JOINT DISTRIBUTION OF L-VALUES AND ORDERS OF SHA GROUPS OF QUADRATIC TWISTS OF ELLIPTIC CURVES

PENG-JIE WONG

To my parents and in loving memory of my grandparents

Abstract. We study the joint distribution of central L-values and orders of Tate-Shafarevich groups of quadratic twists of elliptic curves. In particular, adapting Radziwiłł and Soundararajan's principles of establishing upper and lower bounds for the distribution of central values in families of L-functions, we obtain conditional upper and lower bounds for such a joint distribution for rank zero twists. These lead us to a conjecture on the full asymptotic for the joint distribution.

1. Introduction

As the central L-values of quadratic twists of elliptic curves encode deep arithmetic information (most profoundly, via the Birch and Swinnerton-Dyer conjecture), the study of these L-values has attracted many researchers (see, e.g., [7, 12, 14, 15, 20] and references therein). Notably, as an analogue of Selberg's central limit theorem on the normality of the distribution of $\log|\zeta(\frac{1}{2}+it)|$, a conjecture of Keating and Snaith predicts that the distribution of the logarithm of central L-values of certain quadratic twists of elliptic curves is normal (see [11]). Magnificent progress towards this conjecture has been made by Radziwiłł and Soundararajan [14, 15]. In order to discuss these more precisely, we shall recall some facts for L-functions of elliptic curves as follows.

For an elliptic curve E defined over $\mathbb{Q}$ of conductor $N = N_E$, the associated (normalised) Hasse-Weil L-function is defined by

$$L(s,E) = \sum_{n=1}^{\infty} \frac{\lambda_E(n)}{n^s},$$

for $\mathfrak{Re}(s) > 1$, which extends to an entire function (by the modularity theorem of Wiles, Wiles-Taylor, and Breuil-Conrad-Diamond-Taylor). Here, the coefficients are normalised so that $|\lambda_E(n)| \leq d_2(n)$ for all n, and the centre of the critical strip is $\frac{1}{2}$. Moreover, setting

$$\Lambda(s,E) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s+\tfrac{1}{2}) L(s,E),$$

one has the functional equation

$$\Lambda(s,E) = \epsilon_E \Lambda(1-s,E),$$

where $\epsilon_E = \pm 1$ denotes the root number of E.

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Throughout our discussion, we will let d denote a fundamental discriminant coprime to $2N$, and $\chi_d = (\frac{d}{\cdot})$ be the associated primitive quadratic (Dirichelt) character. In addition, we let E_d stand for the quadratic twist of E by d, and recall the associated twisted L-function is

$$L(s, E_d) = \sum_{n=1}^{\infty} \frac{\lambda_E(n)\chi_d(n)}{n^s}.$$

As $(d, N) = 1$, the conductor of E_d is Nd^2, and the completed L-function

$$\Lambda(s, E_d) = \left(\frac{\sqrt{N}|d|}{2\pi}\right)^s \Gamma(s + \tfrac{1}{2}) L(s, E_d)$$

extends to an entire function and satisfies

$$\Lambda(s, E_d) = \epsilon_E(d)\Lambda(1 - s, E_d)$$

with $\epsilon_E(d) = \epsilon_E \chi_d(-N)$. Famously, Waldspurger's theorem implies that the central values $L(\frac{1}{2}, E_d)$ are always non-negative. Recalling that $L(\frac{1}{2}, E_d) = 0$ if $\epsilon_E(d) = -1$, one would therefore often restrict their attention to the following subset of fundamental discriminants:

$$\mathcal{E} = \{d : d \text{ is a fundamental discriminant with } (d, 2N) = 1 \text{ and } \epsilon_E(d) = 1\}.$$

A conjecture of Keating-Snaith for $L(\frac{1}{2}, E_d)$ predicts that as d ranges, $\log L(\frac{1}{2}, E_d)$ behaves like a normal random variable with mean $-\frac{1}{2}\log\log|d|$ and variance $\log\log|d|$. (Here, as later, $\log L(\frac{1}{2}, E_d)$ is interpreted as $-\infty$ if $L(\frac{1}{2}, E_d) = 0$.) More precisely, for any fixed interval (α, β), as $X \to \infty$,

$$\begin{aligned}
&\#\left\{d \in \mathcal{E}, X < |d| \le 2X : \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \in (\alpha, \beta)\right\} \\
&= (\Psi(\alpha, \beta) + o(1))\#\{d \in \mathcal{E} : X < |d| \le 2X\},
\end{aligned}$$

where

$$\Psi(\alpha, \beta) = \int_\alpha^\beta \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt,$$

and the kernel of the integral is the probability density function of a standard normal random variable (i.e., with mean 0 and variance 1). In [14], Radziwiłł and Soundararajan showed that unconditionally, for any fixed $V \in \mathbb{R}$, as $X \to \infty$,

$$\#\left\{d \in \mathcal{E}, 20 < |d| \le X : \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \ge V\right\} \le (\Psi(V, \infty) + o(1))\#\{d \in \mathcal{E} : |d| \le X\}.$$

More recently, Radziwiłł and Soundararajan [15] further proved the following conditional lower bound towards the Keating-Snaith conjecture for $L(\frac{1}{2}, E_d)$.

Theorem 1.1 (Radziwiłł-Soundararajan)**.** *Assume the generalised Riemann hypothesis (GRH) for all twisted L-functions $L(s, E \otimes \chi)$ with Dirichlet characters χ. Then for any fixed (α, β), as $X \to \infty$,*

$$\begin{aligned}
&\#\left\{d \in \mathcal{E}, X < |d| \le 2X : \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \in (\alpha, \beta)\right\} \\
&\ge \frac{1}{4}(\Psi(\alpha, \beta) + o(1))\#\{d \in \mathcal{E} : X < |d| \le 2X\}.
\end{aligned} \tag{1.1}$$

It is worth noting that the factor $\frac{1}{4}$ appearing in (1.1) coincides with the proportion of non-vanishing $L(\frac{1}{2}, E_d)$ established by Heath-Brown [7] under GRH. Furthermore, recently, Smith [17] established several remarkable unconditional results towards Goldfeld's conjecture. Notably, in [17, Corollary 1.3], Smith proved that for E, with full rational 2-torsion, such that E has no rational cyclic subgroup of order four, if the Birch and Swinnerton-Dyer conjecture holds for all the quadratic twists of E, then Goldfeld's conjecture holds for E, (i.e., 50% of the quadratic twists of E have analytic rank 0, 50% of the twists have analytic rank 1, and 0% have higher analytic rank); consequently, 100% of the quadratic twists of E by $d \in \mathcal{E}$ have non-vanishing central values $L(\frac{1}{2}, E_d)$.

As remarked in [15], compared to Smith's algebraic approach, the analytic principle of Radziwiłł-Soundararajan is capable of establishing that certain proportion of central values are of the typical size predicted by Keating-Snaith's conjecture under GRH. Unfortunately, it seems hard to combine these two approaches to prove Keating-Snaith's conjecture or remove the assumption of GRH. Nonetheless, the recent work of Bui, Evans, Lester, and Pratt [2] proved analogues of Keating-Snaith's conjecture (for mollified central values) with a full asymptotic. As noted in [15], the vanishing central values are assigned a weight equal to zero in [2], which results in a side effect of little control over the weight.

In a slightly different vein, based on the above-mentioned Keating-Snaith conjecture, Radziwiłł and Soundararajan [15, Conjecture 1] formulated the following conjecture regarding the distribution of orders of Tate-Shafarevich groups $Ш(E_d)$ of E_d.

