Improvements to the theoretical estimates of the Schwarz preconditioner with Δ-GenEO coarse space for the indefinite Helmholtz problem

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Abstract

The purpose of this work is to improve the estimates for the Δ-GenEO method from the paper [2] when applied to the indefinite Helmholtz equation. We derive k-dependent estimates of quantities of interest ensuring the robustness of the method.

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1 Introduction

In this work we focus on the indefinite version of Helmholtz boundary value problems with highly variable coefficients defined on $\Omega \subset \mathbb{R}^d$, given by:

$$\begin{aligned} -\operatorname{div}(A\nabla u) - k^2 u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

We use the same setting as in [2], namely we work on a computational domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) which is supposed to be polygonal or Lipschitz polyhedral. The matrix A is positive definite, and we assume that (1.1) has a unique weak solution $u \in H^1(\Omega)$ for all $f \in L^2(\Omega)$.

The global domain Ω is covered by a set of overlapping subdomains Ω_i, $i = 1, \dots, N$, and the classical one-level additive Schwarz preconditioner is built from partial solutions on each subdomain. Since this preconditioner is, in general, not scalable as the number of subdomains grows, an additional global coarse solve is usually added to enhance scalability, as well as robustness with respect to coefficient heterogeneity or the increase in the wavenumber. As in [2] we use local generalised eigenvalue problems that are obtained by a shift of the left-hand side of (1.1). After discretisation by finite elements, the linear systems arising from (1.1) are symmetric but indefinite. For this reason, we use GMRES as the iterative solver, and the convergence analysis relies on the 'Elman theory' [3], which requires an upper bound for the norm of the preconditioned matrix and a lower bound on the distance of its field of values from the origin. As a result, the number of GMRES iterations to achieve a given error tolerance is a function of these two bounds.

The main results of this preprint. Our main theoretical result provides rigorous and k-explicit upper bounds on the coarse mesh diameter H and on the 'eigenvalue tolerance' τ

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(for the local generalised eigenvalue problems) which ensure that GMRES enjoys robust and mesh-independent convergence when applied to the preconditioned problem. This rigorous explicit bound for (1.1) follows the methodology from [2] and provides an improvement of the k-dependent estimates. As a reminder, GenEO coarse spaces are usually based on the m_i (dominant) eigenfunctions corresponding to the smallest eigenvalues $\lambda_1^i \leq \lambda_2^i \leq \cdots \leq \lambda_{m_i}^i$ of the generalised eigenvalue problem on the subdomain Ω_i. To obtain a robust rate of convergence for GMRES that depends only on Λ (the maximum number of times any point is overlapped by the subdomains Ω_i), we need some conditions on H and the eigenvalue tolerance $\tau := \min_{i=1}^N \lambda_{m_i+1}^i$, where N is the number of subdomains. In the estimates $C_{\mathsf{stab}} > 0$ the stability constant for the problem (1.1) also appears. The hidden constants usually depend only on Λ.

More explicitly, in [2, Theorem 4.1] it is proved that a robust rate is achieved if

$$H \lesssim k^{-2} \qquad \text{and} \qquad (1 + C_{\mathsf{stab}})^2\, k^8 \lesssim \tau. \tag{1.2}$$

In the current paper, we show that the bounds (1.2) can be improved:

$$H \lesssim k^{-1} \qquad \text{and} \qquad (1 + C_{\mathsf{stab}})^2 k^4 \lesssim \tau. \tag{1.3}$$

Although, this is a first improvement, these bounds remain quite pessimistic, showing a strong dependence on the wavenumber k.

2 Useful Background

2.1 Problem formulation and discretisation

The weak formulation of (1.1) is to find $u \in H_0^1(\Omega)$ such that

$$b(u, v) = (f, v) \qquad \text{for all } v \in H_0^1(\Omega), \tag{2.1}$$

where $f \in L^2(\Omega)$ and $b(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is defined as

$$b(u, v) = \int_\Omega (A\nabla u \cdot \nabla v - k^2 uv)\, \mathrm{d}x.$$

We shall be making use of the positive definite bilinear form $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$.

$$a(u, v) = \int_\Omega A\nabla u \cdot \nabla v dx,$$

If a and b are defined on a subdomain Ω' of Ω we employ the notations $a_{\Omega'}$ and $b_{\Omega'}$. The following weak regularity assumptions are going to be used throughout this work.

Assumption 2.1. *The coefficient A in problem (1.1) satisfies the following conditions.*

(i) *$A : \Omega \to \mathbb{R}^{d\times d}$ is symmetric with $0 < a_{\mathsf{min}} \leq a_{\mathsf{max}}$ s.t.*

$$a_{\mathsf{min}}|\boldsymbol{\xi}|^2 \leq A(x)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leq a_{\mathsf{max}}|\boldsymbol{\xi}|^2 \qquad \textit{for all } x \in \Omega, \boldsymbol{\xi} \in \mathbb{R}^d. \tag{2.2}$$

(ii) *Without loss of generality, $a_{\mathsf{min}} = 1$ and the diameter, D_Ω, of the domain Ω is such that $D_\Omega \leq 1$. If this is not the case, then the problem can be scaled accordingly.*

Notation 2.2. *For any subdomain $\Omega' \subset \Omega$, we use $(\cdot,\cdot)_{\Omega'}$ to denote the $L^2(\Omega')$ inner product, with the norm denoted by $\|\cdot\|_{\Omega'}$. When $\Omega' = \Omega$, the inner product is written as $(\cdot,\cdot)$ with norm $\|\cdot\|$. The norm induced by the positive bilinear form a is denoted by $\|u\|_{a,\Omega'} = \sqrt{a_{\Omega'}(u, u)}$. When $\Omega' = \Omega$ we abandon the subscript Ω. We see that $b(u, v) = a(u, v) - k^2(u, v)$.*

The bilinear form $a(\cdot,\cdot)$ is positive definite (SPD), whereas $b(\cdot,\cdot)$ is symmetric, but in general indefinite. In all what follows, solvability of (2.1) is assumed.

Assumption 2.3. *For any $f \in L^2(\Omega)$, it is assumed that the problem 2.1 has a unique solution $u \in H_0^1(\Omega)$ and there exists a constant $C_{\mathsf{stab}} > 0$ such that*

$$\|u\|_a \leq C_{\mathsf{stab}}\|f\| \qquad \textit{for all } f \in L^2(\Omega). \tag{2.3}$$

Let $\mathcal{T}_h$ be any shape regular triangular mesh over the domain, Ω. For the purpose of this work, 2 or 3-dimensional simplices are being considered, but this could easily be applied to d-dimensional simplices, where the maximum diameter is h. Let $V^h \subset H_0^1(\Omega)$ be any conforming finite element space. The Galerkin approximation of (2.1) is to find $u_h \in V^h$ such that

$$b(u_h, v) = (f, v) \qquad \text{for all } v \in V^h. \tag{2.4}$$

If n denotes the dimension of V^h, with a basis given by $\{\phi_i\}_{i=1}^n$, then (2.4) can be represented by the linear system

$$\mathbf{B}\mathbf{u} = \mathbf{f}. \tag{2.5}$$

The matrix $\mathbf{B}$ is defined in terms of the basis functions as $(\mathbf{B})_{ij} = b(\phi_j, \phi_i)$ and $(\mathbf{f})_i = (f, \phi_i)$. It is also possible to define the matrix $\mathbf{A}$ using the same basis functions as $(\mathbf{A})_{ij} = a(\phi_j, \phi_i)$.

