ARC-TRANSITIVE MAPS WITH COPRIME EULER CHARACTERISTIC AND EDGE NUMBER

C. H. LI AND LU YI LIU

Abstract. This is one of a series of papers which aim towards a classification of edge-transitive maps of which the Euler characteristic and the edge number are coprime. This one carries out the classification work for arc-transitive maps with nonsolvable automorphism groups, which illustrates how the edge number impacts on the Euler characteristic for maps. The classification is involved with the construction of some new and interesting arc-regular maps.

Key words: arc-regular, maps, Euler characteristic

1. Introduction

A map is a 2-cell embedding of a graph into a closed surface. Throughout the paper, we denote by $\mathcal{M} = (V, E, F)$ a map with vertex set V, edge set E, and face set F. The underlying graph (V, E) of $\mathcal{M}$ is written as Γ, and the supporting surface of $\mathcal{M}$ is denoted by $\mathcal{S}$. We always assume that Γ has no free edges and loops, but multi-edges are permitted. The Euler characteristic of $\mathcal{M}$ is defined to be that of its supporting surface, so

$$\chi(\mathcal{M}) = \chi(\mathcal{S}) = |V| - |E| + |F|.$$

Then a relation between the genus g of $\mathcal{S}$ and the Euler characteristic is given by Euler formula:

$$\chi(\mathcal{S}) = \begin{cases} 2-2g, & \text{if } \mathcal{S} \text{ is orientable;} \\ 2-g, & \text{if } \mathcal{S} \text{ is nonorientable.} \end{cases}$$

This paper explores relations between Euler characteristic $\chi(\mathcal{M})$ and the edge number $|E|$. It is easy to see that usually, the bigger $|E|$, the bigger $\chi(\mathcal{M})$. The result of this paper shows that $\chi(\mathcal{M})$ and $|E|$ should have a large common divisor in general by showing that if $\gcd(\chi(\mathcal{M}), |E|) = 1$ then $\mathcal{M}$ is very restricted.

An *arc* of $\mathcal{M}$ is an incident pair (v, e) of vertex v and edge e, and a *flag* of $\mathcal{M}$ is an incident triple (v, e, f) of vertex v, edge e and face f. Each edge $e = [v, e, v']$ corresponds to two arcs (v, e), (v', e) and four flags (v, e, f), (v, e, f'), (v', e, f) and (v', e, f') . The arc set and flag set of $\mathcal{M}$ are denoted by A and $\mathcal{F}$ respectively, so that $|\mathcal{F}| = 2|A| = 4|E|$.

An automorphism of $\mathcal{M}$ is a permutation of flags that preserve incident relations, and all automorphisms $\mathcal{M}$ form the automorphism group $\mathrm{Aut}(\mathcal{M})$. An automorphism of $\mathcal{M}$ fixing a flag must fix all the flags, and so is the identity. Thus $\mathrm{Aut}\mathcal{M}$ is semiregular on the flags of $\mathcal{M}$. If $\mathrm{Aut}\mathcal{M}$ is transitive on the flags of $\mathcal{M}$, then it is regular, and $\mathcal{M}$ is called a *regular map*. If $\mathrm{Aut}\mathcal{M}$ is transitive on the arc set of $\mathcal{M}$ and intransitive on the flag set, then $\mathrm{Aut}\mathcal{M}$ is regular on the arc set, called an

This work was partially supported by NSFC grant 11931005.

arc-regular map. For an arc-regular map $\mathcal{M}$, the number of arcs equals the order $|\mathrm{Aut}\mathcal{M}|$.

The problem of classifying symmetrical maps has been studied for specific prescribed Euler characteristic, see [7] for negative prime Euler characteristic, [3] for Euler characteristic $-3p$, and [4] for Euler characteristic being $-p^2$. In this paper, we study arc-transitive maps of Euler characteristic coprime to the number of edges of the map. In the following theorem, we classify such maps with nonsolvable automorphism groups.

Theorem 1.1. *Let $\mathcal{M} = (V, E, F)$ be an arc-transitive map such that $\gcd(\chi(\mathcal{M}), |E|) = 1$, and let $G = \mathrm{Aut}\mathcal{M}$ be nonsolvable. Then one of the following holds:*

(1) *$G = \mathrm{PSL}(2, 5)$, $\mathcal{M}$ is type 1 (flag regular), and $\{G_\alpha, G_e, G_f\} = \{\mathrm{D}_6, \mathrm{D}_4, \mathrm{D}_{10}\}$.*
(2) *G is nonsolvable, $\mathcal{M}$ is of type 2^*, and one of the following is true:*
  (i) *$G = \mathrm{PSL}(2, p)$, and $\{G_\alpha, G_f, G_{f'}\} = \{\mathrm{D}_{2p}, \mathrm{D}_{p+1}, \mathrm{D}_{p-1}\}$, where $p \equiv 1 \pmod 4$;*
  (ii) *$G = \mathrm{PGL}(2, p)$, and $\{G_\alpha, G_f, G_{f'}\} = \{\mathrm{D}_{2p}, \mathrm{D}_{2(p+1)}, \mathrm{D}_{2(p-1)}\}$;*
  (iii) *$G = (\mathrm{Z}_n \times \mathrm{PSL}(2, p)).2$, and $\{G_\alpha, G_f, G_{f'}\} = \{\mathrm{D}_{2np}, \mathrm{D}_{2(p+1)}, \mathrm{D}_{2(p-1)}\}$, where $n > 1$ and $p \equiv 3 \pmod 4$.*

Locally finite edge-transitive maps were classified into fourteen types by Graver and Watkins[9] according to combinations of stabilizers of vertices, edges, faces, and Petri walks. Among the fourteen types of edge transitive maps, four are arc-regular: type 2^*, 2^*ex, 2^p, and 2^pex. The maps satisfying Theorem 1.1 are of type 2^*.

For an integer n, let n_p be the p-part, which means that $n = n_pm$ such that n_p is a p-power and $\gcd(p, m) = 1$. Let G be a finite group. For a prime divisor p of $|G|$, denote by G_p a Sylow p-subgroup, and $G_{p'}$ a Hall p'-subgroup of G, where $p' = \pi(G) \setminus \{p\}$. Denote by $G^{(\infty)}$ the smallest normal subgroup of G such that $G/G^{(\infty)}$ is solvable. For a subgroup $H < G$, let $\mathbf{N}_G(H)$ and $\mathbf{C}_G(H)$ be the normalizer and the centralizer of H in G, respectively. By Z_n, we mean a cyclic group of order n, and D_{2n} is a dihedral group of order $2n$.

2. The edge number and the Euler characteristic

Let $\mathcal{M} = (V, E, F)$, and let $\mathcal{M}^* = (F, E, V)$ be the *dual map* of $\mathcal{M}$. Let $\Gamma = (V, E)$, and $\Gamma^* = (F, E)$ be the underlying graph of $\mathcal{M}$ and $\mathcal{M}^*$, respectively. For a vertex $\alpha \in V$, let $\Gamma(\alpha)$ be the neighborhood of α, namely, the set of vertices of Γ which are adjacent to α. Let $E(\alpha)$ be the set of edges incident with α. Then $|\Gamma(\alpha)| \leqslant |E(\alpha)|$, and the size $|E(\alpha)|$ is called the *valency* of the vertex α. The *face length* of a face f in a map $\mathcal{M}$ is the number of edges incident with f, which equals the vertex valency of f in the dual map $\mathcal{M}^*$. A map $\mathcal{M}$ is said to be of *constant valency* if all of its vertices have equal valency and of *constant face length* if all of the faces have equal face length.

