Complete moment convergence of moving average processes for m-widely acceptable sequence under sub-linear expectations

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Abstract: In this article, the complete moment convergence for the partial sum of moving average processes $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ is estabished under some proper conditions, where $\{Y_i, -\infty < i < \infty\}$ is a sequence of m-widely acceptable (m-WA) random variables, which is stochastically dominated by a random variable Y in sub-linear expectations space $(\Omega, \mathcal{H}, \mathbb{E})$ and $\{a_i, -\infty < i < \infty\}$ is an absolutely summable sequence of real numbers. The results extend the relevant results in probability space to those under sub-linear expectations.

Keywords: m-widely acceptabl random variables; Moving average processes; Complete convergence; Complete moment convergence; Sub-linear expectation

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1 Introduction

In order to study the uncertainty in probability, Peng [14, 15, 16] introduced the concepts of the sub-linear expectations space. Motivated by the works of Peng [14, 15, 16], lots of people try to extend the results of classic probability space to those of the sub-linear expectations space. Zhang [28, 29, 30] got the exponential inequalities, Rosenthal's inequalities, and Donsker's invariance principle under sub-linear expectations. Under sub-linear expectations, Xu and Cheng [23] studied how small the increments of G-Brownian motion are. Xu and Zhang [20, 21] got a three series theorem of independent random variables and a law of logarithm for arrays of row-wise extended negatively dependent random variables under the sub-linear expectations. Zhong and Wu [35] obtained the complete convergence and complete moment convergence for weighted sums of extended negatively dependent random variables under sub-linear expectations. For more limit theorems under sub-linear expectations, the interested readers could refer to Wu and Jiang [17], Zhang and Lin [32], Zhong and Wu [35], Hu and Yang [9], Chen [1], Zhang [31], Hu, Chen, and Zhang [8], Gao and Xu [3], Kuczmaszewska [11], Chen and Wu [2], Xu and Cheng [22, 23], Xu et al. [24, 26], Xu and Kong [25], and references therein.

Guan, Xiao and Zhao [4] studied complete moment convergence of moving average processes for m-WOD sequence. For more results about complete moment convergence of moving average processes, the interested reader could refer to Zhang and Ding [34], Hosseini and Nezakati [6] and refercences therein. The main conclusions of Guan, Xiao and Zhao [4] are that under proper conditions the complete moment convergence for the partial sum of moving average processes produced by m-widely orthant dependent random variables holds. Recently, Wu, Deng, and Wang studied capacity inequalities and strong laws for m-widely acceptable (m-WA) random variables under sub-linear expectations. It is natural to wonder whether or not the relevant results of Guan, Xiao and Zhao

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[4] hold for moving average processes produced by m-WA random variables under sub-linear expectations. Here, we try to get the complete moment convergence for the partial sum of moving average processes produced by m-WA random variables under sub-linear expectations, complementing the relevant results obtained in Guan, Xiao and Zhao [4].

We organize the rest of this paper as follows. We give some necessary basic notions, concepts and corresponding properties, and cite necessary lemma under sub-linear expectations in the next section. In Section 3, we give our main results, Theorems 3.1-3.2, the proofs of which are also presented in this section.

2 Preliminaries

As in Xu and Cheng [22], we use similar notations as in the work by Peng [15, 16], Chen [1], and Zhang [29]. Suppose that $(\Omega, \mathcal{F})$ is a given measurable space. Assume that $\mathcal{H}$ is a subset of all random variables on $(\Omega, \mathcal{F})$ such that $X_1, \cdots, X_n \in \mathcal{H}$ implies $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in \mathcal{C}_{l,Lip}(\mathbb{R}^n)$, where $\mathcal{C}_{l,Lip}(\mathbb{R}^n)$ represents the linear space of (local lipschitz) function φ fulfilling

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)(|\mathbf{x} - \mathbf{y}|), \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ.

Definition 2.1 *A sub-linear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \mapsto \bar{\mathbb{R}} := [-\infty, \infty]$ fulfilling the following properties: for all $X, Y \in \mathcal{H}$, we have*

(a) *Monotonicity: If $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;*

(b) *Constant preserving: $\mathbb{E}[c] = c$, $\forall c \in \mathbb{R}$;*

(c) *Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \geq 0$;*

(d) *Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$ whenever $\mathbb{E}[X] + \mathbb{E}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$.*

A set function $V : \mathcal{F} \mapsto [0, 1]$ is named to be a capacity if

(a) $V(\emptyset) = 0$, $V(\Omega) = 1$;

(b) $V(A) \leq V(B)$, $A \subset B$, $A, B \in \mathcal{F}$.

A capacity V is called sub-additive if $V(A \bigcup B) \leq V(A) + V(B)$, $A, B \in \mathcal{F}$.

In this sequel, given a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, set $\mathbb{V}(A) := \inf\{\mathbb{E}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\} = \mathbb{E}[I_A]$, $\forall A \in \mathcal{F}$ (see (2.3) and the definitions of $\mathbb{V}$ above (2.3) in Zhang [28]). $\mathbb{V}$ is a sub-additive capacity. Set

$$C_{\mathbb{V}}(X) := \int_0^\infty \mathbb{V}(X > x)\mathrm{d}x + \int_{-\infty}^0 (\mathbb{V}(X > x) - 1)\mathrm{d}x.$$

As in 4.3 of Zhang [28], throughout this paper, define an extension of $\mathbb{E}$ on the space of all random variables by

$$\mathbb{E}^*(X) = \inf\{\mathbb{E}[Y] : X \leq Y, Y \in \mathcal{H}\}.$$

Then $\mathbb{E}^*$ is a sublinear expectation on the space of all random variables, $\mathbb{E}[X] = \mathbb{E}^*[X]$, $\forall X \in \mathcal{H}$, and $\mathbb{V}(A) = \mathbb{E}^*(I_A)$, $\forall A \in \mathcal{F}$.

Definition 2.2 *Suppose $\{Y_n, n \geq 1\}$ is a sequence of random variables in sub-linear expectations space $(\Omega, \mathcal{H}, \mathbb{E})$. $\{Y_n, n \geq 1\}$ is called to be widely acceptable (WA), if there exists a positive sequence $\{g(n), n \geq 1\}$ of dominating coefficients such that for all $n \in \mathbb{N}$, we have*

$$\mathbb{E} \exp\left(\sum_{i=1}^{n} a_{ni} f_i(Y_i)\right) \leq g(n) \prod_{i=1}^{n} \mathbb{E} \exp\left(a_{ni} f_i(Y_i)\right), \quad 0 < g(n) < \infty, \tag{2.1}$$

where $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of nonnegative constants and $f_i(\cdot) \in C_{b,Lip}(\mathbb{R})$, $i = 1, 2, \ldots$, are all non-decreasing (or all non-increasing) real valued truncation functions.

