Ma1], is the *linear structure* $\mathbb{R}^n$ so that any $C^1$-small perturbation of the identify map can be written in the form

$$\mathrm{id} + v : \mathbb{R}^n \to \mathbb{R}^n$$

where $v : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$-close to the zero map as well as its $C^0$-norm.

For the case of contact space $(\mathbb{R}^{2n+1}, \xi)$, there is no such a simple form of perturbation in the context of contactomorphisms, which prevents one from directly applying Mather's':

- construction of rolling-up operators, or
- utilizing the homothetic transformations.

The upshot of Rybicki's proof in [Ryb2] of perfectness of $\mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0) = \mathrm{Cont}_c^\infty(\mathbb{R}^{2n+1}, \alpha_0)$ is to correctly *contactify* the two operations.

We start with the observation that we have a natural covering projection

$$\mathrm{pr}_{k+1} : \mathbb{R}^{2n+1} \to \mathcal{W}_{k+1}^{2n+1}$$

and that any sufficiently $C^1$-small (and so $C^0$-small) contactomorphism $g \in \mathcal{W}_{k+1}^{2n+1}$ can be uniquely lifted to a contactomorphism $\widetilde{g}$ of $\mathcal{W}_{k+1}^{2n+1}$ that satisfies

$$g = \widetilde{g} \circ \mathrm{pr}_k$$

and that $\Gamma_{\widetilde g}$ is periodic in the variable $\xi_k$.

**Definition 7.1** ($\mathrm{Cont}_c^{T^\ell}(\mathcal{W}_k^{2n+1},\alpha_0)$)**.** For $\ell+1,\cdots n+1$, we define a subset

$$\begin{aligned} & \mathrm{Cont}_c^{T^\ell}(\mathcal{W}_k^{2n+1},\alpha_0) \\ := \quad & \{f \in \mathrm{Cont}_c^{T^\ell}(\mathcal{W}_k^{2n+1},\alpha_0) \mid f - \mathrm{id} \text{ does not depend on } \xi_i \text{ with } 0 \le i \le \ell\}. \end{aligned}$$

It follows from Proposition 6.7 that $\mathrm{Cont}_c(\mathcal{W}_k^{2n+1},\alpha_0)^{(\ell)}$ is a $C^1$-small neighborhood of the identity.

Since $f - \mathrm{id} =: v$ does not depend on $\xi_i$ with $0 \le i \le \ell$, we may abuse the notation $v$ by omitting the projection $\pi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1-\ell}$ from $v \circ \pi$ and just write

$$v \circ \pi(\xi_0, \ldots, \xi_\ell, \ldots \xi_n, p_1, \ldots, p_n) := v(\xi_{\ell+1}, \ldots, \xi_n, p_1, \ldots, p_n).$$

When $\ell = k$, this becomes

$$v(\xi_0, \ldots, \xi_k, \ldots \xi_n, p_1, \ldots, p_n) = v(\xi_{k+1}, \ldots, \xi_n, p_1, \ldots, p_n), \tag{7.1}$$

i.e., $v$ can be identified with a map $v : \mathbb{R}^{n-k} \times \mathbb{R}^n \to \mathcal{W}_k^{2n+1}$ by abusing notation.

**Corollary 7.2.** *Suppose that $f \in \mathrm{Cont}_c^{T^\ell}(\mathcal{W}_k^{2n+1},\alpha_0) \cap \mathcal{U}_1$, i.e., $T^\ell$-equivariant. Then the function $u_f = u_{f;A}$ defined by*

$$\mathscr{G}_A(f) =: u_f \in C^\infty(\mathcal{W}_k^{2n+1}, \mathbb{R})$$

*a $T^k$-invariant function on $\mathcal{W}_k^{2n+1}$ where the $T^k$ acts on the $k$ circle factors of $\mathcal{W}_k^{2n+1} = S^1 \times T^*(T^{k-1} \times \mathbb{R}^{n-k}) \cong (S^1)^k \times \mathbb{R}^{2n-k+1}$ by the standard rotations of circles. In particular, $u_f$ does not depend on the variables $\xi_0, \ldots, \xi_{k-1}$.*

*Proof.* This is an immediate consequence of the discussion given in Section 6. □

## 8. Contact-scaling and shifting of the supports of contactomorphisms

Suppose a positive integer $A > 0$ which will be chosen sufficiently large whose size is to be determined later. For each given such integer, we start with the $(2n+1)$-cube $[-1,1]^{2n+1}$ and consider its shifting by an integer $-(2A-1) \le k_i \le 2A-1$ in the $p_i$-direction. By the choice $0 \le k_i \le 2A-1$, we have $[k_i - 1, k_i + 1] \subset [-2A, 2A]$.

Then we consider the conjugation of $f$

$$\rho_{A,t} \circ f \circ \rho_{A,t}^{-1} \tag{8.1}$$

by the (affine) contactomorphism

$$\rho_{A,t} := \chi_{A^2} \circ \sigma_i^t, \quad i = 1, \cdots, n. \tag{8.2}$$

**Remark 8.1.** We would like to highlight one difference between our definition of $\rho_{A,t}$ and that of [Ryb2]: We do not involve the front scaling transformation $\eta_A$ but only $\chi_A$ by taking the square of $\chi_A$ instead. This turns out to be a crucial change to be made for the purpose of obtaining the optimal power of $A^{4-2r+2(n+1)}$. In this regard, it is crucial for the length of $q$-rectangularpid to become $A^2$ which is responsible for the coefficient 2 in front of $r$ and $(n+1)$, and the number 4 in the constant term in the exponent comes from the power of the $z$ direction for the contact rescaling operation

$$\chi_A^2(z,q,p) = (A^4 z, A^2 q, A^2 p).$$

The map $\chi_A \circ \eta_A$ provides the power 1 of $A$ in the $q$-direction while the power 3 in the $z$ direction, which would give rise to the power $A^{4-r+(n+1)}$ which will give rise to only $r > n+5$. This is the reason why we use the map $\chi_A^2 = \chi_{A^2}$ instead of $\chi_A \eta_A$ used in [Ryb2].

We also define the vector version of the conjugation (8.1)

$$\rho_{A,\mathbf{t}} = \chi_{A^2} \circ \sigma_i^{\mathbf{t}} \quad i = 1, \cdots, n \tag{8.3}$$

where we write $\mathbf{t} := \sum_{i=1}^n t_i \vec{e}_i$ and

$$\sigma_i^{\mathbf{t}} := \sigma_1^{t_1} \circ \sigma_2^{t_2} \circ \cdots \circ \sigma_n^{t_n} :$$

It has the following explicit formula (before applying cutting-off: we recall the discussion given in Subsection 4.2 here)

$$\rho_{A,\mathbf{t}}(z,q,p) = \left(A^4 z + \sum_{i=1}^{n}, A^2 q, A^2(p + t\vec{f_i})\right), \quad \mathbf{t} = (t_1, t_2, \dots, t_n)_p = \sum_{i=1}^{n} t_i \vec{f_i}. \tag{8.4}$$

Observe that the image $\rho_{A;\mathbf{t}}\left([-2,2]^{2n+1}\right)$ is contained in the following

$$A^4\left(\prod_{i=1}^{n}[-2 - t\sum_{i=1}^{n}|q_i|, 2 + t\sum_{i=1}^{n}|q_i|]\right) \times [-2A^2, 2A^2]^n \times A^2\left(\prod_{i=1}^{n}[-2 - |t_i|, 2 + |t_i|]\right). \tag{8.5}$$

**Lemma 8.2.** *Let* $\mathbf{k} = \sum_{i=1}^n k_i \vec{f_i} + (\sum_{i=1}^n k_i) \cdot \vec{e}_z$ *with* $[k_i| \leq 2A - 1$ *for all* $i = 1, \cdots, n$. *Consider* $f \in \mathrm{Cont}_c(J^1\mathbb{R}^n, \alpha_0)$ *satisfying*

$$\operatorname{supp} f \subset [-2,2]^{2n+1} + \mathbf{k}.$$

*Then for any* $\mathbf{t} = (t_1, \dots, t_n)$ *with* $|t_i| \leq 2A - 1$,

$$\operatorname{supp}\left(\rho_{A,\mathbf{t}} f \rho_{A,\mathbf{t}}^{-1}\right) \subset [-3A^5, 3A^5] \times [-2A^2, 2A^2]^n \times [-2A^3, 2A^3]^n \tag{8.6}$$

*provided* $A \geq 2n$.

*Proof.* Consider the map $\rho_{A,t}$ above associated to $i$. We evaluate

$$\rho_{A,\mathbf{t}}^{-1}(z,q,p) = \left(\frac{z - \sum_{i=1}^n t_i q_i}{A^4}, \frac{q}{A^2}, \frac{p - \mathbf{t}}{A^2}\right). \tag{8.7}$$

Therefore if this point is not contained in $\operatorname{supp} f$, then one of the following inequalities holds:

(1) $\frac{z - \sum_{i=1}^n t_i q_i}{A^4} \notin [-2,2] + \sum_{i=1}^n k_i$ for some $i$,
(2) $\left|\frac{q_i}{A^2}\right| > 2$ for some $i$,
(3) $\frac{p_i - t_i}{A^2} \notin [-2 + k_1, 2 + k_i]$ for some $i$,

If (3) holds, then

$$p_i < (-2 + k_i)A^2 + t_i \quad \text{or} \quad p_i > (2 + k_i)A^2 + t_i.$$

Using $|k_i| \leq 2A - 1$ and $|t_i| \leq 2A$, we derive

$$p_i < -2A^3 \quad \text{or} \quad p_i > 2A^3$$

and hence $p_i \in [-2A^2, 2A^2]$.

If (1) holds but (2) fails to hold, we have inequalities

$$\frac{q_i}{A^2} \in [-2,2]$$

for all $i = 1, \cdots, n$ and

$$z < A^4\left(-2 + \sum_{i=1}^n k_i\right) + \sum_{i=1} t_i q_i \quad \text{or} \quad z > A^4\left(2 + \sum_{i=1}^n k_i\right) + \sum_{i=1} t_i q_i$$

for some $i$. The first inequality implies

$$-2A^2 \leq q_i \leq 2A^2$$

Combining this with the second inequality, we have derived that any point $(z,q,p)$ satisfying

$$z < -2A^4 + nA^4(-2A + 1 - 2A) \quad \text{or} \quad z > 2A^4 + nA^4(2A - 1 + 2A)$$

is not in $\operatorname{supp}(\rho_{A,\mathbf{t}} f \rho_{A,\mathbf{t}}^{-1})$. In particular, this holds if $z < -3A^5$ or $z > 3A^5$, provided $A \geq 5n$.

Combining the above altogether, we have finished the proof. □

This leads us to the consideration of the following subsets of $\mathcal{W}_k^{2n+1}$.

**Definition 8.3** ($J_A$)**.** Let $0 \leq t_i \leq 2A$ and $A > 0$. We define

$$\begin{aligned} J_A &:= [-3A^5, 3A^5] \times [-2A^2, 2A^2]^n \times [-2A^3, 2A^3]^n \\ &\subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \cong J^1\mathbb{R}^n. \end{aligned} \tag{8.8}$$

We will always make this choice of $A > 0$ and $t_i$'s from now on.

**Remark 8.4.** We warn the readers that our definition of $J_A$ and many others with the same notations from [Ryb2] are different therefrom in their specific choices of the orders of powers of $A$. Our choices are made to make the relevant constructions and estimates in [Ryb2] as optimal as possible. We encourage readers to compare the differences of the exponents of the power of $A$ appearing in the definitions of various contact cylinders and rectangularpids below. It is important for the power of $A$ to be 2 for the $q$-cube factors appearing in (8.9) below for our purpose of obtaining the optimal thresholds.

We will also need to consider the following families of contact cylinders

$$\begin{aligned} J_A^{(k)} &= S^1 \times (S^1)^{k-1} \times [-2A^2, 2A^2]^{n-k+1} \times [-2A^3, 2A^3]^n \\ K_A^{(k)} &= S^1 \times (S^1)^{k-1} \times [-2,2] \times [-2A^2, 2A^2]^{n-k} \times [-2A^3, 2A^3]^n \end{aligned}$$

for $k = 1, \dots, n$ and

$$\begin{aligned} J_A^{(0)} &= J_A = [-3A^5, 3A^5] \times [-2A^2, 2A^2]^n \times [-2A^3, 2A^3]^n \\ K_A^{(0)} &= K_A = [-2,2] \times [-2A^2, 2A^2]^n \times [-2A^3, 2A^3]^n. \end{aligned} \tag{8.9}$$

of the construction of a family of Rybicki's rolling-up operators

$$\Psi_A^{(k)} : \mathrm{Cont}_{J_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}_{K_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0$$

for which the rolling occurs in the $q_k$-coordinate direction. Note that we have natural covering projections $J_A \to J_A^{(k)}$ and $K_A \to K_A^{(k)}$, respectively. The fiber of $J_A \to J_A^{(k)}$ is isomorphic to

$$\mathbb{Z}_{4A^2} \times (\mathbb{Z}_{4A^3})^{k-1},$$

and the fiber of $K_A \to K_A^{(k)}$ is isomorphic to

$$\mathbb{Z}_4 \times (\mathbb{Z}_{4A^2})^{k-1}.$$

Similarly $J_A^{(k)} \to K_A^{(k)}$ has fiber isomorphic to $\mathbb{Z}_{A^2}$.

## 9. Rybicki's fragmentation of the second kind: Definition

In [Ryb2], Rybicki introduced some fragmentation for the case of contactomorphisms which involves a *fragmentation of the 1-jet potentials* defined in the previous section. Such a fragmentation is uniquely applicable to the case of contactomorphisms because its natural analog which does not preset either in the case of diffeomorphisms [Ma1, E2] nor in that of symplectomorphisms [Ba1]. Rybicki [Ryb2] introduced the following-type of fragmentation which he calls *the fragmentations of the second kind.*

We start with the Rybicki's fragmentation in the direction of $z = \xi_0$.

**Lemma 9.1** (Compare with Proposition 5.6 [Ryb2]). *Let $2A > 1$ be an even integer, $\psi : [0,1] \to [0,1]$ be a boundary-flattening function such that $\psi \equiv 1$ on $[0, \frac{1}{4}]$ and $\psi \equiv 0$ on $[\frac{3}{4}, 1]$, and let*

$$E_{2A} := E_{2A}^{(0)} = [-2A, 2A]^{2n+1}.$$

*Then there exists a $C^1$-neighborhood $\mathcal{U}_{\psi,A}$ of the identity in $\mathrm{Cont}_{E_{2A}}(\mathbb{R}^{2n+1}, \alpha_0)$ such that for any $g \in \mathcal{U}_{\psi,A}$ there exists a factorization*

$$g = g_1 \cdots g_{4k+1}, \tag{9.1}$$

*that satisfies the following properties: The factorization is uniquely determined by $\Phi_A$, $\psi$ and $A$ so that $\mathrm{supp}(f_K)$ is contained in an interval of the form*

$$\left( \left[ k - \frac{3}{4}, k + \frac{3}{4} \right] \times \mathbb{R}^{2n} \right) \cap E_{2A},$$

*with $k \in \mathbb{Z}, |k_i| \leq 2A$.*

*Proof.* We extend $\psi$ to $[-1,1]$ as an even function and then to $\mathbb{R}$ as a 2-periodic function. We lift it to $E_{2A}$. Let $g$ be a sufficiently $C^1$-small contactomorphism with $\operatorname{supp} g \subset E_{2A}$ and consider the function

$$g^\psi := \mathscr{G}_A^{-1}\left(\psi \mathscr{G}_A(g)\right) = \mathscr{G}_A^{-1}(\psi u_g).$$

We now take a factorization $g = g_2^\psi g_1^\psi$ by first defining

$$g_1^\psi := g_{-2A} \circ g_{-2(A-1)} \circ \cdots \circ g_{2(A-1)} \circ g_{2A}, \tag{9.2}$$

and then setting

$$g_2^\psi := g(g_1^\psi)^{-1} \tag{9.3}$$

where $g_i$'s satisfy $\operatorname{supp} g_{2k} \subset [2k-\frac{3}{4}, 2k+\frac{3}{4}] \times \mathbb{R}^{2n}$ and $\operatorname{supp} g_{2k+1} \subset [2k+\frac{1}{4}, 2k+\frac{7}{4}] \times \mathbb{R}^{2n}$. □

We mention that the collections $\{\operatorname{supp} g_{2k}\}_{k=1}^n$ and $\{\operatorname{supp} g_{2k+1}\}_{k=1}^n$ are disjoint from one another respectively. By applying the above construction consecutively to all variables $(z,q,p) = (\xi_0, \xi, p)$, we obtain the following.

**Proposition 9.2** (Compare with Proposition 5.7 [Ryb2])**.** *Let $A$, $\psi$ and $E_{2A}$ be as in Lemma 9.1. Then there exists a $C^1$-neighborhood $\mathcal{U}_{\psi,A}$ of the identity in $\mathrm{Cont}_{E_{2A}}(\mathbb{R}^{2n+1}, \alpha_0)$ such that for any $g \in \mathcal{U}_{\psi,A}$ there exists a factorization*

$$g = g_1 \cdots g_{a_m}, \quad a_m = (4A+1)^m$$

*that satisfies the following properties: The factorization is uniquely determined by $\Phi_A$, $\chi$ and $A$ so that $\operatorname{supp}(f_K)$ is contained in an interval of the form*

$$\left(\left[k_1 - \frac{3}{4}, k_1 + \frac{3}{4}\right] \times \cdots \times \left[k_m - \frac{3}{4}, k_m + \frac{3}{4}\right]\right) \cap E_{2A},$$

*with $k_i \in \mathbb{Z}, |k_i| \leq 2A$.*

## 10. The 'hat' operation: deforming to an $S^1$-equivariant map

Now we take the 'hat' operation of turning the given contactomorphism into one that becomes $S^1$-symmetric in an additional direction of $q_i$s. This is the analog to the construction given in [Ma1, p. 524]. However the direct application of Mather's construction cannot work for the contactomorphisms *because the addition operation + on the vector space $\mathbb{R}^{2n+1}$ does not respect the contact property.* Here enters one of Rybicki's key ideas of exploiting the representation of contactomorphisms $g$ sufficiently $C^1$-close to the identity by the Legendrian graph of $g - id$ in the contact product $(M_{\mathcal{W}_k^{2n+1}}, \widehat{\mathscr{A}})$ followed by their contact potentials $u_g \in C^\infty(\mathcal{W}_k^{2n+1}, \mathbb{R})$ via the following sequence of one-to-one correspondences:

$$g \longleftrightarrow g - \mathrm{id} \longleftrightarrow \Gamma_g \longleftrightarrow \Delta_g \longleftrightarrow \Phi_{U;A}(\Delta_g) \longleftrightarrow \operatorname{Image} j^1 u_g \longleftrightarrow u_g. \tag{10.1}$$

This construction is reversible and respects the $T^{k+1}$-action on $\mathcal{W}_k^{2n+1}$ which are given by the linear rotations of the underlying $(k+1)$ torus $\mathcal{W}_k^{2n+1} \to (S^1)^{k+1} = T^{k+1}$.

The upshot of this step is that both $\mathcal{W}_k^{en+1}$ and the set $C^\infty(\mathcal{W}_k^{2n+1}, \mathbb{R})$ are *is linear*, and hence we can apply the Mather-type constructions thereto and then read back the above diagram to obtain a *contact diffeomorphism* $g$ associated to any given function $v \in C^\infty(\mathcal{W}_k^{2n+1}, \mathbb{R})$ sufficiently $C^2$-close the zero function. The detail of the construction is now in order. (See [Ryb2, p. 3309] for the relevant counterpart.)

Consider the cylinders

$$E_A^{(k)} := (S^1)^k \times [-A, A]^{2n+1-k} \subset \mathcal{W}_k^{2n+1}, \quad k = 1, \ldots, n+1.$$

We start with a $T^k$ equivariant element

$$g \in \mathrm{Cont}^{T^k}_{E_A^{(k+1)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0 \subset \mathrm{Cont}^{T^k}_c(\mathcal{W}_k^{2n+1}, \alpha_0)_0$$

sufficiently $C^1$-close to the identity. By definition, $g$ is equivariant under the $T^k$ action, and so $u_g$ is a $T^k$-invariant real-valued function by Proposition 6.7. Therefore we can express

$$u_g = u'_g \circ \mathrm{pr}_k^c; \quad \mathrm{pr}_k^c : \mathcal{W}_{k+1}^{2n+1} \to \mathbb{R}^{2n-k-1}$$

for some function $u'_g : \mathbb{R}^{2n-k-1} \to \mathbb{R}$. By the identification of

$$\mathbb{R}^{2n-k-1} \cong \{[(0,\ldots,0)]\} \times \mathbb{R}^{2n-k-1} \subset \mathbb{R}^{2n+1},$$

$u'_g$ can be canonically defined from $u_g$ by defining

$$u'_g(\xi_k, \xi_{k+1}, \ldots, \xi_n, p) := u([(0,\ldots,0)], \xi_k, \xi_{k+1}, \ldots, \xi_n, p).$$

Here $[(0,\ldots,0)] \in \mathbb{R}^{k+1}/\mathbb{Z}^{k+1}$ is the identity element.

