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Analytic Approach for Computation of Topological Number of Integrable Vortex on Torus

Kaoru Miyamoto and Atsushi Nakamura

*Department of Physics, School of Science, Kitasato University
Sagamihara 252-0373, Japan*

E-mail: miyamoto.kaoru@st.kitasato-u.ac.jp,
nakamura@sci.kitasato-u.ac.jp, nakamura00@gmail.com

ABSTRACT: An analytic method to calculate the vortex number on a torus is constructed, focusing on analytic vortex solutions to the Chern-Simons-Higgs theory, whose governing equation is the so-called Jackiw-Pi equation. The equation is one of the integrable vortex equations and is reduced to Liouville's equation. The requirement of continuity of the Higgs field strongly restricts the characteristics and the fundamental domain of the vortices. Also considered are the decompactification limits of the vortices on a torus, in which “flux loss” phenomena occasionally occur.

Contents

1	Introduction	1
2	The Jackiw-Pi equation	3
2.1	Integrable vortices	3
2.2	Non-relativistic Chern-Simons-matter theory and the Jackiw-Pi equation	6
2.3	Vortex solutions on torus	8
3	Analytic calculation of vortex number on torus	9
3.1	Strategy for analysis	9
3.2	Vortices from elliptic functions: Examples	13
4	Large period limits of vortices	21
4.1	Cylinder limits of the vortex from Jacobi elliptic function	21
4.2	Planar limit	24
5	Conclusion and discussions	25
A	Derivation of the formula (3.4)	26

1 Introduction

Vortices are ubiquitous structures in various scales of nature. They typically demonstrate a topologically non-trivial configuration of fields both in quantum and classical dynamics [1–3]. We define the vortices here as the localised static solutions to a gauge theory coupled with a matter, or a Higgs field. A characteristic example of the vortices is the magnetic flux in type-II superconductors, which appeared as the topological solitons in the Ginzburg-Landau(GL) model, namely, the static energy functional of the 2+1 dimensional Abelian-Higgs model. Those vortices in GL model emerge as the defects of field configuration with a spontaneous phase transition of the system, in which the ordinary Maxwell electromagnetism governs the dynamics of the vortices described by an Abelian gauge field. In 2+1 dimensions or 3 spatial dimensions, another “dynamics” for the gauge fields is possible: one can incorporate the Chern-Simons three-form, which brings topological degrees of freedom. The physical significance of the Chern-Simons term stems from the fact the theory including it serves as an effective theory of the quantum Hall effect [4, 5]. The vortex solutions may notably exist in the Maxwell-Chern-Simons-matter theories and the pure Chern-Simons-matter theories [6]. In this paper, we consider the Jackiw-Pi vortex equation, which governs the non-relativistic Chern-Simons-matter theory in the static case, and its vortex solutions.

In ordinary Abelian-Higgs vortices such as in type-II superconductors, the flux of the gauge field, i.e., the magnetic flux, is concentrated at the zeroes of the Higgs field. However, another kind of vortices exists for which the magnetic flux is excluded from the Higgs zero, sometimes referred to as exotic vortices [7]. The Jackiw-Pi vortices belong to the latter case.

As with the ordinary field theories, the field equations governing the vortices are also second-order differential equations. However, situations in which they become first-order equations exist if the coupling constants obey critical relations, i.e., the BPS (Bogomolnyi-Prasad-Sommerfield) limits. In such cases, the system of first-order BPS equations is equivalent to the celebrated Liouville equation, a second-order solvable differential equation. In [8], Manton shows that five distinct cases of such integrable vortex equations exist according to the curvature of the background surfaces on which the vortex lives. Among the five equations, the Taubes equation [9], the Ambjørn-Olesen equation [10], and the Bradlow equation [11] are defined on a hyperbolic surface of constant curvature \mathbb{H}^2 , while the Popov equation [12, 13] is defined on a sphere S^2 . The integrable vortex equation on a plane \mathbb{R}^2 is the Jackiw-Pi equation [14, 15] considered in the present paper. There exist some geometrical interpretations behind those integrable vortex equations. In [16], the Higgs fields of the vortices are explained as conformal factors of a metric of the constant curvature surface with isolated singularities. In [17], it is interpreted that these integrable equations can be reduced from four-dimensional Yang-Mills theories, and the relation between the vortices and a flat non-Abelian connection in three-dimensions is shown in [18]. In addition, the higher-order generalizations in terms of the “vortex polynomials” to those integrable vortex equations are considered in [19], which includes the equations from Chern-Simons theories.

Although the integrable vortices except for the Popov vortices are defined on non-compact surfaces, they may also live on compact surfaces of constant curvature, i.e., the surfaces of genus $g \geq 1$. For those vortices, the periodicity of the background surfaces strongly restricts the solution spaces to the vortex equations. For the cases of $g = 1$, the Jackiw-Pi vortices on a torus are considered earlier in [20] and reconsidered in [21], in which the elliptic functions describe the vortices. For the cases of $g = 2$, a special solution to the Taubes equation is constructed with Schwarzian triangular functions [8]. However, a comprehensive understanding of the vortices on compact surfaces has not been achieved. For example, a systematic approach to calculating the characteristics of the vortices such as the vortex number is not established. In this paper, we will show an analytical method for determining the vortex number on a torus, namely the first Chern number, and aim to elucidate the vortices on compact surfaces in detail.

Another topic of solitonic objects on compact spaces such as torus is the possibility of having twisted periodic conditions, which leads to the so-called fractionally charged solitons, e.g., [22]. Those kinds of objects are one of the key research interests for the physics of confinement [23–25], however, we do not consider the twisted periodic conditions here and remain them as a subject of future research.

The study of topological objects such as vortices, skyrmions, etc., is actively performed and still underway. Although the integrability of such systems is essential, soliton-like phenomena also appear in the systems without integrability, e.g., [26]. These are important

interdisciplinary studies of physics, mathematics, and other areas of science. In particular, these solitonic objects on the spaces with periodicity, which may be interpreted as compact spaces, would provide an important contribution to condensed matter physics as well as mathematical physics. In this context, the possibility of interpreting water waves as topological solitons has been proposed in [27]. This direction would open a new window into the research for topological solitons. It is important to clarify the meaning of integrability in nature.

This paper is organized as follows. In the next section, we define the model deriving the Jackiw-Pi equation and give an outline of the integrable vortex equations. In section 3, we construct a method for the analytic calculation of the vortex number on compact surfaces and analyse typical cases with specific examples. In section 4, the decompactification limits of the vortex solutions are considered on a torus. In the final section, we give conclusions and discussions.

2 The Jackiw-Pi equation

In this section, we inaugurate the Jackiw-Pi vortex equation from some field theoretical point of view. We firstly introduce the integrable vortex equations on constant curvature surfaces discussed by Manton [8] and see the Jackiw-Pi equation is one of the five integrable equations derived from the Abelian Higgs model. As an alternative derivation of the Jackiw-Pi equation, we also focus on the non-relativistic Chern-Simons-matter theory. We then give the general solutions to the Jackiw-Pi equation in terms of a meromorphic function on a flat plane \mathbb{R}^2 and a torus T^2 .

2.1 Integrable vortices

We consider a complex scalar field with a quartic potential coupled with an abelian gauge field on a two-dimensional surface with critical coupling constants [3]. Let M_0 be a surface with conformal metric $ds_0^2 = \Omega_0(dx^2 + dy^2)$, where the conformal factor Ω_0 is a function of x and y . The static energy functional E of such the Abelian Higgs model on M_0 is

$$E = \int_{M_0} \left\{ -\frac{4}{\Omega_0^2} F_{z\bar{z}}^2 - \frac{2C}{\Omega_0} |D_{\bar{z}}\phi|^2 + \left(-C_0 + C|\phi|^2 \right)^2 \right\} \Omega_0 dz d\bar{z}, \quad (2.1)$$

where $z = x + iy$, $F_{z\bar{z}} = \partial_z a_{\bar{z}} - \partial_{\bar{z}} a_z$ is the field strength, $D_z = \partial_z + ia_z$ and $D_{\bar{z}} = \partial_{\bar{z}} - ia_{\bar{z}}$ are the covariant derivatives with respect to gauge potentials a_z and $a_{\bar{z}}$, and ϕ is a complex Higgs field. The real constants C_0 and C will be specified later. Applying the Bogomolny completion to this energy functional, we obtain the following formula,

$$E = \int_{M_0} \left\{ \left(-\frac{2i}{\Omega_0} F_{z\bar{z}} + C_0 - C|\phi|^2 \right)^2 - \frac{2C}{\Omega_0} |D_{\bar{z}}\phi|^2 \right\} \Omega_0 dz d\bar{z} - 4\pi C_0 N, \quad (2.2)$$

where

$$N := \frac{1}{2\pi} \int_{M_0} F_{xy} dx dy = \frac{1}{2\pi} \int_{M_0} F_{z\bar{z}} dz d\bar{z} \quad (2.3)$$

is the first Chern number taking an integer value. The integer (2.3) is interpreted as the number of the vortices considering with multiplicity and is a topological invariant because it is independent of the metric of M_0 . From this formula, if the following Bogomolny equations,

$$D_{\bar{z}}\phi = 0, \quad -\frac{2i}{\Omega_0}F_{z\bar{z}} = -C_0 + C|\phi|^2, \quad (2.4)$$

are satisfied, we find the energy is bounded below $E \geq -4\pi C_0 N$ if $C \leq 0$, thus the field configurations satisfying (2.4) are stable. However, we will discuss the case of $C > 0$ below and in the next section from another point of view. Eliminating $F_{z\bar{z}}$ by using the first equation of (2.4), we obtain the following equation,

$$\partial\bar{\partial}\log|\phi|^2 = \frac{\Omega_0}{2}(C_0 - C|\phi|^2), \quad (2.5)$$

where we have defined $\partial := \partial_z$ and $\bar{\partial} := \partial_{\bar{z}}$. We refer to the equation (2.5) as the generalised vortex equation.

We note that the values of the constants C_0 and C can be normalised as either $-1, 0$ or 1 by rescaling the metric and $|\phi|$. In ref.[8], Manton argued that there are nine possible ways to choose C_0 and C , but the four cases of them are invalid: The right-hand-side of (2.5) must be positive since the left-hand side of (2.5) is the magnetic field $F_{z\bar{z}}$ whose integral gives the positive topological number N . Therefore the remaining five cases $(C_0, C) = (-1, -1), (-1, 0), (-1, 1), (0, 1)$, and $(1, 1)$ are acceptable.