For convenience, for a map $\mathcal{M} = (V, E, F)$, let $\chi = \chi(\mathcal{M}) = |V| - |E| + |F|$, the Euler characteristic of $\mathcal{M}$.

Lemma 2.1. *Assume that $\mathcal{M}$ has constant vertex valency k and constant face length ℓ, and $\gcd(\chi, |E|) = 1$. Then $|E|$ divides $k\ell$, $|V|$ divides 2ℓ and $|F|$ divides $2k$.*

Proof. Since each edge is incident with two vertices and each vertex is incident with k edges, we have that $|V|k = |E|2$, and $|V| = \frac{2}{k}|E|$. As each edge is incident with two faces and each face is incident with ℓ edges, we have that $|F|\ell = |E|2$, and $|F| = \frac{2}{\ell}|E|$. Thus $\chi = |V| - |E| + |F| = \frac{2}{k}|E| - |E| + \frac{2}{\ell}|E| = \frac{2\ell - k\ell + 2k}{k\ell}|E|$, and so

$$(k\ell)\chi = (2\ell - k\ell + 2k)|E|.$$

Since $\gcd(\chi, |E|) = 1$, it follows that E divides $k\ell$. As $|E| = \frac{k|V|}{2}$, we conclude that $|V|$ divides 2ℓ. Similarly, $|F|$ divides $2k$. □

This lemma shows that, if $\gcd(\chi, |E|) = 1$ then the edge number $|E|$ is small relative to the valency and the face length. The statements in the next lemma are well-known for maps.

Lemma 2.2. *Let $\mathcal{M} = (V, E, F)$, and let $\mathcal{F}$ be the set of flags of $\mathcal{M}$, and let $G \leqslant \mathrm{Aut}\mathcal{M}$. Then the following statements hold.*

(1) *G is semiregular on $\mathcal{F}$.*
(2) *$|\mathcal{F}| = 4|E|$, and $|G|$ divides $4|E|$.*
(3) *For a flag $(v, e, f) \in \mathcal{F}$, each of the stabilizers G_v and G_f is cyclic or dihedral, and $G_e = 1$, Z_2 or D_4.*

The next lemma characterizes Sylow subgroups of the automorphism group of a map whose Euler characteristic is coprime to the number of edges.

Lemma 2.3. *Let $\mathcal{M} = (V, E, F)$, let $\Omega = V \cup E \cup F$, and let $G \leqslant \mathrm{Aut}\mathcal{M}$. Assume that $\gcd(\chi(\mathcal{M}), |E|) = 1$. Then the following statements are true:*

(1) *$\gcd(\chi, |G|) \neq 1$ if and only if $|E|$ is odd;*
(2) *each Sylow subgroup of G is a subgroup of the stabilizer G_ω, where $\omega \in \Omega$;*
(3) *each Sylow subgroup of G is a cyclic or dihedral subgroup;*
(4) *$|G| = \mathsf{lcm}\{|G_\omega| \mid \omega \in \Omega\}$.*

Proof. Let $\chi = \chi(\mathcal{M})$. Since $|\mathcal{F}| = 4|E|$ and $\gcd(\chi, |E|) = 1$, we obtain that

$$\gcd(\chi, |\mathcal{F}|) = \gcd(\chi, 4|E|) = \gcd(\chi, 4).$$

Since G is semiregular on $\mathcal{F}$, we have that $|G|$ divides $|\mathcal{F}|$. Thus $\gcd(\chi, |G|)$ divides $\gcd(\chi, |\mathcal{F}|) = \gcd(\chi, 4|E|)$, and so $\gcd(\chi, |G|)$ divides 4. It follows that $\gcd(\chi, |G|) \neq 1$ if and only if $|E|$ is odd, as in part (1).

Let $V = V_1 \cup \cdots \cup V_r$, $E = E_1 \cup \cdots \cup E_s$ and $F = F_1 \cup \cdots \cup F_t$ such that each V_i, E_j and F_k is an orbit of G on V, E, F, respectively. Then $|V_i| = \frac{|G|}{|G_{v_i}|}$, $|E_j| = \frac{|G|}{|G_{e_j}|}$, and $|F_k| = \frac{|G|}{|G_{f_k}|}$, where $v_i \in V_i$ for $1 \leqslant i \leqslant r$, $e_j \in E_j$ for $1 \leqslant j \leqslant s$, and $f_k \in F_k$ for $1 \leqslant k \leqslant t$. Thus $|V| = |V_1| + \cdots + |V_r|$, $|E| = |E_1| + \cdots + |E_s|$, and $|F| = |F_1| + \cdots + |F_t|$, and by Euler-Poincare formula

$$\chi = \left(\frac{|G|}{|G_{v_1}|} + \cdots + \frac{|G|}{|G_{v_r}|}\right) - \left(\frac{|G|}{|G_{e_1}|} + \cdots + \frac{|G|}{|G_{e_s}|}\right) + \left(\frac{|G|}{|G_{f_1}|} + \cdots + \frac{|G|}{|G_{f_t}|}\right).$$

Let p be a prime divisor of $|G|$. Assume that p divides χ. Then $p = 2$, and $\gcd(\chi, |G|) \neq 1$. Thus $|E|$ is odd by part (1), and the 2-part $|G|_2$ divides 4 as $|G|$ divides $|\mathcal{F}| = 4|E|$. So a Sylow 2-subgroup of G is cyclic or dihedral.

Now assume that p does not divide χ. Then p does not divide $\frac{|G|}{|G_\omega|}$ for some element $\omega \in \Omega = V \cup E \cup F$. Thus a Sylow p-subgroup of G has order divides $|G_\omega|$, and is conjugate to a subgroup of G_ω, as in part (2). So a Sylow p-subgroup is cyclic or dihedral by Lemma 2.2, as in part (3).

Finally, since $|G_\omega|$ divides $|G|$ for any $\omega \in \Omega$, we have that $\mathsf{lcm}\{|G_\omega| : \omega \in \Omega\}$ divides $|G|$. Conversely, let $p_1, p_2, \dots, p_d$ be the prime divisors of $|G|$, and without loss of generality, assume that G_{ω_i} contains a Sylow p_i-subgroup G_{p_i} of G for $1 \leqslant i \leqslant d$. Then $|G| = |G_{p_1}||G_{p_2}| \dots |G_{p_d}|$ divides $\mathsf{lcm}\{|G_{\omega_i}| \mid 1 \leqslant i \leqslant d\}$. It follows that $|G| = \mathsf{lcm}\{|G_\omega| : \omega \in \Omega\}$, as in part (4). □

This lemma tells us that the automorphism group of a map whose Euler characteristic is coprime to the number of edges is so-called an *almost Sylow-cyclic* group, namely, all of its odd order Sylow subgroups are cyclic, and Sylow 2-subgroups has an index 2 cyclic subgroup. A characterization of non-solvable almost Sylow-cyclic groups in [11] and [13], and see [10] for solvable cases.