Now we define a $T^{k+1}$-invariant function $\widetilde{u}_g$ by setting

$$\widetilde{u}_g(\xi_0, \xi, p) := u'_g(0, \xi_{k+1}, \cdots, \xi_n, p) \tag{10.2}$$

which is now clearly invariant under the translation in the additional direction of $\xi_k$. It follows that $u_g$ is again contained in the given neighborhood $\mathcal{V}_2$. Then we define

$$\widehat{g} := \mathscr{G}_A^{-1}(\widetilde{u}_g) \tag{10.3}$$

which is now $T^{k+1}$-equivariant and hence contained in

$$\widehat{g} \in \mathrm{Cont}^{T^{k+1}}_{E_A^{(k+1)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0 \subset \mathrm{Cont}_c^{T^{k+1}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0.$$

Here we remark that the domains of the contactomorphisms can be naturally identified with

$$S^1 \times T^*(T^k \times \mathbb{R}^{n-k}),$$

respectively for $0 \le k \le n$. In summary, each hat operation adds the $S^1$-equivariance in the one more direction of $\xi$'s.

## 11. Rolling-up operator and unfolding-fragmentation operators

Denote by $\pi_k : \mathcal{W}_k^{2n+1} \to \mathcal{W}_{k+1}^{2n+1}$ the natural projection in the $q_k$ direction induced by the covering projection $\mathbb{R} \to S^1$. We consider a contactomorphism $g$ of $\mathcal{W}_{k+1}^{2n+1}$ contained in a sufficiently $C^1$-small neighborhood $\mathcal{U}_1$ of the identity with $\mathrm{supp}\, g \subset J_A^{(k)}$, i.e., in $\mathrm{Cont}_{J_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)$. More specifically, we will assume $M_1^*(g) < \frac{1}{4}$.

### 11.1. Mather's rolling-up operator.

We first recall Mather's rolling-up operators $\Theta_A^{(k)}$ in the current context

$$\Theta_A^{(k)}(g)(\theta_0, \ldots, \theta_{k-1}, q_{k+1}, \ldots, q_n, p) = \pi_k\left((T_k g)^N(z, q_1, \ldots, q_n, p)\right) \tag{11.1}$$

where we make a choice of $N$ as follows. For any $x = (\theta_0, \cdots, \theta_{k-1}, \ldots, q_n, p) \in \mathcal{W}_{k+1}^{2n+1}$, we choose $\widetilde{x} \in \mathbb{R}^{n+1} \times \mathbb{R}^n$ with $\pi_{k+1}(\widetilde{x}) = x$ with $q_k < -2A^2$ for the covering map $\pi_{k+1} : \mathbb{R}^{2n+1} \to \mathcal{W}_{k+1}^{2n+1}$. Choose a sufficiently large $N \in \mathbb{N}$ so that

$$q_k\left((T_k f)^N(\widetilde{x})\right) > 2A^2.$$

(It is easy to check that it is enough to choose any $N > 4A^2 + 4A$ by starting with $\widetilde{x}$ with $q_k(\widetilde{x}) = -2A^2 - 2A$.)

The following summarizes basic properties of Mather's rolling-up operators [Ma1, Defintiion p. 520] applied to the contactomorphisms.

**Proposition 11.1** (Compare with Proposition 8.1 [Ryb2]). *Let $k = 0, \ldots, k$. After shrinking $\mathcal{U}_1$ if necessary,*

$$\Theta_A^{(k)} : \mathrm{Cont}(\mathcal{W}_k^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}_{K_A^{(k+1)}}(\mathcal{W}_{k+1}^{2n+1}, \alpha_0)$$

*satisfies the following properties:*

*(1) $\Theta_A^{(k)}$ is continuous and preserves the identity.*

*(2)* $\Theta_A^{(k)}(\mathrm{Cont}^{T^k}(\mathcal{W}_k^{2n+1},\alpha_0)_0 \cap \mathcal{U}_1) \subset \mathrm{Cont}^{T^k}\left(\mathcal{W}_{k+1}^{2n+1},\alpha_0\right)_0$, *i.e.,* $\Theta_A^{(k)}(g)$ *is also* $S_k^1$-*equivariant.*

11.2. **Unfolding-fragmentation operators $\Xi_{A;N}^{(k)}$.** Rybicki [Ryb2] also considers the following map (for $N = 2$)

$$\Xi_{A;N}^{(k)} : \mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_{k+1}^{2n+1},\alpha_0) \cap \mathcal{U}_2 \to \mathrm{Cont}_{K_A^{(k)}}(\mathcal{W}_k^{2n+1},\alpha_0)_0, \quad k = 0,\ldots,n,$$

where $\mathcal{U}_2$ is a $C^1$-small neighborhood of id in $\mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_{k+1}^{2n+1},\alpha_0)$. This is a contact counterpart of Mather's operator of 'fragmentation followed by shifting supports' [Ma1, Construction p. 524]. We call the map an *unfolding-fragmentation operator in the $q_k$-direction.* Its construction is now in order. This is where the construction of [Ryb2] makes a stark difference from that of [Ma1] and the Legendrianization followed by taking the contact potential plays a fundamental role it Rybicki's proof.

We need to give the general definition of $\Xi_{A;N}^{(k)}$ applied to the $N$-fragmentation, while [Ryb2, p.3313] gave the construction only for the case of the 2-fragmentation and stop short of giving the general definition associated to the $N$-fragmentation, although he implicitly employed the defintion for the general case in the proof of [Ryb2, Lemma 8.6]. (See the end of the proof in [Ryb2, p. 3318] and Remark 11.2 below of the present paper.)

To make clear the dependence on $N$ of its definition, we denote by

$$\Xi_{A;N}^{(k)}$$

the one associated to the $N$-fragmentation with $N = 2, 3 \ldots$, leaving the corresponding construction for $N = 2$ to [Ryb2, Section 8].

**Remark 11.2.** We would like to remark that the statement like "$[g_a] = [g^{a^{n+2}}]$ with $g = \Xi^{(k)}(f^*)$" in the end of the proof of [Ryb2, Lemma 8.6] is imprecise since the definition of $\Xi_A^{(k)}$ itself depends on $a$ and hence the $g$ on the right hand side is ambiguous. In this regard, the presentation of [Ryb2, Section 8] is rather imprecise and is missing some details, although they are all correctible by utilizing the generalized construction of the operator $\Xi_A^{(k)}$ for the $a$-fragmentation is straightforward which leads to our extended operator $\Xi_{A;a}^{(k)}$ for each choice of integer $a \geq 2$. This imprecise presentation and lack of this general construction in [Ryb2] make the proof of the homological identity $[g] = [g^{a^{n+2}}]$ for $a > 2$ and of its conclusion $[g] = [g_a]$ are rather misleading because apparently its proof should have involved the operator $\Xi_{A;a}^{(k)}$ associated to the general $a$-fragmentation. In this regard, we believe that both the statement and the proof of [Ryb2, Lemma 8.6] should be corrected by taking this dependence on $a$ into account as presented in Section 12 and Section 22.

We denote by $\rho : [0,1] \to [0,1]$ the standard boundary flattening function once and for all that satisfies

$$\begin{aligned} \rho(t) &= \begin{cases} 1 & \text{for } 0 \leq t \leq \frac{1}{4} \\ 1 & \text{for } \frac{3}{4} \leq t \leq 0 \end{cases} \\ \rho'(t) &\geq 0. \end{aligned} \tag{11.2}$$

We then extend the function to the interval $[-1,1]$ as an even function which we still denote by $\rho : [-1,1] \to [0,1]$. We also define $\widetilde{\rho}$ to be the time-reversal $\widetilde{\rho}(t) = \rho(1-t)$. We then extend the function periodically to whole $\mathbb{R}$ with period 2.

Having the identification $S^1 = \mathbb{R}/\mathbb{Z}$ and the covering projection $\pi : \mathbb{R} \to S^1$ in our mind, we consider the $N$ pieces of subintervals of length 2 given by

$$I_j = [-N + j - 1, -N + j], \quad j = 1,\ldots,2N. \tag{11.3}$$

We denote the left and the right boundary points of $I_j$ by $b_j^\pm$ respectively, i.e.,

$$b_j = -N + j - 1, \quad b_j^+ = -N + j. \tag{11.4}$$

We write its concentric subinterval of the length 1

$$I'_j = \left[-N + j - \frac{1}{4}, -N + j - \frac{1}{4}\right] \subset I_j \tag{11.5}$$

centered at

$$-2N + j - \frac{1}{2} =: c_j, \quad j = 1, \ldots, 2N. \tag{11.6}$$

In this regard, we will sometimes write them as

$$I_j = I_j(c_j), \quad I'_j := \frac{1}{2} I_j(c_j)$$

for the clarity of presentation. We also define

$$A_\varepsilon(c_k) = [c_k - \varepsilon, c_k + \varepsilon]. \tag{11.7}$$

**Remark 11.3.** We observe that when $N$ is odd,

- The interval $I_j$ for $j = 0$ or $j = N$ only the half of the intervals are contained in the given big interval $[-N, N]$.
- The interval $I_j$ centered at 0 lies at the eventh order. In fact $0 = c_{2j}$ with $j = \frac{N+1}{2}$,

and that when $N$ is even,

- The interval $I_j$ for $j = 0$ or $j = N$ are fully contained in the given big interval $[-N, N]$.
- The interval $I_j$ centered at 0 lies at the oddth order. In fact $0 = c_{2j+1}$ with $j = \frac{N}{2}$.

Now we scale the intervals $[-N, N]$ down to $[-1, 1]$ and define

$$I_j^N = \frac{1}{N} I_j, \quad (I'_j)^N = \frac{1}{2} I_j^N, \tag{11.8}$$

$$A_\varepsilon^N(c_k) := \frac{1}{N} A_\varepsilon(c_k). \tag{11.9}$$

Since $\rho(t) \equiv 1$ for $t = 0 \mod 2$ or and $\rho(t) \equiv 0$ for $t \equiv 1 \mod 2$, we can extend it to the whole $\mathbb{R}$ 2-periodically. Then we have

$$\operatorname{supp} \rho \subset \bigcup_{j=1}^N (-N + 2j - 1) + \left[-\frac{3}{4}, \frac{3}{4}\right] = \bigcup_{j=1}^N A^N_{\frac{3}{4}}(c_{2j}). \tag{11.10}$$

Later it will be more convenient to enumerate the intervals symmetrically with respect to the origin so that the interval centered at the origin always appears at $j = 0$. In this ordering we can write

$$\bigcup_{j=1}^N A^N_{\frac{3}{4}}(c_{2j}) = \bigcup_{j=-[\frac{N+1}{2}]}^{[\frac{N+1}{2}]} A^N_{\frac{3}{4}}(c_{2(j+[\frac{n+1}{2}]+1)}) \tag{11.11}$$

where $[b]$ is the largest integer smaller than or equal to a real number $b$.

We set

$$\psi_k^N(\xi_0, \xi, p) := \rho(N\xi_k), \quad k = 1, \ldots, n \tag{11.12}$$

respectively, and lift it a function defined on $\mathcal{W}^m_{k+1}$. For each given

$$g \in \operatorname{Cont}_{J_A^{(k+1)}}(\mathcal{W}^m_{k+1}, \alpha_0)_0 \cap \mathcal{U}_2,$$

we define

$$g^{\psi_k} := \mathscr{G}_A^{-1}(\psi_k^N \mathscr{G}_A(g)) = \mathscr{G}_A^{-1}(\psi_k^N u_g).$$

Let $g_1^{\psi_k}$ (resp. $g_2^{\psi_k}$) be the unique lift of $(g^{\psi_k})^{-1} g$ (resp. $g^{\psi_k}$) to $\mathcal{W}^m_k$ which are periodic contactomorphisms on

$$T^k \times \mathbb{R} \times [-2A^2, 2A^2]^{n-k} \times [-2A^3, 2A^3]^n.$$

For the notational convenience, we set

$$\mathcal{E}_{A,n,k} := [-2A^2, 2A^2]^{n-k} \times [-2A^3, 2A^3]^n.$$

By definition, we have
$$g = g_2^{\psi_k} g_1^{\psi_k}. \tag{11.13}$$
Furthermore, for a small enough $C^1$ neighborhood $\mathcal{U}_2$, there is a sufficiently small $\varepsilon > 0$ such that
$$\begin{aligned} g_1^{\psi_k} &= g \quad \text{on } T^k \times I'_{2j-1} \times \mathcal{E}_{A,n,k} \\ g_2^{\psi_k} &= g \quad \text{on } T^k \times I'_{2j} \times \mathcal{E}_{A,n,k} \end{aligned}$$
for all $j = 1, \ldots N$.

We put
$$\begin{aligned} E_k^- &= \{(\xi_0, \xi, y) \in \mathcal{W}_k^m \mid -1 \le \xi_k \le 0\} \\ E_k^+ &= \left\{(\xi_0, \xi, y) \in \mathcal{W}_k^m \,\middle|\, \frac{1}{2} \le \xi_k \le 3/2\right\}. \end{aligned}$$

The following definition is the same as those of [Ma1, Construction of p.524], [E2, p.119] and [Ryb2, Equation (8.2)] (for $N = 2$).

**Definition 11.4** (The map $\Xi^{(k)}_{A;N}$)**.** We define $\Xi^{(k)}_{A;N}(g) := f$ by requiring $f$ to satisfy the requirement:
$$\pi_k f = \begin{cases} g_1^{\psi_k} & \text{on } E_k^-, \quad f(E_k^-) = E_k^- \\ g_2^{\psi_k} & \text{on } E_k^+, \quad f(E_k^+) = E_k^+ \end{cases}$$
and
$$f = \mathrm{id} \quad \text{on } \mathcal{W}_k^m \setminus (E_k^- \cup E_k^+).$$

This map is well-defined since $E_k^+ \cap E_k^- = \emptyset$ and by the 1-periodicity of the multi-bump function $\psi_k^N$. The following conservation of supports from $g$ to $u_g$ and vice versa are important in the comparison study of $g$ and the associated $u_g$.

**Lemma 11.5.**
$$\operatorname{supp}(g - \mathrm{id}) \cup \operatorname{supp} \ell_g = \operatorname{supp} j^1 u_g. \tag{11.14}$$

The following properties of $\Xi^{(k)}_{A;N}$ are immediate to check which will be used later. (See [Ryb2, p.3313] for a simpler relevant discussion thereon for the case of 2-fragmentation.)

**Lemma 11.6.** *For any given sufficiently small $\varepsilon > 0$, there exists some $\delta > 0$ depending only on $\varepsilon$ such that*
$$\pi_k \Xi^{(k)}_{A;N}(g) = g \quad \text{on } A_\varepsilon^N \times \mathbb{R}^{2n} \tag{11.15}$$
*for all $g \in \mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_{k+1}^m, \alpha_0)_0 \cap \mathcal{U}_2$ with $M_1^*(g) \le \delta$. Here $A_\varepsilon^N$ is the union*
$$A_\varepsilon^N = \bigcup_{j=1}^{2N} A_\varepsilon^N(c_j) \tag{11.16}$$
*where the interval $A_\varepsilon^N(c_j)$ is as introduced in* (11.9)*.*

*Proof.* By the definitions of $\mathscr{G}_A(g)$ and $g^{\psi_k^N}$, we have
$$g^{\psi_k^N} = \mathscr{G}_A^{-1}(\psi_k \mathscr{G}_A(g)).$$
If $\xi_k \in A_{2j}^N(\frac{1}{8N})$, then we have $\psi_k^N(\xi_k) = 1$ by definition of $\psi$ and so
$$g^{\psi}(x) = \mathscr{G}_A^{-1}(u_g(x)).$$
Since $\Phi_{U;A} \circ \Delta_{\mathrm{id}}(x) = (x, 0, 0)$ and by the obvious estimates
$$\|Du_g\| \le C_1 \|D(g - id)\|_{C^0} + C_2 \|h_X\|_{C^0}$$
and
$$\max\{\|g - \mathrm{id}\|, \|Df\|, \|\ell_g\|\} \le C_3(\|Du_g\|_{C^0} + \|u_g\|_{C^0})$$

where $C_i$ depend only on the Darboux-Weinstein chart $\Phi_U$, $A$ and $N$ but are independent of $g$, provided $\|M_1^*(g)\|$ is sufficiently small. The above estimates then imply $M_0^*(g) = \max\{\|g - \mathrm{id}\|_{C^0}, \|\ell_g\|_{C^0}$ and $\|j^1 u_g\|_{C^0}$ are comparable to each other. On the other hand, on $A^N_{2j-1}(\frac{1}{8N})$, we have $\psi_k(x) = 0$ on $\cup I'_{2j-1}$ and $\psi_k^N(x) = 1$ on $\cup I'_{2j}$ and so $g^{\psi_k^N}(x) = g$. Then by Lemma 11.5 and combining the above discussion, we have proved the lemma by continuity of the map $\mathscr{G}_A$, the definition of $\psi$ in (11.12) and the support identity (11.14). □

## 12. Rolling-up contactomorphisms

By now, we have constructed the map

$$\Theta_A^{(k)} : \mathrm{Cont}_{J_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_1 \longrightarrow \mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_{k+1}^{2n+1}, \alpha_0)_0$$

and

$$\Xi_{A;N}^{(k)} : \mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_{k+1}^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_2 \longrightarrow \mathrm{Cont}_{K_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0$$

for each integer $N \geq 2$ by suitably choosing $\mathcal{U}_1$ and $\mathcal{U}_2$.

12.1. **Properties of $\Theta_A^{(k)}$ and $\Xi_{A;N}^{(k)}$.** We also have the inclusion map

$$\mathrm{Cont}_{K_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0 \hookrightarrow \mathrm{Cont}_{J_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0$$

since $K_A^{(k)} \subset J_A^{(k)}$, and hence we may canonically regard the map $\Xi_{A;N}^{(k)}$ also as a map

$$\Xi_{A;N}^{(k)} : \mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_{k+1}^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_2 \longrightarrow \mathrm{Cont}_{J_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0,$$

especially $\mathrm{Cont}_{J_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0$ as its codomain. We may choose $\mathcal{U}_2$ so small and then $\mathcal{U}_1$ that the composition of the two maps are defined. We will choose $\mathcal{U}_2$ so small that the map $\Xi_{A;N}^{(k)}$ is defined and that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Cont}_{J_A^k}(\mathcal{W}_k^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_1 \arrow[r, "\Theta_A^{(k)}"] & \mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_{k+1}^{2n+1}, \alpha_0)_0 \\
\mathrm{Cont}_{K_A^k}(\mathcal{W}_k^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_1 \arrow[u] & \mathrm{Cont}_{J_A^{(k+1)}}(\mathcal{W}_{k+1}^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_2 \arrow[l, "\Xi_{A;N}^{(k)}"] \arrow[u, "\Theta_A^{(k)} \circ \Xi_{A;N}^{(k)}"']
\end{tikzcd} \tag{12.1}$$

where the left vertical arrow map is just the inclusion map induced by the inclusion map $K_A^k \subset J_A^k$.

The following, especially the *strict equality* of the composition in Statement (3), plays an important role in Rybicki's construction $\vartheta$ of the contact counterpart of Mather's rolling-up operator $\theta_f$ that appears in the proof of [Ma1, Theorem 2, p.518].