Conjecture 1 (Radziwiłł-Soundararajan). *Let E be given in Weierstrass form $y^2 = f(x)$ for a monic cubic integral polynomial f, and let K denote the splitting field of f over $\mathbb{Q}$. Define $c(g) \in \mathbb{N}$ so that $c(g) - 1$ is the number of fixed points of $g \in \mathrm{Gal}(K/\mathbb{Q})$, and set*

$$\mu(E) = -\frac{1}{2} - \frac{1}{|G|}\sum_{g \in G} \log c(g) \quad \textit{and} \quad \sigma(E) = 1 + \frac{1}{|G|}\sum_{g \in G} (\log c(g))^2.$$

Then, as d ranges over $\mathcal{E}$, the distribution of $\log(|Ш(E_d)|/\sqrt{|d|})$ is approximately Gaussian, with mean $\mu(E)\log\log|d|$ and variance $\sigma(E)^2 \log\log|d|$. Note that denoting n_K the degree of K, one has the following table of explicit values of $\mu(E)$ and $\sigma(E)^2$.

n_K	1	2	3	6
$\mu(E)$	$-\frac{1}{2} - 2\log 2$	$-\frac{1}{2} - \frac{3}{2}\log 2$	$-\frac{1}{2} - \frac{2}{3}\log 2$	$-\frac{1}{2} - \frac{5}{6}\log 2$
$\sigma(E)^2$	$1 + 4(\log 2)^2$	$1 + \frac{5}{2}(\log 2)^2$	$1 + \frac{4}{3}(\log 2)^2$	$1 + \frac{7}{6}(\log 2)^2$

When $L(\frac{1}{2}, E_d) \neq 0$, there is a nature analytic correspondence of $|Ш(E_d)|$ defined by

$$S(E_d) = L(\tfrac{1}{2}, E_d)\frac{|E_d(\mathbb{Q})_{\mathrm{tors}}|^2}{\Omega(E_d)\,\mathrm{Tam}(E_d)}, \tag{1.2}$$

where $|E_d(\mathbb{Q})_{\mathrm{tors}}|$ denotes the order of the rational torsion group of E_d, $\Omega(E_d)$ is the real period of a minimal model for E_d, and $\mathrm{Tam}(E_d) = \prod_p T_p(d)$ is the product of the Tamagawa numbers. (Note that when $L(\frac{1}{2}, E_d) = 0$, one may set $S(E_d) = 0$ so that (1.2) is still valid.) As a support of their conjecture, Radziwiłł and Soundararajan [14, Theorem 3] established an upper bound for the distribution of $\log(S(E_d)/\sqrt{|d|})$ as follows. For any fixed $V \in \mathbb{R}$, as $X \to \infty$,

$$\#\left\{d \in \mathcal{E}, 20 < |d| \le X : \frac{\log(S(E_d)/\sqrt{|d|}) - \mu(E)\log\log|d|}{\sqrt{\sigma(E)^2 \log\log|d|}} > V\right\}$$

is bounded above by

$$(\Psi(V,\infty) + o(1))\#\{d \in \mathcal{E} : |d| \le X\}.$$

Moreover, if the Birch and Swinnerton-Dyer conjecture holds for elliptic curves with analytic rank zero, then the quantity above is also an upper bound for

$$\#\left\{d \in \mathcal{E}, 20 < |d| \le X : L(\tfrac{1}{2}, E_d) \ne 0, \ \ \frac{\log(|\text{Ш}(E_d)|/\sqrt{|d|}) - \mu(E)\log\log|d|}{\sqrt{\sigma(E)^2 \log\log|d|}} > V\right\}.$$

In light of the work of Radziwiłł and Soundararajan [14, 15] mentioned earlier, we shall prove the following theorem that provides a conditional lower bound for the joint distribution of central values and orders of Tate-Shafarevich groups of quadratic twists of E.

Theorem 1.2. *Assume GRH for the family of twisted L-functions $L(s, E\otimes\chi)$ with all Dirichlet characters χ. For any fixed $\underline{\alpha} = (\alpha_1,\alpha_2)$ and $\underline{\beta} = (\beta_1,\beta_2)$, as $X \to \infty$,*

$$\#\left\{d \in \mathcal{E}, X < |d| \le 2X : \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \in (\alpha_1,\beta_1), \right.$$
$$\left. \frac{\log(S(E_d)/\sqrt{|d|}) - \mu(E)\log\log|d|}{\sqrt{\sigma(E)^2\log\log|d|}} \in (\alpha_2,\beta_2)\right\}$$

is greater or equal to

$$\frac{1}{4}(\Xi_E(\underline{\alpha},\underline{\beta}) + o(1))\#\{d \in \mathcal{E} : X < |d| \le 2X\},$$

where

$$\Xi_E(\underline{\alpha},\underline{\beta}) = \int_{(\alpha_1,\beta_1)\times(\alpha_2,\beta_2)} \frac{1}{2\pi\sqrt{\det(\mathfrak{K}_E)}} e^{-\frac{1}{2}\mathbf{v}^{\mathrm{T}}\mathfrak{K}_E^{-1}\mathbf{v}} d\mathbf{v} \quad \textit{with} \quad \mathfrak{K}_E = \begin{pmatrix} 1 & \sigma(E)^{-1} \\ \sigma(E)^{-1} & 1 \end{pmatrix}. \tag{1.3}$$

Here, as later, $\sigma(E)$ is chosen to be $\sqrt{\sigma(E)^2}$, which is always greater than one (see Conjecture 1).

Furthermore, suppose the Birch and Swinnerton-Dyer conjecture holds for elliptic curves with analytic rank zero. Then the above assertion is true with $S(E_d)$ being replaced by $|\text{Ш}(E_d)|$.

Remarks. (i) It is worth noting that the kernel of the integral in (1.3) is the probability density function of a bivariate normal distribution $(\mathcal{Q}_1, \mathcal{Q}_2)$ such that each $\mathcal{Q}_i$ is standard normal, and the correlation between $\mathcal{Q}_1$ and $\mathcal{Q}_2$ equals $\sigma(E)^{-1}$. In particular, when $(\alpha_2,\beta_2) = \mathbb{R}$, $\Xi_E(\underline{\alpha},\underline{\beta}) = \Psi(\alpha_1,\beta_2)$. Therefore, our result recovers Theorem 1.1 for such an instance.
(ii) Our proof of Theorem 1.2 is a combination of the works of Radziwiłł and Soundararajan [14, 15]. A crucial step is to calculate a "weighted" moments of real combinations of $\mathcal{P}(d;x)$ and $\mathcal{P}(d;x) - \mathcal{C}(d;x)$ (see Proposition 2.2). This led us to refine [15, Proposition 3] as in Proposition 2.1, which particularly gives us the flexibility to introduce a sieve parameter v for d (cf. [14, Proposition 1]).

(iii) A key new input of this article is to invoke the Cramér-Wold device (see Proposition 2.5), which allows us to derive the desired bivariate normal distribution by studying weighted moments of real combinations of $\mathcal{P}(d;x)$ and $\mathcal{P}(d;x)-\mathcal{C}(d;x)$. This idea has its root in the joint work [9] with Hsu, where the Cramér-Wold device was used inexplicitly to establish a "log-independence" between Dirichlet L-functions (over the critical line). We shall present a detailed argument in Lemma 3.2.
(iv) Last but not least, we note that in [2, Theorem 1.4], a log-independence result for weighted central L-values of Dirichlet twists of two distinct newforms was established under GRH. As discussed in [2, Sec. 1.4], the authors directly computed an asymptotic formula for the involving joint distribution, which requires a more complicated calculation in contrast to our argument.

A direct consequence of this theorem is the following corollary that yields conditional support towards Conjecture 1.