An explicit classification of almost Sylow cyclic groups with dihedral Sylow 2-subgroups is given in the following proposition.

Proposition 2.4. *Let G be a finite group of which each Sylow subgroup is cyclic or dihedral. Then one of the following statements holds:*

(i) $G = \mathrm{Z}_m{:}\mathrm{Z}_n$ *is metacyclic, where* $\gcd(m, n) = 1$ *or* 2*;*
(ii) $G = (\mathrm{Z}_m \times \mathrm{D}_{2^e}){:}\mathrm{Z}_n$*, where m is odd and $4 \nmid n$;*
(iii) $G = \mathrm{Z}_2^2.\mathrm{D}_{6m} \times \mathrm{Z}_n = (\mathrm{Z}_2^2 \times \mathrm{Z}_m).\mathrm{D}_6 \times \mathrm{Z}_n$*, where mn is odd;*
(iv) $G = (\mathrm{Z}_m{:}\mathrm{Z}_n) \times \mathrm{PSL}(2, p)$ *or* $((\mathrm{Z}_m{:}\mathrm{Z}_n) \times \mathrm{PSL}(2, p)).2$*, where $m, n, |\mathrm{PSL}(2, p)|$ are pairwise coprime.*

Proof. Assume first that G is solvable. If a Sylow 2-subgroup is cyclic, then G is metacyclic of the form $\mathrm{Z}_m{:}\mathrm{Z}_n$, as in part (i). Assume next that a Sylow 2-subgroup $G_2 = \mathrm{D}_{2^{e+1}}$ is dihedral with $e \geqslant 1$. Let F be the Fitting subgroup of G.

(1). Suppose that G/F is abelian. Then, G/F is cyclic since G is almost Sylow cyclic. Let π be the set of odd prime divisors of $|G/F|$. Let H be a Hall π-subgroup of G, and let $F_{\pi'}$ be a Hall π'-subgroup of F.

If $F_2 = G_2$, then $F_{\pi'} = G_2 \times F_{2'}$, and $G = (G_2 \times F_{2'}){:}H = (\mathrm{D}_{2^d} \times \mathrm{Z}_m){:}\mathrm{Z}_n$, with $d = e$, mn odd and $\gcd(m, n) = 1$, as in part (ii).

Suppose that $F_2 < G_2 = \mathrm{D}_{2^{e+1}}$. Then, as G/F is abelian, $|G/F|_2 = |G_2/F_2| = 2$, and $F_2 = \mathrm{Z}_{2^e}$ or D_{2^e}, so that $G_2 = F_2{:}\langle g\rangle$, where g is an involution. If $F_2 = \mathrm{Z}_{2^e}$, then $F_{\pi'} = \mathrm{Z}_m$, and $G = \mathrm{Z}_m{:}\mathrm{Z}_n$, where $\gcd(m, n) = 2$, as in part (i). If $F_2 = \mathrm{D}_{2^e}$, then $F_{\pi'} = \mathrm{Z}_m \times \mathrm{D}_{2^e}$, and $G = (\mathrm{Z}_m \times \mathrm{D}_{2^e}){:}\mathrm{Z}_n$, where the 2-part $n_2 = 2$, as in part (ii).

(2). Suppose that G/F is not abelian. Then $\mathrm{Out}(F_2)$ is nonabelian, and hence $F_2 = \mathrm{D}_4 \cong \mathrm{Z}_2^2$, $G_2 = F_2.2 = \mathrm{D}_8$, and $\mathrm{Out}(F_2) = \mathrm{D}_6$. All Sylow subgroups of G/F_2 are cyclic, and a Sylow 2-subgroup is of order 2. It follows that G/F_2 is metacyclic, and thus $G/F_2 = \mathrm{D}_{2n'} \times \mathrm{Z}_n$. Then, we obtain that

$$G = F_2.(G/F_2) = \mathrm{Z}_2^2.(\mathrm{D}_{2n'} \times \mathrm{Z}_n) = (\mathrm{Z}_2^2 \times \mathrm{Z}_m).\mathrm{D}_6 \times \mathrm{Z}_n,$$

as in part (iii).

Now, assume that G is non-solvable. Let R be the solvable radical of G, the largest solvable normal subgroup of G. Let $N = G^{(\infty)}$ be the solvable residual of G, the smallest normal subgroup N of G such that G/N is solvable. Since a Sylow 2-subgroup of G is dihedral, it follows from Gorenstien's result [8] that

N is a simple group, namely, $N = \mathrm{A}_7$ or $\mathrm{PSL}(2,q)$, where $q = p^f$ with p odd prime. As Sylow subgroups of odd orders are cyclic, we obtain $N = \mathrm{PSL}(2,p)$. Thus, $RN = R \times N \lhd G$, R is of odd order, and so $R = \mathrm{Z}_n{:}\mathrm{Z}_m$ by the previous outcomes of the proof. Therefore, we conclude that $G = (\mathrm{Z}_n{:}\mathrm{Z}_m) \times \mathrm{PSL}(2,p)$ or $((\mathrm{Z}_n{:}\mathrm{Z}_m) \times \mathrm{PSL}(2,p)).2$, as in part (iv). □

3. Automorphism groups of arc-transitive maps

Let $\mathcal{M} = (V, E, F)$, and let $G \leqslant \mathrm{Aut}\mathcal{M}$. Assume that G is transitive on the arc set of $\mathcal{M}$. Then either $\mathcal{M}$ is flag-regular, or $G = \mathrm{Aut}\mathcal{M}$ and $\mathcal{M}$ is arc-regular.

Lemma 3.1. *Let $\alpha, \beta \in V$ be adjacent by an edge e, and let $f, f' \in F$ be the two faces incident with (α, e, β). Then the following hold:*

(1) *$G_\alpha = \langle a \rangle$ or $\langle x, y \rangle$, where x, y are involutions, and $G = \langle a, z \rangle$ or $\langle x, y, z \rangle$ where z is an involution which interchanges α and β and fixes e;*
(2) *One of $\langle G_\alpha, G_\beta \rangle$ or $\langle G_f, G_{f'} \rangle$ is transitive on the edges of $\mathcal{M}$, and either G is flag-regular on $\mathcal{M}$, or one of $|G : \langle G_\alpha, G_\beta \rangle|$ and $|G : \langle G_f, G_{f'} \rangle|$ divide 2.*

By Lemma 2.3 (3), the group G is almost Sylow cyclic, and thus G satisfies Proposition 2.4.

To analyze the structure of G, we need some information regarding subgroups of $G = \mathrm{PSL}(2,p)$ and $\mathrm{PGL}(2,p)$. A list of $\mathrm{PGL}(2,p)$ subgroups and $\mathrm{PSL}(2,p)$ subgroups will be frequently referred to in forthcoming contexts, which is known and listed below, see [2].

