FORBIDDEN COMPLEXES FOR THE 3-SPHERE

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Abstract. A simplicial complex is said to be *critical* (or *forbidden*) for the 3-sphere S^3 if it cannot be embedded in S^3 but after removing any one point, it can be embedded.

We show that if a multibranched surface cannot be embedded in S^3, it contains a critical complex which is a union of a multibranched surface and a (possibly empty) graph. We exhibit all critical complexes for S^3 which are contained in $K_5 \times S^1$ and $K_{3,3} \times S^1$ families. We also classify all critical complexes for S^3 which can be decomposed into $G \times S^1$ and H, where G and H are graphs.

In spite of the above property, there exist complexes which cannot be embedded in S^3, but they do not contain any critical complexes. From the property of those examples, we define an equivalence relation on all simplicial complexes $\mathcal{C}$ and a partially ordered set of complexes $(\mathcal{C}/\sim; \subseteqq)$, and refine the definition of critical. According to the refined definition of critical, we show that if a complex X cannot be embedded in S^3, then there exists $[X'] \subseteqq [X]$ such that $[X']$ is critical for $[S^3]$.

1. Introduction

Throughout this paper we work in the piecewise linear category, consisting of simplicial complexes and piecewise-linear maps.

In [2], the definition of critical multibranched surfaces for the 3-sphere was introduced. More generally, we can define the criticality on simplicial complexes as follows. For two simplicial complexes X and Y, X is said to be *critical* (or *forbidden*) for Y if X cannot be embedded in Y, but for any point $p \in X$, $X - p$ can be embedded in Y. In this paper, the polyhedron $|X|$ is expressed directly using X. Hereafter, we assume the connectivity of simplicial complexes for simplicity.

Let $\Gamma(Y)$ denote the set of critical complexes for Y. By the Kuratowski's and Wagner's theorems ([5], [9]), we will show that $\Gamma(S^2) = \{K_5, K_{3,3}\}$ (Proposition 2.3). In this direction, our major goal in this paper is to characterize $\Gamma(Y)$ for a closed n-manifold Y ($n \leq 3$). To achieve this, first enumerate the complexes X that cannot be embedded in Y. One would think that if we remove as many points as possible from X while maintaining the property that X cannot be embedded in Y, we will obtain a critical complex. However, there are complexes that do not satisfy this requirement (Example 2.16 and Theorem 2.18). Based on these, we refine the definition of the criticality so that X cannot be embedded in Y, but for

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any proper subspace X' of X, which does not contain a subspace homeomorphic to X, X' can be embedded in Y. Then we arrive at the equivalence $X \sim Y$ on simplicial complexes $\mathcal{C}$ as X can be embedded in Y and Y can be embedded in X, and we obtain a partially ordered set of complexes $(\mathcal{C}/\sim; \subseteqq)$. In $(\mathcal{C}/\sim; \subseteqq)$, the definition of the criticality is changed to that $[X]$ is *critical* for $[Y]$ if $[X] \nsubseteqq [Y]$ and for any $[X'] \subsetneqq [X]$, $[X'] \subseteqq [Y]$. Finally we will prove the existence of critical subcomplexes, that is, if $[X] \nsubseteqq [M]$ for a closed n-manifold M $(n \leq 3)$, then there exists $[X'] \subseteqq [X]$ such that $[X']$ is critical for $[M]$. For a typical example, a torus T cannot be embedded in a 2-sphere S^2. By applying this existence theorem, there exist $[K_5], [K_{3,3}] \subseteqq [T]$ such that $[K_5], [K_{3,3}]$ are critical for $[S^2]$.

1.1. **Symbol explanation.** We decompose a 2-dimensional simplicial complex X into the following parts. Let $\triangle^i$ denote an i-dimensional simplex of X, and $N(p; X)$ denote an open neighborhood of p in X. The *2-dimensional part* X_2 of X is decomposed into the *sector* $S(X)$ and the *branch* $B(X)$, where

$$S(X) = \{p \in X \mid \exists N(p; X) \cong \mathbb{R}^2\},$$
$$B(X) = \{\partial \triangle^2 \mid \mathrm{int}\triangle^2 \subset S(X)\} \setminus S(X),$$

and put $\partial X_2 = \{p \in B(X) \mid \exists N(p; X) \cong \mathbb{R}^2_+\}$.

The *1-dimensional part* X_1 of X is decomposed into the *edge* $E(X)$ and the *vertex* $V(X)$, where

$$E(X) = \{p \in X \mid \exists N(p; X) \cong \mathbb{R}^1\},$$
$$V(X) = \{\partial \triangle^1 \mid \mathrm{int}\triangle^1 \subset E(X)\} \setminus E(X).$$

2. Critical complexes

2.1. **Critical complexes for closed manifolds.** In this subsection, we consider critical complexes for closed n-manifolds $(n \leq 3)$.

Lemma 2.1. *Let M be an n-manifold and $X \in \Gamma(M)$ be a critical complex for M. Then* $\dim X \leq n$.

Proof. Suppose that $\dim X > n$ and let B^{n+1} be an open $(n+1)$-ball in X. Then for a point $p \in B^{n+1}$, $X - p$ cannot be embedded in M since $X - p$ contains an open $(n+1)$-ball in $B^{n+1} - p$. □

Let M be a connected compact n-manifold and p be a point in the interior of M. We denote the quotient space obtained from M and the closed interval $[0, 1]$ by identifying p and $\{0\}$ by $M^{\perp}$.

Proposition 2.2. $\Gamma(S^1) = \emptyset$.

Proof. Let $X \in \Gamma(S^1)$. By Lemma 2.1, $\dim X = 1$. Since X cannot be embedded in S^1, X contains $I^{\perp}$, where I denotes a closed interval. However, for a point $p \in I^{\perp}$ with a neighborhood which is homeomorphic to an open interval $(0, 1)$, $X - p$ cannot be embedded in S^1. Hence such complex X does not exist. □

Proposition 2.3. $\Gamma(S^2) = \{K_5, K_{3,3}\}$.

Proof. It can be checked that K_5 and $K_{3,3}$ are critical for S^2. Thus we have $\Gamma(S^2) \ni K_5, K_{3,3}$.

Conversely, let $X \in \Gamma(S^2)$. By Lemma 2.1, $\dim X \leq 2$.

First, suppose that $\dim X = 2$. Then X contains a point p whose open neighborhood is homeomophic to an open disk D. Since X is critical for S^2, $X - p$ can be embedded in S^2 and hence $X - D$ can be embedded in S^2. If $X - D$ is a disk, then X is homeomorphic to S^2. This contradicts to the criticality of X. Otherwise, we can find an embedding of $X - D$ in S^2 such that $\partial N(p; X)$ bounds a disk in $S^2 - (X - D)$. Therefore, by filling with D, we have an embedding of X in S^2. This contradicts to the criticality of X and we have $\dim X = 1$.