The solvability of (2.4) is assured by the following Lemma from [5, Theorem 2]. This is required due to the indefiniteness of (2.4).

Lemma 2.4 (Schatz and Wang, 1996)**.** *Let Assumptions 2.1 and 2.3 hold. Then there exists an $h_0 > 0$ such that, for each h with $0 < h < h_0$, the problem (2.4) has a unique solution $u_h \in V_H$. Moreover, let u be the unique solution of (2.1). Then, for every $\varepsilon > 0$ there exists $h_1 = h_1(\varepsilon) > 0$ such that, for every $h \in (0, h_1)$,*

$$\|u - u_h\| \leq \varepsilon\|u - u_h\|_{H^1(\Omega)} \tag{2.6}$$

and

$$\|u - u_h\|_{H^1(\Omega)} \leq \varepsilon\|f\|. \tag{2.7}$$

Remark 2.5. *The introduction of (2.6) differs from [2], allowing for improvements to be made in later Lemmas.*

By combining the Friedrichs inequality [4, Theorem 13.19] with Assumption 2.1, it possible to state that, for any subdomain $\Omega' \subset \Omega$ with diameter H, we obtain the estimate

$$\|u\|_{\Omega'} \leq \frac{H}{\sqrt{2}}\|\nabla u\|_{\Omega'} \leq \frac{H}{\sqrt{2}}\|u\|_{a_{\Omega'}} \qquad \text{for all } u \in H_0^1(\Omega'). \tag{2.8}$$

This inequality will be used extensively in what follows.

2.2 Domain decomposition

In order to construct the two-level Schwarz preconditioner, the first-level preconditioner needs to be formulated. This is achieved by first partitioning the global domain, Ω, into a set of N subdomains, $\{\Omega_i'\}_{i=1}^N$. It is being assumed that the global mesh, $\mathcal{T}_h$, is sufficiently fine to resolve the subdomains. Then each subdomain, Ω', is expanded by one or more layers of mesh elements, in the sense of Definition 2.6, creating the desired overlapping set of N subdomains, $\{\Omega_i\}_{i=1}^N$.

Definition 2.6. *Given a subdomain $\Omega' \subset \Omega$, which is resolved by the chosen mesh, the extension of Ω' by a layer of elements is*

$$\Omega_{\mathsf{e}} = \mathrm{Int}\left(\bigcup_{\{\ell|\mathrm{supp}(\phi_\ell)\cap\Omega'\neq\emptyset\}} \mathrm{supp}(\phi_\ell)\right)$$

where $\mathrm{Int}(\cdot)$ is the interior of domain. Extension by multiple layers can then be achieved by applying this recursively.

For the domains $i = 1, \dots, N$, it is possible to define the spaces,

$$\widetilde{V}_i = \{v|_{\Omega_i} : v \in V^h\} \subset H^1(\Omega_i) \quad \text{and} \quad V_i = \{v \in \widetilde{V}_i : v|_{\partial\Omega_i} = 0\} \subset H_0^1(\Omega_i). \tag{2.9}$$

For each subdomain, Ω_i, we denote its diameter by H_i, and we set $H := \max\{H_i\}$. For any $u, v \in \widetilde{V}_i$, the local bilinear forms can be defined

$$a_{\Omega_i}(u, v) = \int_{\Omega_i} A\nabla u \cdot \nabla v \, \mathrm{d}x, \qquad b_{\Omega_i}(u, v) = \int_{\Omega_i} (A\nabla u \cdot \nabla v - k^2 uv) \, \mathrm{d}x.$$

Remark 2.7. *Although $b_{\Omega_i}(\cdot, \cdot)$ is generally indefinite, for small enough H the form restricted to V_i is positive definite. This is discussed in Lemma 3.8.*

For any $v_i \in V_i$, let $E_i v_i$ denote its zero extension to the whole of the domain Ω. Then,

$$E_i : V_i \to V^h, \quad i = 1, \dots, N. \tag{2.10}$$

The $L^2(\Omega)$ adjoint of the extension operator is called restriction operator,

$$R_i : V^h \to V_i.$$

By use of the extension operator, the restriction of the bilinear forms to V_i can be given as

$$a_{\Omega_i}(u, v) = a(E_i u, E_i v), \qquad b_{\Omega_i}(u, v) = b(E_i u, E_i v), \qquad (u, v)_{\Omega_i} = (E_i u, E_i v)$$

for all $u, v \in V_i$. The one-level additive Schwarz preconditioner can now be given in matrix form as

$$\mathbf{M}_{AS,1}^{-1} = \sum_{i=1}^{N} \mathbf{E}_i \mathbf{B}_i^{-1} \mathbf{R}_i, \qquad \text{where } \mathbf{B}_i = \mathbf{R}_i \mathbf{B} \mathbf{E}_i. \tag{2.11}$$

Here, $\mathbf{E}_i$ and $\mathbf{R}_i$ denote the matrix representations of E_i and R_i with respect to the basis functions $\{\phi_i\}_{i=1}^n$ and some basis in V_i.

In order to improve the effectiveness of the preconditioner, a coarse space is added. This improves the global exchange of information between the subdomains. Let $V_0 \subset V^h$ be such a coarse space. Let $E_0 : V_0 \to V^h$ be the natural embedding, and let R_0 be the L^2 adjoint of E_0,

$$(v_0, R_0 w) = (E_0 v_0, w) \qquad \text{for all } w \in V^h, v_0 \in V_0.$$

The two-level additive Schwarz preconditioner can now be given in matrix form as,

$$\mathbf{M}_{AS,2}^{-1} = \sum_{i=0}^{N} \mathbf{E}_i \mathbf{B}_i^{-1} \mathbf{R}_i, \qquad \text{where } \mathbf{B}_i = \mathbf{R}_i \mathbf{B} \mathbf{E}_i. \tag{2.12}$$

The preconditioned linear system from (2.5) reads as,

$$\mathbf{M}_{AS,2}^{-1} \mathbf{B} \mathbf{u} = \mathbf{M}_{AS,2}^{-1} \mathbf{f} \tag{2.13}$$

It is now possible to define the projectors used in the analysis. For each $i = 0, \dots, N$, the projectors, $T_i : V^h \to V_i$, are defined by,

$$b_{\Omega_i}(T_i u, v) = b(u, E_i v) \qquad \text{for all } v \in V_i, \tag{2.14}$$

where $\Omega_0 = \Omega$. The existence of the T_i operators are guaranteed by Lemma 3.8. Given the operators T_i, the operator $T : V^h \to V^h$ is defined as

$$T = \sum_{i=0}^{N} E_i T_i. \tag{2.15}$$

This allows for the two-level additive Schwarz preconditioner to be represented in terms of the projector operator, T, as follows. We recall here the result from [2].