**Proposition 12.1** (Compare with Proposition 8.2 [Ryb2])**.** *Taking $\mathcal{U}_2$ and then $\mathcal{U}_1$ sufficiently small, we have the following:*

*(1) $\Xi_{A;N}^{(k)}$ is continuous and preserves the identity.*
*(2) $\Xi_{A;N}^{(k)}\left(\mathrm{Cont}^{T^k}_{J_A^{(k+1)}}(\mathcal{W}_{k+1}^{2n+1}, \alpha_0)_0\right) \subset \mathrm{Cont}^{T^k}_{K_A^{(k)}}(\mathcal{W}_k^{2n+1}, \alpha_0)_0$.*
*(3) We have $\Theta_A^{(k)}\Xi_{A;N}^{(k)}(g) = g$ for any $g \in \mathrm{Dom}(\Xi_{A;N}^{(k)})$.*

*Proof.* Statements (1), (2) are straightforward to check. We focus on the proof of (3). Write

$$f := \Xi_{A;N}^{(k)}(g) \in \mathrm{Cont}_c^k(\mathcal{W}_k^m, \alpha_0)_0 \cap \mathcal{U}_1$$

Let $x = (\xi_0, \xi_1, \ldots, \xi_{k-1}, \xi_k, \xi_{k+1}, \ldots, \xi_n, p) \in \mathcal{W}_{k+1}^m$ and

$$\widetilde{x} = (\xi_0, \xi_1, \ldots, \xi_{k-1}, q_k, \xi_{k+1}, \ldots, \xi_n, p) \in \mathcal{W}_k^m$$

with $q_k \in \mathbb{R}$ satisfying $q_k = \xi_k \mod 1$. We take $q_k < -2A^2$. Then we have

$$\Theta_A^{(k)}(f)(x) := \pi_k(((T_k f)^N(\widetilde{x})))$$

by the definition of $\Theta_A^{(k)}$.

We consider the three cases separately:

$$\widetilde{x} \in E_k^-, \quad \widetilde{x} \in E_k^+, \quad \widetilde{x} \in \mathcal{W}_k^m \setminus E_k^- \cup E_k^+.$$

By definition, this trichotomy is preserved by the application of $f : \mathcal{W}_k^m \to \mathcal{W}_k^m$.

For the last case, we have

$$\begin{aligned} \pi_k(((T_k f)^N(\widetilde{x})))) &= \pi_k(\vec{e}_k + f(T_k f)^{N-1}(\widetilde{x})) \\ &= \pi_k(f(T_k f)^{N-1}(\widetilde{x})) = h_{\star_1}\pi_k((T_k f)^{N-1}(\widetilde{x})) \end{aligned}$$

where $h_{\star_1}$ is the map given by

$$h_{\star_1}(\widetilde{y}) = \begin{cases} h_0(\widetilde{y}) & \text{for } \widetilde{y} \in E_k^- \\ h_1(\widetilde{y}) & \text{for } \widetilde{y} \in E_k^+ \\ \widetilde{y} & \text{for } \widetilde{y} \in \mathcal{W}_k^m \setminus (E_k^- \cup E_k^+) \end{cases} \tag{12.2}$$

depending on the location of $\widetilde{y} := (T_k f)^{N-1}(\widetilde{x})$. (Recall where $h_0 = g_1^{\psi_k}$ and $h_1 = g_2^{\psi_k}$.)

Similarly we define $h_{\star 2}$ so that

$$\pi_k((T_k f)^{N-1}(\widetilde{x})) = h_{\star_2}\pi_k((T_k f)^{N-2}(\widetilde{x})).$$

By repeating this argument inductively, we have obtained

$$\pi_k(((T_k f)^N(\widetilde{x})))) = h_{\star_N} \cdots h_{\star_1}(\widetilde{x}).$$

By the property

$$f(E_k^-) = E_k^-, \quad f(E_k^+) = E_k^+, \quad f(\mathcal{W}_k^m \setminus E_k^+ \cup E_k^-)$$

and $(T_j f(\widetilde{x}))_k > (T_{j-1} f(\widetilde{x}))_k$, it follows that all $h_{\star_\ell} = id$ except possibly for at most two $\ell$'s, and hence $h_{\star_N} \cdots h_{\star_1} = h_2 h_1 = g$. This finishes the proof. ☐

**Remark 12.2.** (1) The property $\Theta_A^{(k)} \circ \Xi_{A;N}^{(k)} = \mathrm{id}$ is a fundamental ingredient in Mather's construction in general. (See [Ma1]-[Ma4], [E2], especially [Ma3] for a detailed analysis on its implication.)

(2) While the composition $\Theta_A^{(k)} \circ \Xi_{A;N}^{(k)}$ is the identity map, we will show that the other composition $\Xi_{A;N}^{(k)} \circ \Theta_A^{(k)}$ is not the identity map on the nose, but will be 'homotopic to the identity'.

We also state the following two lemmata from [Ryb2].

**Lemma 12.3** (Lemma 8.3 [Ryb2])**.** *If $f, g \in \mathrm{Dom}(\Theta^{(k)})$ and $\Theta^{(k)}(f) = \Theta^{(k)}(g)$, then $[f] = [g]$ in $H_1(\mathrm{Cont}_c(\mathcal{W}_{k+1}^{2n+1}, \alpha_0)_0)$.*

*Proof.* See the proof of [Ryb2, Lemma 8.3]. ☐

We remark that when $g_i = \Theta^{(k)}(f_i)$ for $i = 1, \ldots, \ell$, its product $g_1 \cdots g_\ell$ may not be contained in the image of $\Theta^{(k)}$ and the equality

$$\Theta^{(k)}(f_1 \cdots f_\ell) = g_1 \cdots g_\ell$$

fails to hold in general. In other words, the map $\Theta^{(k)}$ is *not multiplicative* on the nose. But the following lemma shows that it is multiplicative in homology.

**Lemma 12.4** (Lemma 8.4 [Ryb2])**.** *Let $k = 1, \ldots, n$.*

*(1) Suppose $g_i = \Theta^{(k)}(f_i)$ for $i = 1, \ldots, \ell$. Then there exists a collection of $\overline{f}_i$ such that $\{\mathrm{supp}\, \overline{f}_i\}$ are disjoint, $[\overline{f}_i] = [f_i]$ in $H_1(\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_0)_0)$, and satisfies*

$$g_1 \cdots g_\ell = \Theta^{(k)}\left(\overline{f}_1 \cdots \overline{f}_\ell\right)$$

*(2) If $g_1$, $g_2$, $g_1 g_2 \in \mathrm{Dom}(\Xi^{(k)})$, then we have*

$$[\Xi^{(k)}(g_1 g_2)] = [\Xi^{(k)}(g_1)\Xi^{(k)}(g_2)]$$

*in $H_1(\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_0)_0)$.*

(3) *If* $[g] = e$ *in* $H_1(\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_0)_0)$*, there is* $f \in \mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_0)_0$ *such that* $\Theta^{(k)}(f) = g$ *and* $[f] = e$ *in* $H_1(\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_0)_0)$.

*Proof.* See the proof of [Ryb2, Lemma 8.4]. □

12.2. **The hat operation applied.** Towards the verification of the claim made in Remark 12.2, we will first use the *hat operation* to deform the composition map

$$\Xi^{(k)}_{A;N} \circ \Theta^{(k)}_A : \mathrm{Cont}_{J^{(k)}_A}\left(\mathcal{W}_k^{2n+1}, \alpha_0\right)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}_{K^{(k)}_A}\left(\mathcal{W}_k^{2n+1}, \alpha_0\right)_0$$

so that we can define our wanted 'auxillary rolling-up operator'

$$\Psi^{(k)}_{A;N} : \mathrm{Cont}^{T^k}_{J^{(k)}_A}\left(\mathcal{W}_k^{2n+1}, \alpha_0\right)_0 \cap \mathcal{U}_1 \to \mathrm{Cont}^{T^k}_{K^{(k)}_A}\left(\mathcal{W}_k^{2n+1}, \alpha_0\right)_0$$

*that is equivariant with respect to the* $T^k$ *actions* on the domain and on the codomain thereof. Recall $\mathrm{Dom}\left(\Xi^{(k)}_{A;N}\right) = \mathrm{Cont}_{J^{(k+1)}_A}(\mathcal{W}_{k+1}^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_2$.

Now, we apply the hat operation explained in Section 10 to the function $\Theta^{(k)}_A(g) = u_g$, and define a map

$$\widehat{\Theta^{(k)}_A} : \mathrm{Cont}^{T^k}_{J^{(k)}_A}\left(\mathcal{W}_k^{2n+1}, \alpha_0\right)_0 \cap \mathcal{U}_1 \longrightarrow \mathrm{Cont}^{T^{k+1}}_{J^{(k+1)}_A}\left(\mathcal{W}_{k+1}^{2n+1}, \alpha_0\right)_0$$

by putting

$$\widehat{\Theta}^{(k)}_A(g) := \widehat{\Theta^{(k)}_A(g)}. \tag{12.3}$$

Then we introduce Rybicki's axillary rolling-up operators $\Psi^{(k)}_{A;N}$.

**Proposition 12.5** (Proposition 8.5 [Ryb2])**.** *Let* $r \geq 2$ *and* $k = 0, \ldots, n$*. There exists a neighborhood* $\mathcal{U}_3 \subset \mathcal{U}_1 \subset \mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_0)_0$ *and a map* $\Psi^{(k)}_{A;N}$ *such that the map*

$$\Psi^{(k)}_{A;N} : \mathrm{Cont}^{T^k}_{J^{(k)}_A}\left(\mathcal{W}_k^{m}, \alpha_0\right)_0 \cap \mathcal{U}_3 \to \mathrm{Cont}^{T^k}_{K^{(k)}_A}\left(\mathcal{W}_k^{m}, \alpha_0\right)_0$$

*satisfies the following:*

(1) $\Psi^{(k)}_{A;N}(\mathrm{id}) = \mathrm{id}$.

(2) *For any* $g \in \mathrm{Dom}(\Psi^{(k)}_{A;N})$*, we have*

$$[\Psi^{(k)}_{A;N}(g) \cdot \Xi^{(k)}_{A;N}\widehat{\Theta}^{(k)}_A(g)] = [g]$$

*in* $H_1(\mathrm{Cont}_c(\mathcal{W}_k^m, \alpha_0)$.

*Proof.* For readers' convenience, we just recall the definition of $\Psi^{(k)}_{A;N}$ leaving the verification of its properties stated in this proposition to the proof of [Ryb2, Proposition 8.5].

Let $g \in \mathrm{Cont}_{J^{(k)}_A}(\mathcal{W}_k^m, \alpha_0) \cap \mathcal{U}_1$. Then we define

$$\Psi^{(k)}_{A;N}(g) := \Xi^{(k)}_{A;N}\left(\Theta^{(k)}_A(g) \cdot (\widehat{\Theta}^{(k)}_A(g))^{-1}\right). \tag{12.4}$$

□

Finally, we are ready to define Rybicki's contact rolling-up operator $\Psi_A = \Psi_{A;N}$. Keeping the definition depends on the integer $N$ in mind, we will sometimes omit $N$ from the notations of $\Xi_{A;N}$ or $\Psi_{A;N}$ unless the dependence on $A$ needs to be emphasized.

For the simplicity of notation, we introduce the following notations:

$$\Theta^{(k>)}_A := \Theta^{(k)}_A \circ \cdots \circ \Theta^{(0)}_A, \quad \widehat{\Theta}^{(k>)}_A := \widehat{\Theta}^{(k)}_A \circ \cdots \circ \widehat{\Theta}^{(0)}_A \tag{12.5}$$

$$\Xi^{(<k)}_{A;N} := \Xi^{(0)}_{A;N} \circ \cdots \circ \Xi^{(k)}_{A;N} \tag{12.6}$$

We state the half of [Ryb2, Lemma 8.6] separately, the proof of which we refer readers thereto.

**Lemma 12.6** (Lemma 8.6 (1) [Ryb2]). *Assume $A$ and $N$ are given. Let $\mathcal{U}_3$ be a sufficiently $C^1$-small neighborhood of the identity in $\mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)_0$. Then for all $g \in \mathcal{U}_3$*

$$\left[\Xi_A^{(<n)} \Theta_A^{(n>)}(g)\right] = [g]$$

*in $H_1(\mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)_0$.*

(Statement of this lemma looks different from that of [Ryb2, Lemma 8.6 (1)] (which is stated only for $a = 2$, though) but equivalent if $\Xi^{(<n)} (= \Xi_A^{(<n)})$ therein is replaced by the current $\Xi_{A;N}^{(<n)}$ which should depend on $N$.)

**Proposition 12.7** (Compare with Proposition 8.7 [Ryb2]). *Let $r \geq 2$. There exists a sufficiently small $C^1$ neighborhood $\mathcal{U}_4 = \mathcal{U}_{\chi,r,A}$ in $\mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)_0$ and a mapping, called the contact rolling-up operator,*

$$\Psi_A : \mathrm{Cont}_{J_A}(\mathbb{R}^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_4 \to \mathrm{Cont}_{K_A}(\mathbb{R}^{2n+1}, \alpha_0)_0$$

*that satisfies the following:*

*(1) $\Psi_A$ is continuous and $g(\mathrm{id}) = \mathrm{id}$.*
*(2) For any $g \in \mathrm{Dom}(\Psi_A)$, $[\Psi_A(g)] = [g]$ in $H_1(\mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)_0$.*

*Proof.* The proof is essentially a duplication of the proof given in the proof of [Ryb2, Proposition 8.7] but we give details of the geometric construction for the self-containedness of the proof here.

Let $g \in \mathrm{Cont}_{J_A}(\mathbb{R}^m, \alpha_0)_0 \cap \mathcal{U}_1$. Define

$$\Psi_A(g) = g_0 g_1 \cdots g_n \tag{12.7}$$

where $g_0 = \Psi_A^{(0)}(g)$ and

$$g_k = \Xi^{(<k-1)} \Psi_A^{(k)} \widehat{\Theta_A}^{(k-1)}(g), \quad k = 1, \ldots, n. \tag{12.8}$$

Statement (1) is apparent by definition.

Now we inductively evaluate

$$\begin{aligned}
[\Psi_A(g)] &= [g_0 g_1 \cdots g_n] \\
&= [g_0 g_1 \cdots g_n \cdot \Xi_A^{(<n)} \widehat{\Theta}_A^{(n>)}(g)] \\
&= \left[g_0 g_1 \cdots g_{n-1} \left(\Xi_A^{(<n-1)} \Psi_A^{(n)} \widehat{\Theta}_A^{(n-1>)}(g)\right) \cdot \left(\Xi_A^{(<n-1)} \Xi_A^{(n)} \widehat{\Theta}_A^{(n)} \widehat{\Theta}_A^{(n-1>)}(g)\right)\right] \\
&= \left[g_0 g_1 \cdots g_{n-1} \cdot \Xi_A^{(<n-1)} \left(\Psi_A^{(n)} \left(\widehat{\Theta}_A^{(n-1>)}(g)\right) \cdot \Xi_A^{(n)} \widehat{\Theta}_A^{(n)} \left(\widehat{\Theta}_A^{(n-1>)}(g)\right)\right)\right] \\
&= \left[g_0 g_1 \cdots g_{n-1} \cdot \cdot \Xi_A^{(<n-1)} \widehat{\Theta}_A^{(n-1>)}(g)\right]
\end{aligned}$$

where we apply Lemma 12.6 for the second equality, the definition (12.8) for the third, and Proposition 12.5 (2) for the fifth equality. By repeating this process inductively downward from $k = n$ to $k = 0$, we have derived

$$[\Psi_A(g)] = \left[g_0 \cdot \Xi_A^{(0)} \widehat{\Theta}_A^{(0)}(g)\right] = \left[\Psi_A^{(0)}(g) \cdot \Xi_A^{(0)} \widehat{\Theta}_A^{(0)}\right] = [g]$$

where we apply the definition of $g_0$ and then again Proposition 12.5 (2) for the third equality. Combining the two, we have finished the proof of Statement (2). ☐

## Part 2. Optimal $C^r$ estimates on contactomorphisms

In this part, we prove all the necessary $C^r$ estimates of the various maps appearing in the proof of the main theorem.

Before starting the estimates, we first recall the remark made by Mather himself in the beginning of [Ma1, Section 6] almost verbatim, except the change of the covering projection $\mathbb{R}^m \to \mathcal{C}_i$ therein by the covering projection $\mathbb{R}^m \to \mathcal{W}_k^m$: "*The projection mapping $\mathbb{R}^{2n+1} \to \mathcal{W}_k^{2n+1}$ with $\mathcal{W}_k^{2n+1} = S^1 \times T^*(T^{k-1} \times \mathbb{R}^{n-k+1})$ gives us a preferred system of coordinates in a neighborhood of any point of $\mathcal{W}_k^{2n+1}$. The transition mappings between different coordinate* *systems which we obtain in this way are all translations. It follows that the $r$th derivative of any $C^r$ mapping of $\mathcal{W}_k^{2n+1}$ into itself is defined independently of the choice of preferred coordinate system. The $r$th derivative of such a mapping $v$ is a mapping $D^r v : \mathcal{W}_k^{2n+1} \to SL^r(\mathbb{R}^{2n+1}, \mathbb{R}^{2n+1})$ of $\mathcal{W}_k^{2n+1}$ into the space of symmetric $r$-linear mappings of $\mathbb{R}^{2n+1}$ into itself.*"

The same kind of practice enables us to work mainly with the Euclidean space $\mathbb{R}^{2n+1}$ in all the following estimates.

## 13. Basic $C^r$ estimates on the spaces $\mathrm{Diff}_c^r(\mathbb{R}^m)$; summary

In this section, we collect some function spaces, norm s and basic estimates of them on the Euclidean spaces or on the cylinders over tori. We start those used by Mather [Ma1] and Epstein [E2] associated with the general diffeomorphism groups, and then go to the case of contactomorphisms as used by Rybicki [Ryb2] afterwards in the next section.

Let $f : U \to \mathbb{R}^m$ be a $C^r$-function, where $U$ is an open subset of $\mathbb{R}^n$. We define

$$\|f\|_r := \sup_{x \in U} \|D^r f(x)\|.$$

We also consider maps between open subsets of spaces like $S^i \times \mathbb{R}^{n-i}$ with $0 \leq i \leq n$.

If $r \geq 1$ and $f$ is a diffeomorphism, we write

$$M_r(f) = \sup\{\|f - \mathrm{id}\,\|, \|f\|_1, \ldots, \|f\|_r\}.$$

If $\mathbf{f} = (f_1, \ldots, f_k)$ is a $k$-tuple of $C^r$-diffeomorphisms, we write $M_r(\mathbf{f}) = \sup_{1 \leq i \leq k} M_r(f_i)$.

We recall the formula

$$D(f \circ g) = (Df \circ g) \cdot (Dg)$$

where the right hand side is a composition of two linear maps, or a matrix multiplication of $n \times n$ matrices. For the higher derivatives, we have

$$\begin{aligned} D^r(f \circ g) &= (D^r f)(Dg \times \cdots \times Dg) + (Df \circ g)(D^r g) \\ &\quad + \sum C(i; j_1, \ldots, j_i)(D^i f \circ g)(D^{j_1} g \times \cdots \times D^{j_i} g) \end{aligned} \tag{13.1}$$

where $C(i; j_1, \ldots, j_i)$ is an integer which is independent of $f$, $g$ and even of dimensions of their domains and codomains for

$$1 < i < r, \quad j_1 + \ldots + j_i = r, \quad j_s \geq 1.$$

We recall that $(D^i f \circ g)$ is a *multilinear map* of $i$ arguments. *This implies that at least one* $j_s \geq 2$. For the simplicity of notation, we write

$$D^J g = (D^{j_1} \times \cdots \times D^{j_i})g, \quad J = (j_1, \ldots, j_i).$$

Then we can write

$$D^r(f \circ g) = (D^r f)(Dg \times \cdots \times Dg) + (Df \circ g)(D^r g) + \sum C(i; J)(D^i f \circ g)(D^J g). \tag{13.2}$$

We see that

$$M_1(f \circ g) \leq M_1(f)(1 + M_1(g)) + M_1(g) \tag{13.3}$$

by writing $f \circ g - \mathrm{id} = (f - \mathrm{id}) \circ g + (g - id)$.

**Definition 13.1** (Admissible polynomial)**.** A polynomial is called *admissible* if its coefficients are non-negative integers, and *has no constant or linear terms*.

We will denote an admissible polynomial by $F_{(*)}(x_1, \ldots x_\ell)$ in general where $(*)$ denotes the set of parameters into the coefficients of the polynomial, e.g.,

$$(*) = \{r, k, \cdots\}$$

where $r$ is the order of differentiation $D^r$ and $k$ ist the order of composition as $f_1 \circ \cdots \circ f_k$ and etc. The polynomial will vary depending on the circumstances that will appear later in the various estimates we carry out. We will (locally) enumerate them when we need to locally introduce several of them at the same time.

We derive the inequality

$$\|f \circ g\|_r \le \|f\|_r (1 + M_1^*(g))^r + \|g\|_r (1 + M_1(f))^r + F_{1;r}(M_{r-1}(f), M_{r-1}(g)) \tag{13.4}$$

where $F_{1,2}$ may be taken to be zero.