Next, since X cannot be embedded in S^2, by [5], X contains K_5 or $K_{3,3}$. If X contains K_5 and $X - K_5 \neq \emptyset$, then for a point $p \in X - K_5$, $X - p$ cannot be embedded in S^2. Hence $X = K_5$. The same holds true for $K_{3,3}$. Thus X is K_5 or $K_{3,3}$. □

Let F_g be a closed orientable surface of genus $g > 0$, and $\Omega(F_g)$ be the set of forbidden graphs for F_g.

Theorem 2.4. $\Gamma(F_g) = \{F_0, \dots, F_{g-1}\} \cup \Omega(F_g)$.

Proof. ($\supset$) $F_i \in \{F_0, \dots, F_{g-1}\}$ cannot be embedded in F_g since it is closed. If we remove a point p from $F_i \in \{F_0, \dots, F_{g-1}\}$, then $F_i - p$ can be embedded in F_g. Thus $F_i \in \Gamma(F_g)$. It follows from the definition that $\Omega(F_g) \subset \Gamma(F_g)$.

($\subset$) Let X be a critical complex. If X has no point whose neighborhood is homeomorphic to $\mathbb{R}^2$, then it is a graph and by the criticality it belongs to $\Omega(F_g)$. Otherwise, for a point p whose neighborhood is homeomorphic to $\mathbb{R}^2$, $X - p$ can be embedded in F_g. If $\partial N(p; X)$ bounds a disk in F_g, then X can be embedded in F_g and we have a contradiction. Otherwise, cutting and pasting F_g along $\partial N(p; X)$, X has an embedding in F_h ($h < g$). If X is closed, then $X = F_h$. Otherwise, by connecting sum F_{g-h} to F_h at any point of $F_h - X$, X has an embedding in F_g and we have a contradiction. □

Theorem 2.5 (Characterization of critical complexes with the same dimension)**.** *Let M be a closed n-manifold and $X \in \Gamma(M)$ be a critical complex for M. Then $\dim X = n$ if and only if X is a closed n-manifold which is homeomorphic to a connected proper summand of M including S^n, namely, $M = X \# M'$ for some closed n-manifold M' which is not homeomorphic to S^n.*

Proof. The "if" part obviously holds and we need to prove the "only if" part.

Since $\dim X = n$, there exists a point $p \in X$ whose neighborhood is homeomorphic to $\mathbb{R}^n$. By the criticality of X, $X - p$ can be embedded in M and we can assume $X - \mathrm{int}B \subset M$, where $B = N(p; X)$ is an n-ball. We divide the proof into two cases.

Case 1: ∂B separates M.

Case 2: ∂B does not separate M.

In Case 1, let M_1 and M_2 be compact submanifolds of M divided by ∂B, where we assume without loss of generality that $X - \mathrm{int}B \subset M_1$. We remark that M_2 is not homeomorphic to an n-ball. If $X - \mathrm{int}B = M_1$, then X is a connected proper summand $\hat{M}_1$ of M, where $\hat{M}_1$ denotes the closed n-manifold obtained from M_1 by capping off ∂M_1. Otherwise, there are a point $q \in M_1 - (X - \mathrm{int}B)$ and a neighborhood $B' = N(q; M_1) \subset M_1 - (X - \mathrm{int}B)$. Note that $\hat{M}_1 - \mathrm{int}B'$ can be embedded in M since $\hat{M}_1 - \mathrm{int}B'$ and $\hat{M}_1 - \mathrm{int}B$ are homeomorphic. Since $X \subset \hat{M}_1 - \mathrm{int}B'$, X can be embedded in M. This is a contradiction.

In Case 2, there exists a simple closed curve C embedded in M such that C intersects ∂B transversely in one point. The curve C cannot be completely contained in the interior of $X - \mathrm{int}B$, so C intersects $\partial(X - \mathrm{int}B)$ transversely. Let α be a subarc of C, such that α is contained in $X - \mathrm{int}B$, one of its endpoints is a point in ∂B, and the other is a point in ∂X. Let $N(\alpha)$ be a neighborhood of α in $X - \mathrm{int}B$. Note that $N(\alpha)$ intersects ∂B in an $(n-1)$-ball and intersects ∂X in another $(n-1)$-ball. Consider $N(\alpha \cup \partial B)$. Note that the closure of $X - N(\alpha \cup \partial B)$ is homeomorphic to X since an $(n-1)$-ball adjacent to ∂X is removed. This implies that X can be embedded in M, which is a contradiction. $\square$

2.2. **Critical multibranched surfaces.** For a 2-dimensional simplicial complex X, we say that X is a *multibranched surface* if $B(X)$ consists of circles and $E(X) = \emptyset$. Eto–Matsuzaki–the second author proved that some family of multibranched surfaces belong to $\Gamma(S^3)$.

Theorem 2.6 ([2], [6])**.** $X_1, X_2, X_3, X_g(p_1, \dots, p_n) \in \Gamma(S^3)$.

2.3. **$K_5 \times S^1$ and $K_{3,3} \times S^1$ families and their critical subcomplexes.**

Theorem 2.7. *If a multibranched surface X cannot be embedded in S^3, then there exists a critical subcomplexes $M \cup H$ of X, where M is a multibranched surface and H is a (possibly empty) graph.*

Proof. Suppose that a multibranched surface X cannot be embedded in S^3. If X is not critical, then there exists a point $p \in X$ such that $X - p$ cannot be embedded in S^3.

Case 1: p is contained in the interior of a sector S.

Case 2: p is contained in a branch B.

In Case 1, $S - p$ is homeomorphic to the interior of a regular neighborhood $N(G \cup \partial S; S)$, where G denotes a spine. Then, the following are equivalent.

- $(X - \mathrm{int}S) \cup N(G \cup \partial S; S)$ cannot be embedded in S^3.
- $(X - \mathrm{int}S) \cup G$ cannot be embedded in S^3.

If $(X - \mathrm{int}S) \cup G$ is critical, then we have the conclusion of Theorem 2.7.

Otherwise, we continue this process on $(X - \mathrm{int}S) \cup G$, eliminating points in G or in other sector S' of X. Eventually we get a critical complex $M \cup G \subset X$ as desired.

In Case 2, for a point $p \in B$, $X - p$ cannot be embedded in S^3. Then for a neighborhood $N(p)$ of p, $X - N(p)$ cannot be embedded in S^3. But $N(p)$ necessarily contains points in a sector, that is, there is a point $q \in S$ such that $q \in N(p)$. Then $X - q$ cannot be embedded in S^3. We can proceed as in the Case 1. Eventually we get a critical complex $M \cup G \subset X$ as desired. $\square$

Let Y_n, P_n, D_n denote $K_{1,n} \times S^1$, an n-punctured sphere, n disks respectively.