Proposition 2.8. *For any $u, v \in V^h$, with corresponding nodal vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$,*

$$\langle \mathbf{M}_{AS,2}^{-1} \mathbf{B} \mathbf{u}, \mathbf{v} \rangle_{\mathbf{A}} = a(Tu, v), \tag{2.16}$$

where $\langle \cdot, \cdot \rangle_{\mathbf{A}}$ is the inner product on $\mathbb{R}^d$, using the matrix $\mathbf{A}$.

2.3 The Δ-GenEO coarse space

In order to use the Δ-GenEO coarse space, we need to recall some definitions from [6].

Definition 2.9 ([6] Definition 3.2)**.** *Given a subdomain Ω_i, which is formed from a union of elements from the whole domain, let*

$$\overline{\mathrm{dof}}(\Omega_i) := \big\{\ell \mid 1 \le \ell \le n \text{ and } \mathrm{supp}(\phi_\ell) \cap \Omega_i \neq \emptyset\big\}$$

denote the set of degrees of freedom that are active in the subdomain Ω_i, including ones on the boundary. In a similar manner, let

$$\mathrm{dof}(\Omega_i) := \big\{\ell \mid 1 \le \ell \le n \text{ and } \mathrm{supp}(\phi_\ell) \subset \overline{\Omega_i}\big\}$$

denote the internal degrees of freedom.

Definition 2.10 (Partition of unity)**.** *Let $\mathrm{dof}(\Omega_i)$ be as in Definition 2.9. For any degree of freedom, $j \in \{1, \dots, n\}$, let μ_j denote the number of subdomains for which j is an internal degree of freedom, i.e.*

$$\mu_j := \#\big\{i \mid 1 \le i \le N,\, j \in \mathrm{dof}(\Omega_i)\big\}.$$

Then, for $i \in \{1, \dots, N\}$, the local partition of unity operator, $\Xi_i : \widetilde{V}_i \to V_i$ is defined by

$$\Xi_i(v) := \sum_{j \in \mathrm{dof}(\Omega_i)} \frac{1}{\mu_j} v_j \phi_j^i \qquad \text{for } v = \sum_{j \in \overline{\mathrm{dof}}(\Omega_i)} v_j \phi_j^i \in \widetilde{V}_i.$$

The local generalised eigenvalue problem that is going to form the basis of the coarse space can now be introduced.

Definition 2.11 ([2], Definition 2.8)**.** *For each $j = 1, \dots, N$, we define the following generalised eigenvalue problem. Find $(p_\ell^j, \lambda_j) \in \widetilde{V}_j \setminus \{0\} \times \mathbb{R}$ such that*

$$a_{\Omega_j}(p_\ell^j, v) = \lambda_j a_{\Omega_j}\big(\Xi_j(p_\ell^j), \Xi_j(v)\big) \qquad \text{for all } v \in \widetilde{V}_j.$$

Remark 2.12. *As $a_{\Omega_j}(\Xi_j(\cdot), \Xi_j(\cdot))$ is symmetric positive definite on V_j, the finite eigenvectors, p_ℓ^j, can be normalised with respect to $a_{\Omega_j}(\Xi_j(\cdot), \Xi_j(\cdot))$ to satisfy the orthogonality conditions*

$$a_{\Omega_j}\big(\Xi_j(p_m^j), \Xi_j(p_\ell^j)\big) = \delta_m^\ell \quad \text{and} \quad a_{\Omega_j}\big(p_m^j, p_\ell^j\big) = \lambda_j \delta_m^\ell. \tag{2.17}$$

For the treatment of the eigenvalue infinity see the discussion in [6].

Definition 2.13 (Δ-GenEO Coarse space)**.** *For each $j \in 1, \dots, N$, let $\{p_\ell^j\}_{\ell=1}^{m_j}$ be the eigenfunctions corresponding to the m_j smallest eigenvalues from Definition 2.11. The value of m_j is to be chosen, with more details on this later. The coarse space, V_0, is given by*

$$V_0 := \mathrm{span}\{E_j \Xi_j(p_\ell^j) \mid l = 1, \dots, m_j \text{ and } j = 1, \dots, N\}. \tag{2.18}$$

We will also need the following notations

$$\Lambda := \max_{T \in \mathcal{T}_h} \left(\#\{\Omega_j \mid 1 \le j \le N, T \subset \Omega_j\}\right), \qquad \tau := \min_{1 \le j \le N} \lambda_{m_j+1}^j.$$

3 Statement of the main result and theoretical tools

We start this section with the statement of the main result and continue by providing a few technical lemmas needed in the proof of this result, which will be given later on. For convenience we set $\Theta := \frac{1}{\tau}$.

Theorem 3.1 (GMRES convergence of the two-level preconditioned system)**.** *Assume that $h \in (0, h_1)$, where h_1 is as in Lemma 3.11, then select H and τ such that*

$$\begin{aligned} s &:= (1+\Lambda^2\Theta)16\sqrt{2}k^2\Lambda^{\frac{3}{2}}\Theta^{\frac{1}{2}}(1+C_{\mathsf{stab}}) < 1, \\ t &:= 32H^2k^2(1+\Lambda^2\Theta)\Lambda < 1. \end{aligned} \tag{3.1}$$

If GMRES is applied in the $\langle\cdot,\cdot\rangle_{\mathbf{A}}$-inner product to solve the preconditioned system given by (2.13), then after m iterations, the norm of the residual, $\mathbf{r}^{(m)}$, is bounded as

$$\|\mathbf{r}^{(m)}\|_{\mathbf{A}}^2 \leq \left(1 - \frac{c_1^2}{c_2^2}\right)^m \|\mathbf{r}^{(0)}\|_{\mathbf{A}}^2, \tag{3.2}$$

where c_1 and c_2 are given by

$$c_1 := bigl(4(1+\Lambda^2\Theta)\big)^{-1}\big(1-\max\{t,s\}\big), \qquad c_2 := 18 + 18\Lambda^2. \tag{3.3}$$

Corollary 3.2. *If (3.1) in Theorem 3.1 are satisfied, then this leads to the conditions*

$$H \lesssim k^{-1} \quad \text{and} \quad (1+C_{\mathsf{stab}})^2k^4 \lesssim \tau. \tag{3.4}$$

If these conditions are satisfied, then c_1 and c_2 are both independent on problem parameters including the heterogeneity and the wavenumber k, leading to a robust GMRES convergence.