Lemma 3.2. *Let $G = \mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$, where $p \geqslant 5$ is a prime. Write $G = \mathrm{PSL}(2,p).\mathrm{Z}_d$, where $d = 1$ or 2 according to $G = \mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$, respectively. Let C be a cyclic subgroup, D be a dihedral subgroup, and P a Sylow subgroup of G. Then either $\mathrm{Z}_p \cong P \cong C \lhd D \cong \mathrm{D}_{2p}$, or*

(1) *$C \leqslant \mathrm{Z}_{\frac{d(p+1)}{2}}$ or $\mathrm{Z}_{\frac{d(p-1)}{2}}$, and*
(2) *$D \leqslant \mathrm{D}_{d(p+1)}$ or $\mathrm{D}_{d(p-1)}$, and*
(3) *$P \leqslant \mathrm{D}_{d(p+1)}$ or $\mathrm{D}_{d(p-1)}$.*

Moreover, $\mathrm{D}_{d(p+1)}$ and $\mathrm{D}_{d(p-1)}$ are maximal subgroups of G.

We are ready to provide a more detailed description of automorphism groups of maps.

Lemma 3.3. *Let $\mathcal{M} = (V, E, F)$ with $\gcd(\chi(\mathcal{M}), |E|) = 1$, and $G \leqslant \mathrm{Aut}\mathcal{M}$ be nonsolvable. Assume that G is arc-transitive on $\mathcal{M}$. Then $G = \mathrm{PSL}(2,p)$, or $(\mathrm{Z}_m \times \mathrm{PSL}(2,p)){:}2$, where $\gcd(m, |\mathrm{PSL}(2,p)|) = 1$.*

Proof. By Proposition 2.4, the nonsolvable group $G = (\mathrm{Z}_m{:}\mathrm{Z}_n) \times \mathrm{PSL}(2,p)$ or $((\mathrm{Z}_m{:}\mathrm{Z}_n) \times \mathrm{PSL}(2,p)).2$, where $m, n, |\mathrm{PSL}(2,p)|$ are pairwise coprime.

The second statement of Lemma 3.1 shows that $\langle G_\omega, G_{\omega'} \rangle$ is an edge-transitive subgroup, where $\{\omega, \omega\} = \{\alpha, \beta\}$ or $\{f.f'\}$. Since G_ω and $G_{\omega'}$ are cyclic or dihedral, we discuss the two cases separately.

(i) Let G_ω be a dihedral group. Note that the index $[G : \langle G_\omega, G_{\omega'}\rangle] = 1$ or 2. It follows that involutions generate G.

Assume first that $G = (\mathrm{Z}_m{:}\mathrm{Z}_n) \times \mathrm{PSL}(2,p)$ with $\gcd(mn, |\mathrm{PSL}(2,p)| = 1$. Then a Sylow 2-subgroup of G is dihedral, which is contained in G_α or G_f for some $\alpha \in V$ and $f \in F$. Since involutions generate G, so $m = n = 1$. Then $G = \mathrm{PSL}(2,p)$.

Now assume that $G = ((\mathrm{Z}_m{:}\mathrm{Z}_n) \times \mathrm{PSL}(2,p)).2$, where $m, n, |\mathrm{PSL}(2,p)|$ are pairwise coprime. Let N be the solvable residual of G. Then $G/N = (\mathrm{Z}_m{:}\mathrm{Z}_n).2$ is a metacyclic group as all its Sylow subgroups are cyclic. Since G/N is generated by dihedral subgroups, we conclude that $G/N = \mathrm{D}_{2m}$, and $G = (\mathrm{Z}_m \times \mathrm{PSL}(2,p)){:}2$.

(ii) Let G_ω be a cyclic group.

Then, G_ω is transitive on $\Gamma(\omega)$. Hence, G is transitive on both faces and vertices. It follows that $|G| = \mathsf{lcm}\{|G_\alpha|, |G_f|, |G_e|\}$ by Lemma 2.3. Again by Lemma 2.3, there is a Sylow 2-subgroup of G which is contained in G_f. Hence, there holds the equation that $|G| = \mathsf{lcm}\{|G_\alpha|, |G_f|\}$. By the list of subgroup of $\mathrm{PSL}(2,p)$ and $\mathrm{PGL}(2,p)$, there do not exist candidates of cyclic subgroup G_α and dihedral subgroup G_f satisfying $|G| = \mathsf{lcm}\{|G_\alpha|, |G_f|\}$.

Above all, we conclude that G_ω is a dihedral subgroup of G, and the group G is as characterized in (i), which completes the proof. □

4. The face-transitive case

In this section, we classify maps $\mathcal{M}$ with $\gcd(\chi(\mathcal{M}), |\mathrm{Aut}\mathcal{M}|) = 1$ which are arc-transitive and face-transitive.

Proposition 4.1. *Let $\mathcal{M} = (V, E, F)$ with $\gcd(\chi(\mathcal{M}), |E|) = 1$, and let $G \leqslant \mathrm{Aut}\mathcal{M}$ be arc-transitive and face-transitive on $\mathcal{M}$. If G is nonsolvable, then $G = \mathrm{A}_5$, and $\{G_\alpha, G_f\} = \{\mathrm{D}_{10}, \mathrm{D}_6\}$, $G_e = \mathrm{D}_4$, and $\mathcal{M}$ is flag-regular on the projective plane with underlying graph being the Peterson graph or K_6.*

Proof. By Lemma 3.3, we have that $G = \mathrm{PSL}(2,p)$, or $(\mathrm{Z}_m \times \mathrm{PSL}(2,p)){:}2$, where $m, |\mathrm{PSL}(2,p)|$ are coprime.

Let (α, e, f) be a flag of $\mathcal{M}$. Without loss of generality, we may assume that p divides $|G_\alpha|$. Since $\gcd(p+1, p-1) = 2$ and $p - 1 > 2$, there exists an odd prime r which divides $p + \varepsilon$, where $\varepsilon = 1$ or -1. Since $p - \varepsilon > 2$, it follows that $\frac{p-\varepsilon}{2}$ divides $|G_e|$, and thus $|G_e| = 4$ and $p \leqslant 9$, so that $p = 5$ or 7. In particular, $\mathcal{M}$ is flag-regular, and a Sylow 2-subgroup of G is of order 4. If $p = 7$, then $N = \mathrm{PSL}(2,7)$, of which a Sylow 2-subgroup is of order 8, which is impossible. Therefore, we have that $p = 5$, and $N = \mathrm{PSL}(2,5)$, and

$$G = \mathrm{Z}_m \times \mathrm{PSL}(2,5),$$

where $\gcd(m, |\mathrm{PSL}(2,5)|) = 1$. Since $\mathcal{M}$ is flag-regular, G is generated by involutions by Lemma 3.3, and so does the quotient $G/N = \mathrm{Z}_m$. So $m = 1$, and $G = \mathrm{PSL}(2,5)$. Now $G_\alpha = \mathrm{D}_{10}$, $G_e = \mathrm{D}_4$ and $G_f = \mathrm{D}_6$. It follows that $\mathcal{M}$ is a map on the projective plane, with the underlying graph being the Peterson graph or K_6. This completes the proof of the lemma. □

5. The face-intransitive case

In this section, we consider the face-intransitive case. Let $\mathcal{M} = (V, E, F)$ be an arc-transitive map that is not flag-regular, and let $G = \mathrm{Aut}\mathcal{M}$. Then G is regular on the arcs, and $\mathcal{M}$ is an arc-regular map. Assume that $\gcd(\chi(\mathcal{M}), |E|) = 1$.