There is a geometrical interpretation of the constants C_0 and C in the generalized vortex equations [16]. Let $ds^2 = |\phi|^2 ds_0^2$, where ds_0^2 is the metric of M_0 . The new metric ds^2 obtained by conformally rescaling the original metric ds_0^2 with the squared Higgs field is referred to as the Baptista metric [16]. Let the Gaussian curvatures of the original metric K_0 and that of the Baptista metric K , respectively. We find

$$K_0 = \frac{-1}{2\Omega_0}\nabla^2\log\Omega_0, \quad K = \frac{-1}{2\Omega_0|\phi|^2}\nabla^2\log\Omega_0|\phi|^2, \quad (2.6)$$

where $\nabla^2 = 4\partial\bar{\partial}$ is the Euclidean Laplace operator. From (2.6), the relation between the two curvatures is

$$\partial\bar{\partial}\log|\phi|^2 = \frac{\Omega_0}{2}(K_0 - K|\phi|^2). \quad (2.7)$$

Because ϕ is a solution to the generalized vortex equation (2.5), we obtain the following relation

$$C_0 - K_0 = (C - K)|\phi|^2, \quad (2.8)$$

by using (2.5).

The relation (2.8) is crucial for the integrability of the generalised vortex equation. Let us consider the curvature K_0 is equal to the constant C_0 , then K equals C from (2.8). If

this is the case, as the curvature $K = C$, the differential equation providing K (the latter of (2.6)) can be written as

$$\nabla^2 \log \Omega = -2C\Omega, \quad (2.9)$$

where $\Omega = \Omega_0 |\phi|^2$. This equation, or the equation with new variable $u := (\log \Omega)/2$

$$\nabla^2 u = -Ce^{2u}, \quad (2.10)$$

is called Liouville's equation, which is one of typical integrable equations. It is known that the general solution to Liouville's equation is

$$e^{2u} = \frac{4}{(1 + Cf(z)g(\bar{z}))^2} \frac{df}{dz} \frac{dg}{d\bar{z}}, \quad (2.11)$$

where f and g are arbitrary holomorphic functions of z and \bar{z} defined on the original surface M_0 , respectively. Hereafter, we restrict ourselves that the case $g = \bar{f}$, namely,

$$\Omega = e^{2u} = \frac{4}{(1 + C|f(z)|^2)^2} \left| \frac{df}{dz} \right|^2, \quad (2.12)$$

for the geometrical interpretation of vortices. Now we consider the case that M_0 is a constant curvature surface, i.e., $K_0 = C_0$ is a constant. Then we find the metric of M_0 takes the form

$$ds_0^2 = \Omega_0 dz d\bar{z} = \frac{4}{(1 + C_0|z|^2)^2} dz d\bar{z}. \quad (2.13)$$

Recalling that $|\phi|^2 = \Omega/\Omega_0$, we obtain a representation of $|\phi|^2$ in terms of f ,

$$|\phi|^2 = \frac{(1 + C_0|z|^2)^2}{(1 + C|f|^2)^2} \left| \frac{df}{dz} \right|^2. \quad (2.14)$$

From (2.14), we find that the squared Higgs field can be considered as the quotient of a metric of constant curvature surface M with complex coordinates $w = f(z)$ and $\bar{w} = \overline{f(z)}$,

$$ds^2 = \Omega dw d\bar{w} = \frac{4}{(1 + C|w|^2)^2} dw d\bar{w}, \quad (2.15)$$

and that of M_0 . Hence, the zeroes of the Higgs field, where the vortex centres are sitting, are the indefinite points of the surface M . This means that M is a constant curvature surface with orbifold-like singularities at the vortex centres.

We introduce the Jackiw-Pi equation as a special case of the generalized vortex equations (2.5). The Jackiw-Pi equation is the integrable vortex equation defined on a flat Euclidean plane, i.e., the background curvature $K_0 = C_0 = 0$ and the conformal factor

$\Omega_0 = 4$. From Manton's classification, the case of $(C_0, C) = (0, 1)$ is the only possible case. Then the target surface is S^2 , and the Jackiw-Pi equation takes the form

$$\partial\bar{\partial}\log|\phi|^2 = -\frac{1}{2}|\phi|^2. \quad (2.16)$$

This is also Liouville's equation. The solution to Liouville's, or the Jackiw-Pi equation is therefore,

$$|\phi(z, \bar{z})|^2 = \frac{4}{(1 + |f|^2)^2} \left| \frac{df}{dz} \right|^2, \quad (2.17)$$

or

$$\phi(z, \bar{z}) = \frac{2}{1 + |f|^2} \frac{df}{dz}, \quad (2.18)$$

where f is a meromorphic function on $\mathbb{C} \simeq \mathbb{R}^2$. The meromorphicity of f is required by the target surface being S^2 .

2.2 Non-relativistic Chern-Simons-matter theory and the Jackiw-Pi equation

As we already mentioned, the Jackiw-Pi equation can be derived from another point of view. In this subsection, we review that the equation is the governing equation to the 2+1 dimensional non-relativistic Chern-Simons-matter theory for its static and (anti-)self-dual solutions. The vortex solutions to the Jackiw-Pi equation are unstable from the perspective of the Abelian-Higgs model, since the positiveness of C , as seen above. We can avoid this instability of the vortices in the Chern-Simons-matter theory.

We introduce the following Lagrangian density,

$$\mathcal{L}_{JP} = i\Psi^* D_0 \Psi - \frac{1}{2m} |\vec{D}\Psi|^2 + \frac{\lambda}{2} |\Psi|^4 + \frac{\kappa}{2} \epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma}, \quad (2.19)$$

where Ψ is a complex scalar field, $D_\mu = \partial_\mu - iqA_\mu$ is the covariant derivative with an Abelian gauge field A_μ , $\vec{D} = (D_1, D_2)$, and $F_{\beta\gamma}$ is the field strength. The metric and the complete anti-symmetric tensor are defined as $g_{\mu\nu} = \text{diag.}(-1, 1, 1)$ and $\epsilon^{012} = 1$, respectively. The Euler-Lagrange equations to this system are

$$iD_0\Psi = -\frac{1}{2m}\vec{D}^2\Psi - \lambda|\Psi|^2\Psi \quad (2.20)$$

$$F_{\alpha\beta} = -\frac{1}{\kappa}\epsilon_{\alpha\beta\gamma}J^\gamma, \quad (2.21)$$

where $J^\gamma = (\rho, \vec{J})$ with

$$\rho = |\Psi|^2, \quad (2.22)$$

$$\vec{J} = -i(\Psi^*\vec{D}\Psi - (\vec{D}\Psi)^*\Psi)/2. \quad (2.23)$$

The field equations (2.21) are decomposed as

$$B := F_{12} = -\frac{q}{\kappa}|\Psi|^2, \quad (2.24)$$

$$E_i := F_{0i} = \frac{1}{\kappa}\epsilon_{ij}J_j, \quad (2.25)$$

where $\epsilon_{ij} := \epsilon_{0ij}$ and $\epsilon_{120} = -\epsilon^{120} = -1$ are understood.

Making use of an identity

$$\vec{D}^2 = 4D_{\pm}D_{\mp} \pm B, \quad (2.26)$$

where $D_{\pm} = (D_1 \mp iD_2)/2$, we can rearrange (2.20) into

$$iD_0\Psi = -\frac{1}{2m}4D_{\pm}D_{\mp}\Psi - \left(\lambda \mp \frac{1}{2m\kappa}\right)|\Psi|^2\Psi. \quad (2.27)$$

We now consider the static solutions to (2.27) with $\partial_0\Psi = \partial_0 A_i = 0$, then (2.27) is reduced to be

$$-\frac{1}{2m}4D_{\pm}D_{\mp}\Psi - \left(\lambda \mp \frac{q}{2m\kappa}\right)|\Psi|^2\Psi + A_0\Psi = 0. \quad (2.28)$$

Taking a gauge

$$A_0 = \pm \frac{q}{2m\kappa}|\Psi|^2, \quad (2.29)$$

(2.28) becomes

$$-\frac{1}{2m}D_{\pm}D_{\mp}\Psi - \left(\lambda \mp \frac{q}{m\kappa}\right)|\Psi|^2\Psi = 0. \quad (2.30)$$

This can be solved by the following “self-dual” or ”anti-self-dual” ansatz

$$D_{\mp}\Psi = 0, \quad (2.31)$$

$$g \mp \frac{q}{m\kappa} = 0, \quad (2.32)$$

the latter is referred to as the (anti-)self-dual coupling. The defining equation for the scalar fields (2.31) is first order in its derivatives, so we refer to it as the BPS equation. From the BPS equation (2.31) with (2.24), we obtain the Jackiw-Pi equation again,

$$\partial\bar{\partial}\log|\Psi|^2 = -\frac{1}{2}|\Psi|^2, \quad (2.33)$$

where we have chosen $q = 1$ and $\kappa = -1$. Thus we find that the magnetic flux density is proportional to the scalar field density

$$B = F_{12} = |\Psi|^2, \quad (2.34)$$

from (2.24) in contrast to the ordinary GL theory.

The static energy with these ansatz turns out to be

$$E_{JP} = \int \left(\frac{1}{2m}|\vec{D}\Psi|^2 - \frac{1}{2}\left(g \mp \frac{1}{m\kappa}\right)|\Psi|^4 \right) dx^2 = \int |D_{\mp}\Psi|^2 dx^2, \quad (2.35)$$

thus the energy takes minimum value for the (anti-)self-dual solutions. Hence the BPS vortex solutions to the non-relativistic Chern-Simons matter theory (2.19) can be regarded as stable configurations.

Hereafter we unify the expression of the scalar field Ψ with ϕ and refer to it as the Higgs field, and consider the Lagrangian (2.19) governs the dynamics of the system.

2.3 Vortex solutions on torus

As shown above, the Jackiw-Pi equation is equivalent to Liouville's equation so that arbitrary holomorphic functions give the local solutions. For the solutions to be regular vortices, we should pay attention to the global behaviour of the field configurations: the finiteness of the flux, or energy, should be imposed. This requirement certainly gives restrictions for the holomorphic functions.

Firstly, we consider the vortex solutions defined on a flat plane \mathbb{R}^2 . It has been shown that the Higgs field ϕ given in (2.17) yields the Jackiw-Pi vortices on \mathbb{R}^2 if, and only if, the meromorphic function f takes the form

$$f(z) = \frac{P(z)}{Q(z)} \quad (2.36)$$

where $P(z)$ and $Q(z)$ are polynomials such that $\deg P < \deg Q$ [21, 28]. In this case, the vortex number N of the solution is proved to be $\deg Q$.

Next, we focus on the vortices on a torus T^2 . In this case, it is sufficient to impose the doubly periodicity for the Higgs field to be regular vortices. It is obvious that imposing the doubly periodicity for the meromorphic functions f leads to that of the Higgs field. The well-known fact is that the general doubly periodic functions are given in terms of the Weierstrass elliptic function $\wp(z)$ and its derivative $\wp'(z)$ on a periodic lattice $\Lambda = 2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2$, where ω_1 and ω_2 are independent complex numbers with positive imaginary part, so-called the half-periods, e.g., [29]. The corresponding doubly periodic meromorphic functions are

$$f(z) = R_1(\wp(z)) + \wp'(z)R_2(\wp(z)), \quad (2.37)$$

where R_1 and R_2 are some rational functions. Although the general form of the meromorphic function $f(z)$, or the Higgs field, is given, the characteristic quantities of vortices such as the vortex number are not obvious from (2.37). As shown in the next section, we consider the concrete examples for $f(z)$ to analyse the structure of individual vortices in detail.