Proof. It is known that $\Lambda \geq 1$; so we have $1 \leq \Lambda \leq \Lambda^{3/2}$ and

$$16\sqrt{2}k^2\Lambda\Theta^{\frac{1}{2}}(1+C_{\mathsf{stab}}) \leq (1+\Lambda^2\Theta)16\sqrt{2}k^2\Lambda^{\frac{3}{2}}\Theta^{\frac{1}{2}}(1+C_{\mathsf{stab}}) < 1,$$

thus leading to the inequality

$$2k^2\Lambda\Theta^{\frac{1}{2}}(1+C_{\mathsf{stab}}) < \frac{1}{8\sqrt{2}}. \tag{3.5}$$

If we use the definition of τ, then (3.5) translates into

$$(1+C_{\mathsf{stab}})^2k^4 \lesssim \tau.$$

Now, using the second inequality from (3.1), we obtain

$$64H^2k^2 \leq 64H^2k^2(1+\Lambda^2\Theta)\Lambda < 1,$$

which leads to the condition $H \lesssim k^{-1}$. □

In practice, conditions (3.4) will introduce a constraint in the size of the subdomains, depending on k and on the number of modes to be added in the coarse space.

Remark 3.3. *Note that the condition (3.1) is different from the one obtained in [2], and allows for the improved conditions on H and τ. These results follow from the modifications made to the later lemmas.*

3.1 Properties of the Δ-GenEO coarse space

The following three lemmas and Proposition 3.7 are also given in [2] and adapted from [6]. The first gives an error estimate for the local projection operator, which is used to approximate a function $v \in \widetilde{V}_i$ in the space being spanned by the eigenfunctions from Definition 2.11.

Lemma 3.4 (Projection operator onto the coarse space)**.** *Let $i \in \{1, \ldots, N\}$ and $\{(p^i_\ell, \lambda^i_\ell)\}$ be the eigenpairs of the generalised eigenproblem, given in Definition 2.11. Suppose that $m_i \in \{1, \ldots, \dim(V_i) - 1\}$ is such that $0 < \lambda^i_{m_i+1} < \infty$, and the eigenvalues can be ordered such that $\lambda^i_1 \leq \lambda^i_2 \leq \ldots \leq \lambda^i_{\dim(\widetilde{V}_i)}$. Then the local projector, $\Pi^i_{m_i}$, defined by*

$$\Pi^i_{m_i} v := \sum_{l=1}^{m_i} a_{\Omega_i}\big(\Xi_i(v), \Xi_i(p^i_\ell)\big) p^i_\ell, \tag{3.6}$$

satisfies

$$\|w\|^2_{a_{\Omega_i}} \leq \|v\|^2_{a_{\Omega_i}} \quad \text{and} \quad \|\Xi(w)\|^2_{a_{\Omega_i}} \leq \frac{1}{\lambda^i_{m_i+1}} \|w\|^2_{a_{\Omega_i}} \qquad \text{for all } v \in \widetilde{V}_i, \tag{3.7}$$

where $w = v - \Pi^i_{m_i} v$.

From these local error estimates it is possible to build a global approximation property.

Lemma 3.5 (Global approximation property)**.** *Assume that the conditions in Lemma 3.4 are satisfied. Let $v \in V^h$; then*

$$\inf_{z \in V_0} \|v - z\|^2_a \leq \|v - z_0\|^2_a \leq \Lambda^2 \Theta \|v\|^2_a, \tag{3.8}$$

where

$$z_0 = \sum_{i=1}^N E_i \Xi_i\big(\Pi^i_{m_i} v|_{\Omega_i}\big) \qquad \text{and} \qquad \Theta = \max_{1 \leq i \leq N} \frac{1}{\lambda^i_{m_i+1}}.$$

The following lemma shows that the GenEO coarse space, in combination with the local finite element space, allows for a stable decomposition. This property is a key component for bounding the condition number in the two-level Schwarz preconditioner for positive definite cases, as used in [6].

Lemma 3.6 (Stable decomposition)**.** *Let $v \in V^h$. Then the decomposition*

$$z_0 = \sum_{i=1}^N E_i \Xi_i\big(\Pi^i_{m_i} v|_{\Omega_i}\big) \quad \text{and} \quad z_i = \Xi_i\big(v|_{\Omega_i} - \Pi^i_{m_i} v|_{\Omega_i}\big) \qquad \text{for } i = 1, \ldots, N$$

satisfies $v = \sum_{i=0}^N E_i z_i$ and

$$\|z_0\|^2_a + \sum_{i=1}^N \|z_i\|^2_{a_{\Omega_i}} \leq 4(1 + \Lambda^2 \Theta) \|v\|^2_a.$$

It is convenient at this time to introduce the projection operators $P_i : V^h \to V_i$ such that, for each $i = 0, \ldots, N$,

$$a_{\Omega_i}(P_i u, v) = a(u, E_i v) \text{for all } v \in V_i. \tag{3.9}$$

In [7, Section 2.2] it is proved that these operators are well defined. Using these operator P_i we can defines $P : V^h \to V^h$ as

$$P = \sum_{i=0}^N E_i P_i. \tag{3.10}$$

Proposition 3.7. *Under the same assumptions as in Lemma 3.6, any $u \in V^h$ satisfies*

$$\|u\|_a^2 \leq 4(1+\Lambda^2\Theta)(Pu,u)_a \tag{3.11}$$

and

$$\sum_{i=0}^{N} \|P_i u\|_{a_{\Omega_i}}^2 \leq (\Lambda+1)\|u\|_a^2. \tag{3.12}$$

3.2 Solvability and stability of $T_i, i = 1, \ldots, N$

The next two lemmas are reformulations of those used in [2] when applied to (1.1), which is a particular case of the problem from [2]. These lemmas are necessary even if there are no improvements of the estimates at this stage.

Lemma 3.8 (T_i is well defined for $i = 1, \ldots, N$)**.** *If $Hk < \sqrt{2}$, then $b_{\Omega_i}(\cdot,\cdot)$ is positive definite on $H_0^1(\Omega_i)$ and the operators T_i, $i = 1, \ldots, N$ are well defined.*

Proof. Using the definition of $b_{\Omega_i}(\cdot,\cdot)$ and the Friedrichs inequality we obtain

$$b_{\Omega_i}(u,u) = a_{\Omega_i}(u,u) - k^2(u,u)_{\Omega_i} \geq \frac{2}{H^2}\|u\|_{\Omega_i}^2 - k^2\|u\|_{\Omega_i}^2 = \frac{2-H^2k^2}{H^2}\|u\|_{\Omega_i}^2.$$

If the condition $Hk < \sqrt{2}$ is satisfied, then $b_{\Omega_i}(\cdot,\cdot)$ is positive definite. By the Lax–Milgram lemma, this ensures that T_i is well defined. □

Remark 3.9. *Whilst this a sufficient condition for T_i to be well defined, it is not a necessary condition. In general, if k^2 is not an eigenvalue of the matrix corresponding to a_{Ω_i}, then T_i is well defined.*

In order to show the robustness of the two-level method additive Schwarz method, it is necessary to prove the stability estimates for the T_i operators.