Lemma 5.1. *Let $\mathcal{M}$ be an arc-regular and face-intransitive map, and let (v, e, f), (v, e, f') be two flags. Then G is generated by three involutions x, y and z such that $\{\langle x, y\rangle, \langle x, z\rangle, \langle y, z\rangle\} = \{G_v, G_f, G_{f'}\}$. In particular, stabilizers of vertices, edges, and faces are all dihedral.*

Proof. Since G is arc-transitive on $\mathcal{M}$, the stabilizer G_v is transitive on the set $E(\alpha)$ of the edges incident with v. Since G is intransitive on the face set F, the pair of incident faces f, f' lie in different orbits of G_v. It follows that G_v is a dihedral group, and so $G_v = \langle x, y\rangle$, where x, y are involutions, such that x fixes one of the faces f, f', say f, and then y fixes the other f'.

Since G is regular on the arcs of $\mathcal{M}$, an involution z interchanges the paired arcs (v, e, v') and (v', e, v) exists. Thus $G_e \cong \mathrm{Z}_2$, and $G = \langle G_v, z\rangle = \langle x, y, z\rangle$ since $\Gamma = (V, E)$ is a connected graph.

Moreover, since G is intransitive on the face set F, it implies that z fixes both faces f and f'. Now $\langle x, z\rangle$ is transitive on all arcs that are incident with f, and as G is regular on the arcs of $\mathcal{M}$, we conclude that $G_f = \langle x, z\rangle$. Similarly, $G_{f'} = \langle y, z\rangle$. This completes the proof. $\square$

The triple of involutions (x, y, z) described in Lemma 5.1 is an arc-regular triple that defines the map $\mathcal{M}$.

5.1. **Stabilizers.**

Using the notation defined above, each Sylow subgroup G is cyclic or dihedral by Lemma 2.3 (3). Thus, G is one of the groups given in Proposition 2.4. Assume that G is a nonsolvable group. By Lemma 3.3 and Lemma 2.3 (4), we have that

$$\begin{aligned} &G = \mathrm{PSL}(2, p) \text{ or } (\mathrm{Z}_m \times \mathrm{PSL}(2, p)){:}\mathrm{Z}_2, \text{ where } \gcd(m, |\mathrm{PSL}(2, p)|) = 1;\\ &|G| = \mathsf{lcm}\{|G_v|, |G_f|, |G_{f'}|\}. \end{aligned}$$

We first establish a useful lemma for determining sets of stabilizers.

Lemma 5.2. *Let $L = \mathrm{PSL}(2, p)$ and $X = \mathrm{PGL}(2, p)$ where $p \geqslant 5$ and $p \equiv \epsilon \pmod 4$, with $\varepsilon = 1$ or -1. For any involutions $u, v \in X$, let $D = \langle u, v\rangle$.*

(1) *If $\langle u, v\rangle = \mathrm{D}_{2(p+\epsilon)}$, then one of u, v lies in L, and the other lies in $G \setminus L$.*
(2) *For $\langle u, v\rangle = \mathrm{D}_{2p}$, either $u, v \in L$ with $p \equiv 1$ $(\mathsf{mod}\ 4)$, or $u, v \notin L$ with $p \equiv 3$ $(\mathsf{mod}\ 4)$.*

Proof. Assume first that $D = \langle u, v\rangle = \mathrm{D}_{2(p+\epsilon)}$. Then $D \cap L = \mathrm{D}_{p+\varepsilon}$, and so one of u, v does not belong to L. Suppose that none of u, v lies in L. Then $D = (D \cap L){:}\langle u\rangle = (D \cap L){:}\langle v\rangle$. Let $\overline{D} = D/D \cap L$, and let $g \mapsto \overline{g}$ be the homomorphism from D to $D/D \cap L$. Thus $\overline{u} = \overline{v}$, and so $uv \in D \cap L$, which is not possible since $D \cap L = \mathrm{D}_{p+\varepsilon}$ does not have an element of order $p + \varepsilon$. Thus one of u, v lies in L, and the other lies in $G \setminus L$.

Now assume that $D = \mathrm{D}_{2p}$. There exists a maximal subgroup $M = \langle a\rangle{:}\langle b\rangle = \mathrm{Z}_p{:}\mathrm{Z}_{p-1}$ of X that contains D. It follows that $u = a^ib^{\frac{p-1}{2}}$ and $v = a^jb^{\frac{p-1}{2}}$, where $i \neq j \in \{1, ..., p\}$. The intersection $M \cap L = \mathrm{Z}_p{:}\mathrm{Z}_{\frac{p-1}{2}}$. If $p \equiv 3 \pmod 4$, then $M \cap L$ is of odd order, and hence we have that $u, v \notin L$. If $p \equiv 1 \pmod 4$, then $b^{\frac{p-1}{2}} \in L$, and therefore, we obtain that $u, v \in L$. □

For convenience, denote $\{\omega_1, \omega_2, \omega_3\}$ by $\{v, f, f'\}$. The next lemma determines the stabilizer triple $\{G_v, G_f, G_{f'}\}$.

Lemma 5.3. *One of the following holds:*

(1) $G = \mathrm{PSL}(2, p)$, *and* $\{G_v, G_f, G_{f'}\} = \{\mathrm{D}_{2p}, \mathrm{D}_{p+1}, \mathrm{D}_{p-1}\}$ *with* $p \equiv 1 \pmod 4$*;*
(2) $G = \mathrm{PGL}(2, p)$, *and either* $\{G_v, G_f, G_{f'}\} = \{\mathrm{D}_{2p}, \mathrm{D}_{2(p+1)}, \mathrm{D}_{2(p-1)}\}$*;*
(3) $G = (\mathrm{Z}_m \times \mathrm{PSL}(2, p)){:}\mathrm{Z}_2$, *and* $\{G_v, G_f, G_{f'}\} = \{\mathrm{D}_{2mp}, \mathrm{D}_{2(p+1)}, \mathrm{D}_{2(p-1)}\}$, *where* $m \neq 1$, *and* $p \equiv 3 \pmod 4$.