**Proposition 13.2** (Proposition 1.6 [E2])**.** *For each $r \ge 2$ and $k \ge 2$, there is an admissible polynomial $F_{2,k,r}$ of one variable with the following property. For $\mathbf{f} = (f_1, \ldots, f_k)$ which is composable, we have*

$$\|f_1, \circ \cdots \circ f_k\|_r \le k\|\mathbf{f}\|_r (1 + M_1(\mathbf{f}))^{r(k-1)} + F_{2;k,r}(M_{r-1}(\mathbf{f})). \tag{13.5}$$

*Moreover*

$$\|f_1 \circ \cdots \circ f_k\|_1 \le k M_1(\mathbf{f})(1 + M_1(\mathbf{f}))^{k-1}. \tag{13.6}$$

The following is a slight variation of [E2, Lemma 1.7] with the replacement of $M_1(f) < \frac{1}{2}$ by $M_1(f) < \frac{1}{4}$.

**Proposition 13.3** (Compare with Lemma 1.7 [E2])**.** *For each $r \ge 2$ and $k \ge 2$, there is an admissible polynomial $F_{3,r}$ of one variable with the following property. Let $f$ be a diffeomorphism of $\mathbb{R}^n$ satisfying $M_1(f) < \frac{1}{4}$ and let $r \ge 2$. Then*

$$\|f^{-1}\|_r \le (1 + M_1(f))^{2(r+1)} \|f\|_r + F_{3;r}(M_{r-1}(f)). \tag{13.7}$$

*Also*

$$\|f^{-1}\|_1 \le M_1(f)(1 + M_1(f))^2 \le 2M_1(f). \tag{13.8}$$

**Remark 13.4.** In general, if we choose $M_1^*(f) < \frac{1}{N}$, then there exists $\delta = \delta(N) \to 1$ as $N \to \infty$ such that

$$\|f^{-1}\|_r \le (1 + M_1(f))^{\delta(N)(r+1)} \|f\|_r + F_{3;r}(M_{r-1}(f)).$$

Also

$$\|f^{-1}\|_1 \le M_1(f)(1 + M_1(f))^2 \le \delta(N) M_1(f).$$

In particular, by letting $N \to \infty$, we can make $\delta(N)$ as close to 1 as we want.

## 14. $C^r$ ESTIMATES OF CONTACTOMORPHISMS OF THE PRODUCTS

For notational convenience, we will use the following notations systematically following those of [Ryb2].

Let $E \subset \mathbb{R}^m$ be a closed subset. We define

$$R_E := \sup_{x \in E} \operatorname{dist}\left(x, \overline{\mathbb{R}^m \setminus E}\right) \le \infty. \tag{14.1}$$

For any $f \in \mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)$ and $r \ge 0$, we put

$$\mu_r^*(f) := \max\{\|D^r(f - id)\|, \|D^r \ell_f\|\} \tag{14.2}$$

and

$$M_r^*(f) = \max\{\mu_0^*(f), \mu_1^*(f), \ldots, \mu_r^*(f)\}. \tag{14.3}$$

We consider the contact cylinders $\mathcal{W}_k^m = T^k \times \mathbb{R}^{m-k}$ with $m = 2n+1$, and

$$E_A^{(k)} = T^k \times [-A, A]^{m-k}, \quad k = 1, \ldots, n+1.$$

By the remark of Mather [Ma1] recalled in the beginning of this part, we can do the estimates on $\mathbb{R}^m$ which we will focus on.

14.1. $C^r$ **estimates of conformal exponents of the products.**

**Notation 14.1.** For a given multi-index $\underline{k} := (1, \ldots, k)$, we write

$$g_{\underline{k}}^\circ := g_k \circ g_{k-1} \circ \cdots \circ g_1.$$

We denote by $\mathbf{g}$ the tuple $(g_1, \cdots, g_k)$ and by $\mathbf{g}_{;\ell}$ the sub-tuple of $\mathbf{g}$ given by

$$\mathbf{g}_{;\ell} := (g_1, \cdots, g_\ell), \quad 1 \leq \ell \leq k.$$

Then we put

$$\|\mathbf{g}\|_r := \max_{i=1}^k \|g_i\|_r$$

for each given such a tuple, and

$$\begin{aligned} \ell_{\mathbf{g}} &:= (\ell_{g_1}, \ldots, \ell_{g_k}), \\ \|\ell_{\mathbf{g}}\|_r &:= \max_{i=1}^k \|\ell_{g_i}\|_r. \end{aligned}$$

With these notations set up, we state the following basic $C^r$ estimates on the conformal exponents of the products.

**Lemma 14.2.** *Let $g_1, \ldots, g_m$ be a set of contactomorphisms associated to the multi-index $I = (1, 2, \cdots, m)$. Then for $r$, $|I| \geq 2$, we have*

$$\begin{aligned} M_r^*(\mathbf{g}) \leq\ & r\|\ell_{\mathbf{g}}\|_r \left(1 + M_1^*(\mathbf{g})\right) + rM_r^*(\mathbf{g}) \left(1 + \|\ell_{\mathbf{g}}\|_1\right)^r \\ & + F_{1,r}(\|\ell_{\mathbf{g}}\|_{r-1}, \{M_{r-1}^*(\mathbf{g}_{;j})\}_j). \end{aligned} \tag{14.4}$$

*a multi-variable admissible polynomial $F_{1,r}$, and*

$$\|\ell_{g^{-1}}\|_r \leq \|\ell_g\|_r \left(1 + M_1^*(g^{-1})\right)^r + \|g^{-1}\|_r(1 + \|\ell_g\|_1)^r \tag{14.5}$$

$$+F_{2,r}\left(\|\ell_g\|_{r-1}, M_{r-1}(g^{-1})\right) \tag{14.6}$$

*for a 2 variable admissible polynomial $F_{2,r}$.*

*Proof.* For (1), we first use (3.7) to write

$$\ell_{g_{\underline{k}}^\circ} = \sum_{j=1}^k \ell_{g_j} \circ g_{\underline{j}}^\circ.$$

Then we derive

$$D^r(\ell_{g_{\underline{m}}^\circ}) = \sum_{j=1}^m D^r(\ell_{g_j} \circ g_{\underline{j}}^\circ).$$

We then apply (13.2) and obtain

$$\begin{aligned} D^r(\ell_{g_j} \circ g_{\underline{j}}^\circ) \leq\ & \|\ell_{g_j}\|_r \left(1 + \|g_{\underline{j}}^\circ\|_1\right)^r + \|g_{\underline{j}}^\circ\|_r(1 + \|\ell_{g_j}\|_1)^r \\ & + F_{1,r,j}\left(\|\ell_{g_j}\|_{r-1}, M_{r-1}(g_{\underline{j}}^\circ)\right). \end{aligned}$$

By summing this over $1 \leq j \leq k$ and using the inequalities $\|\ell_{\mathbf{g}_{;j}}\| \leq \|\ell_{\mathbf{g}}\|$ and

$$\|g_{\underline{j}}^\circ\|_\ell \leq M_r^*(\mathbf{g}_{;j}),$$

we get

$$\begin{aligned} M_r^*(\mathbf{g}) \leq\ & r\|\ell_{\mathbf{g}}\|_r \left(1 + M_1^*(\mathbf{g})\right) + rM_r^*(\mathbf{g}) \left(1 + \|\ell_{\mathbf{g}}\|_1\right)^r \\ & + \sum_{j=1}^k F_{1,r,j}(\|\ell_{\mathbf{g}}\|_{r-1}, M_{r-1}^*(\mathbf{g}_{;j})). \end{aligned}$$

Setting a multi-variable admissible polynomial

$$F_{1,r}(a, b_1, \cdots, b_k) := \sum_{j=1}^k F_{1,r,j}(a, b_j),$$

we have proved (14.4) i.e.,

$$\begin{aligned} M_r^*(\mathbf{g}) &\leq r\|\ell_{\mathbf{g}}\|_r \left(1+M_1^*(\mathbf{g})\right) + rM_r^*(\mathbf{g})\left(1+\|\ell_{\mathbf{g}}\|_1\right)^r \\ &\quad +F_{1,r}(\|\ell_{\mathbf{g}}\|_{r-1}, \{M_{r-1}^*(\mathbf{g}_{;j}\}_j). \end{aligned}$$

To prove (14.5), we first recall $\ell_{g^{-1}} = -\ell_g \circ g^{-1}$. Then we apply (13.2) to obtain

$$\begin{aligned} \|\ell_{g^{-1}}\|_r &\leq \|\ell_g\|_r \left(1+M_1^*(g^{-1})\right)^r + \|g^{-1}\|_r(1+\|\ell_g\|_1)^r \\ &\quad +F_{2,r}\left(\ell_g\|_{r-1}, M_{r-1}(g^{-1})\right) \end{aligned}$$

for a two-variable admissible polynomial $F_{2,r,k}$ of degree less than $r$. Then by substituting $\|g^{-1}\|_1 \leq 2M_1^*(g)$ for $M_1(g) < \frac{1}{4}$, we have finished the proof. ☐

14.2. **Basic $C^r$ estimates on** $\mathrm{Cont}_c^r(\mathbb{R}^{2n+1}, \alpha_0)$**.** The following is the list of basic estimates with respect to the amended norms (14.2) and (14.3)adapted to the case of contactomorphisms of those listed in Section 13 for the case of general diffeomorphisms.

**Lemma 14.3.** *Let $R_E < \infty$ and let $r \geq 0$ be given. Then there exists a constant $C$ independent of $f$ depending only on $R_E$ such that*

$$\mu_r^*(f) \leq C\mu_{r+1}^*(f)$$

*for all $f \in \mathrm{Cont}_E(\mathbb{R}^m, \alpha_0)$.*

*Proof.* Let $x \in E$. Since $R_E < \infty$, there exists $x_0 \in \mathbb{R}^m \setminus E$ with $|x - x_0| \leq R_E$. By the fundamental theorem of calculus, we have

$$D^r(f)(x) - D^r f(x_0) = \int_0^{|x-x_0|} \frac{d}{ds}(D^r f((1-s)x_0 + sx)\, ds$$

where $x_0 \in \mathbb{R}^m \setminus E$ and $x \in E$. The lemma immediately follows from the chain rule

$$\frac{d}{ds}(D^r f((1-s)x_0 + sx) = \sum_{i=1}^m (\xi_i(x) - \xi_i(x_0))\frac{\partial}{\partial \xi_i}(D^r f)((1-s)x_0 + sx)$$

with the choice of constant $C = mR_E$, since $\frac{\partial}{\partial \xi_i}(D^r f)(x_0) = 0$ for any $r \geq 0$ at $x_0 \in \mathbb{R}^m \operatorname{supp} f$. The same estimate also applies to $\ell_f$ since $\ell_f(x_0) = 0$ outside the support of $f$. ☐

The following is [Ryb2, Lemma 3.6] with a slight variation of some numerics in the statements.

**Proposition 14.4.** *Let $f_1, \ldots, f_k \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_0)$ and $\mathbf{f} = (f_1, \ldots, f_k)$.*

*(1) For $r = 1$, we have*

$$\mu_1^*(f_1 \circ \cdots \circ f_k) \leq k\mu_1^*(\mathbf{f})\left((1+M_0^*(\mathbf{f})(1+M_1^*(\mathbf{f}))\right)^{k-1}.$$

*(2) For any $r$, $k \geq 2$, there exists a two variable admissible polynomial $F_{r,k}$ such that*

$$\mu_r^*(f_1 \circ \cdots \circ f_k) \leq k\mu_r^*(\mathbf{f})\left((1+\mu_0^*(\mathbf{f})^{k-1}(1+\mu_1^*(\mathbf{f}))\right)^{r(k-1)} + F_{r,k}(M_{r-1}^*(\mathbf{f}) \tag{14.7}$$

*(3) Suppose $\mu_0^*(f)$, $\mu_1^*(f) < \delta < 1$. Then*

$$\mu_1^*(f^{-1}) \leq \frac{\mu_1^*(f)}{1-\delta}$$

*(4) Suppose $\mu_0^*(f)$, $\mu_1^*(f) < \delta < 1$. Then for any $r \geq 2$, there exists an admissible polynomial $F_r$ such that for any $f \in \mathrm{Cont}_c(\mathbb{R}^m, \alpha_0)$ such that*

$$\mu_r^*(f^{-1}) \leq \left(\frac{1}{1-\delta}\right)^{r+2}\left(1+\frac{\mu_1^*(f)}{1-\delta}\right)^{r+1}\mu_r^*(f) + F_r\left(M_{r-1}^*(f)\right).$$

*Proof.* The proof will be the same as that of [Ryb2, Lemma 3.6], except the fact that we are using the conformal exponent $\ell_f = \log \lambda_f$ which slightly simplifies the estimates thanks to the *more linear* formulae of the exponent of the product, i.e.,

$$\ell_{f_1 \circ \cdots \circ f_k} = \sum_{i=1}^{k} \ell_{f_i} \circ (f_{i+1} \circ \cdots \circ f_k).$$

(Recall that the corresponding formula for the conformal factor $\lambda_f$ are *multiplicative*.) Since we use $\ell_g$ instead of $\lambda_g$ in our estimates, we just focus on this difference of the calculation with the replacement of $\lambda_f$ by the conformal exponents $\ell_f$.

We start with the basic inequality

$$\|Df^{-1}\| \leq \frac{1}{1 - M_1(f)}$$

provided $M_1(f) < 1$, which follows from the inequality $\|M^{-1}\| \leq (1 - \|\operatorname{id} - M\|)^{-1}$ for an invertible matrix $M$.

Therefore we have $\|\ell_{f^{-1}}\| = \|\ell_f\|$, and

$$D\ell_{f^{-1}} = -\left(D\ell_f \circ f^{-1}\right) Df^{-1}$$

which implies

$$\|D\ell_{f^{-1}}\| \leq \|D\ell_f \circ f^{-1}\| \|Df^{-1}\| \leq \frac{\|D\ell_f\|}{1 - M_1^*(f)}.$$

Therefore we obtain

$$\mu_1^*(f^{-1}) = \max\{\|Df\|, \|D\ell_f\|\} \leq \max\left\{\|Df\|, \frac{\|D\ell_f\|}{1 - M_1^*(f)}\right\} \leq \frac{\mu_1^*(f)}{1 - M_1^*(f)}.$$

This proves (1) for the case $r = 1$. For higher $r \geq 2$, we inductively perform the estimates similarly as the proof of [Ryb2, Lemma 3.6] with the replacement of $\lambda_f$ and $\ell_f$.

In particular, if $M_1(f) < \delta$, the inequality is reduced to

$$\mu_1^*(f^{-1}) \leq \frac{\mu_1(f)}{1 - \delta}$$

and

$$\mu_r^*(f^{-1}) \leq \left(\frac{1}{1-\delta}\right)^{r+2} \left(1 + \frac{\mu_1^*(f)}{1-\delta}\right)^{r+1} \mu_r^*(f) + F_r\left(M_{r-1}^*(f)\right).$$

□

## 15. Estimates on derivatives under the Legendrianization

This is the central section of the present paper as well as in [Ryb2]. The materials of the present section are not involved in Mather's case of general diffeomorphisms and other later literature but that relies crucially on some contact geometric aspect related to the geometry of Legendrianization of contactomorphisms laid out in Section 5 and Section 6. In this regard, the following equation is the reason why the estimates given in Proposition 15.1 hold:

$$\Phi_{U;A}^{-1} \circ \Delta_g(\mathcal{W}_k^m) = j^1 u(\mathcal{W}_k^m) \tag{15.1}$$

on $\mathcal{W}_k^{2n+1}$ for some real-valued function $u = u_g$ on $\mathcal{W}_k^m$.

We start with the estimates involving the map $\mathscr{G}_A$ defined in (6.13). Since the chart $\Phi_U$ is fixed and will not be changed, we will suppress the dependence on $\Phi_U$ of various constants appearing below. Essentially all constants depend on this map $\Phi_U$.

The following is a comparison result between the $C^r$-norm of the contactomorphism $f$ and the $C^{r+1}$-norm of its 1-jet potential $u_f$. This is the optimal version of [Ryb2, Proposition 4.6].

**Proposition 15.1** (Compare with Proposition 4.6 [Ryb2]). *Let $A_0 > 2$ be a sufficiently large given constant. For $2 < A \leq A_0$, let $E \subset E_A^{(k)}$ be a given subinterval. For any $r \geq 2$ there is a $C^1$-neighborhood $\mathcal{U}_1$ of the identity in $\mathrm{Cont}_E(\mathcal{W}_k^{2n+1}, \alpha_0)$ such that for any $g \in \mathcal{U}_1$ with $M_1^*(g) < \frac{1}{4}$, the function $u = u_g := \mathscr{G}_A(g)$ satisfies the following:*

(1) *There exists constant $C_1 = C_1(r, \Phi_U)$ such that*
$$M_r^*(g) \leq C_1\|u_g\|_{r+1} + C_1 A(1 + \|u_g\|_2)^r + AF_{2,r}(\|h_X\|_{r-1}, \|u_g\|_r). \tag{15.2}$$
*holds uniformly over $1 < A \leq A_0$.*

(2) *There exists a constant $C_2 = C_2(r, \Phi_U)$ depending only on $r$ and the chart $\Phi_U$ for which*
$$\|u\|_{r+1} \leq C_2 M_r^*(g) + A^2 P_r(M_{r-1}^*(g)) \tag{15.3}$$
*holds uniformly over $1 < A \leq A_0$.*

Before launching on the proof of this proposition, we need some digression into the consequence of the definition $\mathscr{G}_A(g)$. Recall the definition $\Phi_{U;A} = \nu_A \circ \Phi_U \circ \mu_A^{-1}$. We observe
$$\pi_2 \circ \nu_A = \chi_A \circ \pi_2$$
where $\pi_2 : \mathbb{R} \times \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is the projection to the second factor, and that
$$\begin{aligned} \mu_A^{-1} \circ \delta(t, x, X) &= \delta\mu_A^{-1}(x, X, t) = \delta\left(t, \chi_A^{-1}(x), \chi_A^{-1}(X), t\right) \\ &= \delta\left(\chi_A^{-1}(x), \chi_A^{-1}(X), t\right). \end{aligned} \tag{15.4}$$

We regard a function $u : \mathcal{W}_k^{2n+1} \to \mathbb{R}$ as a periodic function on $\mathbb{R}^{2n+1}$ which we denote it by the same letter $u = u(x)$ for $x = (z, q, p)$ in the canonical coordinates of $J^1\mathbb{R}^n \cong \mathbb{R}^{2n+1}$. Then
$$du = \frac{\partial u}{\partial z}dz + \sum_{i=1}^n \frac{\partial u}{\partial q_i}dq_i + \sum_{i=1}^n \frac{\partial u}{\partial p_i}dp_i.$$
We write the coefficient vector of $du$ as
$$Du = (D_z u, D_q u, D_p u), \quad D_q u = (D_{q;1}, \ldots, D_{q;n}),\ D_p u = (D_{p;1}, \ldots, D_{p;n})$$
Then we consider a rescaled Darboux-Weinstein chart $\Phi_{U;A}$.

15.1. **Proof of Statement (1).** For the proof of Statement (1) of Proposition 15.1, we start with (15.1). We can express $\Gamma_g$ as
$$\Gamma_g(x) = \delta\Pi\Phi_{U;A}^{-1}(j^1u(y))$$
for some function $u = u_g$ provided $g$ is sufficiently $C^1$ close to the identity map. It follows from the discussion around (5.15) that $(\delta\Pi\Phi_{U;A}^{-1})^{-1} \circ \Gamma_g$ is a Legendrian submanifold of $\alpha_0$ when $g$ is sufficiently $C^1$-close to the identity.

Then its $p$ and $\eta$ components become
$$g(x) = \mathrm{pr}_2\left(\delta\Pi\Phi_{U;A}^{-1}((j^1u)(y))\right) \tag{15.5}$$
$$\ell_g(x) = \mathrm{pr}_3\left(\delta\Pi\Phi_{U;A}^{-1}((j^1u(y)))\right). \tag{15.6}$$
We also have
$$\Phi_{U;A}^{-1} = \mu_A \circ \Phi_U^{-1} \circ \nu_A^{-1}$$
by the definition (6.8).