Lemma 3.10 (Stability of T_i, $i = 1, \ldots, N$)**.** *Suppose that $Hk \leq 1$. Then, for all $u \in V^h$,*

$$\|T_i u\|_{a_{\Omega_i}} \leq 2\|u\|_{a_{\Omega_i}} + k\sqrt{2}\|u\|_{\Omega_i}. \tag{3.13}$$

Proof. Using the defining relation $b_{\Omega_i}(T_iu, T_iu) = b_{\Omega_i}(u, T_iu)$ and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
\|T_iu\|_{a_{\Omega_i}}^2 &= a_{\Omega_i}(T_iu,T_iu) = b_{\Omega_i}(T_iu,T_iu) + k^2(T_iu,T_iu)_{\Omega_i} \\
&= b_{\Omega_i}(u,T_iu) + k^2(T_iu,T_iu)_{\Omega_i} \\
&= a_{\Omega_i}(u,T_iu) - k^2(u,T_iu)_{\Omega_i} + k^2(T_iu,T_iu)_{\Omega_i} \\
&\leq \|u\|_{a_{\Omega_i}}\|T_iu\|_{a_{\Omega_i}} + k^2\|u\|_{\Omega_i}\|T_iu\|_{\Omega_i} + k^2\|T_iu\|_{\Omega_i}^2 \\
&\leq \|u\|_{a_{\Omega_i}}\|T_iu\|_{a_{\Omega_i}} + k^2\frac{H}{\sqrt{2}}\|u\|_{\Omega_i}\|T_iu\|_{a_{\Omega_i}} + k^2\frac{H^2}{2}\|T_iu\|_{a_{\Omega_i}}^2.
\end{aligned}$$

After simplification, this can be rewritten as

$$\left(1 - k^2\frac{H^2}{2}\right)\|T_iu\|_{a_{\Omega_i}} \leq \|u\|_{a_{\Omega_i}} + k^2\frac{H}{\sqrt{2}}\|u\|_{\Omega_i}.$$

Together with the assumption $Hk \leq 1$, this implies the claimed inequality (3.13). □

3.3 Solvability and stability of T_0

In this section we derive sufficient conditions for solvability and stability of the coarse space operator T_0. The conditions obtained from the subsequent Lemmas 3.11 and 3.12 are an improvement on those found in their counterparts from [2].

First, to ensure that T_0 is well defined, a condition on Θ is required.

Lemma 3.11 (T_0 is well defined)**.** *Suppose that*

$$2k\Lambda\Theta^{\frac{1}{2}}(1 + C_{\mathsf{stab}}) < 1. \tag{3.14}$$

Then there exists $h_1 > 0$ such that, for all $h \in (0, h_1)$, the operator T_0 is well defined.

Proof. Assume that there exists a $w_0 \in V_0 \backslash \{0\}$ such that

$$b(w_0, z) = 0 \qquad \text{for all } z \in V_0. \tag{3.15}$$

Let $w \in H_0^1(\Omega)$ be the solution of

$$b(w, v) = (w_0, v) \qquad \text{for all } v \in H_0^1(\Omega).$$

Let $\varepsilon > 0$ such that $k^2\varepsilon^2 \leq \frac{1}{2}$. We shall impose an additional constraint on ε later. Then Lemma 2.4 implies that there exists $h_1 > 0$ such that, for all $h \in (0, h_1)$, there exists a solution $w_h \in V^h$ of

$$b(w_h, v) = (w_0, v) \qquad \text{for all } v \in V^h,$$

and (2.6) and (2.7) hold. As $w_0 \in V_0 \backslash \{0\}$, we can choose $v = w_0$. Then, for all $z \in V_0$,

$$\begin{aligned}
\|w_0\|^2 &= b(w_h, w_0) = b(w_h, w_0) - b(z, w_0) = b(w_h - z, w_0) \\
&= a(w_h - z, w_0) - k^2(w_h - z, w_0) \leq |a(w_h - z, w_0)| + k^2|(w_h - z, w_0)| \\
&\leq \|w_h - z\|_a \|w_0\|_a + k^2\|w_h - z\|\|w_0\| \leq \|w_h - z\|_a\|w_0\|_a + k^2\varepsilon\|w_h - z\|_{H^1(\Omega)}\|w_0\| \\
&\leq \|w_h - z\|_a\|w_0\|_a + k^2\varepsilon^2\|w_0\|^2 \\
&\leq \|w_h - z\|_a\|w_0\|_a + \frac{1}{2}\|w_0\|^2
\end{aligned}$$

and hence

$$\|w_0\|^2 \leq 2\|w_h - z\|_a\|w_0\|_a.$$

As this is true for all $z \in V_0$, it follows that

$$\|w_0\|^2 \leq 2\|w_0\|_a \inf_{z \in V_0} \|w_h - z\|_a.$$

Combined with Lemma 3.5 this yields

$$\|w_0\|^2 \leq 2\Lambda\Theta^{\frac{1}{2}}\|w_0\|_a\|w_h\|_a. \tag{3.16}$$

Choosing $z = w_0$ in (3.15) we have

$$0 = b(w_0, w_0) = \|w_0\|_a^2 - k^2\|w_0\|^2 \tag{3.17}$$

and hence $\|w_0\|_a = k\|w_0\|$. Together with (3.16) this gives

$$\|w_0\|^2 \leq 2k\Lambda\Theta^{\frac{1}{2}}\|w_0\|\|w_h\|_a. \tag{3.18}$$

We can combine (2.3) and (2.7) to obtain

$$\begin{aligned}
\|w_h\|_a &\leq \|w\|_a + \|w - w_h\|_a \leq \|w\|_a + a_{\mathsf{max}}\|w - w_h\|_{H^1(\Omega)} \\
&\leq C_{\mathsf{stab}}\|w_0\| + a_{\mathsf{max}}\varepsilon\|w_0\| \leq (C_{\mathsf{stab}} + 1)\|w_0\|
\end{aligned} \tag{3.19}$$

if ε is chosen small enough so that $a_{\mathsf{max}}\varepsilon \leq 1$. We now obtain from (3.19) with (3.18) that

$$\|w_0\|^2 \leq 2k\Lambda\Theta^{\frac{1}{2}}(1 + C_{\mathsf{stab}})\|w_0\|^2.$$

With the assumption (3.14) this leads to a contradiction. □

Another requirement is to find the stability conditions for T_0.