Definition 2.3 *Let $m \geq 1$ be fixed integer. A sequence of random variables $\{X_n, n \geq 1\}$ is called to be m-wildely acceptable (m-WA), if for any $n \geq 2$, and $i_1, i_2, \cdots, i_n$ fulfilling $|i_k - i_j| \geq m$ for all $1 \leq k \neq j \leq n$, we have $X_{i_1}, X_{i_2}, \cdots, X_{i_n}$ are WA.*

Definition 2.4 *We say that $\{Y_n; n \geq 1\}$ is stochastically dominated by a random variable Y in $(\Omega, \mathcal{H}, \mathbb{E})$, if there exists a constant C such that $\forall n \geq 1$, for all non-negative $h \in \mathcal{C}_{l,Lip}(\mathbb{R})$, $\mathbb{E}(h(Y_n)) \leq C\mathbb{E}(h(Y))$.*

Assume that $\mathbf{X}_1$ and $\mathbf{X}_2$ are two n-dimensional random vectors defined, respectively, in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. They are called identically distributed if for every function $\psi \in \mathcal{C}_{l,Lip}(\mathbb{R})$ such that $\psi(\mathbf{X}_1) \in \mathcal{H}_1, \psi(\mathbf{X}_2) \in \mathcal{H}_2$,

$$\mathbb{E}_1[\psi(\mathbf{X}_1)] = \mathbb{E}_2[\psi(\mathbf{X}_2)],$$

whenever the sub-linear expectations are finite. $\{X_n\}_{n=1}^{\infty}$ is named to be identically distributed if for each $i \geq 1$, X_i and X_1 are identically distributed.

In the paper we assume that $\mathbb{E}$ is countably sub-additive, i.e., $\mathbb{E}(X) \leq \sum_{n=1}^{\infty} \mathbb{E}(X_n)$, whenever $X \leq \sum_{n=1}^{\infty} X_n$, $X, X_n \in \mathcal{H}$, and $X \geq 0$, $X_n \geq 0$, $n = 1, 2, \ldots$. Hence $\mathbb{E}^*$ is also countably sub-additive. Let C stand for a positive constant which may change from place to place. $I(A)$ or I_A represent the indicator function of A. Write $\log(x) = \ln \max\{\mathrm{e}, x\}$, $x > 0$.

We cite the following lemma (cf. Lemma 2.2 of Xu et al. [26]).

Lemma 2.1 *If for a random variable X on $(\Omega, \mathcal{F})$, $C_{\mathbb{V}}\{|X|\} < \infty$, then*

$$\mathbb{E}^*[|X|] \leq C_{\mathbb{V}}\{|X|\}.$$

Next we cite and give some useful lemmas.

Lemma 2.2 *(cf. Wu et al. [36])Let $\{X_n, n \geq 1\}$ be a sequence of m-WA random variables with dominating coefficients $g(n)$. If $\{f_n(\cdot), n \geq 1\}$ are all non-decreasing (non-increasing), then $\{f_n(X_n), n \geq 1\}$ are still m-WA with dominating coefficients $\{g(n), n \geq 1\}$.*

Lemma 2.3 *Let $0 < t \leq 1$ or $t = 2$ and $\{X_n, n \geq 1\}$ be a sequence of WA random variables in sub-linear expectations space $(\Omega, \mathcal{H}, \mathbb{E})$. Assume further that $\mathbb{E}(X_n) \leq 0$ for each $n \geq 1$ when $t = 2$. Then for all $x > 0$, and $y > 0$,*

$$\mathbb{V}\left(S_n \geq x\right) \leq \sum_{i=1}^{n} \mathbb{V}\left(X_i > y\right) + g(n) \exp\left(\frac{x}{y} - \frac{x}{y} \ln\left(1 + \frac{xy^{t-1}}{\sum_{i=1}^{n} \mathbb{E}|X_i|^t}\right)\right). \tag{2.2}$$

Proof If $0 < t \leq 1$, then we can establish (2.2) by the adapted proof of Theorem 2.1 of Shen [37]. If $t = 2$, (2.2) follows immediately from Lemma 2.1 of Wu et al. [36]. For readers' convenience, here we give detailed proof when $0 < t \leq 1$.

For $y > 0$, write $\bar{X}_i = \min\{X_i, y\}$, $i = 1, 2, \cdots, n$, and $T_n = \sum_{i=1}^n \bar{X}_i$, $n \geq 1$. We easily see that

$$\{S_n \geq x\} = \{T_n \neq S_n\} \bigcup \{T_n \geq x\},$$

which yields that for any positive h,

$$\mathbb{V}(S_n \geq x) \leq \mathbb{V}(T_n \neq S_n) + \mathbb{V}(T_n \geq x) \leq \sum_{i=1}^n \mathbb{V}(X_i > y) + \mathrm{e}^{-hx}\mathbb{E}\mathrm{e}^{hT_n}.$$

It follows that

$$\mathbb{V}(S_n \geq x) \leq \sum_{i=1}^n \mathbb{V}(X_i > y) + g(n)\mathrm{e}^{-hx}\prod_{i=1}^n \mathbb{E}\mathrm{e}^{h\bar{X}_i}. \tag{2.3}$$

For $0 < t \leq 1$, $h > 0$, the function $\frac{\mathrm{e}^{hu}-1}{u^t}$ is increasing on $u > 0$. Hence

$$\begin{aligned}
\mathbb{E}\mathrm{e}^{h\bar{X}_i} &\leq 1 + \mathbb{E}\left(\frac{\mathrm{e}^{h\bar{X}_i}-1}{|\bar{X}_i|^t}|\bar{X}_i|^t\right) \leq 1 + \mathbb{E}\left(\frac{\mathrm{e}^{hy}-1}{|y|^t}|\bar{X}_i|^t\right) \\
&\leq 1 + \frac{\mathrm{e}^{hy}-1}{|y|^t}\mathbb{E}\left(|\bar{X}_i|^t\right) \leq \exp\left\{\frac{\mathrm{e}^{hy}-1}{|y|^t}\mathbb{E}\left(|\bar{X}_i|^t\right)\right\} \\
&\leq \exp\left\{\frac{\mathrm{e}^{hy}-1}{|y|^t}\mathbb{E}\left(|X_i|^t\right)\right\}.
\end{aligned}$$