In the following calculations, for the simiplicity of notation, *we will identify both $J^1\mathbb{R}^{2n+1}$ and $M_{\mathbb{R}^{2n+1}}$ with $\mathbb{R}^{2(2n+1)+1}$ and so regard $\Pi$ as the coordinate swapping self map, $H$ as another self map on $\mathbb{R}^{2(2n+1)+1}$ and the $G$-actions $\mathcal{G}_1$ and $\mathcal{G}_2$ act all on the same space* $\mathbb{R}^{2(2n+1)+1}$. Precisely speaking the expression '$\Pi + \mathsf{H}$' should have been written as
$$\Pi + \mathsf{H} \circ \Pi$$

Then by substituting (5.14) into (15.5), we derive the formula

$$\begin{aligned} g(x) &= \chi_A \mathrm{pr}_2(\Pi + \mathsf{H})(\nu_A^{-1} j^1 u(y)) \\ \ell_g(x) &= \mathrm{pr}_1(\Pi + \mathsf{H})(\nu_A^{-1} j^1 u(y)) \end{aligned}$$

from (5.14), where $\mathsf{H} = (h_x, h_X, h_t)$. We evaluate

$$\begin{aligned} g(x) &= \chi_A \mathrm{pr}_2(\Pi + \mathsf{H})(\nu_A^{-1} j^1 u(y)) \\ &= \chi_A \mathrm{pr}_2(\Pi + \mathsf{H})(\nu_A^{-1}(u(y), y, Du(y)) \\ &= \chi_A \eta_A^{-1} Du(y) + \chi_A h_X(\nu_A^{-1}(j^1 u(y)). \end{aligned} \tag{15.7}$$

Similarly, using (15.6),

$$\ell_g(x) = A^{-2}(u(y) + h_t(\nu_A^{-1}(j^1 u(y)). \tag{15.8}$$

We have

$$\chi_A \eta_A^{-1}(z, q, p) = (Az, q, Ap), \quad \eta_A \chi_A^{-1}(t, x, X) = (A^{-1} z, q, A^{-1} p).$$

Combining them with the above formulae, we derive

$$\|D(\chi_A(h_X \mu_A^{-1} j^1 u)\| \leq A \|D(h_X \eta_A^{-1} j^1 u)\|,$$

and taking further derivatives using the formula (13.2), we have obtained

$$\begin{aligned} \|D\Gamma_g\|_r &\leq A\|D(h_X j^1 u)\|_r \\ &\leq A\|D\chi_A^{-1} j^1 u\|_r (1 + M_1(h_X))^r \\ &\quad + A\|h_X\|_r (1 + M_1^*(\chi_A^{-1} j^1 u))^r + A F_{1,r}(M_{r-1}^*(h_X), M_{r-1}^*(\chi_A^{-1} j^1 u)) \\ &\leq C\|Dj^1 u\|_r (1 + M_1(h_X))^r \\ &\quad + CA\|h_X\|_r (1 + M_1^*(j^1 u))^r + A F_{1,r}(M_{r-1}^*(h_X), M_{r-1}^*(j^1 u)) \end{aligned}$$

for any $r \geq 1$. Here the third inequality holds, since $\|Dj^1 u\| \leq C M_1^*(g)$, $\|u\|_r \leq C\|u\|_{r+1}$, $\|\chi_A^{-1}\| \leq \frac{1}{A} \leq 1$ and $\chi_A$ is a linear invertible map.

15.2. **Proof of Statement (2).** Again we start with (15.7) and (15.8). We rewrite them into

$$\begin{aligned} Du(y) &= \eta_A \chi_A^{-1} g(x) - \eta_A h_X(\nu_A^{-1}(j^1 u(x)) & (15.9) \\ u(y) &= A^{-2}\left(\ell_g(x) - h_t(\nu_A^{-1}(j^1 u(x)))\right). & (15.10) \end{aligned}$$

We will derive the estimate of $\|u\|_{r+1}$ in terms of $\|g\|_r$ inductively over $r$ from these two, remembering that $|x - y| \leq \|h_x\|$.

For $\|u\|_0$ and $\|Du\|_0$, we derive

$$|u(y)| \leq A^{-2}|\ell_g(x)| + |h_t(\nu_A^{-1}(j^1 u(y)))| \leq A^{-2}(\|\ell_g\| + \|h_t\|)$$

and

$$|Du(y)| \leq \|\eta_A \chi_A^{-1} g\|_0 + \|\eta_A h_X\|_0 \leq \|g\|_0 + \|h_X\|_0.$$

Combining the two and using $A \geq 1$, we have derived

$$\begin{aligned} \|u\|_1 &\leq A^{-2}(\|\ell_g\| + \|h_t\|) + \|g\|_0 + \|h_X\|_0 \\ &\leq M_0^*(g) + \|\mathsf{H}\|_0. \end{aligned} \tag{15.11}$$

For the higher derivatives $\|D^r u\|$, $r \geq 2$, we start from

$$\Gamma_g(x) = \delta\Pi\Phi_{U;A}^{-1}(j^1 u(y))).$$

which is equivalent to

$$j^1 u(y) = \Phi_{U;A}(\delta\Pi)^{-1}\Gamma_g(x). \tag{15.12}$$

In particular, we have

$$x = \mathrm{pr}_2 \delta\Pi\Phi_{U;A}^{-1}(j^1 u(y)).$$

We mention that the map

$$y \mapsto \mathrm{pr}_2 \delta\Pi\Phi_{U;A}^{-1} \circ (j^1 u(y)))$$

is invertible as a map to $R_{g,A} = \mathrm{Image}\,\Gamma_g$ from $\mathcal{W}_k^m$ provided $g$ is sufficiently $C^1$-small. By writing the inverse thereof by $\underline{\Upsilon}_{A,g}$, we can write $x = \underline{\Upsilon}_{A,g}(\Gamma_g(y))$ for the map

$$x = \underline{\Upsilon}_{A,g}(\Gamma_g(y)) := (\mathrm{pr}_2 \delta\Pi\Phi_{U;A}^{-1} j_1 u)^{-1}(y) = (\mathrm{pr}_2 \circ \Gamma_g)^{-1}(y) = g^{-1}(y). \tag{15.13}$$

Therefore substituting $x = \Upsilon(y)$ into (15.12), we can express

$$\begin{aligned} j^1u(y) &= \Phi_{U;A}(\delta\Pi)^{-1}\Gamma_g \circ g^{-1}(y) = \Phi_{U;A}(\delta\Pi)^{-1}(g^{-1}(y), y, \ell_g \circ g^{-1}(y)) \\ &= \Phi_{U;A}(\delta\Pi)^{-1}(g^{-1}(y), y, -\ell_{g^{-1}}(y)). \end{aligned}$$

Recalling $\ell_{g^{-1}} = -\ell_g \circ g^{-1}$, we write

$$\aleph_A := \Phi_{U;A}^{-1}(\delta\Pi)^{-1}, \quad K_g(y) := (g^{-1}(y), y, -\ell_{g^{-1}}(y)) \tag{15.14}$$

**Lemma 15.2.** *For $r \geq 1$, we have*

$$\|j^1u\|_r \leq CM_r^*(g) + F_{r,3}\left(\max\{\|\aleph_A\|_{r-1}, M_{r-1}^*(g)\}\right).$$

*Proof.* We decompose

$$j^1u(y) = \aleph_A \circ K_g(y).$$

By applying (13.4) here with $f_1 = \aleph_A$, $f_2 = K_g$,, we obtain the formula

$$\|j^1u\|_r \quad \leq \quad (1 + \|\aleph_A\|_1)M_r^*(g) + (1 + M_1^*(g))^r \|\aleph_A\|_r + F_r\left(\max\{\|\aleph_A\|_{r-1}, M_{r-1}^*(g)\}\right)$$

where the terms $\|\aleph_A\|_r$ measured after the evaluation of $K_g$. In particular for $r \geq 1$, we have

$$\|\aleph_A\|_r \leq C\frac{1}{A^2}\|K_g\| \leq \frac{C}{A^2}M_1^*(g) \leq \frac{CC'}{A^2}M_r^*(g)$$

where the first inequality follows Proposition 6.3 (2). Therefore we can bound

$$(1 + M_1^*(g))^r \|\aleph_A\|_r \ \leq C''$$

uniformly over $1 \leq A \leq A_0$ and hence

$$\|j^1u\|_r \leq (1 + 2C''')\|\aleph_A\|_r + F_r\left(\max\{\|\aleph_A\|_{r-1}, M_{r-1}^*(g)\}\right).$$

□

Therefore we obtain

$$\|u\|_{r+1} \leq CM_r^*(g) + A^2 P_r(M_{r-1}^*(g))$$

inductively over $r \geq 1$, where $P_r$ is a polynomial that has no constant term. This finishes the proof of the second inequality of Proposition 15.1 if we have made

$$\|h_X\|_1,\ M_1^*(g) < \delta$$

for a sufficiently small $\delta$ by choosing the Darboux-Weinstein chart $\Phi_U$ with the neighborhood of $U$ of $\Delta_{\mathcal{W}_k^m}$ sufficiently small, and considering $g$ in a sufficiently small neighborhood of the identity.

## 16. Fragmentation of the second kind: Estimates

Our goal of this section is to establish the following derivative estimates.

**Proposition 16.1** (Proposition 5.7 [Ryb2])**.** *Let $2A \geq 2$ be an even integer, $\rho : [0,1] \to [0,1]$ be a boundary-flattening function such that $\rho \equiv 1$ on $[0, \frac{1}{4}]$ and $\rho \equiv 0$ on $[\frac{3}{4}, 1]$, and let*

$$E_{2A} := E_{2A}^{(0)} = [-2A, 2A]^{2n+1}$$

*Then there exists a $C^1$-neighborhood $\mathcal{U}_{\chi,A}$ of the identity in $\mathrm{Cont}_{E_{2A}}(\mathbb{R}^{2n+1}, \alpha_0)$ such that for any $g \in \mathcal{U}_{\chi,A}$ the factorization*

$$g = g_1 \cdots g_{a_m}, \quad a_m = (4A+1)^m$$

*given in Proposition 9.2 satisfies the following estimates: Whenever* $\operatorname{supp} g \subset E \subset E_{2A}$ *with* $R_E \leq 2$*, the inequalities*

$$\begin{aligned} M_r^*(g_K) &\leq C_\chi M_r^*(g) + AP_{\chi,r}(M_{r-1}^*(g)), & (16.1)\\ M_r^*(g_K) &\leq C_{\chi,r} M_r^*(g) & (16.2)\end{aligned}$$

*hold for all* $K = 1, \ldots, a_m$ *and* $r \geq 2$*.*

We will prove the estimate inductively over the number $1 \leq \ell \leq n+1$ of directions of the fragmentation.

We start with the case $\ell = 1$. In this case, we restate the above proposition as the following lemma.

**Lemma 16.2** (Compare with Proposition 5.6 (2) [Ryb2])**.** *Let* $2A > 1$ *be an even integer,* $\chi : [0,1] \to [0,1]$ *be a boundary-flattening function such that* $\chi \equiv 1$ *on* $[0, \frac{1}{4}]$ *and* $\rho \equiv 0$ *on* $[\frac{3}{4}, 1]$*, and let*

$$E_{2A} := E_{2A}^{(0)} = [-2A, 2A]^{2n+1}$$

*Then there exists a* $C^1$*-neighborhood* $\mathcal{U}_{\chi,A}$ *of the identity in* $\mathrm{Cont}_{E_{2A}}(\mathbb{R}^{2n+1}, \alpha_0)$ *such that for any* $g \in \mathcal{U}_{\chi,A}$

$$\operatorname{supp} g_K \subset \left( \left[k - \frac{3}{4}, k + \frac{3}{4}\right] \times \mathbb{R}^{2n} \right) \cap E_{2A}$$

*with* $k \in \mathbb{Z}$*,* $|k| \leq 2A$*, there exists a factorization*

$$g = g_1 \cdots g_{4A+1}, \tag{16.3}$$

*that satisfies the following estimates: Whenever* $\operatorname{supp} g \subset E \subset E_{2A}$ *with* $R_E \leq 2$*, the inequalities*

$$\begin{aligned} M_r^*(g_K) &\leq C_\chi M_r^*(g) + AP_{\chi,r}(M_{r-1}^*(g)),\\ M_r^*(g_K) &\leq C_{\chi,r} M_r^*(g) & (16.4)\end{aligned}$$

*hold for all* $K = 1, \ldots, 4A+1$ *and* $r \geq 2$*.*

*Proof.* As in the proof of Proposition 9.2, we extend $\chi$ to $[-1,1]$ as an even function and then to $\mathbb{R}$ as a 2-periodic function. For any sufficiently $C^1$-small contactomorphism $g$ with $\operatorname{supp} g \subset E_{2A}$, we consider the function $g^\psi := \mathscr{G}_A^{-1}(\psi u_g)$ and its factorization

$$\begin{aligned} g_1^\psi &:= g_{-2A} \circ g_{-2(A-1)} \circ \cdots \circ g_{2(A-1)} \circ g_{2A},\\ g_2^\psi &:= g(g_1^\psi)^{-1}\end{aligned}$$

with $\operatorname{supp} g_{2k} \subset [2k - \frac{3}{4}, 2k + \frac{3}{4}] \times \mathbb{R}^{2n}$ and $\operatorname{supp} g_{2k+1} \subset [2k + \frac{1}{4}, 2k + \frac{7}{4}] \times \mathbb{R}^{2n}$ constructed in the proof of Proposition 9.2.

Note that $\operatorname{supp} g_{2k}$ is contained in the disjoint union

$$\left[2k + \frac{1}{4}, 2k + \frac{7}{4}\right] \times \mathbb{R}^{2n}$$

Therefore we have

$$M_r^*(g^\psi) \leq \sum_{k=-A}^{A} M_r^*(g_{2k}). \tag{16.5}$$

On the other hand, we have

$$M_r^*(g_{2k}) \leq C M_r^*(g) + AP_r\big(\sup_{0 \leq s \leq r} \|j_1 u\|_s, \|H\|_r\big)$$

and hence have derived

$$M_r^*(g_{2k}) \leq C_{\chi,r} M_r^*(g) + AP_{\chi,r}\left(M_{r-1}^*(h_X), M_{r-1}^*(g - \mathrm{id})\right)$$

after rechoosing the polynomial $P_{\chi,r}$.

$\square$

*Proof of Proposition 16.1.* By applying the above construction consecutively to all variables $(q, p, z)$, we have finished the proof. Whenever $\operatorname{supp} g \subset E \subset E_{2A}$ with $R_E \leq 2$, the inequalities

$$\begin{aligned} M_r^*(g_K) &\leq C_{\chi,r} M_r^*(g) + AP_{\chi,r}(M_{r-1}^*(g)), \\ M_r^*(g_K) &\leq C_{\chi,r} M_r^*(g) \end{aligned} \tag{16.6}$$

hold for all $K = 1, \ldots, 4A+1$ and $r \geq 2$, *provided we consider $g$ from a sufficiently small $C^1$ neighborhood of the identity.* Analogous decompositions can be obtained with respect to other variables $q_i$ and $y_i$. This finishes the proof. □

## 17. The threshold determining optimal scaling estimates

We consider the situation of Section 8. For each $A > 1$, we consider the square

$$I_A = [-2,2] \times [-2,2]^n \times [-2A, 2A]^n \subset M_{\mathbb{R}^{2n+1}} \cong \mathbb{R}^{2(2n+1)+1}. \tag{17.1}$$

The following is the optimal scaling estimates that essentially determines the threshold $r = n+3$ for the dichotomy appearing later in the main theorem of the present paper. This optimal inequality is the contact counterpart of the inequality [Ma1, p.518], [E2, Equation (5.2)].

**Proposition 17.1** (Compare with Proposition 6.1 [Ryb2])**.** *If $|t_i| \leq 2A$ for $i = 1, \ldots, n$ and $g \in \mathrm{Cont}_{I_A}(\mathbb{R}^{2n+1}, \alpha_0)_0$, we have*

$$M_r^*(\rho_{A,\mathbf{t}} \circ g \circ \rho_{A,\mathbf{t}}^{-1}) \leq A^{4-2r}(2n)^{r+1} M_r^*(g).$$

*Proof.* We know

$$\operatorname{supp}(\rho_{A,\mathbf{t}} \circ g \circ \rho_{A,\mathbf{t}}^{-1}) \subset J_A$$

for all $g \in \mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)$ with support in the shifted $I_1$

$$I_1 + (2k-1)\vec{e}_i = [-2,2]^{n+i} \times [k-1, k+1] \times [-2,2]^{n-i}$$

for all $|k| \leq 2A - 1$ so that $-2A < k-1 < 2(A-1)$ and $-2(A-1) < k+1 < 2A$.

We have

$$\rho_{A,\mathbf{t}}^{-1}(z, q, p) = \left( \frac{z - \sum_{i=1}^n t_i q_i}{A^4}, \frac{q}{A^2}, \frac{p - \mathbf{t}}{A^3} \right)$$

from (8.7). Then we compute

$$D\rho_{A,t}^{-1}(z, q, p) = A^{-4}\vec{e}_z\, dz + \sum_{i=1}^n \left( -\frac{\vec{e}_z t_i}{A^4} + \frac{\vec{e}_i}{A^2} \right) dq_i + \frac{1}{A^3} \sum_{j=1}^n \vec{f}_j\, dp_j$$

as a vector valued one-form on $\mathbb{R}^{2n+1}$. Since $|t| \leq 2A$ and $A \geq 1$, we obtain $\|D\rho_{A,t}^{-1}\| \leq A^{-2}$ and

$$\|D_{zz}\rho_{A,t}\| \leq A^4, \quad \|D_{qz}\rho_{A,t}\| \leq A^2 |\mathbf{t}|_\infty \leq 2A^2 \tag{17.2}$$

$$\|D_{q_i q_i}\rho_{A,\mathbf{t}}\| \leq A^3, \quad \|D_{p_i p_i}\rho_{A,\mathbf{t}}\| \leq A^2. \tag{17.3}$$

This implies $\|D\rho_{A,t}\| \leq A^4$. Then we estimate

$$\begin{aligned} M_r^*(\rho_{A,\mathbf{t}} \circ g \circ \rho_{A,\mathbf{t}}^{-1}) &\leq \|D\rho_{A,\mathbf{t}}\|^r A^4 \|D^r g\| A^{-2r} \\ &\leq A^{4-2r} M_r^*(g). \end{aligned}$$

□

## 18. Estimates on the rolling-up and the fragmentation operators

In this section, we establish crucial estimates concerning the rolling-up and the fragmentation operators denoted by $\Theta_A^{(k)}$ and $\Xi_{A;N}^{(k)}$ in [Ryb2]. *We emphasize that to obtain the threshold $r = n + 2$, the order of power of $A$ being 2 is crucial.*

Recall the counterpart (11.1) of Mother's rolling-up operators for which we made the choice

$$q_k(\widetilde{x}) < -2A^2 \tag{18.1}$$

and choose $N > 0$ large enough so that

$$q_k\left((T_k g)^N(\widetilde{x})\right) > 2A^2 \tag{18.2}$$

therein.

**Proposition 18.1** (Compare with Inequality (2.2)[E2])**.** *Let $k = 0, \dots, k$. After shrinking $\mathcal{U}_1$ if necessary, $\Theta_A^{(k)}$ satisfies the following estimates: There exists a constant $K_1 > 0$ such that*

$$M_r^*(\Theta_A^{(k)}(g)) \leq K_1 A^2 (1 + M_1^*(g))^{rK_1A^2} M_r^*(g) + F_{r,A}(M_{r-1}^*(g)), \tag{18.3}$$

*and*

$$M_1^*(\Theta_A^{(k)}(g)) \leq K_1 A^2 M_1^*(g)(1 + M_1^*(g))^{K_1A^2}. \tag{18.4}$$

*for any $g \in \mathrm{Dom}(\Theta_A^{(k)})$. Moreover $F_{1,A} = 0$.*

*Proof.* The proof is similar to that of [E2, Inequality (2.2)]. Similarly as therein, we choose

$$N = 8A^2 + 4. \tag{18.5}$$

Then by the same argument as in the proof of [E2, Inequality (2.2)], we obtain

$$M_r^*(\Theta_A^{(k)}(g)) \leq K_1 A^2 (1 + M_1^*(g))^{rK_1A^2} M_r^*(g) + F_{r,A}(M_{r-1}^*(g)),$$

where $F_{r,A}$ is an admissible polynomial of one variable. We also have

$$M_1^*(\Theta_A^{(k)}(g)) \leq K_1 A^2 M_1^*(g)(1 + M_1^*(g))^{K_1A^2}.$$

This finishes the proof. $\square$

The following is a key corollary of the above proposition.

**Corollary 18.2.** *Assume the same hypotheses as in Proposition 18.1. There exists a constant $K_2 > 0$ such that*

$$M_r^*(\Theta_A^{(k)}(g)) \leq K_2 r A^2 M r_r^*(g) + F_{r,A}(M_{r-1}^*(g)), \tag{18.6}$$

*and*

$$M_1^*(\Theta_A^{(k)}(g)) \leq K_2 A^2 M_1^*(g). \tag{18.7}$$

*for any $g \in \mathrm{Dom}(\Theta_A^{(k)})$.*

*Proof.* The current proof goes along the same line as that of [E2, p.117]. Recall the definition of exponential function

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{nx} = e^x$$

and the function $n \to (1 + \frac{1}{n})^{nx}$ is an increasing function for any fixed $x > 0$. Therefore if we consider $g$'s whose $C^1$-norm $M_1^*(g) < \frac{1}{A^2}$, then

$$K_1 A^2 (1 + M_1^*(g))^{rK_1A^2} \leq A^2 K_1 \left(1 + \frac{1}{A^2}\right)^{rK_1A^2} \leq A^2 e^{rK_1}.$$

By setting $K_2 = K_1 e^{rK_1}$, we have finished (18.7).