Corollary 1.3. *Assume GRH for the family of twisted L-functions $L(s,E\otimes\chi)$ with all Dirichlet characters χ. For any fixed interval (α,β), as $X\to\infty$,*

$$\#\left\{d\in\mathcal{E}, X<|d|\le 2X : \frac{\log(S(E_d)/\sqrt{|d|})-\mu(E)\log\log|d|}{\sqrt{\sigma(E)^2\log\log|d|}}\in(\alpha,\beta)\right\}$$

is greater or equal to

$$\frac{1}{4}(\Psi(\alpha,\beta)+o(1))\#\{d\in\mathcal{E}: X<|d|\le 2X\}.$$

Furthermore, if the Birch and Swinnerton-Dyer conjecture holds for elliptic curves with analytic rank zero, then the quantity above is also a lower bound for

$$\#\left\{d\in\mathcal{E}, X<|d|\le 2X : L(\tfrac{1}{2},E_d)\neq 0,\ \frac{\log(|\text{Ш}(E_d)|/\sqrt{|d|})-\mu(E)\log\log|d|}{\sqrt{\sigma(E)^2\log\log|d|}}\in(\alpha,\beta)\right\}.$$

We note that Corollary 1.3 has an interesting application to "large values" of $|\text{Ш}(E_d)|$ as follows. In [13, Theorem 2], assuming the Birch and Swinnerton-Dyer conjecture holds for elliptic curves with analytic rank zero, Mai and Murty showed that there are infinitely many (square-free) d such that $L(\frac{1}{2},E_d)\neq 0$ and

$$|\text{Ш}(E_d)|\gg_\varepsilon d^{\frac{1}{2}-\varepsilon}$$

(cf. the last two displayed estimates in the proof of [13, Theorem 1]). Under the further assumption of GRH, applying Theorem 1.2 with $(\alpha,\beta)=(0,\infty)$, for example, one can strengthen the work of Mai and Murty to have at least $\frac{1}{8}$ of $d\in\mathcal{E}$ such that $L(\frac{1}{2},E_d)\neq 0$ and

$$|\text{Ш}(E_d)|\ge|d|^{\frac{1}{2}}(\log|d|)^{\mu(E)}.$$

To end the introduction, in light of Theorem 1.2, we impose the following conjecture that presents a common generalisation of Keating-Snaith's conjecture and Radziwiłł-Soundararajan's conjecture mentioned earlier.

Conjecture 2. *In the notation of Conjecture 1, let $\Xi_E(\underline{\alpha},\underline{\beta})$ and $\mathfrak{K}_E$ be as in Theorem 1.3. Then as d ranges over $\mathcal{E}$, the joint distribution of $\log L(\frac{1}{2},E_d)$ and $\log(|\text{Ш}(E_d)|/\sqrt{|d|})$ is approximately* *bivariate with mean $\mathbf{0}_2$ and covariance matrix $\mathfrak{K}_E$. More precisely, as $X \to \infty$,*

$$\#\left\{d \in \mathcal{E}, 20 < |d| \le X : \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} \in (\alpha_1, \beta_1), \right.$$
$$\left. \frac{\log(\text{Ш}(E_d)/\sqrt{|d|}) - \mu(E)\log\log|d|}{\sqrt{\sigma(E)^2\log\log|d|}} \in (\alpha_2, \beta_2)\right\}$$

is asymptotic to $(\Xi_E(\underline{\alpha}, \underline{\beta}) + o(1))\#\{d \in \mathcal{E} : 20 < |d| \le X\}$.

In addition, by slightly modifying an argument of Radziwiłł-Soundararajan in [14], we have the following result supporting this conjecture "from above".

Theorem 1.4. *In the notation as above, for any $V_1, V_2 \in \mathbb{R}$, as $X \to \infty$,*

$$\#\left\{d \in \mathcal{E}, 20 < |d| \le X : \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} > V_1, \right.$$
$$\left. \frac{\log(S(E_d)/\sqrt{|d|}) - \mu(E)\log\log|d|}{\sqrt{\sigma(E)^2\log\log|d|}} > V_2\right\}$$

is bounded above by

$$(\Xi_E((V_1, V_2), (\infty, \infty)) + o(1))\#\{d \in \mathcal{E} : 20 < |d| \le X\}.$$

Furthermore, if the Birch and Swinnerton-Dyer conjecture holds for elliptic curves with analytic rank zero, then the above assertion is valid with $S(E_d)$ being replaced by $|\text{Ш}(E_d)|$.

This article is arranged as follows. We will devote Section 2 to collect the necessary notation and key propositions required for proving Theorem 1.2. The proof of Theorem 1.2 will be given in Section 3. Two key propositions, Propositions 2.1 and 2.2, will be proved in Sections 4 and 5, respectively. Derivation of Theorem 1.4 will be discussed in Section 6. In Section 7, we will provide a sufficient condition on certain zero sums that implies Conjecture 2.

2. Notation and the key propositions

Following [14], we let N_0 denote the lcm of 8 and N. Set $\kappa = \pm 1$, and let $a \pmod{N_0}$ denote a residue class with $a \equiv 1$ or $5 \pmod 8$. In addition, we shall assume that κ and a are such that for any fundamental discriminant d with sign κ, satisfying $d \equiv a \pmod{N_0}$, the root number $\epsilon_E(d) = \epsilon_E\chi_d(-N)$ is equal to 1. For these κ and a, we set

$$\mathcal{E}(\kappa, a) = \{d \in \mathcal{E} : \kappa d > 0,\ \ d \equiv a \pmod{N_0}\}$$

so that $\mathcal{E}$ is the union of $\mathcal{E}(\kappa, a)$. In addition, we note that the imposed congruence condition on d forces $d \equiv 1 \pmod 4$, and thus $d \in \mathcal{E}(\kappa, a)$ must be square-free as d is a fundamental discriminant.

For $\mathfrak{Re}(s) > 1$, we write

$$-\frac{L'}{L}(s, E) = \sum_{n=1}^{\infty} \frac{\Lambda_E(n)}{n^s}$$

with $|\Lambda_E(n)| \leq 2\Lambda(n)$ (so $\Lambda_E(n)$ is supported only on prime powers). Also, for each fundamental discriminant $d \in \mathcal{E}$ and a parameter $x \geq 3$, we define

$$\mathcal{P}(d;x) = \sum_{\substack{p \leq x \\ p \nmid N_0}} \frac{\lambda_E(p)}{\sqrt{p}} \chi_d(p).$$

Recall that E is given by the model $y^2 = f(z)$, where f is a monic cubic integral polynomial. Denoting 1 plus the number of solutions of $f(z) \equiv 0 \pmod p$ by $c(p)$, one knows $c(p) = 1$, 2, or 4; for p not dividing the discriminant $\operatorname{disc}(f)$ of f, one has

$$T_p(d) = \begin{cases} c(p) & \text{if } p \mid d; \\ 1 & \text{if } p \nmid d. \end{cases}$$

In light of this, one may consider

$$\mathcal{C}(d;x) = \sum_{\log X \leq p \leq x} C_p(d) = \sum_{\log X \leq p \leq x} \left(\log T_p(d) - \frac{1}{p+1} \log c(p) \right),$$

for $x > \log X > \max\{N_0, |\operatorname{disc}(f)|\}$, where

$$C_p(d) = \begin{cases} \frac{p}{p+1} \log c(p) & \text{if } p \mid d; \\ -\frac{1}{p+1} \log c(p) & \text{if } p \nmid d. \end{cases}$$

We also recall the following Mertens type estimates, involving $c(p)$, from [14, Lemma 4] (which is a consequence of the Chebotarev density theorem):

$$\sum_{p \leq y} \frac{\log c(p)}{p} = \left(-\mu(E) - \tfrac{1}{2}\right) \log\log y + O(1) \tag{2.1}$$

and

$$\sum_{p \leq y} \frac{(\log c(p))^2}{p} = \left(\sigma(E)^2 - 1\right) \log\log y + O(1). \tag{2.2}$$

Let h be a smooth function with compactly supported Fourier transform $\widehat{h}(\xi) = \int_{\mathbb{R}} h(t) e^{-2\pi i \xi t} dt$, and satisfy $|h(t)| \ll 1/(1+t^2)$ for all real t. In particular, one may take $h(t) = (\frac{\sin(\pi t)}{\pi t})^2$, the Fejér kernel, so that $\widehat{h}(\xi) = \max(1 - |\xi|, 0)$. In addition, let Φ be a smooth non-negative function compactly supported in $[\frac{1}{2}, \frac{5}{2}]$ such that $\Phi(t) = 1$ on $[1,2]$. We set $\check{\Phi}(s) = \int_0^\infty \Phi(t) t^s dt$. It was shown in [14, Proposition 1] that for $n, v \in \mathbb{N}$ coprime to N_0, with $(n,v) = 1$ and v being square-free, such that $v\sqrt{n} \leq X^{\frac{1}{2}-\varepsilon}$, one has

$$\sum_{\substack{d \in \mathcal{E}(\kappa,a) \\ v \mid d}} \chi_d(n) \Phi\left(\frac{\kappa d}{X}\right) = \delta(n = \square) \frac{X}{vN_0} \prod_{p \mid nv} \left(1 + \frac{1}{p}\right)^{-1} \prod_{p \nmid N_0} \left(1 - \frac{1}{p^2}\right) \check{\Phi}(0) + O(X^{\frac{1}{2}+\varepsilon} n^{\frac{1}{2}}), \tag{2.3}$$

where $\delta(n = \square) = 1$ if n is a square, and $\delta(n = \square) = 0$ otherwise. In light of Radziwiłł and Soundararajan's principle and their work [14], we shall prove the following proposition that refines [15, Proposition 2].