Combining the inequality above and (2.3) yields that

$$\mathbb{V}(S_n \geq x) \leq \sum_{i=1}^n \mathbb{V}(X_i > y) + g(n)\exp\left\{\frac{\mathrm{e}^{hy}-1}{y^t}\sum_{i=1}^n \mathbb{E}\left(|X_i|^t\right) - hx\right\}. \tag{2.4}$$

Taking $h = \frac{1}{y}\log\left(1 + \frac{xy^{t-1}}{\sum_{i=1}^n \mathbb{E}(|X_i|^t)}\right)$ in the right-hand side of (2.4), we obtain (2.2). □

Lemma 2.4 *For a positive real number $q \geq 2$, if $\{X_n, n \geq 1\}$ is a sequence of m-WA random variables with dominating coefficients $\{g(n), n \geq 1\}$. If $C_{\mathbb{V}}\{|X_i|^q\} < \infty$ for every $i \geq 1$, then for all $n \geq 1$, there exist positive constants $C_1(m,q)$, $C_2(m,q)$, and $C_3(m,q)$ depending on q and m such that*

$$\begin{aligned}
\mathbb{E}\left(\left|\sum_{i=1}^n X_i\right|^q\right) &\leq C_1(m,q)\sum_{i=1}^n C_{\mathbb{V}}\{|X_i|^q\} \\
&+ C_2(m,q)g(n)\left(\sum_{i=1}^n \mathbb{E}X_i^2\right)^{q/2} + C_3(m,q)\left(\sum_{i=1}^n [|\mathbb{E}(X_i)| + |\mathbb{E}(-X_i)|]\right)^q.
\end{aligned}$$

Proof Note that

$$C_{\mathbb{V}}\{|X^+|^q\} = \int_0^\infty \mathbb{V}\left(|X^+|^q > x\right)\mathrm{d}x = \int_0^\infty qx^{q-1}\mathbb{V}\left\{|X^+| > x\right\}\mathrm{d}x,$$

where $X^+ := \max\{X, 0\}$. We first suppose that $\{X_n, n \geq 1\}$ is a sequence of WA random variables with dominating coefficients $\{g(n), n \geq 1\}$ and $\mathbb{E}(X_n) \leq 0$. Putting $y = x/r$ in (2.3) yields

$$\mathbb{V}\left(S_n^+ \geq x\right) \leq \sum_{i=1}^{n} \mathbb{V}\left(X_i^+ > y\right) + g(n)\mathrm{e}^r \left(\frac{r\sum_{i=1}^{n}\mathbb{E}\left(|X_i|^2\right)}{r\sum_{i=1}^{n}\mathbb{E}\left(|X_i|^2\right) + x^2}\right)^r. \tag{2.5}$$

By the similar proof of (3.4) of Zhang [31], $n \geq 1$, multiplying both sides of (2.2) by qx^{q-1}, and integrating on the half line, we have

$$\begin{aligned}\mathbb{E}\left(\left(S_n^+\right)^q\right) &\leq C_{\mathbb{V}}\left\{\left(S_n^+\right)^q\right\} \\ &\leq \sum_{i=1}^{n} CC_{\mathbb{V}}\left\{|X_i^+|^q\right\} + Cg(n)\left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{q/2}.\end{aligned}$$

Hence when $\{X_n, n \geq 1\}$ is a sequence of WA random variables with dominating coefficients $\{g(n), n \geq 1\}$, by C_r inequality, we see that

$$\begin{aligned}\mathbb{E}\left(\left(S_n^+\right)^q\right) &\leq C\mathbb{E}\left(\left((S_n - \sum_{i=1}^{n}\mathbb{E}X_i)^+\right)^q\right) + C\left(\sum_{i=1}^{n}|\mathbb{E}(X_i)|\right)^q \\ &\leq \sum_{i=1}^{n} CC_{\mathbb{V}}\left\{|(X_i - \mathbb{E}X_i)^+|^q\right\} + Cg(n)\left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{q/2} + +C\left(\sum_{i=1}^{n}|\mathbb{E}(X_i)|\right)^q \\ &\leq .\sum_{i=1}^{n} CC_{\mathbb{V}}\left\{|X_i^+|^q\right\} + C\sum_{i=1}^{n}|\mathbb{E}(X_i)|^q + Cg(n)\left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{q/2} + +C\left(\sum_{i=1}^{n}|\mathbb{E}(X_i)|\right)^q \\ &\leq \sum_{i=1}^{n} CC_{\mathbb{V}}\left\{|X_i^+|^q\right\} + Cg(n)\left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{q/2} + +C\left(\sum_{i=1}^{n}|\mathbb{E}(X_i)|\right)^q,\end{aligned}$$

and

$$\mathbb{E}\left(\left((-S_n)^+\right)^q\right) \leq \sum_{i=1}^{n} CC_{\mathbb{V}}\left\{|X_i^-|^q\right\} + Cg(n)\left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{q/2} + +C\left(\sum_{i=1}^{n}|\mathbb{E}(-X_i)|\right)^q.$$

Therefore, combining the two equations above yields

$$\begin{aligned}\mathbb{E}\left(\left|\sum_{i=1}^{n}X_i\right|^q\right) &\leq C_1(m,q)\sum_{i=1}^{n} C_{\mathbb{V}}\left\{|X_i|^q\right\} \\ &+C_2(m,q)g(n)\left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{q/2} + C_3(m,q)\left(\sum_{i=1}^{n}\left[|\mathbb{E}(X_i)| + |\mathbb{E}(-X_i)|\right]\right)^q.\end{aligned}$$

When $\{X_n, n \geq 1\}$ is a sequence of m-WA random variables with dominating coefficients $\{g(n), n \geq 1\}$, by the equation above, the similar proof of Corollary 3 of Fang et al. [38] (or the adapted proof of Theorem 2.2 of Wu et al. [36]) and C_r inequality, we finish the proof of this theorem. □

3 Main results

Our main results, considered as an extension of Guan et al. [4] in some sense, are as follows.