Substituting this into (18.3), we obtain

$$M_r^*(\Theta_A^{(k)}(g)) \leq K_2 r A^2 M_r^*(g) + F_{r,A}(M_{r-1}^*(g)).$$

This finishes the proof. $\square$

We next prove the following estimates.

**Proposition 18.3** (Compare with Proposition 8.2 [Ryb2]). *By shrinking $\mathcal{U}_2$ and then $\mathcal{U}_1$ sufficiently, there are constants $C_{\chi,A}$, $\beta$ and $K_1$ such that*

$$M_r^*\left(\Xi^{(k)}_{A;N}(g)\right) \leq C_{\chi,r}\, M_r^*(g) + AP_{\chi,r}(M_{r-1}^*(g)). \tag{18.8}$$

*Proof.* This is an immediate consequence of (16.5) and (16.4). □

## Part 3. Proof of the main theorems

In this section, we give the proofs of the main theorems modulo the derivative estimates which will be established in Part II.

### 19. Rybicki's fundamental homological lemma

In this subsection, we explain a key lemma [Ryb2, Lemma 8.6] (2) that plays a fundamental role in Rybicki's proof of perfectness of the $C^\infty$ contactomorphism group, which is shared by many ingredient used in Mather's proof in [Ma1]–[Ma4] and [E2].

We have already stated the first half of the statement of [Ryb2, Lemma 8.6] in Lemma 12.6. Now we separate the second half of the statement of [Ryb2, Lemma 8.6] here, which we need to improve them in two regards to the following statement below.

**Remark 19.1.** Both the statement of [Ryb2, Lemma 8.6 (2)] and its proof are imprecise and need to be made precise and then proved. This is because the proof spelled out therein only provides the detail for the proof of $[g_2] = [g_2^{2^{n+2}}]$, and then just simply saying : *"Observe the above procedure may be repeated for any integer $a > 2$ by making use of $\eta_a$ and suitable translations $\tau_{i,t}$. As a result there exists $g_a \in \mathrm{Cont}_c(\mathbb{R}^n, \alpha_{st})_0$ such that $\widetilde{\Theta}^{(n)}(g_a) = f^*$ and $[g^{a^{n+2}}] = [g_a]$. Moreover by (1) we have $[g_a] = [g]$."*
To ensure that the validity of this novel identity hold true, one needs to generalize the construction of the operator $\Xi^{(k)}_A$ given in [Ryb2, p.3313] associated to the 2-fragmentation to arbitrary $N$-fragmentations as given in Section 11 of the present paper. Probably the author of [Ryb2] might have had this whole process in his mind. However, this is not even mentioned explicitly, while this homological identity is one of the crucial ingredients in his proof. In the present author's opinion, the details of this should have been provided in more details.

Because of this, we need to provide its details with some revision and amplification of the construction of the unfolding-fragmentation operator associated the $N$-fragmentation with $N > 2$ as given in the previous sections. Furthermore we also need to make a finer choice of various numerical constants appearing in the construction.

Wee recall

$$\mathcal{W}^{2n+1}_{n+1} \cong T^{n+1} \times \mathbb{R}^n \cong S^1 \times T^*(T^n \times \mathbb{R}),$$

and more generally

$$\mathcal{W}^{2n+1}_{k} \cong S^1 \times T^*(T^{k-1} \times \mathbb{R}^{n-k+1}) \tag{19.1}$$

for $k = 0, \ldots, n$. Let $f \in \mathcal{U}_3$ and consider the $T^{n+1}$-equivariant map

$$f^* := \widehat{\Theta}^{(n>)}_A(f) : \mathcal{W}^{2n+1}_{n+1} \to \mathcal{W}^{2n+1}_{n+1}. \tag{19.2}$$

**Lemma 19.2.** *The map $f^*$ satisfies the following:*

*(1) It has the form*

$$f^*(\xi_0, \xi, y) = f^*(z, q, p) = (z + f_0^*(p), q + f_1^*(p), p), \quad (\xi_0, \xi, y) = (z, q, p) \tag{19.3}$$

*for some function $(f_0^*, f_1^*) : \mathbb{R}^n_p \to \mathbb{R}^{n+1}_{(z,q)} = \mathbb{R}^z \times \mathbb{R}^n_q$.*

*(2) We have*

$$[\Xi^{(k)}_{A;a}(f^*)] = [\Xi^{(k)}_{A;a'}(f^*)] \tag{19.4}$$

*Proof.* Since $f^*$ is $T^{n+1}$-equivariant, we can write its unique lifting to $\mathbb{R}^{2n+1}$, still denoted by $f^*$ such that $f^* - id$ is $\mathbb{R}^{n+1}$-equivariant. This implies there exists a map $v : \mathbb{R}^n_p \to \mathbb{R}^{n+1}_{(z,q)}$ of the form given by

$$v(p) := (f_0^*(p), f_1^*(p)) \tag{19.5}$$

which is a section of the projection $\mathbb{R}^{2n+1} \to \mathbb{R}^n_p$. (See Appendix A for the proof in a similar context.) This proves Statement (1).

The statement (2) follows from Lemma 12.3 and Proposition 12.1(3). □

We have the following suggestive expression of $f*$

$$f^* = \mathrm{id} + v \circ \pi_3. \tag{19.6}$$

**Lemma 19.3.** *We put the map*

$$g_a := \Xi^{(<n)}_{A;a}(f^*) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1} \tag{19.7}$$

*Then $[g_a] = [g'_a]$ for all pairs $(a, a')$ with $a$, $a' \geq 2$.*

*Proof.* This follows from the definition (19.1) and (19.4). □

This lemma enables us to define the following cohomology class independent of $a$ but depending only on $f$ (and on $A$).

**Definition 19.4.** Let $f \in \mathrm{Cont}_c(\mathcal{W}^{2n+1}_n, \alpha_0)_0$ be given and $f^*$ be as in (19.2). We denote by $\omega(f)$ this common class of $g_a$ above in $H^1(\mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)_0)$.

This class is misleadingly denoted by $[g]$ in [Ryb2, Section 8] without encoding the dependence of the definition of $g$ therein on $a$ even though the definition of $g$ depends on $a$ and so the independence of the class on $a$ should have been proved in advance, but not even mentioned.

In this regard, the following is the correct statement of [Ryb2, Lemma 8.6].

**Proposition 19.5.** *For any integer $a \geq 2$, there exists an element $g'_a \in \mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)$ such that*

$$[g'_a] = \omega(f) \quad \& \quad [g'_a] = [g_a^{a^{n+2}}]$$

*in $H_1(\mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)_0)$ for all $f \in \mathrm{Cont}_c(\mathcal{W}^{2n+1}_k, \alpha_0)$ sufficiently $C^1$-close to the identity.*

Once we have made the above correct statement to prove, its proof will be a consequence of the arguments employed in the proof of [Ryb2, Lemma 8.6] by combining the strict identity $\Theta^{(n>)}_A \Xi^{(<n)}_{A;a}(g_a) = g_a$ and an inductive application of Proposition 12.1 generalized to the case $a > 2$,

The entirety of the next two sections will be occupied by the proof of Proposition 19.5. We divide our discussion into the two cases $a = 2$ and $a > 2$ purely for the simplicity and convenience of presentation since the details of the latter case are not very different from the former case. However as mentioned before, one needs to specify the dependence on $a$ in the identity "$[g] = [g^{a^{n+2}}]$" from [Ryb2] which we will do here.

In the next section, the case for $a = 2$ will be explained in detail, and then in the section after we consider the general case $a > 3$ in Proposition 21.1 and indicate how the proof of the case $a = 2$ can be adapted to the general case of $a \geq 2$.

## 20. Reformulation of Rybicki's identity "$[g] = \left[g^{2^{n+2}}\right]$"

In this section, we will provide a reformulation of the aforementioned Rybicki's identity "$[g] = \left[g^{2^{n+2}}\right]$" and then give its proof closely following his proof from [Ryb2].

We first introduce the following collection of subsets of $\mathbb{R}^{2n+1}$: where

$$\mathcal{I}_{n;A} = \left(\left[-\frac{1}{2}, 0\right] \cup \left[\frac{1}{4}, \frac{3}{4}\right]\right)^n \times [-2A, 2A]^n. \tag{20.1}$$

For each $1 \leq \ell \leq n+1$ and $\delta > 0$, we define

$$\mathcal{J}_{\ell,\delta} := \left(\left[-\frac{1}{4} - \delta, -\frac{1}{4} + \delta\right] \cup \left[\frac{1}{2} - \delta, \frac{1}{2} + \delta\right]\right)^{\ell} \times \mathbb{R}^n. \tag{20.2}$$

and

$$\mathcal{J}_{\ell,\delta;A} := \left(\left[-\frac{1}{4} - \delta, -\frac{1}{4} + \delta\right] \cup \left[\frac{1}{2} - \delta, \frac{1}{2} + \delta\right]\right)^{\ell} \times [-2A, 2A]^n. \tag{20.3}$$

We mention that for $\ell = n+1$, we have

$$\mathcal{I}_{n+1,\delta;A} \subset I_A$$

where we recall $I_A = [-2,2]^{n+1} \times [-2A, 2A]^n$ is the reference rectangularpid.

Then, by the definitions of the map $\Xi^{(k)}$ (Definition 11.4) and of $g$ above in (19.7), the equality

$$g = \mathrm{id} + v \circ \pi_3 \tag{20.4}$$

on the union

$$\begin{aligned} &\left(\left[-\frac{1}{2} - \varepsilon, -\frac{1}{2} + \varepsilon\right] \cup [1-\varepsilon, 1+\varepsilon]\right)^{n+1} \times [-2A, 2A]^n \\ \bigcup\ &\left(\left[-\frac{1}{8} + \varepsilon, \frac{1}{8} + \varepsilon\right] \cup \left[\frac{3}{8} - \varepsilon, \frac{5}{8} + \varepsilon\right]\right) \times \mathcal{I}_{n;A} \end{aligned}$$

for some $\varepsilon > 0$ (Lemma 11.6). Furthermore we have

$$\operatorname{supp} g \subset \left([-1, 0] \cup \left[\frac{1}{2}, \frac{3}{2}\right]\right)^{n+1} \times [-2A, 2A]^n \tag{20.5}$$

and $\Theta^{(n>)}(g) = f^*$ by Proposition 12.1 (3). The following is a key lemma toward the proof of Proposition 19.5, which we call *Rybicki's identity* is one of the crucial element in the proof.

**Proposition 20.1** (Equation (8.7) [Ryb2])**.** *Consider the case $a = 2$ and let $g_2 = \Xi^{(<n)}_{A;2}(f^*)$. Then we have*

$$[g_2] = \left[g_2^{2^{n+2}}\right]$$

*in $H_1(\mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)_0)$.*

We postpone the proof of this proposition till the next section because its details are rather tedious. Our proof closely follows but also fixes some ambiguities of the argument used in the proof of the identity given in [Ryb2, p. 3317- 3318] by clarifying its notations and much amplifying and optimizing the details of the proof thereof.

As an intermediate step, we will define a contactomorphism denoted by $g_2'$ that we will show simultaneously satisfies the following two equalities

$$[g_2] = [g_2'] \quad \& \quad [g_2'] = [g_2^{2^{n+2}}]. \tag{20.6}$$

The definition of $g_2'$ will take $n+1$ steps starting from the zero-th step. (Construction of the final element $g_2'$ is somewhat reminiscent of Mather's construction performed in [Ma2, Section 3].)

As the zero-th step, we start with considering the conjugation

$$h = \eta_2^{-1} g_2 \eta_2 \tag{20.7}$$

by the front scaling map $\eta_2$. Then it follows from (20.5) that

$$\operatorname{supp} h \subset \left(\left[-\frac{1}{2}, 0\right] \cup \left[\frac{1}{4}, \frac{3}{4}\right]\right) \times \mathcal{I}_{n;A}$$

where $\mathcal{I}_{n;A}$ is as in (20.1). We define the map

$$f^*_{\frac{1}{2}} := \mathrm{id} + \frac{1}{2} v \circ \pi_3 \tag{20.8}$$

where $v$ is the map given in (19.5).

**Lemma 20.2.** *We have* $h = f^*_{\frac{1}{2}}$ *on*

$$\mathcal{J}_{n+1,\varepsilon/2,A} \bigcup \left( \left( \left[ -\frac{1}{16} . \frac{1}{16} \right] \cup \left[ \frac{3}{16}, \frac{5}{16} \right] \right) \times \mathcal{J}_{n,\varepsilon/2,A} \right).$$

**Remark 20.3.** Lemma 20.5 and Lemma 20.6 are stated in the course of the proof of [Ryb2, Lemma 8.6] without details of their proofs. This lack of the details makes Rybicki's proof thereof rather hard to digest. Largely for the purpose of convincing the current author himself and for the convenience of the readers, we provide complete details of the proofs of these sublemmata here partially because we need to generalize the arguments for the case of $a > 2$ and also because some of the numerics appearing in the course of the proof of [Ryb2, Lemma 8.6] are not explicitly given but need to be clearly understood for the extension to $a > 2$. The lack of details has prevented the present author from penetrating the details of the proof and delayed the necessary generalization to the case of $a > 2$, until the present author himself rewrites all the details given here. In this sense, the present subsection is largely a duplication of [Ryb2, Lemma 8.6] with some semantic improvement of its presentation.

20.1. **Step 0 of the construction of $g_2'$: 2-fragmentation.** We take a fragmentation $h = \overline{h}_0 \widehat{h}_0$ similarly as in the definition of $\Xi^{(k)}_{A;a}$ so that

$$\overline{h}_0 = \begin{cases} h & \text{on } [-\frac{1}{2}, 0] \times \mathbb{R}^{2n} \\ \mathrm{id} & \text{there off,} \end{cases} \qquad \widehat{h}_0 = \begin{cases} \widehat{h}_0 = h & \text{on } [\frac{1}{4}, \frac{3}{4}] \times \mathbb{R}^{2n} \\ \mathrm{id} & \text{there off} \end{cases} \tag{20.9}$$

Then we define

$$h_0 = \widehat{h}_0 \tau_{0,\frac{1}{2}} \overline{h}_0 \tau^{-1}_{0,\frac{1}{2}} \tag{20.10}$$

and state a list of some technical properties of the map $h_0$ in the following list of lemmata that will enter into the proof of Proposition 20.1.

**Lemma 20.4.** $[h_0] = [h]$ *in* $H^1(\mathrm{Cont}_c(\mathbb{R}^{2n+1}))$.

*Proof.* We compute

$$\begin{aligned} h^{-1} h_0 &= (\overline{h}_0 \widehat{h}_0)^{-1} (\widehat{h}_0 \tau_{0,\frac{1}{2}} \overline{h}_0 \tau^{-1}_{0,\frac{1}{2}}) \\ &= \widehat{h}_0^{-1} \overline{h}_0^{-1} \widehat{h}_0 \tau_{0,\frac{1}{2}} \overline{h}_0 \tau^{-1}_{0,\frac{1}{2}} = \left[ \widehat{h}_0^{-1}, \overline{h}_0^{-1} \right] \left[ \overline{h}_0^{-1}, \tau_{0,\frac{1}{2}} \right]. \end{aligned}$$

This finishes the proof. ☐

**Lemma 20.5.** *We have*

$$h_0 = f^*_{\frac{1}{2}} \quad \text{on} \quad \left[ \frac{1}{4} - \varepsilon, \frac{1}{4} + \varepsilon \right] \times \mathcal{J}_{n,\varepsilon,A},$$

*and*

$$\operatorname{supp} h_0 \subset \left[ 0, \frac{3}{4} \right] \times \mathcal{I}_{n;A}$$

*provided* $M^*_1(f) < \delta$.

*Proof.* We have $\operatorname{supp} \widehat{h}_0 \subset [\frac{1}{4}, \frac{3}{4}]$ by (21.2) and hence

$$\operatorname{supp}(\tau_{0,\frac{1}{2}} \overline{h}_0 \tau^{-1}_{0,\frac{1}{2}}) \subset \left( \left[ -\frac{1}{4}, \frac{1}{4} \right] \times \mathbb{R}^{2n} \right).$$

Let

$$x = (z, q, p) \in \left[ \frac{1}{4} - \varepsilon, \frac{1}{2} + \varepsilon \right] \times \left( \left[ -\frac{1}{4} - \varepsilon, -\frac{1}{4} + \varepsilon \right] \cup \left[ \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon \right] \right)^n \times [-2A, 2A]^n.$$

(This rectangularpid is nothing but $\mathcal{J}_{n,\varepsilon,A}$ from (20.3) for $(\ell, \delta) = (n, \varepsilon)$.) Obviously

$$\tau^{-1}_{0,\frac{1}{2}}(x) = \left( z - \frac{1}{2}, q, p \right), \quad z - \frac{1}{2} \in \left[ -\frac{1}{4} - \varepsilon, \varepsilon \right]$$

on which $\widehat{h}_1 = \mathrm{id}$. Therefore we obtain

$$h_0(z,q,p) = \overline{h}_0\tau_{0,\frac{1}{2}}\left(z - \frac{1}{2}, q, p\right) = \overline{h}_0\left(z - \frac{1}{2}, q, p\right).$$

Furthermore $\overline{h}_0 = h$ on $\left[-\frac{1}{4} - \varepsilon, 0\right]$ by (21.2) and so

$$h_0(z,q,p) = h(z,q,p).$$

Finally, Lemma 20.2 proves $h = f^*_{1/2}$ for $z \in \left[\frac{1}{4} - \varepsilon, \frac{1}{2} + \varepsilon\right] \subset \left[\frac{3}{8}, \frac{5}{8}\right]$, provided $\delta > 0$ is sufficiently small. □

**Lemma 20.6.** *We have:*

(1) $h_0\tau_{0,\frac{1}{2}}h_0 = f^*_{\frac{1}{2}}$ *on* $[0, \frac{1}{4}] \times \mathbb{R}^{2n}$,

(2) $h_0\tau_{0,\frac{1}{2}}h_0 = \eta_2^{-1}(\Xi^{<n-1)}(f^*))\eta_2$ *on* $[0, \frac{1}{4}] \times \mathbb{R}^{2n}$.

*Here* $\Xi^{(<n-1)}(f^*) \in \mathrm{Cont}(\mathcal{W}_1^{2n+1}, \alpha_0)$ *is viewed as an element of* $\mathrm{Cont}(\mathbb{R}^{2n+1}, \alpha_0)$ *with period 1 with respect to z variable.*

*Proof.* By the fragmentation $h = \overline{h}_0\widehat{h}_0$ and the definition of $h_0$, we derive

$$h_0\tau_{0,\frac{1}{2}}h_0 = \overline{h}_0\tau_{0,\frac{1}{2}}h\tau_{0,\frac{1}{2}}\widehat{h}_0\tau^{-1}_{0,\frac{1}{2}}$$

by inserting the definition into the left hand side. On the other hand, on $\left[0, \frac{1}{4}\right] \times \mathcal{J}_{n,\varepsilon}$, we have

$$z - \frac{1}{2} \in \left[-\frac{1}{2}, -\frac{1}{4}\right].$$

Then (21.2) implies

$$\widehat{h}_0\tau^{-1}_{0,\frac{1}{2}}(z,q,p) = \left(z - \frac{1}{2}, q, p\right).$$

This in turn implies

$$\tau_{0,\frac{1}{2}}\widehat{h}_0\tau^{-1}_{0,\frac{1}{2}}(z,q,p) = (z,q,p)$$

and hence

$$\begin{aligned} h_0\tau_{0,\frac{1}{2}}h_0(z,q,p) &= \overline{h}_0\tau_{0,\frac{1}{2}}h(z,q,p) \\ &= \overline{h}_0\tau_{0,\frac{1}{2}}\eta_2^{-1}g\eta_2(z,q,p). \end{aligned} \tag{20.11}$$

Since $g = \Xi^{(<n)}(f^*) = f^*$ for $(z,q,p) \in \left[\frac{3}{8} - \varepsilon, \frac{5}{8} + \varepsilon\right]$, we compute

$$\eta_2^{-1}g\eta_2(z,q,p) = f^*_{\frac{1}{2}}(z,q,p) \tag{20.12}$$

thereon. From (20.8), we derive

$$|z\left(f^*_{\frac{1}{2}}(z,q,p)\right) - z| \leq \frac{1}{2}\|v\|_{C^0} = z + C\, M_1^*(f).$$

Therefore if $M_1^*(f) < \delta$ for asufficiently small $\delta > 0$, we have

$$\left[\frac{3}{8} - \varepsilon, \frac{5}{8} + \varepsilon\right] \cap \mathrm{supp}\, \overline{h}_0 = \emptyset.$$

This implies

$$\begin{aligned} \overline{h}_0\left(\tau_{0,\frac{1}{2}}(h(z,q,p))\right) &= \tau_{0,\frac{1}{2}}h\left(z - \frac{1}{2}, q, p\right) \\ &= \tau_{0,\frac{1}{2}}\left(z + \frac{1}{2}f_0^*(p), q + \frac{1}{2}f_1^*(p), p\right) \\ &= \left(z + \frac{1}{2} + \frac{1}{2}f_0^*(p), q + \frac{1}{2}f_1^*(p), p\right). \end{aligned}$$

For the support property, suppose

$$(z,q,p) \notin \left[0, \frac{3}{4}\right] \times \mathcal{I}_{n;A} = \left[0, \frac{3}{4}\right] \times \left(\left[-\frac{1}{2}, 0\right] \cup \left[\frac{1}{4}, \frac{3}{4}\right]\right)^n \times [-2A, 2A]^n.$$

Again a direct evaluation, which we omit the straightforward details, shows $h_0(z,q,p) = (z,q,p)$. This confirms the required support property. Combining the above discussions, we have finished the proof Statement (1).