Proposition 2.1. *Let h be a smooth function such that $h(t) \ll (1+t^2)^{-1}$, and its Fourier transform is compactly supported in $[-1,1]$. Let $L \geq 1$ be real, and let ℓ and v be positive integers coprime to N_0 such that v is square-free and $(\ell, v) = 1$. Assume further that $e^{\frac{L}{4}}\ell^{\frac{1}{2}} \leq X^{\frac{1}{2}-3\varepsilon}$ and $v \leq X^{\varepsilon}$. Under GRH for all $L(s, E_d \otimes \chi)$ with $d \in \mathcal{E}(\kappa, a)$ and Dirichlet characters χ modulo N_0, we let $\frac{1}{2} + i\gamma_d$ range over the non-trivial zeros of $L(s, E_d)$, and set*

$$\mathcal{S}_{\kappa,a,v} = \sum_{\substack{d \in \mathcal{E}(\kappa,a) \\ v|d}} \sum_{\gamma_d} h\left(\frac{\gamma_d L}{2\pi}\right) \chi_d(\ell) \Phi\left(\frac{\kappa d}{X}\right).$$

If ℓ is not a square nor a prime times a square, then

$$\mathcal{S}_{\kappa,a,v} \ll X^{\frac{1}{2}+3\varepsilon}\ell^{\frac{1}{2}} e^{\frac{L}{4}}.$$

If ℓ is a square, then

$$\mathcal{S}_{\kappa,a,v} = \frac{X}{vN_0} \prod_{p|\ell v}\left(1+\frac{1}{p}\right)^{-1} \prod_{p \nmid N_0}\left(1-\frac{1}{p^2}\right) \check{\Phi}(0)\left(\frac{2\log X}{L}\widehat{h}(0) + \frac{h(0)}{2} + O(L^{-1})\right) + O(X^{\frac{1}{2}+3\varepsilon}\ell^{\frac{1}{2}}e^{\frac{L}{4}}).$$

Lastly, if ℓ is a prime q times a square, then

$$\mathcal{S}_{\kappa,a,v} \ll \frac{X}{vLN_0}\frac{\log q}{\sqrt{q}} \prod_{p|\ell v}\left(1+\frac{1}{p}\right)^{-1} + X^{\frac{1}{2}+3\varepsilon}\ell^{\frac{1}{2}}e^{\frac{L}{4}}.$$

The proof of Theorem 1.2 relies on the "method of moments" asserting roughly that normal random variables are uniquely determined by their moments (see, e.g. [4]). To proceed, we collect some related facts and moment calculations as follows. Let $\mathcal{N}$ be a normal random variable with mean 0 and variance σ^2. The k-th moment M_k of $\mathcal{N}$ is

$$M_k = \mathbb{E}[\mathcal{N}^k] = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ \frac{k!}{2^{k/2}(k/2)!}\sigma^k & \text{if } k \text{ is even,} \end{cases}$$

where $\mathbb{E}[\mathcal{N}^k]$ denotes the mean of $\mathcal{N}^k$. In [15, Proposition 7], Radziwiłł and Soundararajan showed that for any given $k \in \mathbb{Z}^+$ and large X,

$$\sum_{d \in \mathcal{E}(\kappa,a)} (\mathcal{P}(d;x) - \mathcal{C}(d;x))^k \Phi\left(\frac{\kappa d}{X}\right) = \sum_{d \in \mathcal{E}(\kappa,a)} \Phi\left(\frac{\kappa d}{X}\right)\left(\sigma(E)^2 \log\log X\right)^{\frac{k}{2}}(M_k + o(1)), \tag{2.4}$$

with $x = z = X^{1/(\log\log X)^2}$. Moreover, by virtue of the proof of [14, Proposition 7], one can also take $x = X^{1/\log\log\log X}$, which will be our choice throughout our discussion, as well as proving the following generalisation of (2.4):

$$\begin{aligned} &\sum_{d \in \mathcal{E}(\kappa,a)} (b\mathcal{P}(d;x) + c(\mathcal{P}(d;x) - \mathcal{C}(d;x)))^k \Phi\left(\frac{\kappa d}{X}\right) \\ &= \sum_{d \in \mathcal{E}(\kappa,a)} \Phi\left(\frac{\kappa d}{X}\right)\left((b^2 + 2bc + c^2\sigma(E)^2)\log\log X\right)^{\frac{k}{2}}(M_k + o(1)), \end{aligned} \tag{2.5}$$

for any fixed $b, c \in \mathbb{R}$.[1]

Furthermore, we prove the following proposition regarding the "weighted" moments of real linear combinations of $\mathcal{P}(d;x)$ and $\mathcal{P}(d;x) - \mathcal{C}(d;x)$.

Proposition 2.2. *Fix $b, c \in \mathbb{R}$. Then for any $L \geq 1$ such that $e^L \leq X^{2-10\varepsilon}$, we have*

$$\begin{aligned} &\sum_{d \in \mathcal{E}(\kappa,a)} (b\mathcal{P}(d;x) + c(\mathcal{P}(d;x) - \mathcal{C}(d;x)))^k \sum_{\gamma_d} h\left(\frac{\gamma_d L}{2\pi}\right) \Phi\left(\frac{\kappa d}{X}\right) \\ &= \frac{X}{N_0} \prod_{p \nmid N_0} \left(1 - \frac{1}{p^2}\right) \check{\Phi}(0) \left(\frac{2\log X}{L}\widehat{h}(0) + \frac{h(0)}{2} + O(L^{-1})\right) \\ &\times \left((b^2 + 2bc + c^2\sigma(E)^2)\log\log X\right)^{\frac{k}{2}} (M_k + o(1)) + O(X^{\frac{1}{2}+\varepsilon} e^{\frac{L}{4}}), \end{aligned} \tag{2.6}$$

where the implied constants depend on b, c.

In the remaining part of this section, we shall recall some probability theory for the convenience of the reader. For random variables $\mathcal{X}$ and $\mathcal{Y}$, the covariance $\mathrm{Cov}(\mathcal{X}, \mathcal{Y})$ between $\mathcal{X}$ and $\mathcal{Y}$ is defined as $\mathrm{Cov}(\mathcal{X}, \mathcal{Y}) = \mathbb{E}[(\mathcal{X} - \mathbb{E}(\mathcal{X}))(\mathcal{Y} - \mathbb{E}(\mathcal{Y}))]$. The correlation coefficient $\rho(\mathcal{X}, \mathcal{Y})$ between $\mathcal{X}$ and $\mathcal{Y}$ is defined by

$$\rho(\mathcal{X}, \mathcal{Y}) = \mathrm{Cov}(\mathcal{X}, \mathcal{Y}) / \sqrt{\mathrm{Var}(\mathcal{X})\,\mathrm{Var}(\mathcal{Y})}, \tag{2.7}$$

and the variance $\mathrm{Var}(\mathcal{X} + \mathcal{Y})$ of $\mathcal{X} + \mathcal{Y}$ is given by

$$\mathrm{Var}(\mathcal{X} + \mathcal{Y}) = \mathrm{Var}(\mathcal{X}) + \mathrm{Var}(\mathcal{Y}) + 2\,\mathrm{Cov}(\mathcal{X}, \mathcal{Y}). \tag{2.8}$$