We point out here that the Higgs field $\phi(z, \bar{z})$ itself would not be an observable quantity of the theory but it would be the flux density $\rho = |\phi|^2$. If this is the case the Higgs field itself is not necessary to be doubly periodic: it only needs quasi-doubly periodicity concerning the lattice $\Lambda = 2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2$, i.e.,

$$\phi(z + 2\omega_i) = e^{i\theta_i}\phi(z), \quad (i = 1, 2), \quad (2.38)$$

where $\theta_i \in \mathbb{R}$ are some phase angles. In this context, Akerblom *et.al.* [21] found the general doubly periodic solutions for the flux density ρ . From the gauge theoretical perspective, the quasi-periodic field configurations are admissible because the fields are periodic up to gauge transformation in those cases. However, we consider in this paper the Jackiw-Pi vortices on T^2 constructed from the strict doubly periodic Higgs field $\phi(z, \bar{z})$ itself. The reason is that the continuity of the Higgs field as a complex function is necessary for the analytic calculation of the vortex number, which takes integer values. On the other hand,

the quasi-periodic Higgs fields would give rise to the solitonic objects with non-trivial holonomy. For that kind of vortices, the vortex numbers would be able to take fractional values just as in the case of fractional instantons [23, 24, 30]. Although these are interesting solitonic objects, we concentrate here on the trivial holonomy vortices based on the Higgs fields with strict doubly periodicity.

3 Analytic calculation of vortex number on torus

In this section, we establish the analytic calculation method for the vortex number of the Jackiw-Pi vortices on a torus. The crux of the method is reconsidering the integral (2.3) that determines the vortex number: representing it in an expansion of the Higgs field around its zeroes, i.e., the vortex centres. Then we apply the method to several examples including the vortices with simple zero and also multiple zero. Here we give the details of the approach. An outline of the procedure has been reported in [31]¹ by the present authors.

3.1 Strategy for analysis

Let us recall that the vortex number N of the Jackiw-Pi vortices on 2-dimensional space M is given by the first Chern number,

$$N = \frac{1}{2\pi} \int_M F_{z\bar{z}} dz \wedge d\bar{z} = \frac{1}{4\pi} \int_M F. \quad (3.1)$$

We notice that if M is a closed surface such as a torus, this integral seems to vanish as the surface has no boundary due to the Stokes theorem:

$$\int_M F = \int_M da = \int_{\partial M} a = 0 \quad (3.2)$$

However, numerical integration suggests that N does not vanish even if M is a torus, and the value of N takes a certain integer value.

The principal bundle argument has solved this contradiction [21], in which the connection form of the bundle is not defined globally on a compact base space such as that of Dirac monopoles. In such cases, the transition functions of the bundle carry all of the information for the field configurations. Thus we can find the vortex number takes an integer value from the argument. Nevertheless, we face the difficulty: although the vector bundle argument shows us that N is an integer, it does not give the value concretely. However, as mentioned above, we can obtain the vortex number explicitly with numerical integration. We expect that the vortex number on a torus is supposed to be the sum of the multiplicity of Higgs zeroes, or the winding number, as an analogy to the case of vortices on a flat plane. The analytical method constructed in this section will provide the vortex number exactly and prove this expectation.

We begin with reconsidering the background surface, i.e., the torus, on which the flux is defined. Let ϕ be a solution to the Jackiw-Pi equation possessing m zeroes at

¹The contents of this report include inaccuracies.

$z = \eta_i$ ($i = 1, \dots, m$). We have to pay attention that the Liouville equation (2.16) or (2.33) is not defined at the zeroes of ϕ , the vortex centre, due to the logarithmic singularities of the equation. It suggests that we should define the natural domain of ϕ as the torus without singular points:

$$\tilde{T}^2 := T^2 \setminus \{\eta_i \in T^2 | \phi(\eta_i, \bar{\eta}_i) = 0\}, \quad (i = 1, \dots, m). \quad (3.3)$$

The boundaries $\partial\tilde{T}^2$ of \tilde{T}^2 are infinitesimal circles C_{η_i} ($i = 1, \dots, m$) around zeroes of ϕ . Then the integral (3.1) turns out to be

$$\frac{1}{2\pi} \int_{\tilde{T}^2} F_{z\bar{z}} dz \wedge d\bar{z} = \frac{1}{4\pi i} \sum_{i=1}^m \oint_{C_{\eta_i}} \partial \log |\phi|^2 dz - \bar{\partial} \log |\phi|^2 d\bar{z}, \quad (3.4)$$

through Green's theorem and it may not vanish, where the integration contours rotate counterclockwise. The derivation to (3.4) is given in Appendix A.

We now evaluate the contour integral (3.4) around the zeroes of ϕ . We notice from (2.17) that the zeroes of ϕ may arise from the zeroes of $f'(z)$ or the poles of $|f(z)|^2$. Firstly, let us consider the former cases in which the meromorphic function $f'(z)$ has a simple zero at, say η . In such cases, $f(z)$ can be expanded into the form of Taylor expansion around η as

$$f(z) = c_0 + c_2(z - \eta)^2 + O((z - \eta)^3), \quad f'(z) = 2c_2(z - \eta) + O((z - \eta)^2), \quad (3.5)$$

where c_j 's are complex coefficients. Thus one can expand ϕ around η as

$$\phi(z, \bar{z}) = c(z - \eta) + O(2), \quad (3.6)$$

where $c = 2c_2/(1 + |c_0|^2)$ and $O(2)$ is the terms of order 2 and higher in $z - \eta$, $\bar{z} - \bar{\eta}$ and their products. Evaluating z -derivative part of the integrand of (3.4) around simple zero η , we find

$$\begin{aligned} \partial \log |\phi|^2 dz &= \left(\frac{\partial \phi}{\phi} + \frac{\partial \bar{\phi}}{\bar{\phi}} \right) dz \\ &\simeq \left(\frac{c + O(1)}{c(z - \eta) + O(2)} + \frac{O(1)}{\bar{c}(\bar{z} - \bar{\eta}) + O(2)} \right) dz \\ &\simeq \left(\frac{1}{z - \eta} + O(0) + \frac{O(1)}{z - \eta} \right) dz. \end{aligned} \quad (3.7)$$

On the small circle around η , we take the parametrisations $z = \eta + \epsilon e^{i\theta}$, $\bar{z} = \bar{\eta} + \epsilon e^{-i\theta}$, and $dz = i\epsilon e^{i\theta} d\theta$, where $\epsilon \ll 1$. Taking the limit $\epsilon \rightarrow 0$ with these parametrisations, we observe

$$\partial \log |\phi|^2 dz \simeq \left(\frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}} + O(\epsilon) + \frac{O(\epsilon^2)}{\epsilon e^{-i\theta}} \right) d\theta \xrightarrow{\epsilon \rightarrow 0} id\theta. \quad (3.8)$$

Similarly, the \bar{z} -derivative part becomes

$$\begin{aligned} \bar{\partial} \log |\phi|^2 d\bar{z} &= \left(\frac{\bar{\partial} \phi}{\phi} + \frac{\bar{\partial} \bar{\phi}}{\bar{\phi}} \right) d\bar{z} \\ &\simeq \left(\frac{O(1)}{c(z - \eta) + O(2)} + \frac{\bar{c} + O(1)}{\bar{c}(\bar{z} - \bar{\eta}) + O(2)} \right) d\bar{z} \\ &\simeq \left(\frac{O(1)}{z - \eta} + \frac{1}{z - \eta} + O(0) \right) d\bar{z} \xrightarrow{\epsilon \rightarrow 0} -id\theta, \end{aligned} \quad (3.9)$$

where $d\bar{z} = -i\epsilon e^{-i\theta} d\theta$ is applied. Therefore the integrand of the right-hand-side of (3.4) turns out to be

$$\partial \log |\phi|^2 dz - \bar{\partial} \log |\phi|^2 d\bar{z} \xrightarrow{\epsilon \rightarrow 0} 2id\theta, \quad (3.10)$$

on the small circle around a simple zero η . By integrating it, we obtain a unit vortex number for each simple zero. Hence, the contribution to the vortex number from the simple zeroes of ϕ is

$$N(\text{simple zero}) = \frac{1}{4\pi i} \sum_{\eta_i} \oint_{C_{\eta_i}} 2id\theta = \frac{1}{2\pi} \sum_{\eta_i} \oint_{C_{\eta_i}} d\theta = (\text{Number of simple zeroes}), \quad (3.11)$$

where η_i 's are the location of simple zeroes.

We next consider the Higgs zeroes emerging from the zeroes of $f'(z)$ with order $n \geq 2$. In such cases, the meromorphic function has the expansion around the order n zero

$$f(z) = c_0 + c_{n+1}(z - \eta)^{n+1} + O((z - \eta)^{n+2}), \quad f'(z) = (n+1)c_{n+1}(z - \eta)^n + O((z - \eta)^{n+1}). \quad (3.12)$$

Similarly to the simple zero cases, the Higgs field is expanded around η as

$$\phi(z, \bar{z}) = c(z - \eta)^n + O(n+1), \quad (3.13)$$

where c is a complex constant composed of the expansion coefficients of $f(z)$ and $O(n+1)$ is the terms of order $n+1$ and higher in $z - \eta$, $\bar{z} - \bar{\eta}$, and their products as in the simple zero case. Thus we find

$$\begin{aligned} \partial \log |\phi|^2 dz &= \left(\frac{\partial \phi}{\phi} + \frac{\partial \bar{\phi}}{\bar{\phi}} \right) dz \\ &\simeq \left(\frac{nc(z - \eta)^{n-1} + O(n)}{c(z - \eta)^n + O(n+1)} + \frac{O(n)}{\bar{c}(\bar{z} - \bar{\eta})^n + O(n+1)} \right) dz \\ &\simeq \left(\frac{n}{z - \eta} + O(0) + \frac{O(n)}{(\bar{z} - \bar{\eta})^n} \right) dz \xrightarrow{\epsilon \rightarrow 0} ind\theta, \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} \bar{\partial} \log |\phi|^2 d\bar{z} &= \left(\frac{\bar{\partial} \phi}{\phi} + \frac{\bar{\partial} \bar{\phi}}{\bar{\phi}} \right) d\bar{z} \\ &\simeq \left(\frac{O(n)}{c(z - \eta)^n + O(n+1)} + \frac{n\bar{c}(\bar{z} - \bar{\eta})^{n-1} + O(n)}{\bar{c}(\bar{z} - \bar{\eta})^n + O(n+1)} \right) d\bar{z} \\ &\simeq \left(\frac{O(n)}{(z - \eta)^n} + \frac{n}{(\bar{z} - \bar{\eta})^n} + O(0) \right) d\bar{z} \xrightarrow{\epsilon \rightarrow 0} -ind\theta, \end{aligned} \quad (3.15)$$

where the parametrization around η is similar to the simple zero case. From this expansion, the contribution to the vortex number from order n zeroes is (3.4)

$$\begin{aligned} N(\text{order } n \text{ zero}) &= \frac{1}{4\pi i} \sum_{\eta_i} \oint_{C_{\eta_i}} \partial \log |\phi|^2 dz - \bar{\partial} \log |\phi|^2 d\bar{z} \\ &\xrightarrow{\epsilon \rightarrow 0} \frac{1}{4\pi i} \sum_{\eta_i} \oint_{C_{\eta_i}} 2ind\theta = n \times (\text{Number of order } n \text{ zeroes}). \end{aligned} \quad (3.16)$$

Hence the Higgs zero of order n emerged from the n -th zero of f' contributes to the vortex number n from (3.4).