Theorem 3.1 *Suppose $l(x)$ is a function slowly varying at infinity, $p \geq 1$, $\alpha > \frac{1}{2}$, $\alpha p > 1$. Suppose that $\{a_i, -\infty < i < \infty\}$ is an absolutely summable sequence of real numbers. Assume that $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ is a moving average process produced by a sequence $\{Y_i, -\infty < i < \infty\}$ of m-WA random variables with dominating coefficients $g(n) = O(n^{\delta})$ for some $\delta \geq 0$, and $\{Y_i, -\infty < i < \infty\}$ is stochastic dominated by Y in sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. If $\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0$, $i = 1, 2, \cdots$, for $\frac{1}{2} < \alpha \leq 1$, $C_{\mathbb{V}}\left\{|Y|^p l(|Y|^{1/\alpha})\right\} < \infty$ for $p > 1$ and $C_{\mathbb{V}}\left\{|Y|^{1+\lambda}\right\} < \infty$ for $p = 1$ and some $\lambda > 0$, then for any $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) C_{\mathbb{V}}\left\{\left(\left|\sum_{j=1}^{n} X_j\right| - \epsilon n^{\alpha}\right)^{+}\right\} < \infty. \tag{3.1}$$

Proof For $2^{-\alpha} < \mu < 1$, define $\tilde{g}_{\mu}(x) \in \mathcal{C}_{l,Lip}(\mathbb{R})$ such that $I\{|x| \leq \mu\} < \tilde{g}_{\mu}(x) < I\{|x| \leq 1\}$. Define $g_j(x) \in \mathcal{C}_{l,Lip}(\mathbb{R})$, $j \geq 1$ such that $g_j(x)$ is even function, and for x, $0 \leq g_j(x) \leq 1$; $g_j(x/2^{j\alpha}) = 1$ while $2^{(j-1)\alpha} < |x| \leq 2^{j\alpha}$, and $g_j(x/2^{j\alpha}) = 0$ while $|x| \leq \mu 2^{(j-1)\alpha}$ or $|x| > (1+\mu)2^{j\alpha}$. We see that

$$g_j(|Y|/2^{j\alpha}) \leq I\left\{\mu 2^{(j-1)\alpha} < |Y| \leq (1+\mu)2^{j\alpha}\right\}, |Y|^l \tilde{g}_{\mu}\left(\frac{|Y|}{2^{k\alpha}}\right) \leq 1 + \sum_{j=1}^{k} |Y|^l g_j\left(\frac{|Y|}{2^{j\alpha}}\right), \tag{3.2}$$

$$1 - \tilde{g}_{\mu}\left(\frac{|Y|}{2^{k\alpha}}\right) \leq \sum_{j=k}^{\infty} g_j\left(\frac{|Y|}{2^{j\alpha}}\right). \tag{3.3}$$

Let $f(n) = n^{\alpha p-2-\alpha} l(n)$, $Y_{xj}^{(1)} = -xI\{Y_j < -x\} + Y_j I\{|Y_j| \leq x\} + xI\{Y_j > x\}$, $Y_{xj}^{(2)} = Y_j - Y_{xj}^{(1)}$ be the monotone tructions of $\{Y_j, -\infty < j < \infty\}$ for $x > 0$. Write $Y_x^{(1)} = -xI\{Y < -x\} + YI\{|Y| \leq x\} + xI\{Y > x\}$, $Y_x^{(2)} = Y - Y_x^{(1)}$. Then by Lemma 2.2, we see that $\{Y_{xj}^{(1)}, -\infty < j < \infty\}$ and $\{Y_{xj}^{(2)}, -\infty < j < \infty\}$ are two sequences of m-WA random variables. We observe that

$$\begin{aligned}
&\sum_{n=1}^{\infty} f(n) C_{\mathbb{V}}\left\{\left(\left|\sum_{j=1}^{n} X_j\right| - \epsilon n^{\alpha}\right)^{+}\right\} \\
&\leq \sum_{n=1}^{\infty} f(n) \int_{\epsilon n^{\alpha}}^{\infty} \mathbb{V}\left\{\left|\sum_{j=1}^{n} X_j\right| > x\right\} \mathrm{d}x \leq C \sum_{n=1}^{\infty} f(n) \int_{n^{\alpha}}^{\infty} \mathbb{V}\left\{\left|\sum_{j=1}^{n} X_j\right| > \epsilon x\right\} \mathrm{d}x \\
&\leq C \sum_{n=1}^{\infty} f(n) \int_{n^{\alpha}}^{\infty} \mathbb{V}\left\{\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(2)}\right| > \epsilon x/2\right\} \mathrm{d}x \\
&\quad + C \sum_{n=1}^{\infty} f(n) \int_{n^{\alpha}}^{\infty} \mathbb{V}\left\{\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(1)}\right| > \epsilon x/2\right\} \mathrm{d}x =: I_1 + I_2.
\end{aligned} \tag{3.4}$$

Firstly, we establish $I_1 < \infty$. Observe $|Y_{xj}^{(2)}| < |Y_j|\left(1-\tilde{g}_\mu\left(\frac{|Y_j|}{x}\right)\right)$. Then by Markov's inequality under sub-linear expectations, we see that

$$\begin{aligned}
I_1 &\leq C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-1}\mathbb{E}^*\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(2)}\right| \mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-1}\sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} \mathbb{E}^*\left|Y_{xj}^{(2)}\right| \mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-1}\mathbb{E}|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{x}\right)\right) \mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} f(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} x^{-1}\mathbb{E}|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right) \mathrm{d}x \\
&\leq C\sum_{m=1}^{\infty} m^{-1}\mathbb{E}|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)\sum_{n=1}^{m} f(n).
\end{aligned}$$

If $p>1$, $\alpha p-1-\alpha>-1$, we conclude that

$$\begin{aligned}
I_1 &= C\sum_{k=0}^{\infty}\sum_{m=2^k}^{2^{k+1}-1} m^{\alpha p-1-\alpha} l(m)\mathbb{E}|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right) \\
&\leq C\sum_{k=1}^{\infty} 2^{\alpha pk-\alpha k} l(2^k)\mathbb{E}|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{2^{k\alpha}}\right)\right) \\
&\leq C\sum_{k=1}^{\infty} 2^{\alpha pk-\alpha k} l(2^k)\mathbb{E}^*\sum_{j=k}^{\infty}|Y| g_j\left(\frac{|Y|}{2^{j\alpha}}\right) \\
&= C\sum_{j=1}^{\infty}\mathbb{E}^*|Y| g_j\left(\frac{|Y|}{2^{j\alpha}}\right)\sum_{k=1}^{j} 2^{\alpha pk-\alpha k} l(2^k) \\
&\leq C\sum_{j=1}^{\infty} 2^{\alpha pj} l(2^j)\mathbb{V}\left\{|Y|>\mu 2^{(j-1)\alpha}\right\} \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha p-1} l(m)\mathbb{V}\left\{|Y|>\mu 2^{-1} m^\alpha\right\} \leq CC_{\mathbb{V}}\left\{|Y|^p l(|Y|^{1/\alpha})\right\} < \infty.
\end{aligned}$$