Lemma 3.12 (Stability of T_0)**.** *Suppose that* (3.14) *is satisfied. Then there exists* $h_1 > 0$ *such that, for* $h \in (0, h_1)$,

$$\|T_0 u - u\| \leq 2\Lambda\Theta^{\frac{1}{2}}(1 + C_{\mathsf{stab}})\|T_0 u - u\|_a \qquad \text{for all } u \in V_h. \tag{3.20}$$

Suppose, in addition, that

$$2k^2\Lambda\Theta^{\frac{1}{2}}(1 + C_{\mathsf{stab}}) \leq \frac{1}{2}\,. \tag{3.21}$$

Then

$$\|u - T_0 u\| \leq \sqrt{2}\|u\|_a \qquad \text{for all } u \in V^h. \tag{3.22}$$

Proof. Under condition (3.14), Lemma 3.11 ensures the existence of $h_1 > 0$ such that, for $h \in (0, h_1)$, the operator $T_0 : V^h \to V_0$ is well defined. Consider the auxiliary problem

$$\text{find } \; w_h \in V^h \; \text{ such that } \qquad b(w_h, v) = (T_0 u - u, v) \quad \text{for all } v \in V^h, \tag{3.23}$$

which has a unique solution by Lemma 2.4. The definition of T_0 implies that $b(T_0 u - u, z) = 0$ for all $z \in V_0$. We can choose $v = T_0 u - u$ in (3.23), which yields

$$\begin{aligned}
\|T_0 u - u\|^2 &= b(w_h, T_0 u - u) = b(w_h, T_0 u - u) - b(z, T_0 u - u) = b(w_h - z, T_0 u - u) \\
&= a(w_h - z, T_0 u - u) - k^2(w_h - z, T_0 u - u) \\
&\leq |a(w_h - z, T_0 u - u)| + k^2|(w_h - z, T_0 u - u)| \\
&\leq \|w_h - z\|_a \|T_0 u - u\|_a + k^2\|w_h - z\|\,\|T_0 u - u\| \\
&\leq \|w_h - z\|_a \|T_0 u - u\|_a + k^2\varepsilon\|w_h - z\|_{H^1(\Omega)}\|T_0 u - u\| \\
&\leq \|w_h - z\|_a \|T_0 u - u\|_a + k^2\varepsilon^2\|T_0 u - u\|^2,
\end{aligned}$$

where we used Lemma 2.4 for the last two inequalities. Choosing $\varepsilon > 0$ such that $k^2\varepsilon^2 \leq \frac{1}{2}$ we obtain

$$\|T_0 u - u\| \leq 2\|w_h - z\|_a \|T_0 u - u\|_a. \tag{3.24}$$

As this is true for all $z \in V_0$, it follows Lemma 3.5 that

$$\|T_0 u - u\|^2 \leq 2\|T_0 u - u\|_a \inf_{z \in V_0} \|w_h - z\|_a \leq 2\Lambda\Theta^{\frac{1}{2}}\|T_0 u - u\|_a \|w_h\|_a. \tag{3.25}$$

If we choose ε small enough so that $a_{\mathsf{max}}\varepsilon \leq 1$, we can use a similar calculation as in (3.19) to get $\|w_h\|_a \leq (1 + C_{\mathsf{stab}})\|T_0 u - u\|$. Combined with (3.25) this proves (3.20).

We now come to the proof of (3.22). From the definition of P_0 in (3.9) we have

$$a(P_0 u - u, v) = 0 \qquad \text{for all } u \in V^h, v \in V_0.$$

Taking $v = T_0 u \in V_0$, we get $a(T_0 u, P_0 u - u) = 0$. Moreover, since $P_0 u - T_0 u \in V_0$, the definition of T_0 yields

$$b(u - T_0 u, P_0 u - T_0 u) = 0. \tag{3.26}$$

Using (3.26), the link between the bilinear forms a and b, and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\|u - T_0 u\|_a^2 &= a(u - T_0 u, u - T_0 u) = b(u - T_0 u, u - T_0 u) - k^2(u - T_0 u, u - T_0 u) \\
&= b(u - T_0 u, u - T_0 u) - b(u - T_0 u, P_0 u - T_0 u) - k^2(u - T_0 u, u - T_0 u) \\
&= b(u - T_0 u, u - P_0 u) - k^2(u - T_0 u, u - T_0 u) \\
&= a(u - T_0 u, u - P_0 u) + k^2(u - T_0 u, u - P_0 u) - k^2(u - T_0 u, u - T_0 u) \\
&= a(u - T_0 u, u - P_0 u) + k^2(u - T_0 u, T_0 u - P_0 u) \\
&= a(u - T_0 u, u - P_0 u) + a(T_0 u, u - P_0 u) + k^2(u - T_0 u, T_0 u - P_0 u) \\
&= a(u, u - P_0 u) + k^2(u - T_0 u, T_0 u - P_0 u) \\
&\leq \|u\|_a \|u - P_0 u\|_a + k^2\|u - T_0 u\|\,\|P_0 u - T_0 u\|.
\end{aligned} \tag{3.27}$$

Since our domain satisfies $D_\Omega \le 1$, we can use the definition of P_0 and the Friedrichs inequality to get

$$\|u - P_0 u\|_a \le \|u\|_a,$$
$$\|P_0 u - T_0 u\| \le \|P_0 u - T_0 u\|_a \le \|P_0(u - T_0 u)\|_a \le \|u - T_0 u\|_a.$$

These relations, together with (3.27) and (3.20) lead to

$$\begin{aligned}\|u - T_0 u\|_a^2 &\le \|u\|_a^2 + k^2 \|u - T_0 u\| \, \|u - T_0 u\|_a \\ &\le \|u\|_a^2 + k^2 2\Lambda\Theta^{\frac{1}{2}}(1 + C_{\mathsf{stab}}) \, \|T_0 u - u\|_a^2 \,. \end{aligned} \tag{3.28}$$

Rearranging (3.28) we arrive at

$$\Big(1 - 2k^2\Lambda\Theta^{\frac{1}{2}}(1 + C_{\mathsf{stab}})\Big) \|T_0 u - u\|_a^2 \le \|u\|_a^2.$$

Using assumption (3.21) we can finish the proof of (3.22). □

4 Proof of the main result

Before we can prove Theorem 3.1, we need to state and prove the following lemma.

Lemma 4.1. *Suppose that the assumptions in Theorem 3.1 are satisfied. Then, for all $u \in V_h$,*

$$c_1 \|u\|_a^2 \le (Tu, u)_a, \tag{4.1}$$

and

$$\|Tu\|_a \le c_2 \|u\|_a^2, \tag{4.2}$$

where c_1 and c_2 are as in Theorem 3.1.