Finally, we consider the Higgs zeroes that emerged from the poles of the meromorphic function $f(z)$. Assuming that $f(z)$ has an order n pole at $z = \eta$ with $n \geq 1$, the Laurent expansion of $f(z)$ around η is

$$f(z) = \frac{c}{(z - \eta)^n} + O((z - \eta)^{-n+1}), \quad (3.17)$$

then its derivative is

$$f'(z) = -\frac{nc}{(z - \eta)^{n+1}} + O((z - \eta)^{-n}), \quad (3.18)$$

where c is a complex coefficient and $O((z - \eta)^{-n})$ is the terms of order $-n$ and higher in $z - \eta$. Note that the singularities of $f(z)$ are only poles since it is meromorphic. To find the behaviour of the Higgs zeroes, we observe that

$$\begin{aligned} 1 + |f(z)|^2 &= 1 + \left(\frac{c}{(z - \eta)^n} + \tilde{O}(-n+1) \right) \left(\frac{\bar{c}}{(\bar{z} - \bar{\eta})^n} + \tilde{O}(-n+1) \right) \\ &= 1 + \frac{|c|^2}{(z - \eta)^n (\bar{z} - \bar{\eta})^n} (1 + \tilde{O}(1)) = \frac{|c|^2}{(z - \eta)^n (\bar{z} - \bar{\eta})^n} (1 + \tilde{O}(1)), \end{aligned} \quad (3.19)$$

where $\tilde{O}(n)$ is the terms of order $n \in \mathbb{Z}$ and higher in $z - \eta$, $\bar{z} - \bar{\eta}$, and their products. Note that the $\tilde{O}(n)$ terms may include negative powers of $z - \eta$ and $\bar{z} - \bar{\eta}$ such as $(z - \eta)^{n+m}/(z - \eta)^m$ in contrast to the former cases. Thus the behaviour of the Higgs field ϕ around η is

$$\begin{aligned} \phi(z, \bar{z}) &= \frac{f'(z)}{1 + |f(z)|^2} \simeq \left(-\frac{nc}{(z - \eta)^{n+1}} + \tilde{O}(-n) \right) \frac{(z - \eta)^n (\bar{z} - \bar{\eta})^n}{|c|^2} (1 + \tilde{O}(1)) \\ &= \left(-\frac{n(\bar{z} - \bar{\eta})^n}{\bar{c}(z - \eta)} + \tilde{O}(n) \right) (1 + \tilde{O}(1)) \\ &= -\frac{n(\bar{z} - \bar{\eta})^n}{\bar{c}(z - \eta)} + \tilde{O}(n), \end{aligned} \quad (3.20)$$

and similarly,

$$\bar{\phi}(z, \bar{z}) = -\frac{n(z - \eta)^n}{c(\bar{z} - \bar{\eta})} + \tilde{O}(n). \quad (3.21)$$

We find from (3.20) that the zeroes of the Higgs field ϕ emerge from the cases $n \geq 2$, while the $n = 1$ case gives a non-zero point. The contour integral around η for the $n = 1$ case does not contribute to the vortex number and the point $z = \eta$ becomes a saddle point of ϕ because there remains no radius dependence in its leading order at this point, as we will see later in some examples. We thus observe that

$$\frac{\partial \phi}{\phi} \simeq \frac{\frac{n(\bar{z} - \bar{\eta})^n}{\bar{c}(z - \eta)^2} + \tilde{O}(n-1)}{-\frac{n(\bar{z} - \bar{\eta})^n}{\bar{c}(z - \eta)} + \tilde{O}(n)} = -\frac{1}{z - \eta} + \tilde{O}(0), \quad (3.22)$$

and

$$\frac{\partial \bar{\phi}}{\bar{\phi}} \simeq \frac{-\frac{n^2(z-\eta)^{n-1}}{c(z-\eta)} + \tilde{O}(n-1)}{-\frac{n(z-\eta)^n}{c(z-\eta)} + \tilde{O}(n)} = \frac{n}{z-\eta} + \tilde{O}(0), \quad (3.23)$$

around η , respectively. The z -derivative part of the contour integral of (3.4) and its \bar{z} -derivative counterpart for these cases become

$$\begin{aligned} \oint_{C_\eta} \partial \log |\phi|^2 dz &= \oint_{C_\eta} \left(\frac{\partial \phi}{\phi} + \frac{\partial \bar{\phi}}{\bar{\phi}} \right) dz \\ &= \oint_{C_\eta} \left(\frac{(n-1)}{z-\eta} + \tilde{O}(0) \right) dz \xrightarrow{\epsilon \rightarrow 0} 2(n-1)\pi i, \end{aligned} \quad (3.24)$$

and

$$\oint_{C_\eta} \bar{\partial} \log |\phi|^2 d\bar{z} = \oint_{C_\eta} \left(\frac{\bar{\partial} \phi}{\phi} + \frac{\bar{\partial} \bar{\phi}}{\bar{\phi}} \right) d\bar{z} \xrightarrow{\epsilon \rightarrow 0} -2(n-1)\pi i, \quad (3.25)$$

respectively, where the parametrization $z = \eta + \epsilon e^{i\theta}$, etc., is applied as in the former cases. Note that the $\tilde{O}(0)$ term such as $(\overline{z-\eta})dz/(z-\eta)$ does not contribute to the contour integral around η because it has order ϵ . Hence the contour integral is evaluated as in the previous cases

$$\begin{aligned} N(\text{order } n \text{ pole}) &= \frac{1}{4\pi i} \sum_{\eta_i} \oint_{C_{\eta_i}} \partial \log |\phi|^2 dz - \bar{\partial} \log |\phi|^2 d\bar{z} \\ &\xrightarrow{\epsilon \rightarrow 0} \frac{1}{4\pi i} \sum_{\eta_i} \oint_{C_{\eta_i}} 2i(n-1)d\theta = (n-1) \times (\text{Number of order } n \text{ poles}). \end{aligned} \quad (3.26)$$

Therefore, we find that the Higgs zero emerged from the order n poles of f contributes to the vortex number $n-1$. We note that the order 1 poles of f do not contribute to the vortex number since they do not give the zero points of ϕ , namely, not the vortex centres.

To summarize this subsection, we have proposed an analytical method to calculate the vortex number of the Jackiw-Pi vortex on a torus. In this approach, the defining region of the flux is regarded as a torus without the singular points corresponding to the centres of the vortex, namely, the Higgs zeroes. The evaluation of the flux integration is given through the expansion around each singular point and contour integration. It is shown that there are two types of Higgs zeroes: One of them arises from the zeroes of the derivative of a meromorphic function f' , and the other emerges from the poles of f . The contribution to the vortex number is n from the order n zeroes of f' , and $n-1$ from the order n poles of f .

3.2 Vortices from elliptic functions: Examples

Let us examine typical examples of the Jackiw-Pi vortices on a torus. We choose simple elliptic functions as a meromorphic function $f(z)$ and see the characteristic aspects of the

Higgs fields for individual vortices. In particular, we perform analytic calculations for the vortex number developed in the previous subsection and see the consistency with the numerical integration. It is well known that there are two kinds of fundamental doubly periodic functions, namely, the Weierstrass \wp function and the Jacobi elliptic function. We will find that the characteristic difference appears between the vortices constructed from these fundamental elliptic functions.

Example 1: Weierstrass \wp function First of all, let us consider simply the Weierstrass \wp function $\wp(z; \omega_1, \omega_2)$ for the meromorphic function f , then the Higgs field becomes

$$\phi_\wp(z, \bar{z}) = \frac{\wp'(z; \omega_1, \omega_2)}{1 + |\wp(z; \omega_1, \omega_2)|^2}, \quad (3.27)$$

where $\omega_1, \omega_2 \in \mathbb{C}$ are the half-periods of the lattice ² $\Lambda = 2\mathbb{Z} \omega_1 + 2\mathbb{Z} \omega_2$, and $\wp(z; \omega_1, \omega_2)$ is doubly periodic with respect to the lattice. The \wp -function enjoys the differential equation

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3, \quad (3.28)$$

where g_2 and g_3 are constants and the dependence of the ω_1 and ω_2 is omitted. The values of \wp -function at its half-periods are denoted as $e_j = \wp(\omega_j)$, ($j = 1, 2, 3$) with $\omega_3 := \omega_1 + \omega_2$, and there exist relations $e_1e_2 + e_2e_3 + e_3e_1 = -g_2/4$, and $e_1e_2e_3 = g_3/4$. It is known that the \wp function has a double pole at the origin such as

$$\wp(z) = \frac{1}{z^2} + O(z^0). \quad (3.29)$$

Hence the Higgs field has a zero with a unit vortex number at the origin from the discussion of the previous subsection. We find from (3.20) that the phase angle rotates three times around the origin,

$$\begin{aligned} \phi_\wp(z, \bar{z}) &\simeq -\frac{2\bar{z}^2}{z} + \tilde{O}(2) \\ &= -2\epsilon e^{-3\theta i} + O(\epsilon^2), \end{aligned} \quad (3.30)$$

where $z = \epsilon e^{i\theta}$ and $\bar{z} = \epsilon e^{-i\theta}$ are employed.