If $p=1$, $C_{\mathbb{V}}\left\{|Y|^{1+\lambda}\right\}<\infty$ yields $C_{\mathbb{V}}\left\{|Y|^{1+\lambda'} l(|Y|^{1/\alpha})\right\}<\infty$ for any $0<\lambda'<\lambda$, then by Lemma 2.3 of Xu [27], we see that

$$\begin{aligned}
I_1 &\leq C\sum_{m=1}^{\infty} m^{-1}\mathbb{E}|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)\sum_{n=1}^{m} n^{-1} l(n) \\
&\leq C\sum_{m=1}^{\infty} m^{-1}\mathbb{E}|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)\sum_{n=1}^{m} n^{-1+\alpha\lambda'} l(n) \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha\lambda'-1} l(n)\mathbb{E}|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{k=0}^{\infty}\sum_{m=2^k}^{2^{k+1}-1} m^{\alpha\lambda'-1} l(m)\mathbb{E}|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)\\
&\le C\sum_{k=1}^{\infty} 2^{k(\alpha\lambda')} l(2^k)\mathbb{E}|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{2^{k\alpha}}\right)\right)\\
&\le C\sum_{k=1}^{\infty} 2^{k(\alpha\lambda')} l(2^k)\mathbb{E}^*\left(|Y|\sum_{l=k}^{\infty} g_l\left(\frac{|Y|}{2^{l\alpha}}\right)\right)\\
&\le C\sum_{l=1}^{\infty}\mathbb{E}^*\left(|Y|g_l\left(\frac{|Y|}{2^{l\alpha}}\right)\right)\sum_{k=1}^{l} 2^{k(\alpha\lambda')} l(2^k)\\
&\le C\sum_{l=1}^{\infty}\mathbb{E}\left(|Y|g_l\left(\frac{|Y|}{2^{l\alpha}}\right)\right) 2^{l(\alpha\lambda')} l(2^l)\\
&\le C\sum_{l=1}^{\infty}\mathbb{V}\left\{|Y|>\mu 2^{(l-1)\alpha}\right\} 2^{l\alpha(\lambda'+1)} l(2^l) < \infty.
\end{aligned}
$$

Hence, we get

$$
I_1 < \infty. \tag{3.5}
$$

Next we establish $I_2 < \infty$. From Markov's inequality under sub-linear expectations, Hölder's inequality and Lemma 2.4, follows that

$$
\begin{aligned}
I_2 &\le C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-r}\mathbb{E}^*\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(1)}\right|^r \mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-r}\mathbb{E}^*\left|\sum_{i=-\infty}^{\infty} |a_i|^{\frac{r-1}{r}}\left(|a_i|^{\frac{1}{r}}\left|\sum_{j=i+1}^{i+n} Y_{xj}^{(1)}\right|\right)\right|^r \mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-r}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{r-1}\left(\sum_{i=-\infty}^{\infty}|a_i|\mathbb{E}^*\left|\sum_{j=i+1}^{i+n} Y_{xj}^{(1)}\right|^r\right)\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-r}\left(\sum_{i=-\infty}^{\infty}|a_i|\mathbb{E}\left|\sum_{j=i+1}^{i+n} Y_{xj}^{(1)}\right|^r\right)\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n} C_{\mathbb{V}}\left\{|Y_{xj}^{(1)}|^r\right\}\mathrm{d}x\\
&\quad + C\sum_{n=1}^{\infty} f(n)g(n)\int_{n^\alpha}^{\infty} x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\mathbb{E}\left|Y_{xj}^{(1)}\right|^2\right)^{r/2}\mathrm{d}x\\
&\quad + C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\left(\sum_{j=i+1}^{i+n}\left|\mathbb{E}(Y_{xj}^{(1)})\right| + \left|\mathbb{E}(-Y_{xj}^{(1)})\right|\right)^r \mathrm{d}x\\
&=: I_{21}+I_{22}+I_{23},
\end{aligned} \tag{3.6}
$$

where $r \ge 2$ is given later.

For I_21, if $p>1$, taking $r>\max\{2,p\}$, then by C_r inequality, similar proof of (2.8) of Zhang [33], Lemma 2.3 of Xu [27], we see that

$$\begin{aligned}
I_{21} &\leq C\sum_{n=1}^{\infty} f(n)\int_{n^\alpha}^{\infty} x^{-r}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n} C_{\mathbb{V}}\left\{|Y_x^{(1)}|^r\right\}\mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} f(n)n\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} x^{-r}C_{\mathbb{V}}\left\{|Y_x^{(1)}|^r\right\}\mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} f(n)n\sum_{m=n}^{\infty} m^{\alpha(1-r)-1}C_{\mathbb{V}}\left\{|Y_{(m+1)^\alpha}^{(1)}|^r\right\} \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha(1-r)-1}\int_0^{(m+1)^\alpha}\mathbb{V}\left(|Y|>x\right)x^{r-1}\mathrm{d}x\sum_{n=1}^{m} f(n)n \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha(1-r)-1}\int_0^{(m+1)^\alpha}\mathbb{V}\left(|Y|>x\right)x^{r-1}\mathrm{d}x m^{\alpha p-\alpha}l(m) \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha(p-r)-1}l(m)\sum_{k=1}^{m+1}\int_{(k-1)^\alpha}^{k^\alpha}\mathbb{V}\left(|Y|>x\right)x^{r-1}\mathrm{d}x \\
&\leq C\sum_{k=1}^{\infty}\int_{(k-1)^\alpha}^{k^\alpha}\mathbb{V}\left(|Y|>x\right)x^{r-1}\mathrm{d}x\sum_{m=1\bigvee(k-1)}^{\infty} m^{\alpha(p-r)-1}l(m) \\
&\leq C\sum_{k=2}^{\infty}\mathbb{V}\left(|Y|>(k-1)^\alpha\right)k^{r\alpha-1}k^{\alpha(p-r)}l(k)+C\sum_{m=1}^{\infty} m^{\alpha(p-r)-1}l(m) \\
&\leq C\sum_{k=2}^{\infty}\mathbb{V}\left(|Y|>(1/2)^\alpha k^\alpha\right)k^{\alpha p-1}l(k)+C<\infty. \qquad (3.7)
\end{aligned}$$