For Statement (2), we start with (15.1)

$$h_0\tau_{0,\frac{1}{2}}h_0(z,q,p) = \overline{h}_0\tau_{0,\frac{1}{2}}\eta_2^{-1}g\eta_2(z,q,p).$$

Writing $\Xi_A^{(n)} = \Xi_{A;2}^{(n)}$ and substituting of $g = \Xi_A^{(<n)}(f^*) = \Xi_A^{(<n-1)} \circ \Xi_A^{(n)}(f^*)$ thereinto makes the right hand side thereof become

$$\begin{aligned} h_0\tau_{0,\frac{1}{2}}h_0(z,q,p) &= (\overline{h}_0\tau_{0,\frac{1}{2}}\eta_2^{-1}) \circ \Xi_A^{(<n-1)} \circ (\Xi_A^{(n)}(\eta_2(z,q,p))) \\ &= (\overline{h}_0\tau_{0,\frac{1}{2}}\eta_2^{-1}) \circ \Xi_A^{(<n-1)}(\eta_2(z,q,p))) \\ &= \left((\overline{h}_0\tau_{0,\frac{1}{2}}\eta_2^{-1}) \circ \Xi_A^{(<n-1)} \circ \eta_2\right)(z,q,p) \end{aligned}$$

If $z \in [0, \frac{1}{4}]$, $(z, 2q, 2p) \notin \operatorname{supp} \Xi_A^{(n)}$ and hence

$$\Xi_A^{(n)}(\eta_2(z,q,p))) = \eta_2(z,q,p).$$

On the other hand, Statement (1) proves $h_0\tau_{0,\frac{1}{2}}h_0(z,q,p) = f^*_{\frac{1}{2}}(z,q,p)$ when $z \in [0, \frac{1}{4}]$. Combining the two, we have derived

$$\eta_2^{-1} \circ \Xi_A^{(<n-1)} \circ \eta_2(z,q,p) = (\overline{h}_0\tau_{0,\frac{1}{2}})^{-1}(f^*_{\frac{1}{2}}(z,q,p)).$$

We compute

$$(\overline{h}_0\tau_{0,\frac{1}{2}})^{-1}(f^*_{\frac{1}{2}}(z,q,p)) = (\tau_{0,\frac{1}{2}})^{-1}\overline{h}_0^{-1}\left(z + \frac{1}{2} + \frac{1}{2}f_0^*(p), q + \frac{1}{2}f_1^*(p), p\right).$$

Since $z + \frac{1}{2} + \frac{1}{2}f_0^*(p) \in [-\frac{1}{2}\|f_0^*\|, 1 + \frac{1}{2}\|f_0^*\|]$ and $\overline{h} \equiv \mathrm{id}$ on $[0,2] \setminus [-\frac{1}{2}+\varepsilon, \varepsilon]$,

$$\overline{h}_0^{-1}\left(z + \frac{1}{2} + \frac{1}{2}f_0^*(p), q + \frac{1}{2}f_1^*(p), p\right) = \left(z + \frac{1}{2} + \frac{1}{2}f_0^*(p), q + \frac{1}{2}f_1^*(p), p\right).$$

Then we derive

$$(\tau_{0,\frac{1}{2}})^{-1}\overline{h}_0^{-1}\left(z + \frac{1}{2} + \frac{1}{2}f_0^*(p), q + \frac{1}{2}f_1^*(p), p\right) = \left(z + \frac{1}{2}f_0^*(p), q + \frac{1}{2}f_1^*(p), p\right) = f^*_{\frac{1}{2}}(z,q,p)$$

on $[0, \frac{1}{4}] \times \mathbb{R}^{2n}$. Combining the last 4 equalities, we have finished the proof of Statement (2). □

20.2. **Downward induction for $k_\ell$ with $\ell$ from $n$ to $0$.** Now we define

$$k_0 = h_0\tau_{0,\frac{1}{2}}h_0\tau_{0,\frac{1}{2}}^{-1} = h_0^2[h_0^{-1}, \tau_{0,\frac{1}{2}}] \tag{20.13}$$

by considering the variable $\xi_0$. By the similar arguments used in the study of $h_0$ above, verification of the following list of properties is straightforward,

(1) $\operatorname{supp}(k_0) \subset [0, \frac{5}{4}] \times \mathcal{J}_{n,\varepsilon,A}$,
(2) $k_0 = h$ on $[\frac{1}{4} - \varepsilon, 1 + \epsilon] \times \mathcal{J}_{n,\varepsilon,A}$,
(3) $k_0\tau_{0,1}k_0 = f^*_{\frac{1}{2}}$ on $[0, \frac{1}{4}] \times \mathcal{J}_{n,\varepsilon,A}$,
(4) $k_0\tau_{0,1}k_0 = \eta_2^{-1}\Xi^{(<n-1)}(f^*)\eta_2$ on $[0, \frac{1}{4}] \times \mathbb{R}^{2n}$,
(5) $\Theta^{(0)}(k_0) = f^*_{\frac{1}{2}}$ on $S^1 \times \mathcal{J}_{n,\varepsilon,A}$,
(6) $\Theta^{(0)}(k_0) = \eta_2^{-1}\Xi^{(<n-1)}(f^*)\eta_2$ on $[0, \frac{1}{4}] \times \mathbb{R}^{2n}$ on $\mathcal{W}_1^{2n+1}$,
(7) $[k_0] = [h_0^2] = [h^2] = [g_2^2]$.

The last equality of Statement (7) follows from Lemma 20.5.

Next, starting with $k_0$, by replacing $h$ by $k_0$, we define $\overline{h}_1$, $\widehat{h}_1$, $h_1$ and $k_1$ analogously as above, but now with respect to the variable $\xi_1 = q_1(\mod 1)$. Then we define

$$k_1 = h_1\tau_{1,\frac{1}{2}}h_1\tau_{1,\frac{1}{2}}^{-1} = h_1^2[h_1^{-1}, \tau_{1,\frac{1}{2}}].$$

It satisfies

(1) $\Theta^{(1)}(k_1) = f^*_{\frac{1}{2}}$ on $T^2 \times \mathcal{J}_{n,\varepsilon,A}$,
(2) $\Theta^{(1>)}(k_1) = f^*_{\frac{1}{2}}$ on $T^2 \times \mathcal{J}_{n-1,\varepsilon,A}$,
(3) $\Theta^{(1>)}(k_1) = \eta_2^{-1}\Xi^{<n-2)}(f^*)\eta_2 = f^*_{\frac{1}{2}}$ on $\mathcal{W}^{2n+1}_2$.
(4) $[k_1] = [k_0]^2 = [h^4] = [g_2^4]$.

Continuing this procedure inductively, we obtain a sequence

$$h_2, k_2, \dots, h_n, k_n \in \mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)_0$$

such that it satisfies

(1) $\Theta^{(n)}(k_n) = f^*_{\frac{1}{2}}$ on $T^{n+1} \times \mathcal{J}_{1,\varepsilon,A}$,
(2) $\Theta^{(n>)}(k_n) = f^*_{\frac{1}{2}}$ on $T^{n+1} \times \mathcal{J}_{0,\varepsilon,A}$,
(3) $\Theta^{(n>)}(k_n) = \eta_2^{-1}\Xi^{(0)}(f^*)\eta_2 = f^*_{\frac{1}{2}}$ on $\mathcal{W}^{2n+1}_{n+1}$.
(4) $[k_n] = [k_{n-1}]^2 = [g_2^{2^{n+1}}]$.

Finally we define

$$g_2' := \tau k_n \tau^{-1} k_n \tag{20.14}$$

for a suitable translation in the direction of $(\xi_0, \xi)$ as in Section 11 so that $g_2'$ has its support that is a disjoint union of connected intervals of the same size. (See the proof of [Ryb2, Lemma 8.4].) Then we summarize the above discussion into the following.

**Lemma 20.7.** *$g_2'$ satisfies the second equality of* (20.6).

This finishes the proof of Proposition 20.1 for the case $a = 2$.

## 21. The identity $[g_a] = [g_a^{a^{n+2}}]$ for $a > 2$

The above process of defining $g_a'$ for $a > 2$ starting from $g_a = \Xi^{(<n)}_{A;a}(f^*)$ can be applied verbatim for any integer $a \geq 3$ utilizing the $a$-fragmentation operator $\Xi^{(k)}_{A;a}$ defined in Section 12 with the replacement of $N = 2$ by $N = a$.

As the first step, we will again construct an element $g_a'$ that satisfies $[g_a'] = [g_a^{a^{n+2}}]$. We will briefly indicate necessary changes to be made from (20.6). Let $a > 2$ be any given integer. Our goal is to prove the following.

**Proposition 21.1** (Equation (8.7) [Ryb2])**.** *Let $a \geq 2$ and consider $g_a := \Xi^{(<n))}_{A;a}(f^*)$. Then*

$$[g_a] = \left[g_a^{a^{n+2}}\right]$$

*in $H_1(\mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)_0)$.*

The same kind of proof with the replacement of 2 by an arbitrary integer $a \geq 2$ with some changes in its details can be given to generalize this lemma as follows.

Again we will define a contactomorphism denoted by $g_a'$ that satisfies the two equalities

$$[g_a'] = [g_a] \quad \& \quad [g_a'] = [g_a^{a^{n+2}}] \tag{21.1}$$

and the definition of $g_a'$ from $g_a$ will take $n + 1$ steps. As the zero-th step, we start with considering the conjugation

$$h = \eta_a^{-1} g_a \eta_a$$

by the front scaling map $\eta_a$ similarly as in (20.7).

We consider the interval $[-a, a]$ into $a$ pieces of subintervals of length 2 and then scale them back by the ratio $a$

$$I_j^a = \frac{1}{a}[-a + 2(j-1), -a + 2j] = \left[-1 + \frac{2(j-1)}{a}, -1 + \frac{2j}{a}\right], \quad j = 1, \dots,$$

and take the union of their 'halves'

$$(I_j^a)' = \frac{1}{2} I_j^a,$$

and the union

$$(I')^a := \bigcup_{j=1}^{a} (I_j^a)'.$$

Then we consider the bump function $\psi_k$ defined on $[-a, a]$ in (11.12) and extend periodically to whole $\mathbb{R}$.

With these preparations, the process of defining an element $g_a' \in \mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)$ satisfying $[g_a'] = [g_a^{a^{n+2}}]$ for general $a > 3$ is entirely similar to the case of $a = 2$. After its construction, we will also have established

$$\omega(f) = [g_a^{a^{n+2}}].$$

We start with the following

**Lemma 21.2.** $[g_a'] = \omega(f)$

*Proof.* In the course of the proof of Lemma 19.3, we have established

$$\Theta^{(n>)}(g_a) = f^*.$$

By applying Lemma 12.6 and the definition of $g$ in turn, we obtain

$$[g_a'] = [\Xi_{A;a}^{(<n)}(f^*)] = [g_a] = \omega(f)$$

where the last equality comes from Lemma 19.3 and the definition of $\omega(f) \in \mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)$. $\square$

Again the definition of $g_a'$ will take $n + 1$ steps starting from the zero-th step. As the zero-th step, we start with considering the conjugation

$$h = \eta_a^{-1} g_a \eta_a.$$

Then the proof of the following lemma is similar to that of Lemma 20.2

**Lemma 21.3.** *We define the map* $f_{\frac{1}{a}}^*$ *by*

$$f_{\frac{1}{a}}^*(z, q, p) := \left( z + \frac{1}{a} f_0^*(p), q + \frac{1}{a} f_1^*(p), p \right).$$

*Then*

$$h(z, q, p) = f_{\frac{1}{a}}^*(z, q, p)$$

*on*

$$\mathcal{J}_{n+1,\varepsilon/2,A} \bigcup \left( \left( A_{\frac{1}{2a}}^a(\mathrm{ev}) \cup \left( A_{\frac{1}{2a}}^a(\mathrm{ev}) + \frac{1}{2a} \right) \right) \times \mathcal{J}_{n,\varepsilon/2,A} \right)$$

*where we define*

$$A_{\mathrm{ev}}^a := \bigcup_{i=1}^{a} A_{\frac{1}{2a}}^a \left( \frac{c_{2j}}{a} \right).$$

*Recall the definition* (11.9) *for the interval* $A_{\frac{1}{2N}}^N$ *in general.*

21.1. **Step 0 of the construction of $g_a'$: $a$-fragmentation.** We take an $a$-fragmentation $h = \overline{h}_0 \widehat{h}_0$ similarly as in the definition of $\Xi_{A;N}^{(k)}$ so that

$$\overline{h}_0 = \begin{cases} h & \text{on } \frac{1}{2} I^a \times \mathbb{R}^{2n} \\ \mathrm{id} & \text{there off,} \end{cases} \qquad \widehat{h}_1 = \begin{cases} \widehat{h}_0 = h & \text{on } \left( \left( A_{\mathrm{ev}}^a + \frac{1}{2a} \right) + \frac{1}{2a} \right) \times \mathbb{R}^{2n} \\ \mathrm{id} & \text{there off} \end{cases} \tag{21.2}$$

where by definition we have

$$\frac{1}{2} I^a = \bigcup_{i=1}^{N} \left[ -1 + \frac{2i-1}{a}, -1 + \frac{2i+1}{a} \right], \quad A_{\mathrm{ev}}^a + \frac{1}{2a} = \bigcup_{j=1}^{a} A_{\frac{1}{2a}}^a \left( \frac{c_{2j}}{a} + \frac{1}{2a} \right)$$

In particular, we can further decompose

$$\overline{h}_0 = \overline{h}_{0,1} \cdots \overline{h}_{0,a}$$

so that their supports are pairwise disjoint. Then we define the counterpart of (20.10) for the $a$-fragmentation

$$h_0 = \widehat{h}_0 \left(\tau_{0,\frac{1}{a}} \overline{h}_{0,1} \tau_{0,\frac{1}{a}}^{-1}\right) \left(\tau_{1,\frac{1}{a}} \overline{h}_{0,2} \tau_{1,\frac{1}{a}}^{-1}\right) \cdots \left(\tau_{n,\frac{1}{a}} \overline{h}_{0,n} \tau_{n,\frac{1}{a}}^{-1}\right). \tag{21.3}$$

We and state a list of the counterparts of the properties of the map $h_0$ for the case of $a > 2$ in the following list of lemmata that will enter into the proof of Lemma 19.3. We omit their proofs since they are entirely similar to the case of $a = 2$ once the correct statements for $a > 2$ are made.

**Lemma 21.4.** *We have* $[h_0] = [h]$.

**Lemma 21.5.** *We have*

$$h_0 = f^*_{\frac{1}{a}} \quad \text{on } \frac{1}{4} I^a \times \mathcal{J}_{n,\varepsilon,A},$$

*and*

$$\operatorname{supp} h_0 \subset \frac{3}{4} I^a \times \mathcal{J}_{n,\varepsilon,A}$$

*provided* $M_1^*(f) < \delta$ *for a sufficiently small.*

**Lemma 21.6.** *We have:*

*(1)* $h_0 \tau_{0,\frac{1}{a}} h_0 \tau_{0,\frac{2}{a}} \cdots \tau_{0,\frac{a-1}{a}} h_0 = f^*_{\frac{1}{a}}$ *on* $\frac{1}{4} A^a_{\mathrm{ev}} \times \mathbb{R}^{2n}$,

*(2)* $h_0 \tau_{0,\frac{1}{a}} h_0 \tau_{0,\frac{2}{a}} \cdots \tau_{0,\frac{a-1}{a}} h_0 = \eta_a^{-1} (\Xi^{<n-1)}(f^*)) \eta_a$ *on* $\frac{1}{4} A^a_{\mathrm{ev}} \times \mathbb{R}^{2n}$.

*Here* $\Xi^{(<n-1)}(f^*) \in \mathrm{Cont}(\mathcal{W}_1^{2n+1}, \alpha_0)$ *is viewed as an element of* $\mathrm{Cont}(\mathbb{R}^{2n+1}, \alpha_0)$ *with period 1 with respect to z variable.*

21.2. **Downward induction for** $a > 2$**.** Now we define

$$k_0 = h_0 \tau_{0,\frac{1}{a}} h_0 \tau_{0,\frac{2}{a}} \cdots \tau_{0,\frac{a-1}{a}} h_0$$

using the variable $\xi_0$. By the similar arguments used in the study of $h_0$ for the case $a = 2$ above, verification of the following list of properties is straightforward,

(1) $\operatorname{supp}(k_0) \subset \bigcup_{j=-[\frac{a+1}{2}]}^{[\frac{a+1}{2}]} \left(\frac{2j}{a} + \frac{1}{a}[0, \frac{5}{4}]\right) \times \mathcal{J}_{n,\varepsilon,A}$,

(2) $k_0 = h$ on $\bigcup_{j=-[\frac{a+1}{2}]}^{[\frac{a+1}{2}]} \left(\frac{2j}{a} + \frac{1}{a}[\frac{1}{4} - \varepsilon, 1 + \epsilon]\right) \times \mathcal{J}_{n,\varepsilon,A}$,

(3) $k_0 \tau_{0,1} k_0 = f^*_{\frac{1}{2}}$ on $\bigcup_{j=-[\frac{a+1}{2}]}^{[\frac{a+1}{2}]} \left(\frac{2j}{a} + \frac{1}{a}[0, \frac{1}{4}]\right) \times \mathcal{J}_{n,\varepsilon,A}$,

(4) $k_0 \tau_{0,1} k_0 = \eta_2^{-1} \Xi^{(<n-1)}(f^*) \eta_2$ on $\bigcup_{j=-[\frac{a+1}{2}]}^{[\frac{a+1}{2}]} \left(\frac{2j}{a} + \frac{1}{a}[0, \frac{1}{4}]\right) \times \mathbb{R}^{2n}$,

(5) $\Theta^{(0)}(k_0) = f^*_a$ on $S^1 \times \mathcal{J}_{n,\varepsilon,A}$,

(6) $\Theta^{(0)}(k_0) = \eta_2^{-1} \Xi^{(<n-1)}(f^*) \eta_2$ on $\bigcup_{j=-[\frac{a+1}{2}]}^{[\frac{a+1}{2}]} \left(\frac{2j}{a} + \frac{1}{a}[0, \frac{1}{4}]\right) \times \mathbb{R}^{2n} \subset \mathcal{W}_1^{2n+1}$,

(7) $[k_0] = [h_0^a] = [h^a] = [g_2^a]$.

Now we define

$$g_2' := \left(\tau_{0,\frac{1}{a}} k_n \tau_{0,\frac{1}{a}}^{-1}\right) \left(\tau_{0,\frac{2}{a}} h_n \tau_{0,\frac{2}{a}}^{-1}\right) \cdots \left(\tau_{0,\frac{a-1}{a}} k_n \tau_{0,\frac{a-1}{a}}^{-1}\right) k_n. \tag{21.4}$$

Then we summarize the above discussion into the following.

**Lemma 21.7.** $g_a'$ *satisfies* $[g_2'] = [g_a^{a^{n+2}}]$.

This finishes the proof of Lemma 21.1 for the general cases $a > 2$.