On the other hand, the \wp' has simple zeroes at the half-period points $z = \omega_1, \omega_2$ and ω_3 , so that there exist three simple zeroes of the Higgs field at that points. We observe that the phase angle rotates once around these simple zeroes,

$$\phi_\wp(z, \bar{z}) \simeq c(z - \omega_j) + O(2) = c\epsilon e^{i\theta} + O(\epsilon^2), \quad (3.31)$$

from (3.6), where $c = (6e_j^2 - g_2/2)/(1 + |e_j|^2)$ with $j = 1, 2$, and 3. Thus we find these zeroes, namely the vortex centres, contribute to the vortex number 3 from (3.11) so that the total vortex number of this solution is 4. This is consistent with the numerical integration for the first Chern number (3.1) performed with Mathematica,

$$\frac{1}{2\pi} \int B dx dy = \frac{1}{2\pi} \int |\phi_\wp|^2 dx dy = 4, \quad (3.32)$$

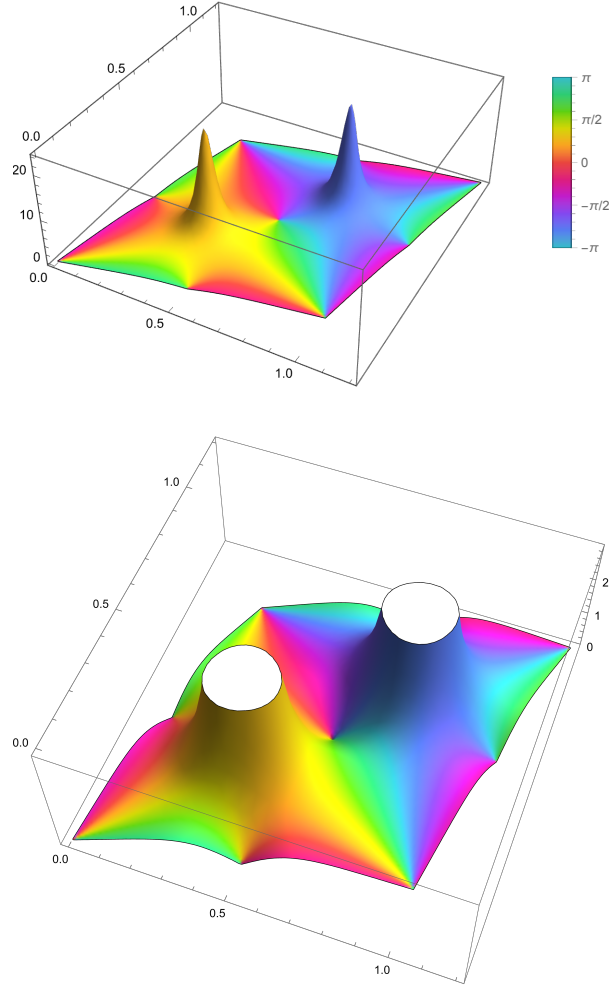


Figure 1: Profiles of ϕ_φ on the fundamental lattice (top), and its enlarged plot with cutoff (bottom) on its fundamental lattice. The vertical axis is the absolute value of the Higgs field $|\phi_\varphi|$ and the colour indicates the phase angle.

where (2.34) is applied.

Figure 1 shows the profile of the Higgs field ϕ_φ with half-periods $\omega_1 = 0.5 + 0.1i$ and $\omega_2 = 0.1 + 0.5i$. The four zero points exist at $0, \omega_1, \omega_2$, and ω_3 in the fundamental lattice. The phase angle structure shows the three times rotation around $z = 0$, while the rotations are once around the other zeroes at $z = \omega_i$, ($i = 1, 2, 3$), as expected.

The absolute value of the Higgs field indicates that the flux is localised at the twin peaks on the fundamental lattice. This localisation structure is not the common feature of ϕ_φ but depends on the choice of half-periods as illustrated in Figure 2. As the region of the fundamental lattice tends to be rectangular, the twin peaks eventually merge into a

²In some literatures these are defined as the whole-periods.

“volcano-like” structure surrounding the zero at ω_3 .

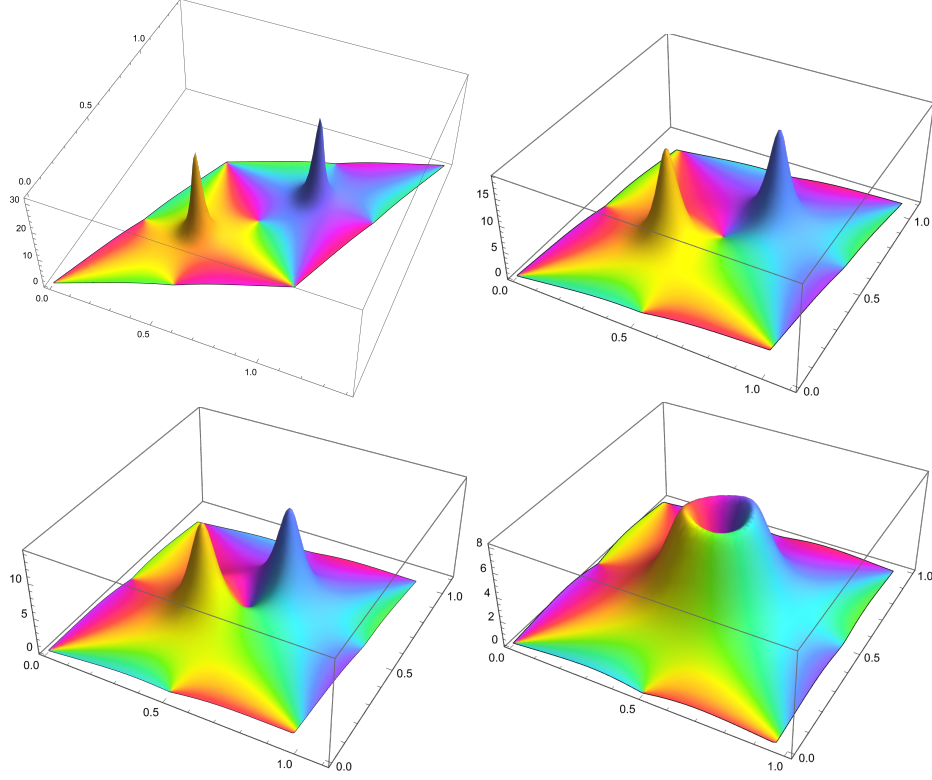


Figure 2: Profiles of ϕ_φ with $\omega_1 = 0.5 + 0.2i, \omega_2 = 0.2 + 0.5i$ (upper left), $\omega_1 = 0.5 + 0.05i, \omega_2 = 0.05 + 0.5i$ (upper right), $\omega_1 = 0.5 + 0.025i, \omega_2 = 0.025 + 0.5i$ (lower left), $\omega_1 = 0.5 + 0.001i, \omega_2 = 0.001 + 0.5i$ (lower right). The vertical scale is not unified, and the colour scale legend is the same as Figure 1.

Example 2: Jacobi sn function The other fundamental elliptic function is the Jacobi elliptic function. Now we apply the Jacobi sn function simply as the meromorphic function f , then the Higgs field takes the form

$$\phi_{\text{sn}}(z, \bar{z}) = \frac{\text{cn}(z; k) \text{dn}(z; k)}{1 + |\text{sn}(z; k)|^2}, \quad (3.33)$$

where $\text{sn}'(z; k) = \text{cn}(z; k) \text{dn}(z; k)$ and k is the modulus. The fundamental lattice on which the sn function defined is $\Lambda = 4K(k)\mathbb{Z} + 2iK'(k)\mathbb{Z}$, where $K(k)$ and $K'(k) := K(k')$ are the complete elliptic integrals of first kind with $k'^2 = 1 - k^2$. We take the modulus $k \in [0, 1)$ as usual, for which $K(k), K'(k) > 0$ so that the fundamental lattice is rectangular.

This vortex solution has four simple zeroes emerging from the simple zeroes of the numerator in (3.33), i.e., K and $3K$ are the zeroes of cn , and $3K + iK'$ and $3K - iK'$ are those of dn . In contrast to the φ -function case, the sn-function in the denominator has only simple poles at iK' and $2K + iK'$ so they do not give zero points but saddle points from (3.20).

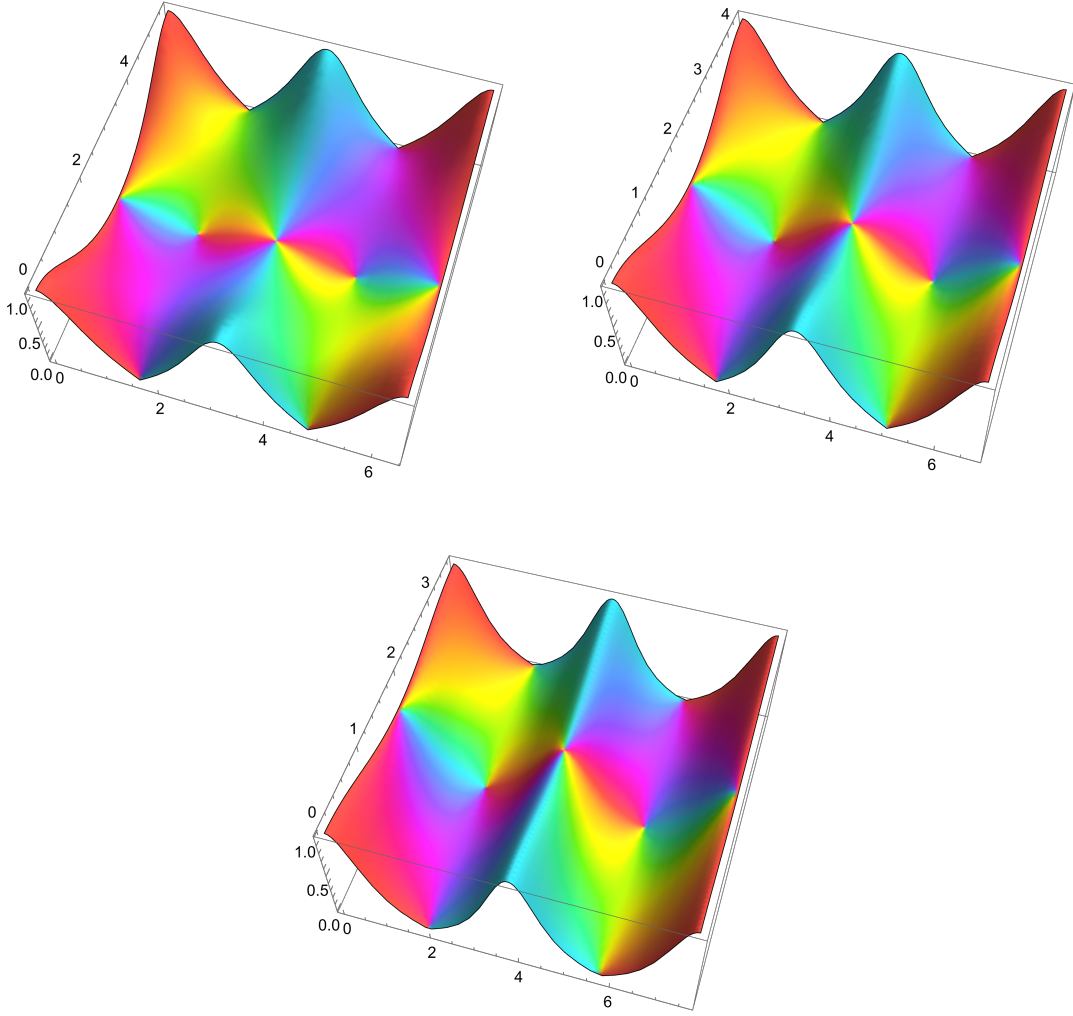


Figure 3: Profiles of ϕ_{sn} with modulus $k = 1/4$ (left), $k = 1/2$ (right) and $k = 3/4$ (lower centre). The domain of these plots is the fundamental lattice spanned by $4K$ and $2iK'$. The colour indicates the phase angle as in the previous Figures.