For I_{21}, if $p=1$, taking $r>\max\{1+\lambda',2\}$, where $0<\lambda'<\lambda$, then by the similar discussion as above, we see that

$$\begin{aligned}
I_{21} &\leq C\sum_{m=1}^{\infty} m^{\alpha(1-r)-1}\int_0^{(m+1)^\alpha}\mathbb{V}\left(|Y|>x\right)x^{r-1}\mathrm{d}x\sum_{n=1}^{m} f(n)n \qquad (3.8) \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha(1-r+\lambda')-1}\int_0^{(m+1)^\alpha}\mathbb{V}\left(|Y|>x\right)x^{r-1}\mathrm{d}x l(m) \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha(1-r+\lambda')-1}l(m)\sum_{k=1}^{m+1}\int_{(k-1)^\alpha}^{k^\alpha}\mathbb{V}\left(|Y|>x\right)x^{r-1}\mathrm{d}x \\
&\leq C\sum_{k=1}^{\infty}\int_{(k-1)^\alpha}^{k^\alpha}\mathbb{V}\left(|Y|>x\right)x^{r-1}\mathrm{d}x\sum_{m=1\bigvee(k-1)}^{\infty} m^{\alpha(1-r+\lambda')-1}l(m) \\
&\leq C\sum_{k=2}^{\infty}\mathbb{V}\left(|Y|>(k-1)^\alpha\right)k^{r\alpha-1}k^{\alpha(1-r+\lambda')}l(k)+C\sum_{m=1}^{\infty} m^{\alpha(1-r+\lambda')-1}l(m) \\
&\leq C\sum_{k=2}^{\infty}\mathbb{V}\left(|Y|>(1/2)^\alpha k^\alpha\right)k^{\alpha(1+\lambda')-1}l(k)+C<\infty. \qquad (3.9)
\end{aligned}$$

For I_{22}, if $1\leq p<2$, observing that $g(n)=O(n^\delta)$, taking $r>2$ fulfilling that $\alpha p+r/2-$

$\alpha pr/2-1+\delta=(\alpha p-1)(1-r/2)+\delta<0$, then by C_r inequality, we conclude that

$$\begin{aligned}
I_{22} &\leq C\sum_{n=1}^{\infty} n^{r/2} f(n)g(n)\int_{n^\alpha}^{\infty} x^{-r}\left(\mathbb{E}\left|Y_x^{(1)}\right|^2\right)^{r/2}\mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} n^{r/2} f(n)g(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} \\
&\quad\times\left[x^{-r}\left(\mathbb{E}\left(|Y|^2\tilde{g}_\mu\left(\frac{\mu|Y|}{x}\right)\right)\right)^{r/2}+\left(\mathbb{E}\left(1-\tilde{g}_\mu\left(\frac{|Y|}{x}\right)\right)\right)^{r/2}\right]\mathrm{d}x \\
&\leq C\sum_{m=1}^{\infty}\left[m^{\alpha(1-r)-1}\mathbb{E}\left(|Y|^2\tilde{g}_\mu\left(\frac{\mu|Y|}{(m+1)^\alpha}\right)\right)^{r/2}+m^{\alpha-1}\left(\mathbb{E}\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)\right)^{r/2}\right] \\
&\quad\times\sum_{n=m}^{\infty} n^{r/2} f(n)g(n) \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha(p-r)+r/2+\delta-2}l(m)\left[\mathbb{E}\left(|Y|^p|Y|^{2-p}\tilde{g}_\mu\left(\frac{\mu|Y|}{(m+1)^\alpha}\right)\right)\right]^{r/2} \\
&\quad+C\sum_{m=1}^{\infty} m^{\alpha p+r/2+\delta-2}l(m)\left(\mathbb{E}\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)\right)^{r/2} \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha p+r/2-\alpha pr/2+\delta-2}l(m)\left(\mathbb{E}|Y|^p\right)^{r/2}<\infty. \qquad (3.10)
\end{aligned}$$

For I_{22}, if $p\geq 2$, observing that $g(n)=O(n^\delta)$, taking $r>(\alpha p-1)/(\alpha-1/2)\geq p$ satisfying $\alpha(p-r)+r/2+\delta-1<0$, then by C_r inequality, and similar proof of (3.10), we have

$$\begin{aligned}
I_{22} &\leq C\sum_{m=1}^{\infty}\left[m^{\alpha(1-r)-1}\mathbb{E}\left(|Y|^2\tilde{g}_\mu\left(\frac{\mu|Y|}{(m+1)^\alpha}\right)\right)^{r/2}+m^{\alpha-1}\left(\mathbb{E}\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)\right)^{r/2}\right] \\
&\quad\times\sum_{n=m}^{\infty} n^{r/2} f(n)g(n) \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha(p-r)+r/2+\delta-2}l(m)\left(\mathbb{E}|Y|^2\tilde{g}_\mu\left(\frac{\mu|Y|}{(m+1)^\alpha}\right)\right)^{r/2} \\
&\quad+C\sum_{m=1}^{\infty} m^{\alpha p+r/2+\delta-2}l(m)\left(\mathbb{E}\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)\right)^{r/2} \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha(p-r)+r/2+\delta-2}l(m)\left(\mathbb{E}|Y|^2\right)^{r/2}<\infty. \qquad (3.11)
\end{aligned}$$

For I_{23}, we take $r>2$. By $\mathbb{E}(Y_i)=\mathbb{E}(-Y_i)=0$, Proposition 1.3.7 of Peng [16] and Lemma 4.5 (iii) of Zhang [28], we get

$$\begin{aligned}
I_{23} &\leq C\sum_{n=1}^{\infty} f(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} x^{-r}\left(\sup_i\sum_{j=i+1}^{i+n}\mathbb{E}\left|Y_{xj}^{(1)}-Y_j\right|\right)^r\mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} f(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} x^{-r}n^r\left(\mathbb{E}\,|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{x}\right)\right)\right)^r\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty} f(n)n^r \sum_{m=n}^{\infty} m^{\alpha(1-r)-1}\left(\mathbb{E}\,|Y|\left(1-\tilde{g}_\mu\left(\frac{|Y|}{m^\alpha}\right)\right)\right)^r \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha(1-r)-1}\frac{\mathbb{E}\left(|Y|^p l(|Y|^{1/\alpha})\right)^r}{m^{\alpha(p-1)r}l^r(m)}\sum_{n=1}^{m} f(n)n^r \\
&\leq C\sum_{m=1}^{\infty} m^{-(\alpha p-1)(r-1)-1}/l^r(m)\left(C_{\mathbb{V}}\left\{|Y|^p l(|Y|^{1/\alpha})\right\}\right)^r < \infty. \qquad (3.12)
\end{aligned}
$$

Hence, (3.1) is established by (3.4)-(3.12). □

We next investigate the case $\alpha p = 1$.