21.3. **Wrap-up of the proof of** $\omega(f) = e$**.** After the homological identity from Lemma 21.1 is established, we wrap up the proof of Proposition 19.5 utilizing a simple number theoretic argument as in [Ryb2]. It follows from Lemma 21.1 that

$$[g_a] = [g_a^{a^{n+2}}] = [g]^{a^{n+2}}$$

which implies $\omega(f) = \omega(f)^{a^{n+2}}$. As mentioned before this identity does not make sense as it is unless we replace $g = g_2$ in his case. The correct identity to show is

$$\omega(f) = \omega(f)^{a^{n+2}}.$$

Therefore we have derived that either $\omega(f) = e$, which will finish the proof, or otherwise $\mathrm{ord}(\omega(f)) = \ell_0 > 1$ and $\omega(f)$ satisfies

$$(\omega(f))^{a^{n+2}-1} = e.$$

From now on, suppose the latter holds for $\ell_0 > 1$. Since this holds for every integer $a \geq 2$, we also have

$$\omega(f)^{a^{n+2}-b^{n+2}} = e$$

and hence $\ell_0 \mid a^{n+2} - b^{n+2}$ for every pair of positive integer $(a, b)$ with $a,\, b \geq 1$. In particular it also holds for $(a, b) = (\ell_0, 1)$, i.e., $\ell_0$ divides

$$\ell_0^{n+2} - 1^{n+2} = \ell_0^{n+2} - 1.$$

This contradicts to the simple fact $\ell_0$ is not a divisor of $\ell_0^{n+2} - 1$, since we assume $\ell_0 > 1$. Therefore we conclude that $\omega(f) = e$ which finishes the proof of Proposition 19.5.

**Remark 21.8.** The way how this homological identity is used in the proof is rather peculiar which the author feels deserves more scrutiny on its meaning. It seems to the author that it is a replacement of the more common practice of infinite repetition construction which is also used by Tsuboi [T3] in his perfectness proof of $\mathrm{Cont}_c^r(M, \alpha)$ for the opposite case of $r < n + \frac{3}{2}$ of the threshold.

## 22. Wrap-up of the proofs of Theorem 1.4 for $r > n + 2$

We are now ready to wrap up the proof of perfectness of $\mathrm{Cont}_c^*(\mathbb{R}^{2n+1}, \alpha_0)$ combining the arguments used by Mather [Ma1, Ma2, Section 3], [E2] and [Ryb2, Section 9] for all $(r, \delta)$ with $r \geq 1$, $0 < \delta \leq 1$ for $(r, \delta) \neq (n + 1, \frac{1}{2})$. Let $f_0 \in \mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)$.

We need to show that $f_0$ belongs to the commutator subgroup thereof. By the fragmentation lemma, Lemma 1.3, we may assume that

$$\mathrm{supp}(f_0) \subset I_A = [-2, 2] \times [-2, 2]^n \times [-2A, 2A]^n. \tag{22.1}$$

Furthermore by a contact conformal rescaling, which does not change its conjugacy class, we may assume that $M_r^*(f_0)$ is as small as we want. Let $A > 0$ be a sufficiently large positive integer which is to be fixed later, and let $I_A$, $J_A$ and $K_A$ be the intervals in $\mathbb{R}^{2n+1}$ given in (17.1), (8.8) and (8.9) respectively. Let $r \geq 1$ be given and define

$$\mathcal{L}_r(\varepsilon, A) := \{u \in C_{I_A}^{r+1}(\mathbb{R}^{2n+1}) \mid \|D^{r+1}u\| \leq \varepsilon\} \tag{22.2}$$

where $\varepsilon > 0$ is a sufficiently small constant which will be fixed in the course of the proof. We observe that $\mathcal{L}_r(\varepsilon, A)$ is a convex and compact subset of a locally convex space $C_{I_A}^{r+1}(\mathbb{R}^{2n+1})$.

**Lemma 22.1.** *Suppose that $r > n+2$. Then there exist constants, a sufficient small $\epsilon_0 > 0$ and a sufficiently large $A > 0$, for which there exists a continuous map $\vartheta : \mathcal{L}_r(\varepsilon, A) \to \mathcal{L}_r(\varepsilon, A)$.*

*Proof.* The proof of this lemma duplicates the 10 steps laid out by Rybicki [Ryb2, Section 9]. (This is the contact replacement of Mather's strategy [Ma1, Section 3] that was applied to the case of diffeomorphisms $\mathrm{Diff}_c(M)_0$).

We may assume

$$\mathrm{pr}_{1,2}(U) \supset I_A$$

as mentioned before again after contact conformal rescaling of $f_0$. Then here are the aforementioned Rybicki's ten steps with some changes of various numerics appearing in the construction and *with the change of the form of the contact scaling from $\chi_A \eta_A$ by $\chi_{A^2}$* in Step (5):

(1) For any $u \in \mathcal{L}_r(\varepsilon, A)$, consider $f \in \mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)$ given as $f = \mathscr{G}_A^{-1}(u)$.

(2) Set $g = ff_0$. Then we have the inequality
$$M_r^*(g) \leq C\|u\|_{r+1} \tag{22.3}$$
from (15.2).

(3) Use a fragmentation of the second kind for $g = ff_0$ (Proposition 9.2) and obtain a fragmentation $g = g_1 \circ \cdots \circ g_{a_n}$ with $a_n = (8A^2 + 8)$, and each $g_K$ is supported in
$$\big([-2,2]^{n+1} \times [k_1 - 1, k_1 + 1] \times \cdots \times [k_1 - 1, k_n + 1]\big) \cap I_A$$
with integers $k_i$ such that $|k_i| \leq 2A - 1$, $i = 1, \dots, n$.

(4) Use the operation of shifting supports of contactomorphisms described Section 8. For any $K = 1, \dots, a_n$, we define
$$\begin{aligned} \widetilde{g}_K &:= \sigma_{A,\mathbf{t}} \circ g_K \circ \sigma_{A,\mathbf{t}}^{-1} \\ &= \left(\sigma_{n,t_n}\left(\sigma_{n-1,t_{n-1}}\left(\cdots\left(\sigma_{1,t_1} g_K \sigma_{1,t_1}^{-1}\right)\cdots\right)\sigma_{n-1,t_{n-1}}^{-1}\right)\sigma_{n,t_n}^{-1}\right) \end{aligned}$$
for a suitable $\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{R}^n$ depending on $K$ in such a way that
$$\operatorname{supp}(\widetilde{g}_K) \subset [-A^5, A^5] \times [-2,2]^{2n}$$
for all $K$. Here we take $|t_i| \leq 2A - 1$, $i = 1, \cdots, n$ and $A > 5n$.

(5) For each $K = 1, \dots, a_n$, define the conjugation
$$h_K = (\chi_{A^2})\widetilde{g}_K(\chi_{A^2})^{-1}, \quad \operatorname{supp}(h_K) \subset J_A.$$
Then we have
$$[h_K] = [\widetilde{g}_K] \tag{22.4}$$
and the inequality
$$M_r^*(\widetilde{h}_K) \leq CA^{4-2r} M_r^*(g) \tag{22.5}$$
from Proposition 17.1.

(6) Apply the rolling-up operator $\Psi_A$ described in Proposition 12.5 to define $\overline{h}_K = \Psi_A(h_K)$. We have $\operatorname{supp}(\overline{h}_K) \subset K_A$. We have
$$M_r^*(\overline{h}_K) \leq CA^2\, M_r^*(h_K) \tag{22.6}$$
from Corollary 18.2 and Proposition 18.3.

(7) Apply a fragmentation of the second kind in $K_A$ in the directions $i = 1, \dots, n$. We write $\overline{a}_n = a_n^3$ and get the fragmentation of $\overline{h}_K$,
$$\overline{h}_K = \overline{h}_{K;1} \circ \cdots \circ \overline{h}_{K;\overline{a}_n}.$$
*In each step of taking the conjugation by $\sigma_{i,t_i}$ for $i = 1, \cdots, n$,* the power of $A$ moves up by 2. Therefore we have
$$M_r^*(\overline{h}_K) \leq CA^{2n} M_r^*(h). \tag{22.7}$$

(8) Apply the operation of shifting supports of contactomorphisms in the $q_i$-directions by the translations $\tau_i$, $i = 1, \dots, n$. For each pair $K$, $i$, we define $\widetilde{h}_{K;i}$ instead of $\overline{h}_{K;i}$ with support
$$\operatorname{supp}(\widetilde{h}_K) \subset I_A.$$
*All the norms of the latter map are the same as those of $\overline{h}_{K;i}$.*

(9) Take the product and write
$$h = \prod_{K=1}^{a_n} \prod_{i=1}^{\overline{a}_n} \widetilde{h}_{K;i}.$$
Then we have
$$M_r^*(h) \leq CA^{4-4r+2n} M_r^*(g) \tag{22.8}$$

(10) Take $u_h := \mathscr{G}_A(h)$. Then
$$\|u_h\|_{r+1} \leq CM_r^*(h) \tag{22.9}$$
from Proposition 15.1 (1).

Then we define the map

$$\vartheta(u) := u_h. \tag{22.10}$$

Combining the inequalities given in the above 10 steps, we have obtained

$$\|\vartheta(u)\|_{r+1} \leq CA^{2(n+2-r)}\|u\|_{r+1}. \tag{22.11}$$

Therefore if $r > n+2$, we can choose $A > 0$ sufficiently large (recalling that we also choose $\delta > 0$ and the Darboux-Weinstein chart $\Phi_U : U \to V$ sufficiently small), we can make the inequality

$$CA^{2(n+2-r)} < 1$$

holds. This finishes the construction of the map $\vartheta : \mathcal{L}_r(\varepsilon, A) \to \mathcal{L}_r(\varepsilon, A)$. □

Once this lemma is established, Schauder-Tychonoff theorem implies that any such continuous map $\vartheta : \mathcal{L}_r(\varepsilon, A) \to \mathcal{L}_r(\varepsilon, A)$ carries a fixed point. The rest of the proof is the same as Rybicki's laid out in [Ryb2, Section 9], especially the first half thereof, except that we again need to incorporate the fact that the map $\vartheta$ itself depends on the integer $a \geq 2$. Since we will fix $a$ in the following paragraph, we just write $g_a = g$.

Let $u \in \mathcal{L}_r(\varepsilon, A)$ be a fixed point of $\vartheta$, i.e., $\vartheta(u) = u$. Denote by $f = \mathscr{G}_A^{-1}(u) \in \mathcal{U}_1 \subset \mathrm{Cont}_c(\mathcal{W}_k^{2n+1}, \alpha_0)$ and $u = u_f$. By definition of the map $\vartheta = \vartheta_{f_0}$ associated to $f_0 \in \mathrm{Cont}_c^r(\mathbb{R}^{2n+1}, \alpha_0)$ defined by the above 10 steps, we obtain the following sequence of identities:

$$\begin{aligned} [ff_0] &= [g] = [g_1 \cdots g_{a_n}] = [g_1] \cdots [g_{a_n}] = [\widetilde{g}_1] \cdots [\widetilde{g}_{a_n}] \\ &= [h_1] \cdots [h_{a_n}] = [\overline{h}_1 \cdots \overline{h}_{a_n}] \\ &= [\overline{h}_{11}] \cdots [\overline{h}_{a_n \overline{a}_n}] = [\widetilde{h}_{11}] \cdots [\widetilde{h}_{a_n \overline{a}_n}] \\ &= [\widetilde{h}_{11} \cdots \widetilde{h}_{a_n \overline{a}_n}] = [h] = [f]. \end{aligned}$$

Here the 5th equality follows from (22.4) and the 8th equality from Proposition 12.7 (3). The last equality is a consequence of the definition $f = \mathscr{G}_A^{-1}(u)$ for the fixed point $u$ of the map $\vartheta$ by the standing hypothesis $\vartheta(u) = u$. For by definition of $\vartheta$, we also have $\vartheta(u) = \mathscr{G}_A(h)$. Since $\mathscr{G}_A$ is a bijective map, this implies $h = f$. All other equalities are either trivial or consequences of Lemma 12.4 and Proposition 12.5

Therefore we have proved $[f_0] = e$ in $H_1(\mathrm{Cont}_c(\mathbb{R}^{2n+1}, \alpha_0)_0)$. This completes the proof of Theorem 1.4 for $r > n+2$.

## 23. Proof of Theorem 1.6 and Theorem 1.7

As for the diffeomorphism case of Mather [Ma1, Ma2], Theorem 1.4 is a consequence of Theorem 1.6 which involves the function $\alpha$ of modulus of continuity, e.g., the function $\alpha(x) = x^\beta$ for the Hölder regularity $(k, \delta)$ with $0 < \delta \leq 1$. We refer readers to [CKK] for a detailed study of the set of modulus of continuity which helps the authors thereof systematically analyse the threshold case $r = n+1$ in [Ma1, Ma2].

23.1. **Proof of Theorem 1.6 and Theorem 1.4 for** $r = n+2$**.** Finally, we explain how we can extend the proof of Theorem 1.4 to the Hölder regularity class $\mathrm{Cont}_c^{(r,\delta)}(\mathcal{W}_k^{2n+1}, \alpha_0)$ for all $(r, \delta)$ with $r = n+1$ and $\frac{1}{2} < \delta \leq 1$. In fact, the proofs of the two theorems do not make difference, observing that all the estimates performed in Part II can be equally carried out for the Hölder class $(r, \delta)$ without change: The only change needed to make the following estimate

$$\mu_{r,\delta}\left(\sigma_{1,t_1} g_K \sigma_{1,t_1}^{-1}\right) \leq CA^{2(1-r-2\delta)} \mu^*_{r,\delta}(g_K). \tag{23.1}$$

(See [Ma1, p.518] for the similar change made to handle the case of $\mathrm{Diff}_c^{r,\delta}(M)_0$.) Now this change will make Proposition 12.7 into one such that the map

$$\Psi_A : \mathrm{Cont}^{r,\delta}_{J_A}(\mathbb{R}^{2n+1}, \alpha_0)_0 \cap \mathcal{U}_4 \to \mathrm{Cont}^{r,\delta}_{K_A}(\mathbb{R}^{2n+1}, \alpha_0)_0$$

that satisfies the estimate

$$M^*_r(\Psi_A(g)) \leq CK_r A^{2(1-r-2\delta+n)} M^*_r(g) + P_{\chi,r}(M^*_{r-1}(g))$$

which in turn gives rise to the same inequality of the map $\varepsilon = \varepsilon_{f_0}$. This proves Theorem 1.6 for the case of $r = n+1$ and $\frac{1}{2} < \delta \leq 1$. Finally the case for $r = n+2$ follows by the same argument of Mather [Ma1] by noticing the equality

$$\mathrm{Cont}_c^{n+2}(\mathbb{R}^{2n+1}, \alpha_0) = \bigcup_{0 \leq \delta < 1} \mathrm{Cont}_c^{(n+2,\delta)}(\mathbb{R}^{2n+1}, \alpha_0).$$

23.2. **Proof of Theorem 1.7.** The case $r = n+1$ and $1 \leq r + \delta < n + \frac{3}{2}$ was previously proved by Tsuboi in [T3] and in particular for $1 \leq r \leq n+1$ for integer $r$. His result follows from our proof by dualizing the construction similarly as Mather did for the diffeomorphism case.

Here are the key points of changes to be made in the estimates for this dual construction from the case of lower threshold are the following:

(1) We just replace $A$ by $A^{-1}$ in the construction, which in particular reverse the direction of the map $\Theta_A^{(k)}$ so that we now have the map

$$\Theta_A^{(k)} : \mathrm{Cont}_{J_A^k}(\mathcal{W}_k^{2n+1}, \alpha_0) \cap \mathcal{U}_1 \to \mathrm{Cont}_{K_A^k}(\mathcal{W}_k^{2n+1}, \alpha_0) \cap \mathcal{U}_1.$$

See Diagram 5.8.

(2) As Mather put it in [Ma2, Section 4, p.37], *".... estimate (1) is essentially a special case of (1)* [Ma1, Section 6]. *Here* $\mathrm{supp}\, u \subset \mathrm{int} D_{i-1,A}$*, whereas there, we have only the weaker condition* $\mathrm{supp}\, u \subset D_{i,A}$*. This explains why we may omit A from the right hand side of the inequality here: the width of* $D_{i-1,A}$ *in the ith coordinate is* 4*, while the width of* $D_{i,A}$ *is* $4A$*."*, The outcome is that we do not need the $A^2$ in (22.7) and so the corresponding equality becomes

$$M_r^*(\overline{h}_K) \leq C M_r^*(h_K).$$

(3) Recall we have used the contact scaling map $\chi_{A^2} = \chi_A^2$ the norm of which is bounded by $A^4$ while the norm of its inverse is bounded by $A^{-2}$. This asymmetry is responsible for the appearance of $2\delta$ for the case of lower threshold and $\delta$ for the case of upper threshold below.

(4) We remind the readers that the domain of the map $\vartheta$ is

$$I_A = [-2,2] \times [-2,2]^n \times [-2A, 2A]^n$$

which plays the role of the reference space that normalizes the conformal factor of the front projection $[-2,2] \times [-2,2]^n$ throughout the constructions.

The final outcome is that the inequality (22.11) is then transformed by

$$\|\vartheta(u)\|_{r+1,\delta} \leq C A^{2(-2-r-n)}$$

on the $C^r$ space, and

$$\|\vartheta(u)\|_{r+1,\delta} \leq C A^{2(-2-r-n+2\delta)}$$

on $C^{r,\delta}$ space. (See [p. 516]mather for the relevant Hölder estimates.) We need either $r < n+1$ or $r = n+1$ which precisely gives rise to the bound for $\delta$ given by

$$-1 + 2\delta < 0$$

which shows that the Hölder regularity $(n+1, \delta)$ be in the required range stated in Theorem 1.6.

Combining the above all, we have finished the proof of Theorem 1.7.

## Appendix A. Proof of Corollary (5.8): equivariant contactomorphisms

In this section, we give the proof of Corollary 5.8 for completeness' sake. We state the corollary here.

**Corollary A.1.** *We have the expression*

$$\Phi_U^{-1}(t,x,X) = (t + h_t(t,X), x + h_x(t,X), x + h_X(t,X)) \in \mathbb{R}^{2(2n+1)+1} \cong J^1\mathbb{R}^{2n+1}$$

*such that* $h_t(0,0) = 0$*,* $h_x(0,0) = 0$ *and* $h_X(0,0) = 0$.

*Proof.* Recall that $\Phi_U$ is $(\mathcal{G}_1, \mathcal{G}_2)$-equivariant, i.e, $\Phi_U^{-1} =: \varphi$ satisfies

$$\varphi(t, x+g, X) = (\varphi_t(t,x,X), g + \varphi_x(t,x,X), g + \varphi_X(t,x,X)), \quad \varphi(0,x,0) = (0,x,x)$$

where we write $\varphi = (\varphi_r, \varphi_x, \varphi_X)$ componentwise. Then we obtain the following system of equations

$$\begin{cases} \phi_t(t, x+g, X) = \phi_t(t,x,X) \\ \phi_x(t, x+g, X) = \phi_x(t,x,X) + g \\ \phi_x(t, x+g, X) = \phi_X(t,x,X) + g \end{cases}$$

for all $t$, $x\,X$ and $g \in \mathbb{R}^{2n+1}$. In particular, by plugging $x = 0$ into the equations, we obtain

$$\phi_t(t,g,X) = \phi_t(t,0,X),\ \phi_x(t,g,X) = \phi_x(t,0,X) + g\ \phi_x(t,g,X) = \phi_X(t,,X) + g.$$

Since $g$ is arbitrary, we can put $g = x$ and then set

$$h_t(t,X) = \varphi_t(t,0,X),\ h_x(t,X) = \varphi_x(t,0,X),\ h_X(t,X) = \varphi(t,0,X).$$

This, $varphi(0,x,0) = (0,x,x)$ and

$$T\varphi|_{(0,x,0)} = \mathrm{id} : \mathbb{R} \oplus \mathbb{R}^{2n+1}_x \oplus \mathbb{R}^{2n+1}_X \to \mathbb{R} \oplus \mathbb{R}^{2n+1}_x \oplus \mathbb{R}^{2n+1}_X$$

imply that we can write $\varphi$ in the form of

$$\varphi(t,x,X) = (t + h_t(t,X), x + h_x(t,X), x + h_X(t,X))$$

with $h_t(0,X) = 0$, $h_x(0,X) = 0$, $h_X(0,X) = 0$. Here we identity

$$T_{(0,x,0)}(J^1\mathcal{W}^m_k) \cong T_{(0,x,x)}(M_{\mathcal{W}^m_k})$$

by parallel translations on $\mathbb{R}^{2(2n+1)+1}$. This finishes the proof. □

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