The profiles of ϕ_{sn} with various values of k are shown in Figure 3. The four simple zeroes appear, for which the phase angle rotates once around them as expected from (3.6). On the other hand, the phase angle rotates twice around the two saddle points, which is consistent with (3.20) with $n = 1$, namely, the behaviour is

$$\phi_{\text{sn}}(\text{saddle pts.}) \sim -\frac{1}{c}e^{-2\theta i} + O(\epsilon). \quad (3.34)$$

The vortex number of these solutions is also four because all four zeroes are simple, which agrees with the numerical integration as with the previous example.

We note that if the phase factor of the Higgs field is ignored the fundamental domain of the solution would be halved in the real axis as shown in Figure 4. This can be understood

by the periodicity of the Jacobi elliptic functions $\text{sn}(z + 2K) = -\text{sn}(z)$, $\text{cn}(z + 2K) = -\text{cn}(z)$, and $\text{dn}(z + 2K) = \text{dn}(z)$ from (3.33). However, the continuity of the Higgs field as a function of z and \bar{z} is lost on this half-domain, on which the phase angle is not periodic at the boundaries so that the formula for the vortex number (3.4) will be invalid. Therefore, we require the strict doubly periodicity of the Higgs field itself in our analysis for the vortices. We will comment on this issue again in the final example.

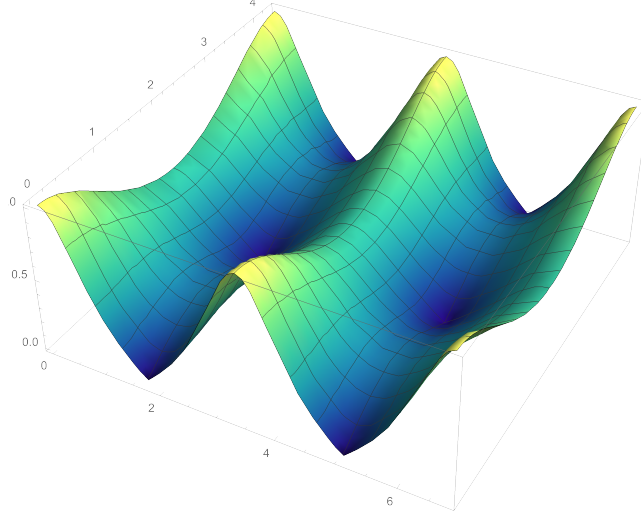


Figure 4: Profiles of $|\phi_{\text{sn}}|$ with modulus $k = 1/2$. The colour indicates the absolute value of the Higgs field.

Example 3: Powers of Jacobi sn function For a vortex solution with multiple zeroes, we consider a solution constructed from the multiple powers of the Jacobi sn function. As an illustration, we choose $\text{sn}^3(z; k)$ as the meromorphic function f , the Higgs field takes the form

$$\phi_{\text{sn}^3}(z, \bar{z}) = \frac{3 \text{cn}(z; k) \text{dn}(z; k) \text{sn}^2(z; k)}{1 + |\text{sn}^3(z; k)|^2}, \quad (3.35)$$

whose fundamental lattice is the same as in the previous example. There are three types of Higgs zeroes in this case: the simple zeroes from $\text{cn}z$ and $\text{dn}z$ and the double zeroes from sn^2z in the numerator, and the triple zeroes from the poles of sn^3z in the denominator. The positions of simple zeroes are $K, 3K, K + iK'$ and $3K + iK'$ of which the first two are from $\text{cn}z$ and the others are from $\text{dn}z$. The double zeroes from sn^2z are located at 0 and $2K$, and the triple zeroes emerging from the poles of sn^3z are at iK' and $2K + iK'$. We observe that the vortex number of this solution is 12, namely, the four simple zeroes, the two double zeroes, and the two triple poles contribute to 4, $4 = 2 \cdot 2$, and $4 = 2 \cdot 2$, respectively. This is consistent with the result of numerical integration.

We remark that the zeroes of the Higgs field can also be identified with rewriting it in terms of elliptic theta functions as,

$$\phi_{\text{sn}^3}(u, \bar{u}) = \frac{3k'k\sqrt{k} \vartheta_1^2(u)\vartheta_2(u)\vartheta_3(u)|\vartheta_4(u)|^2 \arg \vartheta_4^{-4}(u)}{|\vartheta_1(u)|^6 + k^3|\vartheta_4(u)|^6} \quad (3.36)$$

where the variable $u = z/2K$. Note that the absolute values of the elliptic theta functions do not diverge, we find the zeroes of ϕ_{sn^3} only come from the zeroes of the theta functions in the numerator. Let $\tau = iK'/K$, then the simple zeroes of elliptic theta functions $\vartheta_1(u)$, $\vartheta_2(u)$, $\vartheta_3(u)$, and $\vartheta_4(u)$ are located at $0, 1/2, (1 + \tau)/2$, and $\tau/2$, respectively. Together with the quasi-periodicity of the elliptic theta functions, $\vartheta_j(u + 1) = -\vartheta_j(u)$, ($j = 1, 2$) and $\vartheta_j(u + 1) = \vartheta_j(u)$, ($j = 3, 4$), the zeroes of the Higgs field with accurate multiplicities are found in the fundamental lattice.

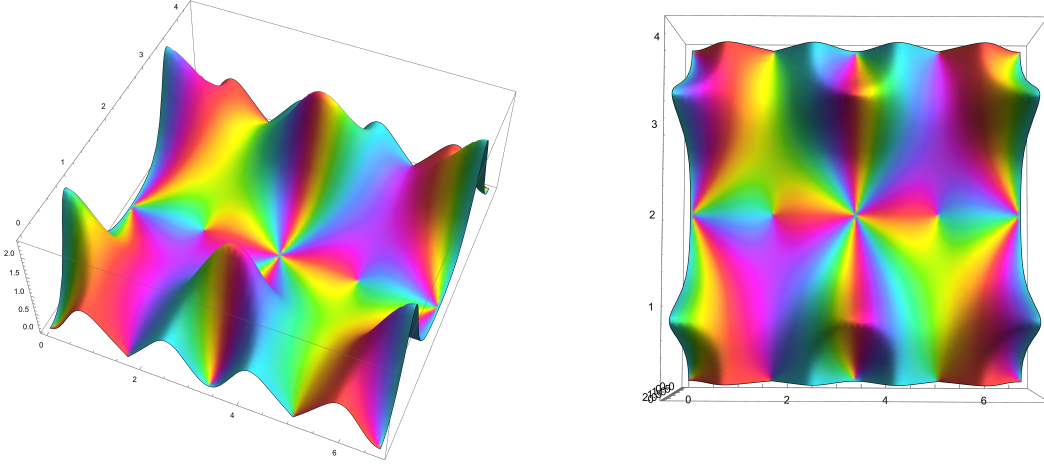


Figure 5: Profile of ϕ_{sn^3} of modulus $k = 1/2$ on the fundamental lattice spanned by $4K$ and $2iK'$. The right is the view from the top of the left profile. The colour scale legend is the same as the previous Figures.

Figure 5 shows the profile of this solution with the phase angle dependence. The simple zeroes are located at $K, 3K, K + iK'$ and $3K + iK'$ with phase angle rotation 2π . The double zeroes from the numerator at 0 and $2K$ show the phase angle rotation twice, i.e., 4π as expected from (3.13) with $n = 2$, whereas those from the poles of the denominator at iK' and $2K + iK'$ demonstrate the rotation angle 8π as expected from (3.20) with $n = 3$.

Example 4: Concerning Olesen's solution In the early 1990s, Olesen constructed a Jackiw-Pi vortex on a torus with unit vortex number [20] in the context of the Chern-Simons-Higgs theory discussed in the previous section. The meromorphic function defining the solution is

$$f(z) = \frac{\wp(z) - e_3}{\sqrt{(e_3 - e_1)(e_2 - e_3)}}. \quad (3.37)$$

Although the solution is similar to the first example (3.27), a constant shift and an overall scaling are assembled for $\wp(z)$. This adjustment makes the periodicity of the flux density $|\phi(z, \bar{z})|^2$, or $|\phi(z, \bar{z})|$, a quarter of the fundamental lattice region of the \wp function, i.e., $|\phi(z + \omega_j, \bar{z} + \bar{\omega}_j)| = |\phi(z, \bar{z})|$, ($j = 1, 2, 3$). For instance, we show a profile of Olesen's solution for the square fundamental region $\omega_1 = 1/2$, $\omega_2 = i/2$ for which $e_1 = -e_2 \simeq 6.87519$ and $e_3 = 0$ in Figure 6.

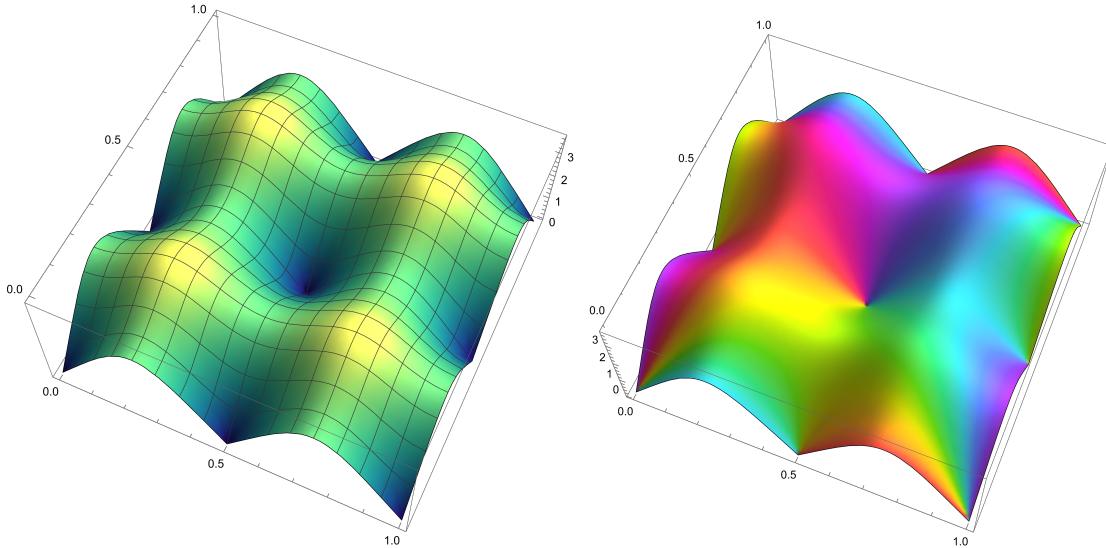


Figure 6: Profile of Olesen's solution of the square fundamental lattice. The left is the profile of $|\phi(z, \bar{z})|$ and the right is $\phi(z, \bar{z})$ with phase angle dependence. The colour scale legend is the same as the previous Figures.