Proof. To prove (4.1), let $u \in V^h$. We proceed in several steps.

Step 1. This is the same procedure as in [2], but adapted for application to (1.1) directly. We start with (3.11) and then use (3.10), (3.11), and the definition of the T_i projection operators to get a preliminary estimate:

$$\begin{aligned}\big(4(1 + \Lambda^2\Theta)\big)^{-1} \|u\|_a^2 \le a(Pu, u) = a\left(\sum_{i=0}^N E_i P_i u, u\right) &= \sum_{i=0}^N a(E_i P_i u, u) \\ &= \sum_{i=0}^N \Big(b(u, E_i P_i u) + k^2 (u, E_i P_i u)\Big) \\ &= \sum_{i=0}^N \Big(b(E_i T_i u, E_i P_i u) + k^2 (u, E_i P_i u)\Big) \\ &= \sum_{i=0}^N \Big(a(E_i T_i u, E_i P_i u) - k^2 (E_i T_i u, E_i P_i u) + k^2 (u, E_i P_i u)\Big) \\ &= \sum_{i=0}^N a(E_i T_i u, E_i P_i u) - k^2 \sum_{i=0}^N \big(E_i T_i u - u, E_i P_i u\big).\end{aligned}$$

Since, by definition $a(E_i T_i u, E_i P_i u) = a_{\Omega_i}(T_i u, P_i u)$ and $a_{\Omega_i}(P_i u, v) = a(u, E_i v)$ for all $v \in V_i$, we have $a(E_i T_i u, E_i P_i u) = a_{\Omega_i}(T_i u, P_i u) = a(E_i T_i u, u)$, which, in turn, implies

$$\begin{aligned}\big(4(1 + \Lambda^2\Theta)\big)^{-1} \|u\|_a^2 &\le \sum_{i=0}^N a(E_i T_i u, u) - k^2 \sum_{i=0}^N (E_i T_i u - u, E_i P_i u) \\ &= a(Tu, u) - k^2 \sum_{i=0}^N (E_i T_i u - u, E_i P_i u).\end{aligned}$$

This can be rewritten as (the sum can be split into terms related to $i = 0$ and $i \geq 1$)

$$\|u\|_a^2 \leq 4(1+\Lambda^2\Theta)\left(a(Tu,u) - k^2(E_0T_0u - u, E_0P_0u) - k^2\sum_{i=1}^{N}(E_iT_iu - u, E_iP_iu)\right). \tag{4.3}$$

Step 2. The next stage is to bound the second and third terms inside the bracket of the preliminary estimate (4.3). It is at this stage that we take advantage of the improved estimates obtained in Lemmas 3.11 and 3.12. We start with the second term in (4.3), where E_0 should be considered as a natural embedding like the identity operator ($E_iT_0u = T_0u$ and $E_iP_0u = P_0u$). We also use (3.20) and (3.22) to get

$$\begin{aligned}
-k^2(E_0T_0u - u, E_0P_0u) &\leq k^2\|E_0T_0u - u\|\,\|E_0P_0u\| = k^2\|T_0u - u\|\|P_0u\| \\
&\leq k^2\|T_0u - u\|\,\|P_0u\|_a \leq k^2 2\Lambda\Theta^{\frac{1}{2}}(1+C_{\mathsf{stab}})\|T_0u - u\|_a\|P_0u\|_a \\
&\leq k^2 2\sqrt{2}\Lambda\Theta^{\frac{1}{2}}(1+C_{\mathsf{stab}})\|u\|_a\|P_0u\|_a.
\end{aligned}$$

With this the second term in (4.3) can be estimated as follows, where s is as in Theorem 3.1,

$$-4(1+\Lambda^2\Theta)k^2\big(E_0T_0u - u, E_0P_0u\big) \leq \frac{s}{2\Lambda^{\frac{1}{2}}}\|u\|_a\|P_0u\|_a \leq \frac{s}{\sqrt{2}(\Lambda+1)^{\frac{1}{2}}}\|u\|_a\|P_0u\|_a. \tag{4.4}$$

Let us now consider the sum in (4.3). Note that, for each $i \in \{1,\dots,N\}$, the operator E_i is just an extension by 0 outside the domain ($a(u, E_iv) = a_{\Omega_i}(u,v)$ for $u \in V^h, v \in V_i$). Using Lemma 3.10 we obtain

$$\begin{aligned}
&- k^2(E_iT_iu - u, E_iP_iu) \leq \big|k^2(E_iT_iu - u, P_iu)_{\Omega_i}\big| \leq k^2\|E_iT_iu - u\|_{\Omega_i}\|P_iu\|_{\Omega_i} \\
&\leq k^2\|E_iT_iu - u\|_{\Omega_i}\|P_iu\|_{\Omega_i} \leq k^2\big(\|T_iu\|_{\Omega_i} + \|u\|_{\Omega_i}\big)\|P_iu\|_{\Omega_i} \\
&\leq k^2\left(\frac{H}{\sqrt{2}}\|T_iu\|_{a_{\Omega_i}} + \|u\|_{\Omega_i}\right)\|P_iu\|_{\Omega_i} \leq k^2\left(H\sqrt{2}\|u\|_{a_{\Omega_i}} + kH\|u\| + \|u\|_{\Omega_i}\right)\|P_iu\|_{\Omega_i} \\
&\leq Hk^2\left(H\sqrt{2}\|u\|_{a_{\Omega_i}} + 3\|u\|_{\Omega_i}\right)\|P_iu\|_{a_{\Omega_i}} \leq Hk^2\left(H\sqrt{2}\|u\|_{a_{\Omega_i}} + \frac{3H}{\sqrt{2}}\|u\|_{a_{\Omega_i}}\right)\|P_iu\|_{a_{\Omega_i}} \\
&< 4H^2k^2\|u\|_{a_{\Omega_i}}\|P_iu\|_{a_{\Omega_i}}.
\end{aligned}$$

Taking the sum over i and applying the Cauchy–Schwarz inequality and the overlap property, $\sum_{i=1}^{N}\|u\|_{a_{\Omega_i}}^2 \leq \Lambda\|u\|_a^2$, we obtain

$$\begin{aligned}
&- k^2\sum_{i=1}^{N}(E_iT_iu - u, E_iP_iu) \leq \sum_{i=1}^{N}4H^2k^2\|u\|_{a_{\Omega_i}}\|P_iu\|_{a_{\Omega_i}} \\
&\qquad \leq 4H^2k^2\left(\sum_{i=1}^{N}\|u\|_{a_{\Omega_i}}^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{N}\|P_iu\|_{a_{\Omega_i}}^2\right)^{\frac{1}{2}} \leq 4H^2k^2\Lambda^{\frac{1}{2}}\|u\|_a\left(\sum_{i=1}^{N}\|P_iu\|_{a_{\Omega_i}}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Using assumption (3.1) and t as in Theorem 3.1 we arrive at

$$\begin{aligned}
-4(1+\Lambda^2\Theta)k^2\sum_{i=1}^{N}\big(E_iT_iu - u, E_iP_iu\big) &\leq 16(1+\Lambda^2\Theta)H^2k^2\Lambda^{\frac{1}{2}}\|u\|_a\left(\sum_{i=1}^{N}\|P_iu\|_{a_{\Omega_i}}^2\right)^{\frac{1}{2}} \\
&\leq \frac{t}{2\Lambda^{\frac{1}{2}}}\|u\|_a\left(\sum_{i=1}^{N}\|P_iu\|_{a_{\Omega_i}}^2\right)^{\frac{1}{2}} \leq \frac{t}{\sqrt{2}(\Lambda+1)^{\frac{1}{2}}}\|u\|_a\left(\sum_{i=1}^{N}\|P_iu\|_{a_{\Omega_i}}^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.5}$$