Suppose that a multibranched surface X contains Y_n as a sub-multibranched surface. We replace Y_n with $P_i \cup D_j$ $(n = i + j)$, where ∂P_i and ∂D_j are attached by degree 1 maps to the branches of degree 1 in Y_n. Note that the degree of each branch remains the same. Make this replacement as recursive as possible into $K_5 \times S^1$ and $K_{3,3} \times S^1$ and get the $K_5 \times S^1$ family (1) – (5) and $K_{3,3} \times S^1$ family (6) – (9).

(1) $K_5 \times S^1$

(2) $(K_4 \times S^1) \cup P_4$

(3) $(K_4 \times S^1) \cup P_3 \cup D_1$
(4) $(K_4 \times S^1) \cup D_4$
(5) $(K_3 \times S^1) \cup P_3 \cup D_3$
(6) $K_{3,3} \times S^1$
(7) $(K_{2,3} \times S^1) \cup P_3$
(8) $(K_{2,3} \times S^1) \cup D_3$
(9) $(K_{1,3} \times S^1) \cup P_3 \cup D_3$

To obtain (5) $(K_3 \times S^1) \cup P_3 \cup D_3$, attach the three boundary components of P_3 (or D_3) to three different branches of $K_3 \times S^1$. Similarly, to obtain $(K_{1,3} \times S^1) \cup P_3 \cup D_3$, attach the three boundary components of P_3 (or D_3) to three different branches of $K_{1,3} \times S^1$.

Theorem 2.8. *All members of $K_5 \times S^1$ and $K_{3,3} \times S^1$ families cannot be embedded in S^3, and they contain critical subcomplexes of the form $M \cup H$ of X as in the following list, where M is a multibranched surface and H is a (possibly empty) graph.*

(1) $K_5 \times S^1 \supset (K_4 \times S^1) \cup K_{1,4}$
(2) $(K_4 \times S^1) \cup P_4 \supset (K_4 \times S^1) \cup K_{1,4}$
(3) $(K_4 \times S^1) \cup P_3 \cup D_1 = (K_4 \times S^1) \cup P_3 \cup D_1$
(4) $(K_4 \times S^1) \cup D_4 \supset (K_4 - K_3) \times S^1 \cup D_4 \cup K_3$
(5) $(K_3 \times S^1) \cup P_3 \cup D_3 = (K_3 \times S^1) \cup P_3 \cup D_3$
(6) $K_{3,3} \times S^1 \supset (K_{2,3} \times S^1) \cup K_{1,3}$
(7) $(K_{2,3} \times S^1) \cup P_3 \supset (K_{2,3} \times S^1) \cup K_{1,3}$
(8) $(K_{2,3} \times S^1) \cup D_3 \supset (K_{1,3} \times S^1) \cup D_3 \cup K_{1,3}$
(9) $(K_{1,3} \times S^1) \cup P_3 \cup D_3 \supset (K_{1,3} \times S^1) \cup D_3 \cup K_{1,3}$

To prove Theorem 2.8, we need two lemmas below.

Let X be a complex embedded in a trivial bundle $F \times S^1$ with the projection $p : F \times S^1 \to F$. We say that X is *vertical* in $F \times S^1$ if $p^{-1}(p(X)) = X$.

Lemma 2.9. *Let G be a connected graph and $f : G \times S^1 \to S^3$ be an embedding. Then f is one of the following type.*

(1) There exist a knot K in S^3 and a trivial bundle structure $D^2 \times S^1$ of $N(K)$ such that $f(G \times S^1)$ is contained in $N(K)$ and $f(G \times S^1)$ is vertical in $N(K)$.
(2) There exist a cable knot K with a cabling annulus A and a trivial bundle structure $D^2 \times S^1$ of $N(K)$ such that $f(G \times S^1)$ is contained in $N(K) \cup N(A)$, $f(G \times S^1) \cap N(K)$ is vertical in $N(K)$ and $f(G \times S^1) \cap N(A)$ consists of mutually disjoint annuli parallel to A.

Proof. Let T be a spanning tree of G. Then $N(f(T \times S^1))$ is a solid torus with a trivial bundle structure $D^2 \times S^1$ in which $f(T \times S^1)$ is vertical. Put $K = \{0\} \times S^1 \subset D^2 \times S^1$.

Let $e_1, \cdots, e_n$ be the edges of $E(G) - E(T)$, and $A_i = f(e_i \times S^1) \cap E(K)$ $(i = 1, \cdots, n)$ be an annulus. Since the boundary slope of A_i is integral and $E(K)$ is a knot exterior in S^3, there are only two possibilities.

(i) A_i is boundary parallel in $E(K)$.
(ii) A_i is a cabling annulus of K.

It is known that a cabling annulus of a knot is unique up to isotopy.

If all annuli $A_1, \cdots, A_n$ are boundary parallel in $E(K)$, then we can isotope them into $N(K)$. Then we have a conclusion (1). Otherwise, there is a cabling annulus A_i and all other cabling annuli are parallel to A_i. Similarly all boundary parallel annuli can be isotoped into $N(K)$. Then we have a conclusion (2). □

We say that an embedding $f : G \times S^1 \to S^3$ is *standard* if it is of type (1) in Lemma 2.9, K is the trivial knot and $p \times S^1$ bounds a disk in $E(K)$ for a point $p \in \partial D^2$.

Let X be a multibranched surface. A *circular permutation system* for X is a choice of a circular ordering of the sectors attached to each branch. See [6, Section 2] for details.

Lemma 2.10. *Let G be a connected graph and $f : G \times S^1 \to S^3$ be an embedding. Then there exists a standard embedding $f_0 : G \times S^1 \to S^3$ with the same circular permutation system as f.*

Proof. First suppose that f is of type (1) in Lemma 2.9. By re-embedding $N(K) = D^2 \times S^1$ in S^3, we have that K is the trivial knot. Moreover, by Dehn twists along D^2, we have that $p \times S^1$ bounds a disk in $E(K)$ for a point $p \in \partial D^2$. Thus we have a standard embedding $f_0 : G \times S^1 \to S^3$. Since the rotation sysytem does not change during the above two operations, we have a standard embedding f_0 with the same circular permutation system as f.

Next let f be of type (2) in Lemma 2.9. By re-embedding cabling annuli contained in $N(A)$ into $N(K)$, we will obtain another embedding f' which is of type (1). Let A^+ and A^- be two annuli which are obtained from $\partial N(K)$ by cutting along ∂A. We replace the cabling annulus A with one of those annuli A^+ and A^-, say A^-, and slightly push it into $\mathrm{int}N(K)$. By repeating this process on all mutually disjoint annuli parallel to A, we obtain another embedding f' of type (1). We note that the rotation sysytem does not change during the above process. □

Remark 2.11. By Lemma 2.10, if there exists an embedding $f : G \times S^1 \to S^3$ for a connected graph G, then as the embedding is vertical, it induces an embedding of G in a disk D^2, and hence G is a planar graph and the rotation system of f is the same as one of a planar embedding of G. Since the circular permutation system determines the regions of $S^3 - f(G \times S^1)$, there is a one-to-one correspondence between the regions of f and f_0.