Theorem 3.2 *Assume that l is a function slowly varying at infinity, $1 \leq p < 2$. Suppose that $\sum_{i=-\infty}^{\infty}|a_i|^\theta < \infty$, where $\theta \in (0,1)$ if $p = 1$ and $\theta = 1$ if $1 < p < 2$. Assume that $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ is a moving average process produced by a sequence $\{Y_i, -\infty < i < \infty\}$ of m-WA random variables with dominating $g(n) = O(n^\delta)$ for some $0 \leq \delta < (2-p)/p$, stochastically dominated by a random variable Y. While $p = 1$, assume that $0 < \delta < 1$. If $\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0$ and $C_{\mathbb{V}}\left\{|Y|^{p(1+\delta)}l(|Y|^p)\right\} < \infty$, then for any $\varepsilon > 0$,*

$$
\sum_{n=1}^{\infty} n^{-1-1/p}l(n)C_{\mathbb{V}}\left\{\left(\left|\sum_{j=1}^{k} X_j\right| - \varepsilon n^{1/p}\right)^+\right\} < \infty. \qquad (3.13)
$$

Proof Let $h(x) = n^{-1-1/p}l(n)$. As in the proof of (3.4), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} h(n)C_{\mathbb{V}}\left\{\left(\left|\sum_{j=1}^{k} X_j\right| - \varepsilon n^{1/p}\right)^+\right\} \\
\leq\ & C\sum_{n=1}^{\infty} h(n)\int_{n^{1/p}}^{\infty}\mathbb{V}\left\{\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(2)}\right| > \varepsilon x/2\right\}\mathrm{d}x \\
&+C\sum_{n=1}^{\infty} h(n)\int_{n^{1/p}}^{\infty}\mathbb{V}\left\{\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(1)}\right| > \varepsilon x/2\right\}\mathrm{d}x \\
=\ & : J_1 + J_2. \qquad (3.14)
\end{aligned}
$$

For J_1, take $\varepsilon' = \delta$. By Markov's inequality under sub-linear expectations, C_r inequality, and Lemma 2.3 of Xu [27], we see that

$$
\begin{aligned}
J_1 &\leq C\sum_{n=1}^{\infty} h(n)\int_{n^{1/p}}^{\infty} x^{-\theta}\mathbb{E}^*\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_{xj}^{(2)}\right|^2 \mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} nh(n)\int_{n^{1/p}}^{\infty} x^{-\theta}\mathbb{E}|Y_x^{(2)}|^\theta \mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} nh(n)\sum_{m=n}^{\infty}\int_{m^{1/p}}^{(m+1)^{1/p}} x^{-\theta}\mathbb{E}|Y|^\theta\left(1-\tilde{g}_\mu\left(\frac{|Y|}{x}\right)\right)\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty} nh(n)\sum_{m=n}^{\infty} m^{(1-\theta)/p-1}\mathbb{E}|Y|^{\theta}\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{1/p}}\right)\right)\\
&= C\sum_{m=1}^{\infty} m^{(1-\theta)/p-1}\mathbb{E}|Y|^{\theta}\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{1/p}}\right)\right)\sum_{n=1}^{m} nh(n)\\
&\leq \begin{cases} C\sum_{m=1}^{\infty} m^{-1/p}l(m)\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{1/p}}\right)\right), & 1<p<2\\ C\sum_{m=1}^{\infty} m^{(1-\theta)/p-1}\mathbb{E}|Y|^{\theta}\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{1/p}}\right)\right)\sum_{n=1}^{m} n^{\varepsilon'-1}l(n), & p=1\end{cases}\\
&\leq \begin{cases} C\sum_{k=0}^{\infty}\sum_{m=2^k}^{2^{k+1}-1} m^{-1/p}l(m)\mathbb{E}|Y|\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m^{1/p}}\right)\right), & 1<p<2\\ C\sum_{k=0}^{\infty}\sum_{m=2^k}^{2^{k+1}-1} m^{-\theta+\varepsilon'}l(m)\mathbb{E}|Y|^{\theta}\left(1-\tilde{g}_{\mu}\left(\frac{|Y|}{m}\right)\right), & p=1\end{cases}\\
&\leq \begin{cases} C\sum_{k=1}^{\infty} 2^{k(-1/p+1)}l(2^k)\mathbb{E}^*\left(|Y|\sum_{j=k}^{\infty} g_j\left(\frac{|Y|}{2^{j/p}}\right)\right), & 1<p<2\\ C\sum_{k=1}^{\infty} 2^{k(-\theta+\varepsilon'+1)}l(2^k)\mathbb{E}^*\left(|Y|^{\theta}\sum_{j=k}^{\infty} g_j\left(\frac{|Y|}{2^{j}}\right)\right), & p=1\end{cases}\\
&\leq \begin{cases} C\sum_{j=1}^{\infty}\mathbb{E}^*\left(|Y|g_j\left(\frac{|Y|}{2^{j/p}}\right)\right)\sum_{k=1}^{j} 2^{k(-1/p+1)}l(2^k), & 1<p<2\\ C\sum_{j=1}^{\infty}\mathbb{E}^*\left(|Y|^{\theta}g_j\left(\frac{|Y|}{2^{j}}\right)\right)\sum_{k=1}^{j} 2^{k(-\theta+\varepsilon'+1)}l(2^k), & p=1\end{cases}\\
&\leq \begin{cases} C\sum_{j=1}^{\infty}\mathbb{V}\left\{|Y|>\mu 2^{(j-1)/p}\right\}2^{j}l(2^j)<\infty, & 1<p<2\\ C\sum_{j=1}^{\infty}\mathbb{V}\left\{|Y|>\mu 2^{(j-1)}\right\}2^{j(\varepsilon'+1)}l(2^k)<\infty, & p=1,\end{cases}
\end{aligned}
\tag{3.15}
$$

where $\tilde{g}_{\mu}(\cdot)$, $g_j(\cdot)$ here are defined as those of (3.2) and (3.3) with only $1/p$ here in place of α there.