Proof. (1). First, assume that $G = \mathrm{PSL}(2, p)$. Then a maximal dihedral subgroup of G is conjugate to D_{2p}, D_{p+1} or D_{p-1}. Recall that each Sylow subgroup of G is a subgroup of G_v, G_f or $G_{f'}$ by Lemma 2.3. Then p divides $|G_{\omega_1}|$ for some $\omega_1 \in \{v, f, f'\}$, and so

$$G_{\omega_1} = \mathrm{D}_{2p} < \mathrm{Z}_p{:}\mathrm{Z}_{\frac{p-1}{2}}.$$

We thus have that $\frac{p-1}{2}$ is even, and $p \equiv 1 \pmod 4$. Then a Sylow 2-subgroup of G is contained in the stabilizer G_{ω_2} for $\omega_2 \in \{v, f, f'\} \setminus \{\omega_1\}$, and hence $G_{\omega_2} \leqslant \mathrm{D}_{p-1}$. Suppose that $G_{\omega_2} < \mathrm{D}_{p-1}$. Since G_{ω_2} contains a Sylow 2-subgroup of D_{p-1}, there exists an odd prime r of $p-1$ such that $|G_{\omega_2}|_r < |p-1|_r$. It follows that a Sylow r-subgroup of G is contained in G_{ω_3} for $\omega_3 \in \{v, f, f'\} \setminus \{\omega_1, \omega_2\}$. By Lemma 3.2, we have that $\gcd(|G_{\omega_3}|, p+1) \leqslant 2$. This is not possible since $|G| = \mathsf{lcm}\{|G_v|, |G_f|, |G_{f'}|\} = \mathsf{lcm}\{|G_{\omega_1}|, |G_{\omega_2}|, |G_{\omega_3}|\}$. Thus

$$G_{\omega_2} = \mathrm{D}_{p-1}.$$

Since $\frac{1}{2}p(p-1)(p+1) = |G| = \mathsf{lcm}(|G_v|, |G_f|, |G_{f'}|) = \mathsf{lcm}\{|G_{\omega_1}|, |G_{\omega_2}|, |G_{\omega_3}|\}$ as mentioned above, it follows that $|G_{\omega_3}|$ is divisible by $\frac{p+1}{2}$. As G_{ω_3} is dihedral by Lemma 5.1, we conclude that

$$G_{\omega_3} = \mathrm{D}_{p+1},$$

so that $\{G_v, G_f, G_{f'}\} = \{|G_{\omega_1}|, |G_{\omega_2}|, |G_{\omega_3}|\} = \{\mathrm{D}_{2p}, \mathrm{D}_{p+1}, \mathrm{D}_{p-1}\}$, as in part (1).

(2). Assume that $G = \mathrm{PGL}(2, p)$. Let D be a maximal dihedral subgroup of G. Then D is conjugate to D_{2p}, $\mathrm{D}_{2(p+1)}$ or $\mathrm{D}_{2(p-1)}$. Arguing similarly to Case (1), we have that

$$G_{\omega_1} = \mathrm{D}_{2p} < \mathrm{Z}_p{:}\mathrm{Z}_{p-1},$$

for some element $\omega_1 \in \{v, f, f'\}$.

Let $p \equiv \varepsilon \pmod 4$, where $\varepsilon = 1$ or -1. Then a Sylow 2-subgroup G_2 of G is a subgroup of $\mathrm{D}_{2(p-\varepsilon)}$. Since G_2 is a subgroup of the stabilizer G_{ω_2} for some $\omega_2 \in \{v, f, f'\} \setminus \{\omega_1\}$. Arguing as in Case (1) shows that

$$G_{\omega_2} = \mathrm{D}_{2(p-\varepsilon)}.$$

Finally, as $p(p-1)(p+1) = |G| = \mathsf{lcm}(|G_v|, |G_f|, |G_{f'}|) = \mathsf{lcm}\{2p, 2(p-\varepsilon), |G_{\omega_3}|\}$, we conclude that $|G_{\omega_3}|$ is divisible by $\frac{p+\varepsilon}{2}$. Since G_{ω_3} is a dihedral group by

Lemma 5.1, we obtain that $G_{\omega_3} = \mathrm{D}_{p+\varepsilon}$ or $\mathrm{D}_{2(p+\varepsilon)}$. Suppose that $G_{\omega_3} = \mathrm{D}_{p+\varepsilon}$. Then $G_{\omega_3} < L$. If $p \equiv 1 \pmod 4$, then $G_{\omega_1} < L$, and so $G = \langle G_{\omega_1}, G_{\omega_3}\rangle \leqslant L$, which is a contradiction. Thus $p \equiv 3 \pmod 4$, and $G_{\omega_3} = \mathrm{D}_{p-1} < L = \mathrm{PSL}(2,p)$. In this case, $G_{\omega_1} \cap L = \mathrm{Z}_p$. However, there exist involutions $u, v, w \in G$ such that $G_{\omega_1} = \langle u, v\rangle$ and $G_{\omega_3} = \langle v, w\rangle$. This is not possible. Thus $G_{\omega_3} = \mathrm{D}_{2(p-1)}$, as listed in part (2).

(3). Assume that $G = (\mathrm{Z}_m \times \mathrm{PSL}(2,p))$:$\mathrm{Z}_2$. Let $R = \langle c\rangle = \mathrm{Z}_m$, and let $\overline{G} = G/R$. Then $\overline{G} = \mathrm{PGL}(2,p)$, and by part (2),

$$\{\overline{G}_\alpha, \overline{G}_f, \overline{G}_{f'}\} = \{\mathrm{D}_{2p}, \mathrm{D}_{2(p+1)}, \mathrm{D}_{2(p-1)}\}.$$

Let $G_{\omega_1} = \langle u, v\rangle$, $G_{\omega_2} = \langle u, w\rangle$ and $G_{\omega_3} = \langle v, w\rangle$, where u, v, w are involutions of G. Without loss of generality, we assume that $G_{\omega_1} = \langle u, v\rangle$ is of order divisible by p. Then $G_{\omega_1} = \langle u, v\rangle = \mathrm{D}_{2m'p}$ where $m'|m$.

Suppose that $p \equiv 1 \pmod 4$. It follows from Lemma 5.2 that $u, v \in L = \mathrm{PSL}(2,p)$, and $m' = 1$, namely, $G_{\omega_1} = \langle u, v\rangle = \mathrm{D}_{2p}$. Since $|G| = \mathsf{lcm}\{|G_{\omega_1}|, |G_{\omega_2}|, |G_{\omega_3}|\}$, we may assume, without loss of generality, that $G_{\omega_2} \cap \langle c\rangle = \langle c'\rangle \neq 1$, and so $G_{\omega_2} = \mathrm{D}_{2m'(p+\varepsilon)}$ with $|c'| = m'$ and $\varepsilon = 1$ or -1. However, $G_{\omega_2} = \langle u, w\rangle$ and u centralizes c, which is a contradiction.

We therefore conclude that $p \equiv 3 \pmod 4$. Then $, u, v \notin L$, $G_{\omega_2} = \langle u, w\rangle$ is of even order divisible by $p+\varepsilon$ and $G_{\omega_3} = \langle v, w\rangle$ is of even order divisible by $p-\varepsilon$, where $\varepsilon = 1$ or -1. By Lemma 5.2 that $u, v \in G \setminus (\langle c\rangle \times L)$, and $w \in L$. Thus w centralizes c. It follows that $|c|$ is coprime to both $|G_{\omega_2}|$ and $|G_{\omega_3}|$, and so

$$\{G_{\omega_2}, G_{\omega_3}\} = \{\mathrm{D}_{2(p+1)}, \mathrm{D}_{2(p-1)}\}.$$

Since $|G| = \mathsf{lcm}\{|G_{\omega_1}|, |G_{\omega_2}|, |G_{\omega_3}|\}$, we conclude that $G_{\omega_1} = \mathrm{D}_{2mp}$. □

5.2. **Constructions of arc-regular triples.**

In this subsection, we shall determine arc-regular triples (x, y, z) for each of the candidates $(G, G_v, G_f, G_{f'})$ described in Lemma 5.3. Let

$$X = \mathrm{PGL}(2,p), \text{ where } p \geqslant 5 \text{ is a prime.}$$

The conclusion in the following lemma is well-known.