Proof of Theorem 2.8. First we show that each member of $K_5 \times S^1$ and $K_{3,3} \times S^1$ families cannot be embedded in S^3.

(1) By Remark 2.11, $K_5 \times S^1$ cannot be embedded in S^3 since K_5 is not planar.

(2) Suppose that there exists an embedding $f : (K_4 \times S^1) \cup P_4 \to S^3$. Then by Lemma 2.10 and Remark 2.11, $f(K_4 \times S^1)$ divides S^3 into four regions as K_4 divides S^2 into four regions. We note that each region of $f(K_4 \times S^1)$ contains three branches except for one branch. Now $f(P_4)$ is contained in one of those regions, but in this case, one component of $f(\partial P_4)$ cannot be attached to a branch. This is a contradiction.

(3) Suppose that there exists an embedding $f : (K_4 \times S^1) \cup P_3 \cup D_1 \to S^3$. Since one branch of $f(K_4 \times S^1)$ bounds a disk of D_1, $f|_{K_4 \times S^1}$ is a standard embedding. There are four regions of $f|_{K_4 \times S^1}$, say R_1, R_2, R_3 and R_4, which are all solid tori. Exactly one region, say R_4, contains three branches as meridians, and other three regions R_1, R_2 and R_3 contain three branches as longitudes. Therefore only R_4 can

contain a disk of D_1 as meridian disks. Then P_3 is contained in another region, say R_1, which does not contain ∂D_1. However, ∂P_3 consists of three longitudes of R_1, hence it is impossible. This is a contradiction.

(4) Suppose that there exists an embedding $f : (K_4 \times S^1) \cup D_4 \to S^3$. In this case, each branch of $K_4 \times S^1$ bounds a disk of D_4, hence $f|_{K_4 \times S^1}$ is a standard embedding. There are four regions of $f|_{K_4 \times S^1}$, say R_1, R_2, R_3 and R_4, which are all solid tori. Exactly one region, say R_4, contains three branches as meridians, and other three regions R_1, R_2 and R_3 contain three branches as longitudes. Therefore only R_4 can contain disks of D_4 as meridian disks, but at most three meridian disks. Then at most one disk of D_4 cannot be attached to a branch. This is a contradiction.

(5) Suppose that there exists an embedding $f : (K_3 \times S^1) \cup P_3 \cup D_3 \to S^3$. Similar to (3), $f|_{K_3 \times S^1}$ is a standard embedding, and there are two solid torus regions R_1 and R_2, where R_2 contains three disks of D_3 as meridian disks. Since R_2 is divided by D_3 into three ball regions, P_3 cannot be contained in it. Therefore P_3 is contained in R_1. Then we obtain a 2-sphere S consiting of P_3 and D_3. By observing three annuli of $f(e \times S^1)$ for $e \in E(K_3)$, the 2-sphere S is non-separating in S^3. This is a contradiction.

(6), (7), (8) are similar to (1), (2), (4) respectively.

(6) By Remark 2.11, $K_{3,3} \times S^1$ cannot be embedded in S^3 since $K_{3,3}$ is not planar.

(7) Suppose that there exists an embedding $f : (K_{2,3} \times S^1) \cup P_3 \to S^3$. Then by Lemma 2.10 and Remark 2.11, $f(K_{2,3} \times S^1)$ divides S^3 into three regions as $K_{2,3}$ divides S^2 into three regions. We note that each region of $f(K_{2,3} \times S^1)$ contains two branches except for one branch among three branches to which P_3 attaches. Now $f(P_3)$ is contained in one of those regions, but in this case, one component of $f(\partial P_3)$ cannot be attached to a branch. This is a contradiction.

(8) Suppose that there exists an embedding $f : (K_{2,3} \times S^1) \cup D_3 \to S^3$. In this case, each of degree two branches of $K_{2,3} \times S^1$ bound a disk of D_3, hence $f|_{K_{2,3} \times S^1}$ is a standard embedding. There are three regions of $f|_{K_{2,3} \times S^1}$, say R_1, R_2 and R_3, which are all solid tori. Exactly one region, say R_3, contains two branches of degree one as meridians, and other two regions R_1 and R_2 contain two branches of degree one as longitudes. Therefore only R_3 can contain disks of D_3 as meridian disks, but at most two meridian disks. Then at most one disk of D_3 cannot be attached to a branch. This is a contradiction.

(9) Suppose that there exists an embedding $f : (K_{1,3} \times S^1) \cup P_3 \cup D_3$. Then $f((K_{1,3} \times S^1) \cup D_3)$ divides S^3 into three regions, say R_1, R_2 and R_3. $f(P_3)$ is contained in one of those regions, say R_1. However, since R_1 contains only two branches of degree one in $f((K_{1,3} \times S^1) \cup D_3)$, one component of ∂P_3 cannot be attached to a branch of degree one in $f((K_{1,3} \times S^1) \cup D_3)$. This is a contradiction.

It is straightforward to check along the proof of Theorem 2.7 that in the list of Theorem 2.8, each subcomplex of each member of $K_5 \times S^1$ and $K_{3,3} \times S^1$ is critical for S^3. We leave it to the reader. □

2.4. **Classifying critical subcomplexes in the $K_5 \times S^1$ and $K_{3,3} \times S^1$ families.** In the list of Theorem 2.8, all critical complexes except for (3) and (5) have a form $M \cup G$, where $M = B \cup S$ denotes a multibranched surface with a branch B and a sector S and G denotes a graph. In cases (1), (2), a graph $K_{1,3}$ is attached to a multibranched surface in such a way that the degree one vertices of the graph are attached to different branches, while in cases (4), (6), (7), (8), (9), the graph K_3 or $K_{1,3}$ is attached in such a way that the degree one vertices of the graph are attached to different sectors.

We classify these critical complexes $M \cup G$ ($G \neq \emptyset$) as follows. We assume all sectors are attached to branches by degree one maps. We assume that all sectors are orientable surfaces, and that the multibranched surfaced does not contain a non-orientable surface.

(I) K_4-type — The branch B is divided into four parts B_i ($i = 1, \ldots, 4$) and the sector S is divided into six parts S_{ij} ($i < j$, $i = 1, 2, 3$, $j = 2, 3, 4$), where $\partial S_{ij} = B_i \cup B_j$. The multibranched surface M can be embedded in S^3 so that it divides S^3 into four regions R_k ($k = 1, \ldots, 4$), where $\partial R_k = \bigcup_{i \neq k, j \neq k} S_{ij}$. Furthermore, we assume that M has a unique embedding in S^3 up to homeomorphism. (Thus, the branch B and the sector S are corresponding to the vertices and the edges of K_4.) The graph G is $K_{1,4}$ or a tree which has a $K_{1,4}$-minor and each vertex v_i ($i = 1, \ldots, 4$) of degree one is attached to a point in B_i. We call this complex $M \cup G$ a K_4-*type*. In the above list, (1), (2) are of K_4-type.