Let $\mathbf{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_N)$ be a random vector in $\mathbb{R}^N$. For each j, we set $\sigma_j^2 = \mathrm{Var}(\mathcal{X}_j)$. The random vector $\mathbf{X}$ is called a bivariate normal distribution with mean $\mathbf{0}_N$ and covariance matrix $\mathfrak{K}$ if its probability density function $f_{\mathbf{X}}(\mathbf{v})$ satisfies

$$f_{\mathbf{X}}(\mathbf{v}) = \frac{1}{2\pi\sqrt{\det(\mathfrak{K})}} e^{-\frac{1}{2}\mathbf{v}^{\mathrm{T}}\mathfrak{K}^{-1}\mathbf{v}},$$

where the vector $\mathbf{v}^{\mathrm{T}}$ denotes the transpose of $\mathbf{v}$, and $\mathfrak{K} = (\sigma_{ij})$ is a real $N \times N$ symmetric positive definite matrix with $\sigma_{ii} = \sigma_i^2$ and $\sigma_{ij} = \rho(\mathcal{X}_i, \mathcal{X}_j)\sigma_i\sigma_j$. Also, the characteristic function $\phi_{\mathbf{X}}(\mathbf{v})$ of $\mathbf{X}$, with mean $\mathbf{0}_N$ and covariance matrix $\mathfrak{K}$, is

$$\phi_{\mathbf{X}}(\mathbf{v}) = \exp\left(-\tfrac{1}{2}\mathbf{v}\mathfrak{K}\mathbf{v}^{\mathrm{T}}\right). \tag{2.9}$$

We shall require the following results for bivariate normal distributions.

Proposition 2.3 ([8, Theorem 2.1])**.** *Let $\mathfrak{K}$ be a 2×2 non-singular symmetric positive definite matrix. Then there exists a bivariate normal distribution $\mathbf{X}$ with mean $\mathbf{0}_2$ and covariance matrix $\mathfrak{K}$.*

We also recall the following properties regarding normal random variables.

---

[1]Indeed, the presence of b and c would add the factor $c^j(b+c)^{k-j}$ to the sum in [14, Eq. (15)]. Nonetheless, it can be easily checked that such an extra factor does not affect the proof but appears in the first displayed asymptotics in [14, p. 1051]. This leads to the change of the corresponding terms of $\log\log X$ from $((\sigma(E)^2-1)\log\log X)^{j/2}(\log\log X)^{(k-j)/2}$ to $(c^2(\sigma(E)^2-1)\log\log X)^{j/2}((b+c)^2\log\log X)^{(k-j)/2}$ (note that both j and $k-j$ are even here). From which (and applying the binomial theorem), one obtains the claimed generalisation.

Proposition 2.4 ([3, Theorem 5.5.32]). *Let $(\mathcal{X}_1, \mathcal{X}_2)$ be a pair of normal distributions. Then $(\mathcal{X}_1, \mathcal{X}_2)$ is a bivariate normal distribution if and only if any real linear combination of $\mathcal{X}_1$ and $\mathcal{X}_2$ is a normal distribution.*

Also, we invoke the following important necessary and sufficient condition regarding the convergence in distribution for random vectors, the Cramér-Wold device (see, e.g., [1, Theorem 29.4]).

Proposition 2.5 (Cramér-Wold device). *Let $\mathbf{X}_T = (\mathcal{X}_{1,T}, \mathcal{X}_{2,T})$ and $\mathbf{X} = (\mathcal{X}_1, \mathcal{X}_2)$ be random vector vectors in $\mathbb{R}^2$. Then $\mathbf{X}_T$ converges to $\mathbf{X}$ in distribution, as $T \to \infty$, if and only if $\sum_{j=1}^2 a_j \mathcal{X}_{j,T}$ converges to $\sum_{j=1}^2 a_j \mathcal{X}_j$ in distribution, as $T \to \infty$, for every $(a_1, a_2) \in \mathbb{R}^2$.*

Last but not least, we shall state precisely the method of moments as follows.

Proposition 2.6 (Fréchet and Shohat). *Suppose that the distribution of a random variable $\mathcal{X}$ is determined by its moments and that each $\mathcal{X}_n$ has moments of all orders. If $\mathbb{E}(\mathcal{X}^k) = \mathbb{E}(\mathcal{X}_n^k) + o(\mathbb{E}(\mathcal{X}_n^k))$, as $n \to \infty$, for all $k \in \mathbb{N}$, then $\mathcal{X}_n$ converges to $\mathcal{X}$ in distribution.*

3. Proof of the main theorem

To prove our main theorem, we shall adapt the strategy of Radziwiłł and Soundararajan as in [15], which requires the following proposition.

Proposition 3.1 ([15, Proposition 1]). *Let $d \in \mathcal{E}$, and let $3 \le x \le |d|$. Assume GRH for $L(s, E_d)$, and let $\frac{1}{2} + i\gamma_d$ run over the non-trivial zeros of $L(s, E_d)$. If $L(\frac{1}{2}, E_d)$ is non-vanishing, then*

$$\log L(\tfrac{1}{2}, E_d) = \mathcal{P}(d; x) - \tfrac{1}{2} \log\log x + O\bigg(\frac{\log |d|}{\log x} + \sum_{\gamma_d} \log\bigg(1 + \frac{1}{(\gamma_d \log x)^2}\bigg)\bigg).$$

Recalling the definition of $S(E_d)$ in (1.2), since $\Omega(E_d) \asymp 1/\sqrt{|d|}$, and $|E_d(\mathbb{Q})_{\mathrm{tors}}|$ is bounded, we derive

$$\log S(E_d) = \log L(\tfrac{1}{2}, E_d) + \log \sqrt{|d|} - \log \mathrm{Tam}(E_d) + O(1). \tag{3.1}$$

Moreover, as shown in [15, pp. 1047-1048], one has

$$\begin{aligned}&\sum_{\substack{d \in \mathcal{E} \\ X/\log X \le |d| \le X}} |\log \mathrm{Tam}(E_d) + (\mu(E) + \tfrac{1}{2}) \log\log X - \mathcal{C}(d; x)| \\ &\ll X \log\log\log X + X \sum_{p < \log X} \frac{\log c(p)}{p} + X \sum_{x < p \le X} \frac{\log c(p)}{p} \\ &\ll X \log\log\log X + X \log\log\log\log X,\end{aligned}$$

where the last estimate follows from (2.1). Thus, the number of $d \in \mathcal{E}$, with $X/\log X \le |d| \le X$, such that

$$|\log \mathrm{Tam}(E_d) + (\mu(E) + \tfrac{1}{2}) \log\log X - \mathcal{C}(d; x)| \ge (\log\log\log X)^2 \tag{3.2}$$

is at most $\ll X/\log\log\log X$. Therefore, the number of $d \in \mathcal{E}$ with $20 \le |d| \le X$ (and thus $X \le |d| \le 2X$) satisfying (3.2) is $\ll X/\log\log\log X$. Hence, except for at most $\ll X/\log\log\log X$

$d \in \mathcal{E}$ with $X \le |d| \le 2X$, we have

$$\log S(E_d) = \log L(\tfrac{1}{2}, E_d) + \log \sqrt{|d|} + (\mu(E) + \tfrac{1}{2}) \log\log X - \mathcal{C}(d;x) + O((\log\log\log X)^2).$$

Moreover, by Proposition 3.1, for every $d \in \mathcal{E}$ with $X \le |d| \le 2X$, one has

$$\log L(\tfrac{1}{2}, E_d) = \mathcal{P}(d;x) - \tfrac{1}{2}\log\log X + O\Bigg(\log\log\log X + \sum_{\gamma_d} \log\left(1 + \frac{1}{(\gamma_d \log x)^2}\right)\Bigg),$$

and thus

$$\begin{aligned} &\log(S(E_d)/\sqrt{|d|}) - \mu(E)\log\log|d| \\ &= \mathcal{P}(d;x) - \mathcal{C}(d;x) + O\Bigg((\log\log\log X)^2 + \sum_{\gamma_d} \log\left(1 + \frac{1}{(\gamma_d \log x)^2}\right)\Bigg) \end{aligned} \tag{3.3}$$

for all but at most $\ll X/\log\log\log X$ $d \in \mathcal{E}$ with $X \le |d| \le 2X$.