Step 3. We are now ready to combine (4.4) and (4.5) in order to get an estimate for the second and third terms in (4.3) in terms of $\|u\|_a^2$. This again follows the same procedure as used in [2] with no further refinements (note that we also use the well-known inequality $(a+b)^2 \leq 2a^2+2b^2 \;\Rightarrow\; a+b \leq \sqrt{2a^2+2b^2}$),

$$
\begin{aligned}
&-4(1+\Lambda^2\Theta)\left(k^2(E_0T_0u-u,E_0P_0u)+k^2\sum_{i=1}^{N}(E_iT_iu-u,E_iP_iu)\right)\\
&\leq \frac{s}{\sqrt{2}(\Lambda+1)^{\frac{1}{2}}}\|u\|_a\|P_0u\|_a+\frac{t}{\sqrt{2}(\Lambda+1)^{\frac{1}{2}}}\|u\|_a\left(\sum_{i=1}^{N}\|P_iu\|_{a_{\Omega_i}}^2\right)^{\frac{1}{2}}\\
&\leq \frac{\max\{s,t\}}{\sqrt{2}(\Lambda+1)^{1/2}}\|u\|_a\left[\|P_0u\|_a+\left(\sum_{i=1}^{N}\|P_iu\|_{a_{\Omega_i}}^2\right)^{\frac{1}{2}}\right]\\
&\leq \frac{\max\{s,t\}}{(\Lambda+1)^{1/2}}\|u\|_a\left(\sum_{i=0}^{N}\|P_iu\|_{a_{\Omega_i}}^2\right)^{\frac{1}{2}}\\
&\leq \frac{\max\{s,t\}}{(\Lambda+1)^{1/2}}\|u\|_a\big(\|u\|_a^2(\Lambda+1)\big)^{\frac{1}{2}}=\max\{s,t\}\|u\|_a^2. \qquad (4.6)
\end{aligned}
$$

In the last inequality we used (3.12).

Step 4. Although no further improvements to the results are obtained at this stage, (3.22) is a direct result of earlier enhancements to the estimations being found. Following the same procedure as used in [2], we exploit the estimate (4.6) and proceed to the final result. In a first instance we use (4.6) together with (4.3) to get

$$\|u\|_a^2 \leq 4(1+\Lambda^2\Theta)a(Tu,u)+\max\{s,t\}\|u\|_a^2,$$

which leads to the desired estimate

$$c_1\|u\|_a^2=\big(4(1+\Lambda^2\Theta)\big)^{-1}\big(1-\max\{s,t\}\big)\|u\|_a^2 \leq a(Tu,u).$$

We now come to the proof of (4.2). We start with the following relation, where we use $E_0T_0u=T_0u$,

$$\|Tu\|_a^2=\left\|T_0u+\sum_{i=1}^{N}E_iT_iu\right\|_a^2 \leq 2\|T_0u\|_a^2+2\left\|\sum_{i=1}^{N}E_iT_iu\right\|_a^2. \qquad (4.7)$$

Using the Cauchy–Schwarz inequality and (3.22) we find an upper bound for the first term on the right-hand side of (4.7):

$$
\begin{aligned}
\|T_0u\|_a^2 &= a(T_0u,T_0u)=a(T_0u,T_0u)-a(u,T_0u)+a(u,T_0u)\\
&\leq a(T_0u-u,T_0u)+a(u,T_0u) \leq \|T_0u-u\|_a\|T_0u\|_a+\|u\|_a\|T_0u\|_a\\
&\leq 3\|u\|_a\|T_0u\|_a,
\end{aligned}
$$

which yields

$$\|T_0u\|_a^2 \leq 9\|u\|_a^2. \qquad (4.8)$$

For the second term on the right-hand side of (4.7) we use Lemma 3.10 to get

$$\begin{aligned}\left\|\sum_{i=1}^{N} E_i T_i u\right\|_a^2 &\le \Lambda \sum_{i=1}^{N} \|T_i u\|_{a_{\Omega_i}}^2 \le \Lambda \sum_{i=1}^{N} \Big(2\|u\|_{a_{\Omega_j}} + k\sqrt{2}\|u\|_{\Omega_j}\Big)^2 \\ &\le \Lambda \sum_{i=1}^{N} \Big(2\|u\|_{a_{\Omega_j}} + kH\|u\|_{a_{\Omega_j}}\Big)^2 \\ &\le 9\Lambda \sum_{i=1}^{N} \|u\|_{a_{\Omega_j}}^2 \le 9\Lambda^2 \|u\|_a^2. \end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9) with (4.7) we arrive at

$$\|Tu\|_a \le 2(9\|u\|_a^2) + 2(9\Lambda^2\|u\|_a^2) = (18 + 18\Lambda^2)\|u\|_a^2, \tag{4.10}$$

which finishes the proof of (4.2). □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Using (4.1) with (2.16) we obtain

$$c_1\|u\|_a^2 \le a(Tu, u) = \langle \mathbf{M}_{AS,2}^{-1}\mathbf{B}\mathbf{u}, \mathbf{u}\rangle_{\mathbf{A}},$$

which can be written as

$$c_1 \le \frac{\langle \mathbf{M}_{AS,2}^{-1}\mathbf{B}\mathbf{u}, \mathbf{u}\rangle_{\mathbf{A}}}{\|\mathbf{u}\|_{\mathbf{A}}^2}.$$

In a similar way we use (4.2), again with (2.16), to get

$$\|\mathbf{M}_{AS,2}^{-1}\mathbf{B}\mathbf{u}\|_{\mathbf{A}}^2 = \|Tu\|_a^2 \le c_2\|u\|_a^2 = c_2\|\mathbf{u}\|_{\mathbf{A}}^2,$$

which implies

$$\|\mathbf{M}_{AS,2}^{-1}\mathbf{B}\|_{\mathbf{A}}^2 \le c_2.$$

Now the result follows directly from Elman theory [3]. □

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