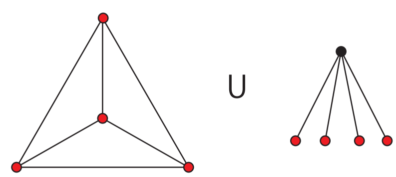

FIGURE 1. K_4-type

(II) Θ_4-type — The sector S is divided into four parts S_i ($i = 0, \ldots, 3$) and $\partial S_i = B$. The multibranched surface M can be embedded in S^3 so that it divides S^3 into four regions R_j ($j = 1, \ldots, 4$), where $\partial R_j = S_{j-1} \cup S_j$ for $j = 1, 2, 3$ and $\partial R_4 = S_3 \cup S_0$. Moreover, we assume that M can be embedded in S^3 so that the sector S takes any circular permutation like Θ_4. The graph G has three edges e_k ($k = 1, 2, 3$) and each edge is attached to M so that e_k connects a point in intS_k and a point in intS_{k+1} for $k = 1, 2$ and e_3 connects a point in intS_3 and a point in intS_1. Think of G as K_3 or as a union of three disjoint edges. We call this complex $M \cup G$ a Θ_4-*type*. In the above list, (4) are of Θ_4-type.

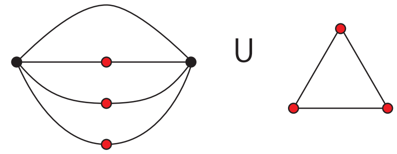

FIGURE 2. Θ_4-type

(III) $K_{2,3}$-type — The sector S is divided into three parts S_i ($i = 1, 2, 3$) and $\partial S_i = B$. The multibranched surface M can be embedded in S^3 so that it divides S^3 into three regions R_j ($j = 1, 2, 3$), where $\partial R_j = S_j \cup S_{j+1}$

for $j = 1, 2$ and $\partial R_3 = S_3 \cup S_1$. The graph G is $K_{1,3}$ and each vertex v_i $(i = 1, 2, 3)$ of degree one is attached to a point in $\mathrm{int}S_i$. We call this complex $M \cup G$ a $K_{2,3}$-*type*. In the above list, (6), (7), (8), (9) are of $K_{2,3}$-type.

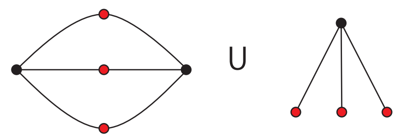

FIGURE 3. $K_{2,3}$-type

It is straightforward to check that critical complexes of those types are critical.

Theorem 2.12. *K_4-type, Θ_4-type and $K_{2,3}$-type are critical for S^3.*

2.5. **Critical complexes which have a form** $(G\times S^1)\cup H$**.** Let X be a simplicial complex such that the 2-dimensional part X_2 of X is a product $G \times S^1$ for a graph G. Then X can be expressed as $X = (G \times S^1) \cup H$, where H is the 1-dimensional part X_1 of X.

We define a *reduction* of $X = (G\times S^1)\cup H$ to $\hat{X} = G\cup H$ as follows. We regard S^1 as the quotient space $[0, 1]/\{0\} \sim \{1\}$. By a map $f : (G\times S^1)\cup H \to (G\times \{0\})\cup H$, we obtain a reduction $\hat{X} = G \cup H$ of $X = (G \times S^1) \cup H$.

Theorem 2.13. *Let X be a critical complex for S^3 such that the 2 dimensional part X_2 of X is a product $G \times S^1$ for a graph G. Put $X = (G \times S^1) \cup H$, where H is the 1-dimensional part X_1 of X. Then a reduction $\hat{X} = G \cup H$ has a minor $G' \cup H'$ which is one of the following.*

(1) $G' \cup H'$ is K_5, where $H' = K_{1,4}$.
(2) $G' \cup H'$ is K_5, where $H' = K_3$.
(3) $G' \cup H'$ is $K_{3,3}$, where $H' = K_{1,3}$.

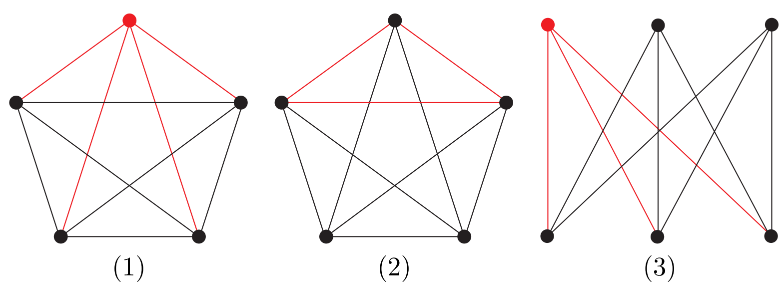

FIGURE 4.

The characterization (1), (2) and (3) in Theorem 2.13 coincide with three types (I), (II) and (III) in Section 2.

Lemma 2.14. *Let $e \in E(G)$ be an edge and $p \in \mathrm{int}(e \times S^1)$ be a point. Suppose that there exists an embedding $f : X - p \to S^3$. Then there exists an embedding $f' : X-p \to S^3$ with the same circular permutation system as f such that $f'((G\times S^1)-p)$ is contained in a standard embedding $f_0 : G \times S^1 \to S^3$.*

Proof. We divide the proof into two cases.

Case 1: e is not a cut edge for G.
Case 2: e is a cut edge for G.

Case 1. By Lemma 2.10, there exists a standard embedding $f_0 : (G-e)\times S^1 \to S^3$ with the same circular permutation system as f. We can regard a once punctured annulus $(e\times S^1)-p$ as a union of $\mathrm{int}N(\partial e\times S^1; e\times S^1)$ and $\mathrm{int}N(e; e\times S^1)$. Since e is not a cut edge for G, we may assume that e is contained in a region R of $G-e$ in D^2, where $f_0((G-e)\times S^1)\subset D^2\times S^1$. We embed $(e\times S^1)-p$ in $R\times S^1$ so that it is contained in $e\times S^1$. Then we obtain the desired embedding $f' : X-p\to S^3$ with the same circular permutation system as f.