Similar to [15, Lemma 1], we prove the following lemma regarding a lower bound for the joint distribution of $\mathcal{P}(d;x)$ and $\mathcal{P}(d;x) - \mathcal{C}(d;x)$.

Lemma 3.2. *Let $\alpha_i < \beta_i$ be real numbers, and set $\underline{\alpha} = (\alpha_1, \alpha_2)$ and $\underline{\beta} = (\beta_1, \beta_2)$. Let $\mathcal{H}_X(\underline{\alpha}, \underline{\beta})$ be the set of discriminants $d \in \mathcal{E}$, with $X \le |d| \le 2X$, such that*

$$\mathcal{Q}_1(d;X) = \frac{\mathcal{P}(d;x)}{\sqrt{\log\log X}} \in (\alpha_1, \beta_1) \quad \text{and} \quad \mathcal{Q}_2(d;X) = \frac{\mathcal{P}(d;x) - \mathcal{C}(d;x)}{\sqrt{\sigma(E)^2 \log\log X}} \in (\alpha_2, \beta_2),$$

while $L(s, E_d)$ has no zeros $\frac{1}{2} + \gamma_d$ with $|\gamma_d| \le ((\log X)(\log\log X))^{-1}$. Then for any $\delta > 0$, we have

$$\mathcal{H}_X(\alpha, \beta) \ge \left(\frac{1}{4} - \delta\right)(\Xi_E(\underline{\alpha}, \underline{\beta}) + o(1)) \#\{d \in \mathcal{E} : X \le |d| \le 2X\},$$

where $\Xi_E(\underline{\alpha}, \underline{\beta})$ is defined as in (1.3).

Proof. Let Φ be a smooth approximation to the indicator function of $[1,2]$, and let κ and $a \pmod{N_0}$ be as in Section 2. It follows from (2.5) and the method of moments (more precisely, Proposition 2.6) that for any $a_1, a_2 \in \mathbb{R}$, $a_1\mathcal{Q}_1(d;X) + a_2\mathcal{Q}_2(d;X)$ converges to a normal random variable $\mathcal{Z}_{a_1,a_2}$, with mean 0 and variance $a_1^2 + 2a_1a_2\sigma(E)^{-1} + a_2^2$, in distribution. (Here, as later, we choose $\sigma(E) = \sqrt{\sigma(E)^2} > 0$.) Note that by (2.9), the characteristic function of $\mathcal{Z}_{a_1,a_2}$ is

$$\phi_{\mathcal{Z}_{a_1,a_2}}(u) = \exp\left(-\tfrac{1}{2}\left(a_1^2 + 2a_1a_2\sigma(E)^{-1} + a_2^2\right)u^2\right). \tag{3.4}$$

On the other hand, as $0 < \sigma(E)^{-1} < 1$, Sylvester's criterion (see, e.g., [5]) tells us that the matrix

$$\mathfrak{K}_E = \begin{pmatrix} 1 & \sigma(E)^{-1} \\ \sigma(E)^{-1} & 1 \end{pmatrix} \tag{3.5}$$

is always positive definite. Hence, by Proposition 2.3, there is a bivariate normal distribution $(\mathcal{Q}_1, \mathcal{Q}_2)$ such that each $\mathcal{Q}_i$ is standard normal, and the correlation between $\mathcal{Q}_1$ and $\mathcal{Q}_2$ equals $\sigma(E)^{-1}$. Therefore, by Proposition 2.4, for any $a_1, a_2 \in \mathbb{R}$, $a_1\mathcal{Q}_1 + a_2\mathcal{Q}_2$ is a normal distribution. In addition, as $\mathrm{Var}(\mathcal{Q}_i) = 1$ for each i, (2.7) implies that $\mathrm{Cov}(\mathcal{Q}_1, \mathcal{Q}_2) = \sigma(E)^{-1}$, and thus (2.8) gives

$$\mathrm{Var}(a_1\mathcal{Q}_1 + a_2\mathcal{Q}_2) = a_1^2\,\mathrm{Var}(\mathcal{Q}_1) + a_2^2\,\mathrm{Var}(\mathcal{Q}_2) + 2a_1a_2\,\mathrm{Cov}(\mathcal{Q}_1, \mathcal{Q}_2) = a_1^2 + 2a_1a_2\sigma(E)^{-1} + a_2^2.$$

From which, we derive that the characteristic function of $a_1\mathcal{Q}_1 + a_2\mathcal{Q}_2$ is the same as the right of (3.4), i.e., the characteristic function of $\mathcal{Z}_{a_1,a_2}$. Recalling that the characteristic function uniquely determines the distribution (see, e.g., [16, Theorem 9.5.1]), we then conclude that

$$\mathcal{Z}_{a_1,a_2} = a_1\mathcal{Q}_1 + a_2\mathcal{Q}_2$$

in distribution. Consequently, $a_1\mathcal{Q}_1(d;X) + a_2\mathcal{Q}_2(d;X)$ converges to $a_1\mathcal{Q}_1 + a_2\mathcal{Q}_2$ in distribution for any real a_i. From this and Proposition (2.5), it follows that $(\mathcal{Q}_1(d;X), \mathcal{Q}_2(d;X))$ converges to $(\mathcal{Q}_1, \mathcal{Q}_2)$, in distribution, whose joint probability distribution is given by $\frac{1}{2\pi\sqrt{\det(\mathfrak{K}_E)}}e^{-\frac{1}{2}\mathbf{v}^{\mathrm{T}}\mathfrak{K}_E^{-1}\mathbf{v}}$, with $\mathfrak{K}_E$ defined in (3.5). In other words, we have shown that

$$\sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ \mathcal{Q}_i(d;X)\in(\alpha_i,\beta_i)\,\forall i}} \Phi\left(\frac{\kappa d}{X}\right) = (\Xi_E(\underline{\alpha},\underline{\beta}) + o(1)) \sum_{d\in\mathcal{E}(\kappa,a)} \Phi\left(\frac{\kappa d}{X}\right). \tag{3.6}$$

Now, following [15], we choose h to be the Fejér kernel while taking $L = (2-\eta)\log X$. By Proposition 2.2 and an analogous argument as above, we have

$$\begin{aligned}\sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ \mathcal{Q}_i(d;X)\in(\alpha_i,\beta_i)\,\forall i}} \sum_{\gamma_d} h\left(\frac{\gamma_d L}{2\pi}\right)\Phi\left(\frac{\kappa d}{X}\right) &= (\Xi_E(\underline{\alpha},\underline{\beta}) + o(1)) \sum_{d\in\mathcal{E}(\kappa,a)}\sum_{\gamma_d} h\left(\frac{\gamma_d L}{2\pi}\right)\Phi\left(\frac{\kappa d}{X}\right)\\ &= (\Xi_E(\underline{\alpha},\underline{\beta}) + o(1))\left(\frac{1}{1-\frac{\eta}{2}} + \frac{1}{2} + o(1)\right)\sum_{d\in\mathcal{E}(\kappa,a)} \Phi\left(\frac{\kappa d}{X}\right).\end{aligned} \tag{3.7}$$