We find the quarter periodicity in the absolute value of the Higgs field or the flux density, and the quarter region having only one zero. Therefore the vortex number of Olesen's solution is one from the perspective of the flux density. The general form of the Jackiw-Pi vortices whose flux density has a periodicity of a quarter of the fundamental region of the associated elliptic functions is given in [21]. This interpretation is reasonable because the physically observable quantity is the flux density and the phase angle of the Higgs field is not significant. However, we find from the right of Figure 6 that the phase angle of the Higgs field is not periodic in the quarter region, so it is not a continuous function in this restricted cell. The continuity of the Higgs field itself is critical for the analytic calculation of the vortex number developed in this paper, so we advocate here the interpretation that the periodicity of this solution is identical to that of the \wp function and the vortex number is 4 since it has four zeroes as of the first example. This interpretation will be acceptable from the point of view of the Aharonov-Bohm-like effects, in which the phase angle of the field plays a critical role.

4 Large period limits of vortices

If we take a fundamental period of the elliptic functions to be infinite, the fundamental lattice expands in that direction and the torus turns out to be a cylinder. Furthermore, the fundamental lattice becomes planar if both of the periods are set to be infinite. In this section, we consider the vortex solutions defined on such large period limits of the fundamental lattice. We will find characteristic “flux loss” phenomena of vortices in those limiting cases.

4.1 Cylinder limits of the vortex from Jacobi elliptic function

To consider the cylinder limit, the vortex made from the Jacobi elliptic functions is favourable because they become elementary functions in the limits.

We reconsider the second example in the last section, i.e., the case of $f(z) = \text{sn}(z)$ with a modulus k . The cylinder limits are obtained by taking the limit $k \rightarrow 0$ or 1 , for which $\text{sn}(z) \rightarrow \sin(z)$ or $\tanh(z)$, respectively. Here we consider the profiles of the Higgs field in these limits and calculate the vortex number of such cases. Then we confirm that they are consistent with numerical integration.

Firstly, we consider the trigonometric function limit $k \rightarrow 0$. The Higgs field ϕ_{sn} then becomes

$$\phi_{\text{sn}}(z, \bar{z}) \xrightarrow{k \rightarrow 0} \phi_{\sin}(z, \bar{z}) = \frac{\cos z}{1 + |\sin z|^2}, \quad (4.1)$$

which has simple zeroes at $z = (2m + 1)\pi/2$ with $m \in \mathbb{Z}$. The fundamental region of $\sin z$ has an infinite period in the imaginary direction and a period 2π in the real direction, as shown in Figure 7. As a result, the point at infinity is excluded from the fundamental region, in other words, the region becomes open and can be thought of as a cylinder. We parametrise the fundamental region of $\sin z$ with $z = x + iy$ as $0 \leq x \leq 2\pi$ and $-\infty < y < \infty$, for which the simple zeroes of $\cos z$ are located at $z = \pi/2$ and $3\pi/2$.

The numerical integration for the vortex number in this cylinder limit is 4 as in the case of a general value of $k < 1$. We now derive this vortex number analytically through (3.4), in that the integration along boundaries at infinity contributes to the vortex number. We evaluate the contour integral

$$N_{k \rightarrow 0} = \frac{1}{2\pi} \int_{\widetilde{\text{Cyl}}} F_{z\bar{z}} dz \wedge d\bar{z} = \frac{1}{4\pi i} \int_{\partial \widetilde{\text{Cyl}}} \partial \log |\phi_{\sin}|^2 dz - \bar{\partial} \log |\phi_{\sin}|^2 d\bar{z}, \quad (4.2)$$

where $\widetilde{\text{Cyl}}$ is the cylinder with the Higgs zeroes removed as in the case of the torus. The difference from the torus cases is that the boundary $\partial \widetilde{\text{Cyl}}$ includes the edges at the infinities in its imaginary coordinate. Thus, the contour integral in the right-hand-side of (4.2) is composed of the small circles around the two Higgs zeroes and the two edges at infinities. Since the two zeroes at $z = \pi/2$ and $3\pi/2$ are simple, the integration around these zeroes contributes to 2 from (3.6), so that it can be expected that the rest comes from the integration along the edges $E_{y \rightarrow \pm\infty} := \{z = x + iy \mid 0 \leq x \leq 2\pi, y \rightarrow \pm\infty\}$. We observe

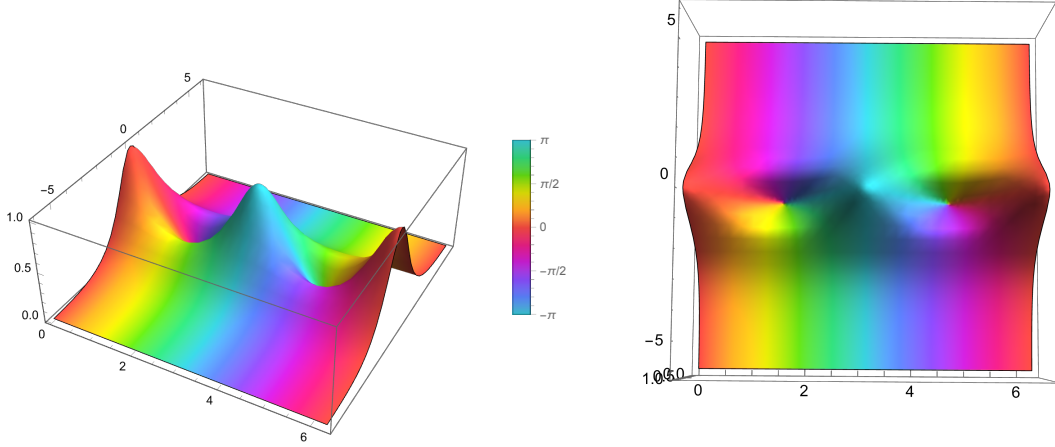


Figure 7: Profile of ϕ_{sin} with the phase angle dependence. The right figure is the view from the top of the left figure. The periodicity persists along the real axis, yet vanishes along the imaginary axis. Hence the fundamental region is an infinitely long strip that contains two zeroes.

the integrands of (4.2) are

$$\partial \log |\phi_{\text{sin}}|^2 = -\frac{2 \cos \bar{z} \sin z}{1 + |\sin z|^2} - \tan z, \quad \bar{\partial} \log |\phi_{\text{sin}}|^2 = -\frac{2 \cos z \sin \bar{z}}{1 + |\sin z|^2} - \tan \bar{z}, \quad (4.3)$$

and

$$\partial \log |\phi_{\text{sin}}|^2 \xrightarrow{y \rightarrow \pm \infty} \pm i. \quad (4.4)$$

The limits of $\bar{\partial} \log |\phi_{k \rightarrow 0}|^2$ are complex conjugates of (4.4). Hence, $N_{k \rightarrow 0}$ can be calculated as follows.

$$\begin{aligned} N_{k \rightarrow 0} &= \frac{1}{4\pi i} \left\{ \int_{E_{y \rightarrow +\infty}} (idz - (-i)d\bar{z}) + \int_{E_{y \rightarrow -\infty}} ((-i)dz - (i)d\bar{z}) + (\text{Contribution from zeroes}) \right\} \\ &= \frac{1}{4\pi i} \left\{ \int_0^{2\pi} 2idx + \int_{2\pi}^0 (-2i)dx \right\} + 2 = \frac{1}{4\pi i} \{4i \times 2\pi\} + 2 = 4. \end{aligned} \quad (4.5)$$

Here the orientation of the integral at the edges is set to ensure that the circle rotates counterclockwise around zeroes at infinities, and we should interpret that the Higgs zeroes sent away to infinity still contribute to the integration. This is consistent with numerical integration as mentioned above.

We consider next the hyperbolic function limit $k \rightarrow 1$, for which $\text{sn}(z; k)$ turns into $\tanh z$. Then the Higgs field becomes

$$\phi_{\text{sn}}(z, \bar{z}) \xrightarrow{k \rightarrow 1} \phi_{\tanh}(z, \bar{z}) = \frac{\text{sech}^2(z)}{1 + |\tanh(z)|^2}. \quad (4.6)$$

In contrast to the former case, the fundamental region of this function has an infinite period in the real direction and a period 2π in the imaginary directions, as shown in Figure 8. Hence the fundamental region is also considered as a cylinder. We take the parametrisation of the fundamental region $z = x + iy$ as $-\infty < x < \infty$ and $0 \leq y \leq \pi$.

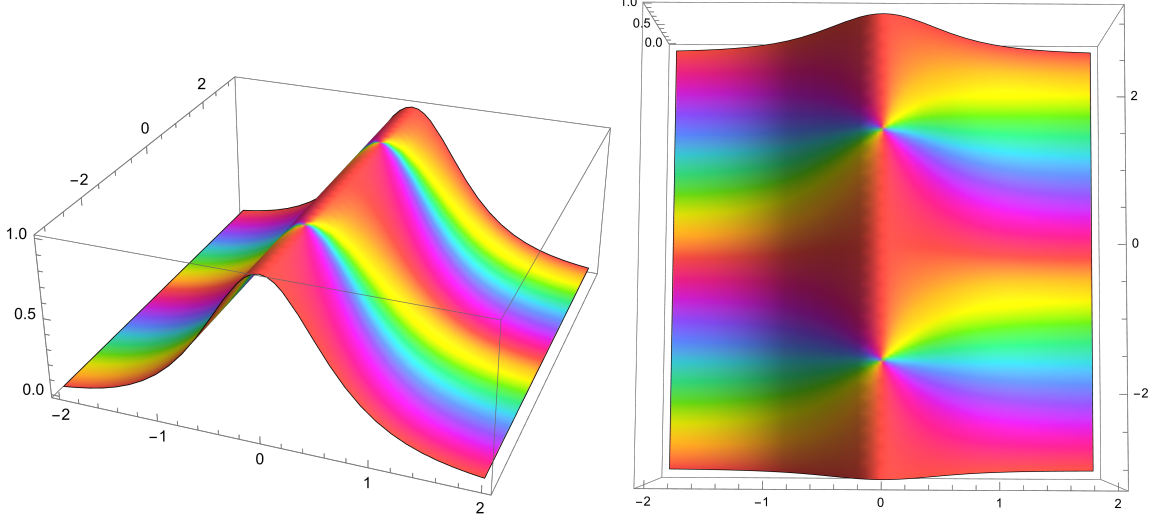


Figure 8: Profile of ϕ_{\tanh} . The right figure is the view from the top of the left figure. The fundamental region is an infinitely long strip that contains no zeroes. Note that these figures are drawn for double periods in the imaginary axis

The numerical integration for the vortex number in this limit is 2, in contrast to the trigonometric function case. In the fundamental region, ϕ_{\tanh} has no zeroes, thus the contribution to the vortex number of ϕ_{\tanh} will come from the integration on the edges of the region. We now give the analytic calculation for this integration similarly to the former.