Case 2. Put $G-e = G_1\cup G_2$. By Lemma 2.10, there exist standard embeddings $f_0 : G_1\times S^1\to S^3$ and $g_0 : G_2\times S^1\to S^3$ with the same circular permutation system as f. We combine those two embeddings into one embedding $h_0 : (G_1\cup G_2)\times S^1\to S^3$ so that $G_1\cup G_2\subset D^2$, there is an arc α properly embedded in D^2 which separates G_1 and G_2, $\partial\alpha\times S^1$ is the trivial link. Then we may assume that e is contained in the outside region R of $G_1\cup G_2$ in D^2. We embed $(e\times S^1)-p$ in $R\times S^1$ so that it is contained in $e\times S^1$. Then we obtain the desired embedding $f' : X-p\to S^3$ with the same circular permutation system as f. □

Lemma 2.15. *If $X=(G\times S^1)\cup H$ is critical, then a reduction $\hat{X}=G\cup H$ is also critical for S^2.*

Proof. First suppose that $\hat{X}$ can be embedded in S^2. Then $\hat{X}$ is contained in a disk $D^2\subset S^2$ and by embedding $D^2\times S^1$ in S^3, $X=(G\times S^1)\cup H$ can be embedded in S^3. This contradicts the criticality of X.

Next we will show that for any edge e in $G\cup H$, $(G\cup H)-e$ can be embedded in S^2.

Let $e\in E(G)$ be an edge and $p\in \mathrm{int}(e\times S^1)$ be a point. Then there exists an embedding $f : X-p\to S^3$. By Lemma 2.14, there exists an embedding $f' : X-p\to S^3$ with the same circular permutation system as f such that $f'((G\times S^1)-p)$ is contained in a standard embedding $f_0 : G\times S^1\to S^3$. This shows that a reduction $\hat{X}=(G-e)\cup H$ can be embedded in S^2.

Let $e'\in E(H)$ be an edge. Then there exists an embedding $f' : (G\times S^1)\cup(H-e')\to S^3$. By Lemma 2.10, there exists a standard embedding $f'_0 : G\times S^1\to S^3$ with the same circular permutation system as f'. We embed $H-e'$ into the corresponding regions of G in D^2. Then we obtain an embedding $G\cup(H-e')$ in S^2. □

Proof of Theorem 2.13. By Lemma 2.15, a reduction $\hat{X}=G\cup H$ is critical for S^2. Hence by Kuratowski's and Wagner's Theorem ([5], [9]), $\hat{X}$ has a minor of K_5 or $K_{3,3}$. It is straightforward to check that if $\hat{X}=G\cup H$ has a minor K_5, then we have the conclusions (1) or (2), and if $\hat{X}=G\cup H$ has a minor $K_{3,3}$, then we have the conclusion (3). We leave it to the reader. □

2.6. **Complexes which do not contain critical complexes.** Suppose that a complex X cannot be embedded in the 3-sphere S^3. Then we expect that there is a subspace $X'\subset X$ which is critical. However, there are many complexes which cannot be embedded in S^3 but do not contain any critical complexes. Let's start with a simple example.

Example 2.16. Let S^2 be the 2-sphere and α_1, α_2 be arcs embedded in S^2 so that they intersect 1 point transversely. Let D_1, D_2 be two disks and p_1, p_2 be two points in ∂D_1, ∂D_2 respectively. We obtain a complex X from S^2, D_1, D_2 and an arc γ by gluing a subarc of $\partial D_i - p_i$ to α_i for $i = 1, 2$, and connecting p_1 and p_2 by γ.

This complex X cannot be embedded in S^3 since if it can be, then D_1 and D_2 lie in different sides of S^2, but then γ cannot connect p_1 and p_2. However, X does not contain any critical complex. Suppose that $X' \subset X$ is a critical complex. Then X' contains both of S^2 and γ since if we remove any point from S^2 or γ, then it can be embedded in S^3. Since X' cannot be embedded in S^3, for any small neighborhood $N(x)$ of a point $x = \alpha_1 \cap \alpha_2$, $N(x)$ must contain two parts D_1' and D_2' of D_1 and D_2 respectively, and there exists a path connecting D_1' and D_2' and containning γ. However, for any point q in $\mathrm{int}D_i'$, $X' - q$ cannot be embedded in S^3 since it contains a subcomplex which is homeomorphic to X'.

Example 2.16 leads us to the next lemma. We say that a point p in X is an *boundary point* if it has a neighborhood in X which is homeomorphic to $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ for some integer n and p is corresponding to the origin. We define the *boundary* ∂X of X as the set of boundary points.

Lemma 2.17. *Let X be a complex which is critical for Y. Then $\partial X = \emptyset$.*

Proof. Suppose that $\partial X \neq \emptyset$. Let p be a point in $\mathrm{int}X$ close to ∂X. Then $X - p$ can be embedded in Y. However, X can be contained in $X - p$ and hence X can be also embedded in Y. This is a contradiction. □

Theorem 2.18. *The cone over K_5 cannot be embedded in S^3. But, it does not contain any critical complex.*

Proof. First we observe that the cone over K_5 cannot be embedded in S^3. If it can be, then for the vertex v of it, a sufficiently small neighborhood $N(v)$ is homeomorphic to the cone over K_5. This shows that the 2-sphere $\partial N(v)$ contains a non-planar graph K_5, and we have a contradiction.

Next we show that the cone over K_5 does not contain any critical complexes. By Lemma 2.17, it is sufficient to check only subcomplexes X with $\partial X = \emptyset$. However, such a subcomplex of the cone over K_5 is only the vertex v. This completes the proof. □

Example 2.19. The *octahedron obstruction* is introduced in [1]. It is obtained from the octahedron with its eight triangular faces by adding 3 more faces of size 4 orthogonal to the three axis. We remark that after removing upper 4 triangular faces, the octahedron obstruction still cannot be embedded in S^3, and that after removing 2 opposite triangular faces, the octahedron obstruction can be embedded in S^3, where the boundaries of 2 opposite triangular faces must form a Hopf link. By removing non-adjacent 4 triangular faces from the octahedron obstruction, we obtain a closed surface, which is homeomorphic to the real projective plane as Boy's surface.

Example 2.20. Let X be a complex which is obtained from the octahedron obstruction by removing a face of size four, and adding an arc connecting two points p_1, p_2 in the interior of two faces D_1, D_2 of size four. We remove D_1 and add a graph G. Let D_3 and D_4 be disks in the octahedron separated by ∂D_2. Since G

contains two paths connecting p_1 and two points in D_3 and D_4, eventually we obtain a critical subcomplex $X' \subset X$ with $\partial X' = \emptyset$, which is of $K_{2,3}$-type in Theorem 2.12.

3. Refined critical complexes

3.1. **Partially ordered set of complexes.** From Example 2.16 and Theorem 2.18, we derive the following refined definition of critical. For two connected simplicial complexes X and Y, X is said to be *refined critical* for Y if X cannot be embedded in Y, but for any proper subspace X' of X, which does not contain a subspace homeomorphic to X, X' can be embedded in Y. This refined definition of critical leads us the following equivalence relation.

Let $\mathcal{C}$ be the set of simplicial complexes. We define an equivalence relation on $\mathcal{C}$ as follows. This equivalence relation coincides with Fréchet dimension type ([3], [8]). Two simplicial complexes X and Y are *equivalent* $X \sim Y$ if X can be embedded in Y and Y can be embedded in X. Then we say that $[X] \in \mathcal{C}$ is *critical* for $[Y] \in \mathcal{C}$ if $X \in [X]$ cannot be embedded in $Y \in [Y]$, but for any proper subcomplex X' of X with $[X'] \neq [X]$, X' can be embedded in Y.