As noted in [15, Proof of Lemma 1], the "weight" $\sum_{\gamma_d} h(\frac{\gamma_d L}{2\pi})$ is ≥ 0, and if $L(s,E_d)$ has a zero with $|\gamma_d| \leq ((\log X)(\log\log X))^{-1}$ then such a weight has to be $\geq 2 + o(1)$ (since there would either be a complex conjugate pair of such zeros, or a double zero at $s = \frac{1}{2}$ as $\Lambda'(s,E_d) = -\Lambda'(1-s,E_d)$ for $d\in\mathcal{E}$). Denote $\mathcal{Z}$ the set of fundamental discriminants $d\in\mathcal{E}$ such that $L(s,E_d)$ has no zeros with $|\gamma_d| \leq ((\log X)(\log\log X))^{-1}$. Using (3.6) and (3.7), we then deduce that

$$\left(\frac{1}{1-\frac{\eta}{2}} + \frac{1}{2} + o(1)\right) \sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ \mathcal{Q}_i(d;X)\in(\alpha_i,\beta_i)\,\forall i}} \Phi\left(\frac{\kappa d}{X}\right) = \sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ \mathcal{Q}_i(d;X)\in(\alpha_i,\beta_i)\,\forall i}} \sum_{\gamma_d} h\left(\frac{\gamma_d L}{2\pi}\right)\Phi\left(\frac{\kappa d}{X}\right)$$

is greater or equal to

$$0 + 2\sum_{\substack{d\in\mathcal{E}(\kappa,a)\backslash\mathcal{Z}\\ \mathcal{Q}_i(d;X)\in(\alpha_i,\beta_i)\,\forall i}} \Phi\left(\frac{\kappa d}{X}\right) = 2\sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ \mathcal{Q}_i(d;X)\in(\alpha_i,\beta_i)\,\forall i}} \Phi\left(\frac{\kappa d}{X}\right) - 2\sum_{\substack{d\in\mathcal{E}(\kappa,a)\cap\mathcal{Z}\\ \mathcal{Q}_i(d;X)\in(\alpha_i,\beta_i)\,\forall i}} \Phi\left(\frac{\kappa d}{X}\right)$$

as $\mathcal{E}(\kappa,a)$ is the disjoint union of $\mathcal{E}(\kappa,a)\cap\mathcal{Z}$ and $\mathcal{E}(\kappa,a)\backslash\mathcal{Z}$. Therefore, we arrive at

$$\sum_{\substack{d\in\mathcal{E}(\kappa,a)\cap\mathcal{Z}\\ \mathcal{Q}_i(d;X)\in(\alpha_i,\beta_i)\,\forall i}} \Phi\left(\frac{\kappa d}{X}\right) \geq \left(1 - \frac{1}{2-\eta} - \frac{1}{4} + o(1)\right) \sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ \mathcal{Q}_i(d;X)\in(\alpha_i,\beta_i)\,\forall i}} \Phi\left(\frac{\kappa d}{X}\right),$$

which completes the proof upon summing over all possible pairs (κ, a). □

Remarks. By (2.4) and (2.5), with $(b,c)=(0,1)$, as established in [15], each $\mathcal{Q}_i(d;X)\to\mathcal{Q}_i$ in distribution as $X\to\infty$. However, it should be noted that in general, these convergence do not imply that the random vector $(\mathcal{Q}_1(d;X),\mathcal{Q}_2(d;X))$ converges to $(\mathcal{Q}_1,\mathcal{Q}_2)$ in distribution. Moreover, each $\mathcal{Q}_i$ being standard normal does not imply that $\mathcal{Q}_i(d;X)\to\mathcal{Q}_i$ forms a Gaussian random vector. These technical issues are the very reason why we invoked the existence theorem of bivariate normal random variables as well as the Cramér-Wold device.

To continue, we recall the following result from [15, Lemma 2].

Lemma 3.3 (Radziwiłł-Soundararajan)**.** *The number of discriminants $d\in\mathcal{E}$, with $X\le|d|\le 2X$, such that*

$$\sum_{|\gamma_d|\ge((\log X)(\log\log X))^{-1}}\log\left(1+\frac{1}{(\gamma_d\log x)^2}\right)\ge(\log\log\log X)^3$$

is $\ll X/\log\log\log X$.

Now, we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. For $d\in\mathcal{H}_X(\underline{\alpha},\underline{\beta})$, Lemma 3.2 allows us to make the arrangement

$$\frac{\mathcal{P}(d;x)}{\sqrt{\log\log X}}\in(\alpha_1,\beta_1)\quad\text{and}\quad\frac{\mathcal{P}(d;x)-\mathcal{C}(d;x)}{\sqrt{\sigma(E)^2\log\log X}}\in(\alpha_2,\beta_2),$$

while $L(s,E_d)$ has no zeros with $|\gamma_d|\le((\log X)(\log\log X))^{-1}$. Furthermore, applying Lemma 3.3, we can discard $\ll X/\log\log\log X$ elements from $\mathcal{H}_X(\underline{\alpha},\underline{\beta})$ so that the contribution of zeros with $|\gamma_d|\ge((\log X)(\log\log X))^{-1}$ to the last sum in (3.3) is $<(\log\log\log X)^3$. Consequently, there are at least

$$\left(\frac{1}{4}-\delta\right)(\Xi_E(\underline{\alpha},\underline{\beta})+o(1))\#\{d\in\mathcal{E}:X\le|d|\le 2X\}$$

fundamental discriminants $d\in\mathcal{E}$ with $X\le|d|\le 2X$ such that

$$\frac{\log L(\frac{1}{2},E_d)+\frac{1}{2}\log\log X}{\sqrt{\log\log X}}=\frac{\mathcal{P}(d;x)}{\sqrt{\log\log X}}+O\left(\frac{(\log\log\log X)^3}{\sqrt{\log\log X}}\right)\in(\alpha_1,\beta_1).$$

and

$$\frac{\log(S(E_d)/\sqrt{|d|})-\mu(E)\log\log|d|}{\sqrt{\sigma(E)^2\log\log X}}=\frac{\mathcal{P}(d;x)-\mathcal{C}(d;x)}{\sqrt{\sigma(E)^2\log\log X}}+O\left(\frac{(\log\log\log X)^3}{\sqrt{\log\log X}}\right)\in(\alpha_2,\beta_2).$$

Combined with $X\le|d|\le 2X$, these complete the proof. □

4. Proof of Proposition 2.1

Recall that the explicit formula [15, Eq. (16)] states

$$\sum_{\gamma_d}h\left(\frac{\gamma_d L}{2\pi}\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}h\left(\frac{tL}{2\pi}\right)\left(\log\frac{Nd^2}{(2\pi)^2}+2\mathfrak{Re}\frac{\Gamma'}{\Gamma}(1+it)\right)dt$$
$$-\frac{1}{L}\sum_{n=1}^{\infty}\frac{\Lambda_E(n)}{\sqrt{n}}\chi_d(n)\left(\widehat{h}\left(\frac{\log n}{L}\right)+\widehat{h}\left(-\frac{\log n}{L}\right)\right).$$

This allows us to write

$$\mathcal{S}_{\kappa,a,v}=\sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ v|d}}\sum_{\gamma_d}h\left(\frac{\gamma_d L}{2\pi}\right)\chi_d(\ell)\Phi\left(\frac{\kappa d}{X}\right)=S_1-S_2,$$

where

$$S_1=\frac{1}{2\pi}\int_{-\infty}^{\infty}h\left(\frac{tL}{2\pi}\right)\sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ v|d}}\left(\log\frac{Nd^2}{(2\pi)^2}+2\mathfrak{Re}\frac{\Gamma'}{\Gamma}(1+it)\right)\chi_d(\ell)\Phi\left(\frac{\kappa d}{X}\right)dt$$

and

$$S_2=\frac{1}{L}\sum_{n=1}^{\infty}\frac{\Lambda_E(n)}{\sqrt{n}}\left(\widehat{h}\left(\frac{\log n}{L}\right)+\widehat{h}\left(-\frac{\log n}{L}\right)\right)\sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ v|d}}\chi_d(\ell n)\Phi\left(\frac{\kappa d}{X}\right).$$

As remarked in [15], it is relatively easy to calculate S_1 by using (2.3), which yields

$$S_1=\delta(\ell=\Box)\frac{X}{vN_0}\prod_{p|\ell v}\left(1+\frac{1}{p}\right)^{-1}\prod_{p\nmid N_0}\left(1-\frac{1}{p^2}\right)\check{\Phi}(0)(2\log X+O(1))\frac{\widehat{h}(0)}{L}+O(X^{\frac{1}{2}+\varepsilon}\ell^{\frac{1}{2}}).$$