The vortex number $N_{k \rightarrow 1}$ is also given by (3.4),

$$N_{k \rightarrow 1} = \frac{1}{4\pi i} \int_{\partial \widetilde{\text{Cyl}}} \partial \log |\phi_{\tanh}|^2 dz - \bar{\partial} \log |\phi_{\tanh}|^2 d\bar{z} \quad (4.7)$$

where $\partial \widetilde{\text{Cyl}}$ are only the edges of the cylinder $E_{x \rightarrow \pm\infty} := \{z = x + iy \mid x \rightarrow \pm\infty, 0 \leq y \leq \pi\}$ in this case. The integrands take the following form

$$\partial \log |\phi_{\tanh}|^2 = \bar{\partial} \log |\phi_{\tanh}|^2 = -2 \tanh(z + \bar{z}), \quad (4.8)$$

thus, at the edges $x \rightarrow \pm\infty$,

$$-2 \tanh(z + \bar{z}) \rightarrow \mp 2. \quad (4.9)$$

Hence, $N_{k \rightarrow 1}$ can be evaluated as,

$$\begin{aligned} N_{k \rightarrow 1} &= \frac{1}{4\pi i} \left\{ \int_{E_{x \rightarrow \infty}} (-2) dz - (-2) d\bar{z} + \int_{E_{x \rightarrow -\infty}} 2 dz - 2 d\bar{z} \right\} \\ &= \frac{1}{4\pi i} \left\{ -4 \int_{\pi}^0 idy + 4 \int_0^{\pi} idy \right\} = \frac{1}{4\pi i} \{4i \times 2\pi\} = 2. \end{aligned} \quad (4.10)$$

We should interpret here that the two of four Higgs zeroes fled away from the integration in contrast to the former case. This is also consistent with the numerical integration and illustrates the flux loss phenomena.

4.2 Planar limit

In the examples of the cylinder limit considered so far, the vortex number is not conserved in the limit $k \rightarrow 1$, while it remains in the limit $k \rightarrow 0$. We note that a similar flux loss phenomenon has been reported in [21], where the authors constructed the vortex on a torus from the elliptic function

$$f(z) = \frac{\wp'(z; t/2, it/2)}{\wp(z; t/2, it/2)}. \quad (4.11)$$

Here $\wp(z; t/2, it/2)$ is the Weierstrass \wp function defined on the fundamental lattice $\Lambda = \mathbb{Z}t + \mathbb{Z}it$. They have taken the planar limit $t \rightarrow \infty$ of the lattice, for which the function (4.11) tends to $-2/z$, and shown that the vortex number in this limit equals half of that of the torus. Although the meromorphic function $f \sim 1/z$ does not give a Higgs zero from (3.20) as in the last example, this case also has a vortex number since the integration contour can be taken around infinity in the planar limit.

Here we illustrate another simple case of a vortex in the planar limit. Let us consider the first example of the last section $f(z) = \wp(z; \omega_1, \omega_2)$, which leads to a vortex of the vortex number 4 on a torus. If we let both periods be infinite, then the Weierstrass function becomes a rational function, i.e., $f(z) \rightarrow 1/z^2$ defined on a plane. On first inspection, this vortex gives the vortex number 1 because the meromorphic function has only a double pole at the origin from the point of view of (3.26). However, it is known that the meromorphic functions $1/z^n$ give radially symmetric vortices of the vortex number $2n$ [28] so that the vortex number will be 4 in this case. We can confirm that no flux loss is observed by evaluating the flux integral as follows. The Higgs field in the planar limit becomes from (4.11)

$$\phi(z, \bar{z}) = 2 \frac{-2/z^3}{1 + (1/|z|^4)} = -\frac{4\bar{z}^2/z}{1 + |z|^4}, \quad (4.12)$$

which is rewritten in the polar coordinates

$$\phi(z, \bar{z}) = \phi(r, \theta) = -\frac{4re^{-3i\theta}}{1 + r^4}, \quad (4.13)$$

where $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$. The vortex number, or the flux integral, is easily calculated as

$$N_{\text{planar}} = \frac{1}{2\pi} \int_{\mathbb{R}^2} |\phi|^2 dx dy = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^\infty \frac{16r^2}{(1 + r^4)^2} r dr = 4, \quad (4.14)$$

which indicates no flux loss in the planar limit of the vortex. This observation shows that some relics of the vortex on a torus survive at infinity after taking the planar limit from the perspective of the plane as a decompactified torus. The scenario would be similar to the cylinder limit of a torus considered in the last subsection.

As we have seen in this section, the vortices on a torus occasionally give rise to the flux loss phenomena at the large period limits. We have confirmed this fact in some cases through an analytic manner.

5 Conclusion and discussions

In this paper, we have considered the aspects of the Jackiw-Pi vortices on a torus and, in particular, developed an analytical calculation method to determine the vortex number. The Jackiw-Pi equation can be derived from the Abelian Higgs model on \mathbb{R}^2 with critical coupling constants, which is also one of the integrable vortex equations proposed by [8]. The equation can also be regarded as the “(anti-)self-dual” equation to the Chern-Simons-Higgs theory in 2+1 dimensions. A meromorphic function characterizes the vortex solution to the Jackiw-Pi equation, and the first Chern number of the gauge field is interpreted as the vortex number. If one chooses an elliptic function as the meromorphic function then the Jackiw-Pi vortex can be thought of as defined on a torus. To determine analytically the value of the vortex number, we propose a calculation method using the expansion of the Higgs field around its zeroes. We advocate that the continuity of the Higgs fields is crucially important for determining the vortex number on a torus. The continuity strongly restricts the fundamental domain of the vortices, as shown in this paper.

In the case of a vortex on a plane, the vortex number is equivalent to the degree of mapping, namely the order of zeroes [28], but whether such interpretation applies to the vortex on a torus has not been clear. In this paper, we have shown that the vortex number is given as the sum of the degree of zeroes just as in the planar case even on a torus through the analytical calculation method. It will provide a general procedure for computing the vortex number on a compact surface.

We have also examined some concrete examples of the vortex on a torus and its cylinder or planar limits. Here we discuss the facet of the vortices with the “flux loss” at such decompactification limits. An elliptic function determines a certain elliptic curve, i.e., an algebraic curve of third or fourth order. For example, the Weierstrass \wp -function parametrizes the third order elliptic curve $Y^2 - 4X^3 + g_2X + g_3 = 0$, where $X = \wp(z)$ and $Y = \wp'(z)$. In the planar limit of the torus considered in the last section, the \wp -function degenerates into the rational function $1/z^2$. This means that the elliptic curve degenerates into the third-order curve $Y^2 - X^3 = 0$, which is singular at the origin. The situation is similar to the flux loss case $f = \wp'(z)/\wp(z)$, which degenerates into $-1/z$ and the degenerate curve is $Y^2 - X^4 = 0$. The difference between them is that the latter is reducible or factorizable, namely, $Y^2 - X^4 = (Y - X^2)(Y + X^2) = 0$, while the former is irreducible. Similarly, the cylinder limits of the vortex from the Jacobi elliptic function cases share this characteristic. The meromorphic function $f(z) = \text{sn}(z)$ parametrizes the fourth order elliptic curve $Y^2 - (1 - X^2)(1 - k^2X^2) = 0$, where $X = \text{sn}(z)$ and $Y = \text{sn}'(z) = \text{cn}(z)\text{dn}(z)$. The trigonometric function limit $k \rightarrow 0$ reduces the curve into a quadratic curve $Y^2 - 1 + X^2 = 0$, while the hyperbolic function limit $k \rightarrow 1$ does into $Y^2 - (1 - X^2)^2 = 0$. The latter is reducible to $(Y - 1 + X^2)(Y + 1 - X^2) = 0$. From the discussion in the last section, we notice the trigonometric function limit does not induce the flux loss, while the hyperbolic function

limit induces it. As can be inferred from these observations of the decompactification limits, a conjecture might be possible that the flux loss phenomena occur if the corresponding algebraic curve is factorizable into lower-order curves. Whether this conjecture holds or is rejected, we expect that further study unveils the true mechanism of the flux loss.

Finally, we comment on the solutions with twisted periodic conditions mentioned in the Introduction. The vortex solutions considered in this paper have strict periodicity so the vortex numbers are integers. However, there could be the vortices on a torus with twisted periodicity, which have fractional vortex numbers as in the cases of the fractional instantons [22, 24, 30]. We consider such fascinating objects a subject of future research.

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A Derivation of the formula (3.4)

In this Appendix, we derive the key formula (3.4) for the analysis developed in this paper. We rewrite the field strength $F_{z\bar{z}}$ in terms of the Higgs field ϕ , satisfying the Jackiw-Pi equation. Firstly, we solve the gauge field $a_{\bar{z}}$ from $D_{\bar{z}}\phi = 0$,

$$a_{\bar{z}} = i\bar{\partial}\log\phi. \quad (\text{A.1})$$

Similarly, a_z is given by the complex conjugate of (A.1). Hence $F_{z\bar{z}}$ can be expressed as

$$\begin{aligned} F_{z\bar{z}} &= \partial a_{\bar{z}} - \bar{\partial} a_z \\ &= \partial(i\bar{\partial}\log\phi) - \bar{\partial}(-i\partial\log\bar{\phi}) = i\partial\bar{\partial}\log|\phi|^2. \end{aligned} \quad (\text{A.2})$$

The surface integration of the field strength on the punctured torus \tilde{T}^2 where the Jackiw-Pi vortices are defined then becomes

$$\begin{aligned} \frac{1}{2\pi} \int_{\tilde{T}^2} F_{z\bar{z}} dz \wedge d\bar{z} &= \frac{1}{2\pi} \int_{\tilde{T}^2} i\partial\bar{\partial}\log|\phi|^2 dz \wedge d\bar{z} \\ &= \frac{1}{4\pi} \int_{\tilde{T}^2} (\partial_x^2 + \partial_y^2) \log|\phi|^2 dx \wedge dy \\ &= \frac{1}{4\pi} \int_{\partial\tilde{T}^2} -\partial_y \log|\phi|^2 dx + \partial_x \log|\phi|^2 dy \\ &= \frac{1}{4\pi i} \int_{\partial\tilde{T}^2} \partial \log|\phi|^2 dz - \bar{\partial} \log|\phi|^2 d\bar{z} \\ &= \frac{1}{4\pi i} \sum_i \oint_{C_{\eta_i}} \partial \log|\phi|^2 dz - \bar{\partial} \log|\phi|^2 d\bar{z}, \end{aligned} \quad (\text{A.3})$$

where in the third equality we applied Green's theorem, e.g., [32]. Here $\partial\tilde{T}^2$ denotes the boundaries of \tilde{T}^2 , which are infinitesimal circles around the punctures $z = \eta_i$. We determine the orientation of the circle C_{η_i} so that they rotate counterclockwise as shown in Figure 9. We accentuate that the continuity of the Higgs field ϕ is critical and necessary in the derivation of (A.3) for applying Green's theorem.

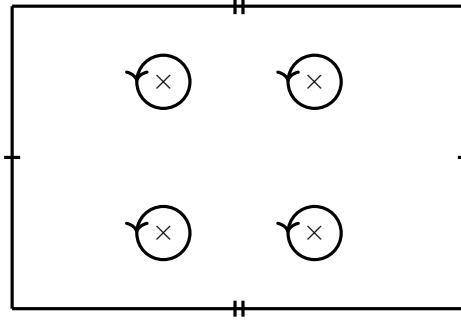


Figure 9: A sketch of integration on a punctured torus. Opposite edges of the rectangle are identified: only circles around singular points (\times) serve as boundaries.

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