Let $\Gamma([Y])$ denote the set of equivalence classes of critical complexes for $[Y]$. Then it holds that $\Gamma(S^3) \subset \Gamma([S^3])$ and that the complex X of Example 2.16, the cone over K_5 belong to $\Gamma([S^3])$. To see this, just note that if a subcomplex of X contains the cone point of X, then it is equivalent to X. But if the cone point is removed, then the new complex can be embedded in S^3.

For $[X], [Y] \in \mathcal{C}/\sim$, we define a relation $[X] \subseteqq [Y]$ if X can be embedded in Y. Then we have a partially ordered set $(\mathcal{C}/\sim; \subseteqq)$. We denote $[X] \subsetneqq [Y]$ if $[X] \subseteqq [Y]$ and $[X] \neq [Y]$.

3.2. **Refined critical complexes for closed manifolds.**

Proposition 3.1. *If $X \in \Gamma(Y)$, then $[X] \in \Gamma([Y])$.*

Proof. Suppose that $X \in \Gamma(Y)$. Then X cannot be embedded in Y, but for any point $p \in X$, $X - p$ can be embedded in Y. This implies that $[X] \not\subseteqq [Y]$, but for any proper subcomplex X' of X, X' can be embedded in Y. Hence, $[X] \not\subseteqq [Y]$, but for any $[X'] \subsetneqq [X]$, $[X'] \subseteqq [Y]$. Thus $[X] \in \Gamma([Y])$. □

Proposition 3.2. *Let M be a closed n-manifold and $[X] \in \Gamma([M])$ be a critical element such that $[X] \not\subseteqq [M']$ for any closed n-manifold M'. Then $[X] = [B^{n\perp}]$.*

Proof. Let $[X] \in \Gamma([M])$ be a critical element such that $[X] \not\subseteqq [M']$ for any closed n-manifold M'. This implies that there exists a point $p \in X$ such that for any neighborhood $N(p; X)$, $N(p; X)$ cannot be embedded in B^n. Let B^k be a k-ball in X such that $p \in B^k$ and k is maximal. Then $k \geq n$ and there exists B^n in $N(p; X)$ such that $p \in B^n$. Since $N(p; X) \setminus B^n \neq \emptyset$, there exists a point $q \in N(p; X) \setminus B^n$ and there exists an arc $\gamma \in N(p; X)$ connecting p and q. Hence a union of B^n and γ forms $B^{n\perp}$ and we have $B^{n\perp} \subseteqq X$. Since $B^{n\perp}$ cannot be embedded in M and $[X]$ is critical for M, we have $B^{n\perp} = X$. □

Proposition 3.2 shows the next proposition.

Proposition 3.3. $\Gamma([S^1]) = \{B^{1\perp}\}$.

Mardešić–Segal essentially proved the next theorem.

Theorem 3.4 ([7])**.** $\Gamma([S^2]) = \{[K_5], [K_{3,3}], [B^{2^\perp}]\}$.

We generalize Theorem 3.4 to the next theorem.

Theorem 3.5. $\Gamma([F_g]) = \{[F_0], \dots, [F_{g-1}], [B^{2^\perp}]\} \cup \{[G] \mid G \in \Omega(F_g)\}$.

Proof. By Theorem 2.4 and Proposition 3.1, 3.2, we have

$$\Gamma([F_g]) \supset \{[F_0], \dots, [F_{g-1}], [B^{2^\perp}]\} \cup \{[G] \mid G \in \Omega(F_g)\}.$$

Conversely, let $[X] \in \Gamma([F_g])$. First, suppose that $[X] \not\subseteqq [F']$ for any closed surface F'. Then, by Proposition 3.2, we have $[X] = [B^{2^\perp}]$. Next, suppose that $[X] \subseteqq [F']$ for some closed surface F'. This implies that X can be embedded in F'. In the case that $X = F'$, we have $[X] \in \{[F_0], \dots, [F_{g-1}]\}$. In the case that $X \subsetneqq F'$, we will show that $[X] \in \{[G] \mid G \in \Omega(F_g)\}$ as follows. Since X is a compact subspace of F', the 2-dimensional part X_2 of X is a compact subsurface of F' and the 1-dimensional part X_1 of X is a graph embedded in $X - \mathrm{int}X_2$. Let α be an arc properly embedded in X_2. We deform X_2 along α as shown in Figure 5. If the resultant complex X' can be embedded in F_g, then X can be

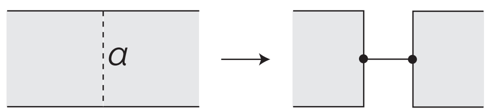

Figure 5. Topological deformations

also embedded in F_g. This contradicts that $[X] \in \Gamma([F_g])$. Hence $[X'] \not\subseteqq [F_g]$ and $[X'] \subsetneqq [X]$, therefore $[X]$ is not critical for $[F_g]$. It follows from this argument that each component of X_2 is a disk. Next we replace each disk of X_2 with a "very large" grid. Geelen–Richter–Salazar ([4]) showed that if a very large grid is embedded in a surface, then a large subgrid is embedded in a disk in the surface. Similarly this contradicts that $[X] \in \Gamma([F_g])$. Hence $X_2 = \emptyset$ and $\dim X = 1$. It follows that $[X] \in \{[G] \mid G \in \Omega(F_g)\}$. □

3.3. **Existence of critical subcomplexes.** As we have seen Example 2.16 and Theorem 2.18, those examples do not satisfy the natural property. However, by considering the equivalence relation above, we obtain the next natural property.

Theorem 3.6 (Existence of critical subcomplexes)**.** *Suppose that a 2-dimensional complex X cannot be embedded in a closed n-manifold M ($n \leq 3$). Then there exists an element $[X'] \subseteqq [X]$ such that $[X']$ is critical for $[M]$.*

Proof. Suppose that X cannot be embedded in M. While maintaining this property, perform the following topological operations (I), (II) and (III) as much as possible. We remark that the complex obtained by such deformations can be embedded in the original complex.

(I) Remove an edge or a sector.
(II) Remove an open disk from a sector without boundary.
(III) Deform as (a), (b) or (c) of Figure 6 along an essential arc α, where if an inequivalent complex is obtained by deforming along α, then α is said to be *essential*.