Lemma 5.4. *For any involution $z \in X$, there exist involutions $u, v \in X$ such that $\langle z, u\rangle \cong \mathrm{D}_{2(p+1)}$ and $\langle z, v\rangle = \mathrm{D}_{2(p-1)}$.*

Our constructions of arc-regular triples depend on analyzing the action of X on the projective space of $\mathbb{F}_p^2$.

Let $\mathcal{P}$ be the set of projective points of the vector space $\mathbb{F}_p^2$, and we denote $\{\delta_0, ..., \delta_p\}$ by $\mathcal{P}$ and write $(\sigma, \tau) = (\delta_0, \delta_1)$, for convenience. Then X is a sharply 3-transitive permutation group on $\mathcal{P}$, and

- $X_\sigma = \mathrm{Z}_p$:Z_{p-1} is sharply 2-transitive on $\{\delta_1, ..., \delta_p\}$, and
- $X_{\sigma\tau} := X_\sigma \cap X_\tau = \mathrm{Z}_{p-1}$ is sharply transitive on $\{\delta_2, ..., \delta_p\}$.
- Each cyclic subgroup of X of order $p+1$ is sharply transitive on $\mathcal{P}$.

Lemma 5.5. *Let $H \cong \mathrm{Z}_{p+1}$ be a subgroup of X. For the involution $z \in H$, there exist $\delta, \delta' \in \{\delta_2, \ldots, \delta_p\}$ such that X_δ contains u and $X_{\delta'}$ contains v satisfy that $\langle z, u\rangle \cong \mathrm{D}_{2(p+1)}$ and $\langle z, v\rangle \cong \mathrm{D}_{2(p-1)}$.*

Proof. By Lemma 5.2, the involutions u, v appeared in Lemma 5.4 fix some points $\delta, \delta' \in \mathcal{P} \setminus \{\sigma, \tau\}$, respectively. The proof then follows. $\square$

Further to Lemma 5.5, we need to find elements $u, v \in X$ such that $\langle u, v\rangle = \mathrm{D}_{2p}$. For an involution $z \in X_{\sigma\tau}$ and $2 \leqslant i \leqslant p$, let

$$S_i = \{w \in X_{\delta_i} \mid \langle z, w\rangle \cong \mathrm{D}_{2(p+1)}\}, \text{ and } T_j = \{w \in X_{\delta_j} \mid \langle z, w\rangle \cong \mathrm{D}_{2(p-1)}\}.$$

Then Lemma 5.5 tells us that $S_i \neq \emptyset$ and $T_j \neq \emptyset$ for some $i, j \in \{2, \ldots, p\}$. The next lemma shows that we may take $i = j$.

Lemma 5.6. *For each $i \in \{2, \ldots, p\}$, we have that $S_i \neq \emptyset$ and $T_i \neq \emptyset$.*

Proof. Since H centralizes z and transitive on $\mathcal{P}$, Lemma 5.5 implies that $S_i \neq \emptyset$ and $T_i \neq \emptyset$ for all $i \in \mathcal{P}$. $\square$

With the preparations, we may now state our constructions. The first construction is for $\mathrm{PGL}(2, p)$.

Construction 5.7. Let $G = \mathrm{PGL}(2, p)$, and

(1) let z be the unique involution of a cyclic subgroup H of order $p + 1$,
(2) let $x_i, y_i \in G_{\delta_i}$ be involutions such that $\langle z, x_i\rangle \cong \mathrm{D}_{2(p+1)}$ and $\langle z, y_i\rangle \cong \mathrm{D}_{2(p-1)}$, where $i \in \mathcal{P}$.

Lemma 5.8. *Let (x_i, y_i, z) be a triple of involutions produced in Construction* 5.7. *Then (x_i, y_i, z) is an arc-regular triple for $G = \mathrm{PGL}(2, p)$ such that $\langle x_i, y_i\rangle \cong \mathrm{D}_{2p}$, and $\langle z, x_i\rangle \cong \mathrm{D}_{2(p+1)}$ and $\langle z, y_i\rangle \cong \mathrm{D}_{2(p-1)}$.*

Proof. Since x_i, y_i are two involutions of G_{δ_i}, the subgroup $\langle x_i, y_i\rangle$ is a dihedral subgroup of $G_{\delta_i} \cong \mathrm{Z}_p{:}\mathrm{Z}_{p-1}$, so that $\langle x_i, y_i\rangle \cong \mathrm{D}_{2p}$. The other statements then follow from the construction. $\square$

Similarly, we can construct arc-regular triples for the simple group $\mathrm{PSL}(2, p)$.

Construction 5.9. Let $G = \mathrm{PSL}(2, p)$, with $p \equiv 1 \pmod 4$, and

(1) let z be the unique involution of $G_{\sigma\tau} = \mathrm{Z}_{\frac{p-1}{2}}$,
(2) let x_i, y_i be two involutions of G_{δ_i} such that $\langle z, x_i\rangle \cong \mathrm{D}_{p+1}$ and $\langle z, y_i\rangle \cong \mathrm{D}_{p-1}$, where $i \in \{2, \ldots, p\}$.

Lemma 5.10. *Let (x_i, y_i, z) be a triple of involutions produced in Construction* 5.9. *Then (x_i, y_i, z) is an arc-regular triple for $G = \mathrm{PSL}(2, p)$ such that $\langle x_i, y_i\rangle \cong \mathrm{D}_{2p}$, and $\langle z, x_i\rangle \cong \mathrm{D}_{p+1}$ and $\langle z, y_i\rangle \cong \mathrm{D}_{p-1}$.*

Proof. The involutions x_i, y_i of G_{δ_i} generate a dihedral subgroup $\langle x_i, y_i\rangle$ of $G_{\delta_i} \cong \mathrm{Z}_p{:}\mathrm{Z}_{\frac{p-1}{2}}$. It follows that $\langle x_i, y_i\rangle \cong \mathrm{D}_{2p}$. The other statements then follow from Construction 5.10. $\square$

Finally, we construct arc-regular triples for $\mathrm{Z}_m{:}\mathrm{PGL}(2,p)$ for certain primes p.