For J_2, as in the proof of I_2, observing that $g(n)=O(n^{\delta})$, for some $0\leq\delta<(2-p)/p$, taking $q=2$ by Lemma 2.4 and similar proof of (2.8) of Zhang [33], we get

$$
\begin{aligned}
J_2 &\leq C\sum_{n=1}^{\infty} h(n)\int_{n^{1/p}}^{\infty} x^{-2}\mathbb{E}^*\left|\sum_{i=-\infty}^{\infty} a_i\sum_{j=i+1}^{i+n} Y_{xj}^{(1)}\right|^2 \mathrm{d}x\\
&\leq C\sum_{n=1}^{\infty} nh(n)(1+g(n))\sum_{m=n}^{\infty}\int_{m^{1/p}}^{(m+1)^{1/p}} x^{-2}C_{\mathbb{V}}\left\{|Y_x^{(1)}|^2\right\}\mathrm{d}x\\
&\quad +C\sum_{n=1}^{\infty} h(n)\sum_{m=n}^{\infty}\int_{m^{1/p}}^{(m+1)^{1/p}} x^{-2}\left[\sum_{i=1}^{n}|\mathbb{E}(Y_{xj}^{(1)})|+|\mathbb{E}(-Y_{xj}^{(1)})|\right]^2\mathrm{d}x\\
&\leq C\sum_{n=1}^{\infty} nh(n)(1+g(n))\sum_{m=n}^{\infty} m^{-1/p-1}\int_0^{(m+1)^{2/p}}\mathbb{V}\left\{|Y|^2>y\right\}\mathrm{d}y\\
&\quad +C\sum_{n=1}^{\infty} h(n)\sum_{m=n}^{\infty} m^{-1/p-1}\left[\sum_{i=1}^{n}\mathbb{E}(|Y_{m^{1/p}j}^{(2)}|)\right]^2\\
&\leq C\sum_{m=1}^{\infty} m^{-1/p-1}\int_0^{(m+1)^{2/p}}\mathbb{V}\left\{|Y|^2>y\right\}\mathrm{d}y\sum_{n=1}^{m} n^{-1/p}l(n)(1+g(n))\\
&\quad +C\sum_{m=1}^{\infty} m^{-1/p-1}\left[\mathbb{E}(|Y_{m^{1/p}}^{(2)}|)\right]^2\sum_{n=1}^{m} n^{1-1/p}l(n)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{m=1}^{\infty} m^{-2/p+\delta} l(m) \int_0^{(m+1)^{2/p}} \mathbb{V}\left\{|Y|^2 > y\right\} \mathrm{d}y \\
&\quad + C\sum_{m=1}^{\infty} m^{-2/p+1} l(m) \left[\mathbb{E}(|Y^{(2)}_{m^{1/p}}|)\right]^2 \\
&\leq C\sum_{m=1}^{\infty} m^{-2/p+\delta} l(m) \sum_{\ell=1}^{m+1} \int_{(\ell-1)^{2/p}}^{(\ell)^{2/p}} \mathbb{V}\left\{|Y|^2 > y\right\} \mathrm{d}y \\
&\quad + C\sum_{m=1}^{\infty} m^{-2/p+1} l(m) \left[\mathbb{E}|Y| \left(1 - \tilde{g}_\mu\left(\frac{|Y|}{m^{1/p}}\right)\right)\right]^2 \\
&\leq C\sum_{\ell=1}^{\infty} \int_{(\ell-1)^{2/p}}^{(\ell)^{2/p}} \mathbb{V}\left\{|Y|^2 > y\right\} \mathrm{d}y \sum_{m=1 \bigvee (\ell-1)}^{\infty} m^{-2/p+\delta} l(m) \\
&\quad + C\sum_{m=1}^{\infty} m^{-2/p+1} l(m) \frac{\left(\mathbb{E}\left(|Y|^{p(1+\delta)} l(|Y|^p)\right)\right)^2}{\left(m^{(p(1+\delta)-1)/p} l(m)\right)^2} \\
&\leq C\sum_{\ell=2}^{\infty} \int_{(\ell-1)^{2/p}}^{(\ell)^{2/p}} \mathbb{V}\left\{|Y|^2 > y\right\} \ell^{1+\delta-2/p} l(\ell) \mathrm{d}y + C \\
&\quad + C\sum_{m=1}^{\infty} m^{-2\delta-1}/l(m) \left(C_{\mathbb{V}}\left\{|Y|^{p(1+\delta)} l(|Y|^p)\right\}\right)^2 \\
&\leq C\sum_{\ell=2}^{\infty} \int_{(\ell-1)^{2/p}}^{\ell^{2/p}} \mathbb{V}\left\{|Y|^2 > y\right\} y^{(1+\delta-2/p)p/2} l(y^{p/2}) \mathrm{d}y + C \\
&\leq C\int_1^{\infty} \mathbb{V}\left\{|Y|^p > y\right\} \mathrm{d}(l(y) y^{1+\delta}) + C \leq C C_{\mathbb{V}}\left\{|Y|^{p(1+\delta)} l(|Y|^p)\right\} < \infty. \quad (3.16)
\end{aligned}
$$

Therefore, combining (3.14)-(3.16) results in (3.13). This completes the proof. □

By Theorems 3.1, 3.2, we conclude that the following Corollary holds.

Corollary 3.1 *Under the same conditions of Theorem 3.1, for any $\epsilon > 0$, we have*

$$
\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{V}\left\{\left|\sum_{j=1}^{n} X_j\right| > \epsilon n^{\alpha}\right\} < \infty. \quad (3.17)
$$

Under the conditions of Theorem 3.2, for any $\epsilon > 0$, we have

$$
\sum_{n=1}^{\infty} n^{-1} l(n) \mathbb{V}\left\{\left|\sum_{j=1}^{n} X_j\right| > \epsilon n^{1/p}\right\} < \infty. \quad (3.18)
$$

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The authors declare that they have no competing interests.
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Both authors contributed equally and read and approved the final manuscript.

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