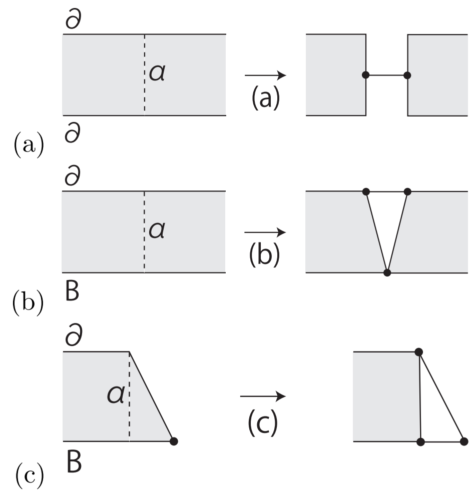

FIGURE 6. Topological deformations

Let X' be the resultant complex. We will show that $[X']$ is critical for $[M]$. Let $[X''] \subsetneqq [X']$, then X'' can be embedded in X'. Hence there exists a triangulation of X' such that X'' is a subcomplex of X'. We take the barycentric subdivision on this triangulation. Then there exist two subcomplexes X'_0 and X''_0 of X' such that

$$X'' \subseteqq X''_0 \subsetneqq X'_0 \subseteqq X',$$

X''_0 can be obtained from X'_0 by removing a single simplex Δ, and $X'_0 \sim X'$. We will show that X''_0 can be embedded in M, and it follows that $[X''] \subseteqq [M]$.

By an operation (I), $\dim \Delta \neq 1$. Thus $\dim \Delta = 2$. Since we take the barycentric subdivision, $\partial\Delta$ contains at most one component of $B(X'_0)$. We need to consider the following cases (1)-(7).

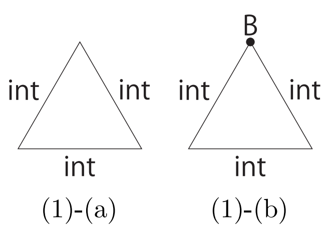

FIGURE 7. Case (1)

In Case (1)-(a), by an operation (II), X''_0 can be embedded in M.

In Case (1)-(b), if Δ is contained in a sector without boundary, then similarly to (1)-(a), X''_0 can be embedded in M. Otherwise, by an operation (III)-(a) along an arc α which connects a point in the boundary of the sector with a point $\partial\Delta$, we have the same resultant as (III)-(b) on X'_0. Hence X''_0 can be embedded in M.

In Case (2)-(a), $X'_0 \sim X''_0$. This contradicts to $X'_0 \not\sim X''_0$.

In Case (2)-(b), by an operation (III)-(b), X''_0 can be embedded in M.

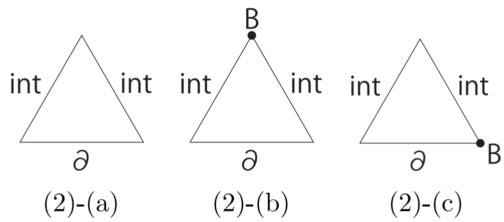

Figure 8. Case (2)

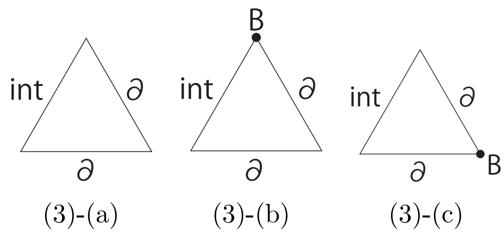

Figure 9. Case (3)

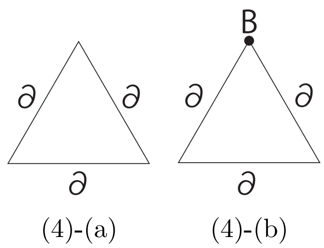

Figure 10. Case (4)

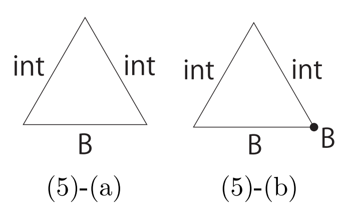

Figure 11. Case (5)

In Case (2)-(c), $X_0' \sim X_0''$. This contradicts to $X_0' \not\sim X_0''$.
In Case (3)-(a), $X_0' \sim X_0''$. This contradicts to $X_0' \not\sim X_0''$.
In Case (3)-(b), $X_0' \sim X_0''$. This contradicts to $X_0' \not\sim X_0''$.
In Case (3)-(c), $X_0' \not\sim X_0''$. But it holds that X_0'' is *embeddability equivalent* to X_0', that is, X_0'' can be embedded in M if and only if X_0' can be embedded in M.

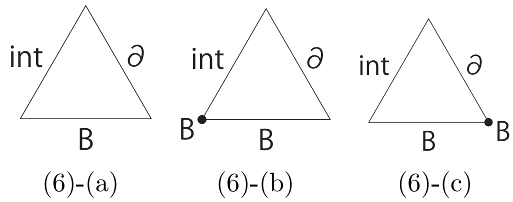

FIGURE 12. Case (6)

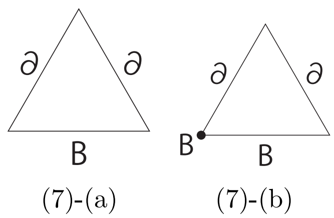

FIGURE 13. Case (7)

We observe that X_0'' can be obtained from X_0' by an operation (III)-(a) and (I). By the assumption, X_0'' can be embedded in M. It follows that X_0' can be embedded in M, and this is a contradiction.

In Case (4)-(a), X_0' is disconnected, a contradiction.

In Case (4)-(b), it holds that X_0'' is embeddability equivalent to X_0'. Note that X_0'' can be obtained from X_0' by operations (III)-(a) and (I). Similarly to (3)-(c), we have a contradiction.

In Case (5)-(a), if Δ is contained in a sector without boundary, then by an operation (II), X_0'' can be embedded in M. Otherwise, a complex which obtained from X_0' by operations (III)-(b), (III)-(c) and (I) is embeddability equivalent to X_0'' as shown in Figure 14. Hence X_0'' can be embedded in M.

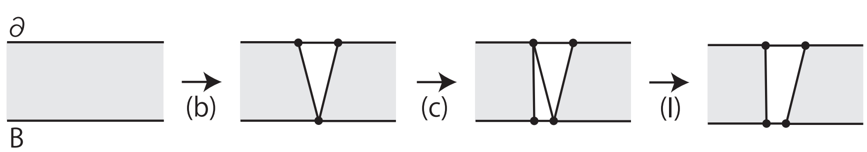

FIGURE 14. Topological deformations

In Case (5)-(b), similarly to (5)-(a), X_0'' can be embedded in M.

In Case (6)-(a), by an operation (III)-(c), X_0'' can be embedded in M.

In Case (6)-(b), similarly to (6)-(a), X_0'' can be embedded in M.

In Case (6)-(c), similarly to (6)-(a), X_0'' can be embedded in M.

In Case (7)-(a), X_0'' is embeddability equivalent to X_0'. But X_0'' can be obtained from X_0' by operations (I)'s. This is a contradiction.

In Case (7)-(b), similarly to (7)-(a), we have a contradiction.

This completes the proof. □

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