Construction 5.11. Let $p \equiv 3 \pmod 4$ be a prime, and let $L = \mathrm{PSL}(2,p)$, and $X = L{:}\langle w\rangle = \mathrm{PGL}(2,p)$. Let $G = \langle c\rangle{:}X = (\langle c\rangle \times L){:}\langle w\rangle$ where $|c| = m$ and $c^w = c^{-1}$. Furthermore,

(1) fix an involution $z \in L = \mathrm{PSL}(2,p)$;
(2) let $u \in X_\delta$ and $u' \in X_{\delta'}$ such that $\langle z,u\rangle \cong \mathrm{D}_{2(p+1)}$ and $\langle z,u'\rangle \cong \mathrm{D}_{2(p-1)}$;
(3) let $g \in \mathbf{C}_X(z) \cong \mathrm{D}_{2(p+1)}$ be such that $(\delta')^g = \delta$, so that $v := (u')^g \in X_\delta$;
(4) let $x = cu$, and $y = v$.

Lemma 5.12. *Let $G = \langle c\rangle{:}X$, and (x,y,z) a triple as constructed in Construction* 5.11. *Then $\langle x,y\rangle = \mathrm{D}_{2mp}$, $\langle x,z\rangle = \mathrm{D}_{2(p+1)}$ and $\langle y,z\rangle = \mathrm{D}_{2(p-1)}$.*

Proof. By Construction 5.11, the triple of involutions $u,v,z \in X$ are such that $\langle u,v\rangle = \mathrm{D}_{2p}$, $\langle u,z\rangle = \mathrm{D}_{2(p+1)}$ and $\langle v,z\rangle = \mathrm{D}_{2(p-1)}$. Since $p \equiv 3 \pmod 4$ and $z \in L$, we have that $u,v \in X \setminus L$. Thus $G = (\langle c\rangle \times L){:}\langle u\rangle = (\langle c\rangle \times L){:}\langle v\rangle$. Let $x = cu$, and $y = v$. Then we have that

$$\langle y,z\rangle = \langle v,z\rangle = \mathrm{D}_{2(p-1)}.$$

As $uv \in L$, we have that $|uv| = p$, and

$$\langle x,y\rangle = \langle cu,v\rangle = \mathrm{D}_{2mp}.$$

Further, $\langle x,z\rangle = \langle cu,z\rangle$ is a dihedral group. Thus z inverts $xz = cuz$. Since z centralizes c, it follows that $\langle cuz\rangle \cap \langle c\rangle = 1$. So $\langle cu,z\rangle \cap \langle c\rangle = 1$, and

$$\langle x,z\rangle = \langle cu,z\rangle \cong \mathrm{D}_{2(p+1)}.$$

This proves the lemma. □

6. Proof of Theorem 1.1

We summarize the arguments for the proof of Theorem 1.1.

Let $\mathcal{M} = (V,E,F)$ be an arc-transitive map, and let $G = \mathrm{Aut}\mathcal{M}$. Assume that $\gcd(\chi(\mathcal{M}),|E|) = 1$, and that G is a nonsolvable group. Then each Sylow subgroup of G is cyclic or dihedral by Lemma 2.3 (3). Thus, by Lemma 3.3,

$$G = \mathrm{PSL}(2,p) \text{ or } (\mathrm{Z}_m \times \mathrm{PSL}(2,p){:}\mathrm{Z}_2.$$

If $\mathcal{M}$ is flag-regular, then $G = \mathrm{PSL}(2,5)$, and $\mathcal{M}$ is a map on the projective plane with the underlying graph being the Peterson graph or K_6, which are dual to each other.

Assume that $\mathcal{M}$ is not flag-regular. Then, $\mathcal{M}$ is an arc-regular map.

First, if $G = \mathrm{PSL}(2,p)$, then $p \equiv 1 \pmod 4$, and $\{G_v,G_f,G_{f'}\} = \{\mathrm{D}_{2p},\mathrm{D}_{p+1},\mathrm{D}_{p-1}\}$ by Lemma 5.3 (1), and the existence of arc-regular triples are confirmed by Construction 5.9 and Lemma 5.10.

Next, for $G = \mathrm{PGL}(2,p)$, we have that $\{G_v,G_f,G_{f'}\} = \{\mathrm{D}_{2p},\mathrm{D}_{2(p+1)},\mathrm{D}_{2(p-1)}\}$ by Lemma 5.3 (2), and the existence of arc-regular triples are confirmed by Construction 5.7 and Lemma 5.8.

Finally, for the case where $G = \mathrm{Z}_m{:}\mathrm{PGL}(2,p)$ with $m \neq 1$, we have that the prime $p \equiv 3 \pmod 4$, and $\{G_v, G_f, G_{f'}\} = \{\mathrm{D}_{2mp}, \mathrm{D}_{2(p+1)}, \mathrm{D}_{2(p-1)}\}$ by Lemma 5.3 (3). The existence of arc-regular triples are justified by Construction 5.11 and Lemma 5.12.

This completes the proof of Theorem 1.1 □

References

[1] H. R. Brahana, Regular Maps and Their Groups. *Amer. J. Math.*, **49** (1927), 268—284.

[2] J. N. Bray, D. F. Holt, C. M. Roney-Dougal, The maximal subgroups of the low-dimensional finite classical groups. *London Mathematical Society Lecture Note Series*, **407** (2013), xiv+438.

[3] M. Conder, R. Nedela, J.Širáň, Classification of regular maps of Euler characteristic $-3p$. *J. Combin. Theory Ser. B*, **102** (2012), 967–981.

[4] M. Conder, P. Potočnik, J. Širáň, Regular maps with almost Sylow-cyclic automorphism groups, and classification of regular maps with Euler characteristic $-p^2$. *J. Algebra*, **324** (2010), 2620–2635.

[5] M. Conder, Regular maps and hypermaps of Euler characteristic −1 to −200. *J. Combin. Theory Ser. B*, **99** (2009), 455-459.

[6] A. B. d'Azevedo, D.A. Catalano, J. Širáň, Bi-rotary Maps of Negative Prime Characteristic. *Ann. Comb.* **23** (2019), 27—50.

[7] A. B. d'Azevedo, R. Nedela, J. Širáň, Classification of regular maps of negative prime Euler characteristic. *Trans. Amer. Math. Soc.*, **357** (2004), 4175–4190.

[8] D. Gorenstein, J. H. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups. I. *J. Algebra*, **2** (1965), 218–270.

[9] J. E. Graver, M. E. Watkins, Locally finite, planar, edge-transitive graphs. *Mem. Amer. Math. Soc*, **126** (1997), vi+75.

[10] A. W. Joseph, Spaces of constant curvature. *AMS Chelsea*, (2011), xviii+424.

[11] M. Suzuki, On finite groups with cyclic Sylow subgroups for all odd primes. *Amer. J. Math.*, **77** (1955), 657–691.

[12] J. Širáň, How symmetric can maps on surfaces be? *London Math. Soc. Lecture Note Ser*, **409** (2013), 161–238

[13] W. J. Wong, On finite groups with semi-dihedral Sylow 2-subgroups. *J. Algebra*, **4** (1966), 52–63.

SUSTech International Center for Mathematics, Department of Mathematics, Southern University of Science and Technology, Shenzhen, Guangdong 518055, P. R. China

Email address: `lich@sustech.edu.cn`

Department of Mathematics, Southern University of Science and Technology, Shenzhen, Guangdong 518055, P. R. China

Email address: `12031108@mail.sustech.edu.cn`