Now, we turn our attention to S_2. We start by noting that as $v\mid d$, $\chi_d(\ell n)=0$ if $(n,v)>1$. Hence, we may assume $(n,v)=1$ throughout our argument (especially, for n appearing in S_2). Moreover, as argued in the paragraph leading to [15, Eq. (22)], since d is fixed in a residue class $(\operatorname{mod} N_0)$, if n is a prime power dividing N_0, then $\chi_d(n)$ is determined by the congruence condition on d. Consequently, as $v\sqrt{\ell}\le X^{\frac{1}{2}-2\varepsilon}$ by our assumption, it follows from (2.3) that adding the condition $(n,N_0)=1$ to the involving sum in S_2 contributes an error at most

$$\ll\frac{1}{L}\sum_{(n,N_0)>1}\frac{\Lambda(n)}{\sqrt{n}}\left|\sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ v|d}}\chi_d(\ell)\Phi\left(\frac{\kappa d}{X}\right)\right|\ll\delta(\ell=\Box)\frac{X}{vL}+X^{\frac{1}{2}+\varepsilon}\ell^{\frac{1}{2}}.$$

Recall that $d\in\mathcal{E}(\kappa,a)$ has to be square-free. So, for d coprime to N_0 such that $v\mid d$, since v is also square-free and coprime to N_0, one has

$$\sum_{\beta|(v,d/v)}\mu(\beta)\sum_{\substack{(\alpha,vN_0)=1\\ \alpha^2|d/v}}\mu(\alpha)=\begin{cases}1 & \text{if } d \text{ is square-free and } v\mid d;\\ 0 & \text{otherwise.}\end{cases}$$

From this, as in [14, Eq. (20)], writing $d=kv\beta\alpha^2$, one then obtains

$$\sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ v|d}}\chi_d(\ell n)\Phi\left(\frac{\kappa d}{X}\right)=\sum_{\beta|v}\sum_{(\alpha,\ell nvN_0)=1}\mu(\beta)\mu(\alpha)\left(\frac{v\beta\alpha^2}{\ell n}\right)\sum_{k\equiv\overline{av\beta\alpha^2}(\operatorname{mod} N_0)}\left(\frac{k}{\ell n}\right)\Phi\left(\frac{\kappa kv\beta\alpha^2}{X}\right).$$

For a positive parameter $A\le X$ to be chosen later, bounding the sum over k trivially, one can see that the contribution of the terms in the expression with $\alpha>A$ is

$$\ll\sum_{\beta|v}\sum_{\alpha>A}\frac{X}{v\beta\alpha^2}\ll\frac{Xv^{\varepsilon}}{vA}.$$

Therefore, under GRH, we know the terms with $\alpha > A$ in S_2 the contributes an error at most

$$\ll \frac{1}{L}\sum_{n=1}^{\infty}\frac{\Lambda_E(n)}{\sqrt{n}}\left(\widehat{h}\left(\frac{\log n}{L}\right)+\widehat{h}\left(-\frac{\log n}{L}\right)\right)\frac{Xv^{\varepsilon}}{vA} \ll \frac{XLv^{\varepsilon}}{vA} \ll \frac{X\log X}{v^{1-\varepsilon}A}. \tag{4.1}$$

To handle remaining terms in S_2, setting

$$H(\xi) = \widehat{h}(\xi) + \widehat{h}(-\xi),$$

we can write the terms with $\alpha \leq A$ in S_2 as

(4.2)

$$\frac{1}{L}\sum_{n=1}^{\infty}\frac{\Lambda_E(n)}{\sqrt{n}}H\left(\frac{\log n}{L}\right)\sum_{\beta|v}\sum_{\substack{(\alpha,\ell nvN_0)=1\\ \alpha\leq A}}\mu(\beta)\mu(\alpha)\left(\frac{v\beta\alpha^2}{\ell n}\right)\sum_{k\equiv a\overline{v\beta\alpha^2}(\mathrm{mod}\, N_0)}\left(\frac{k}{\ell n}\right)\Phi\left(\frac{\kappa kv\beta\alpha^2}{X}\right).$$

Applying the Poisson summation, as stated in [14, Lemma 7], one can express the inner sum over k in (4.2) as

$$\frac{X}{\ell nN_0v\beta\alpha^2}\left(\frac{\kappa N_0}{\ell n}\right)\sum_{m}\check{\Phi}\left(\frac{Xm}{\ell nN_0v\beta\alpha^2}\right)e\left(\frac{ma\overline{v\beta\alpha^2\ell n}}{N_0}\right)\tau_m(\ell n), \tag{4.3}$$

where $\tau_m(\ell n)$ is a Gauss sum defined by

$$\tau_m(\ell n) = \sum_{b(\mathrm{mod}\,\ell n)}\left(\frac{b}{\ell n}\right)e\left(\frac{mb}{\ell n}\right) = \left(\frac{1+i}{2}+\left(\frac{-1}{\ell n}\right)\frac{1-i}{2}\right)G_m(\ell n),$$

and

$$G_m(\ell n) = \left(\frac{1-i}{2}+\left(\frac{-1}{\ell n}\right)\frac{1+i}{2}\right)\sum_{b(\mathrm{mod}\,\ell n)}\left(\frac{b}{\ell n}\right)e\left(\frac{mb}{\ell n}\right).$$

We first consider the terms with $m \neq 0$, and note that we may assume $n \leq e^L$ as $\widehat{h}$ is compactly supported in $[-1,1]$ (and so is H). In addition, since $\check{\Phi}(s) \ll_K \frac{1}{|s|^K}$ for any $K > 0$, the terms with $|m| > B$ in (4.3) contributes at most

$$\ll_K \frac{X}{\ell nN_0v\beta\alpha^2}\sum_{|m|>B}\left(\frac{\ell nN_0v\beta\alpha^2}{Xm}\right)^K(\ell n) \ll \ell n\left(\frac{\ell nN_0v\beta\alpha^2}{X}\right)^{K-1}\frac{1}{B^{K-1}} \ll \ell nX^{-(K-1)\varepsilon}$$

where we used the trivial bound $|\tau_m(\ell n)| \leq \ell n$, provided that $B \geq \ell e^L A^2 X^{-1+3\varepsilon}$ and $K > 1$. Hence, choosing K to be sufficiently large (depending on $\varepsilon > 0$), the contribution of the terms with $|m| > \ell e^L A^2 X^{-1+3\varepsilon}$ from (4.3) to S_2 is $\ll 1$.

So, remaining terms (with $0 < |m| \leq \ell e^L A^2 X^{-1+3\varepsilon}$) in S_2 becomes

$$\frac{1}{L}\sum_{n=1}^{\infty}\frac{\Lambda_E(n)}{\sqrt{n}}H\left(\frac{\log n}{L}\right)\sum_{\beta|v}\sum_{\substack{(\alpha,\ell nvN_0)=1\\ \alpha\leq A}}\mu(\beta)\mu(\alpha)\left(\frac{v\beta\alpha^2}{\ell n}\right)$$
$$\times\frac{X}{\ell nN_0v\beta\alpha^2}\left(\frac{\kappa N_0}{\ell n}\right)\sum_{0<|m|\leq\ell e^LA^2X^{-1+3\varepsilon}}\check{\Phi}\left(\frac{Xm}{\ell nN_0v\beta\alpha^2}\right)e\left(\frac{ma\overline{v\beta\alpha^2\ell n}}{N_0}\right)\tau_m(\ell n).$$

Upon a rearrangement, it is

$$
\begin{aligned}
(4.4)\qquad & \frac{X}{\ell L N_0} \sum_{0<|m|\le \ell e^L A^2 X^{-1+3\varepsilon}} \sum_{\beta | v} \sum_{\substack{(\alpha,\ell v N_0)=1\\ \alpha\le A}} \frac{\mu(\beta)\mu(\alpha)}{v\beta\alpha^2} \sum_{(n,\alpha N_0)=1} \frac{\Lambda_E(n)}{n\sqrt{n}} H\left(\frac{\log n}{L}\right) \\
& \times \left(\frac{\kappa v\beta N_0}{\ell n}\right) \check{\Phi}\left(\frac{Xm}{\ell n N_0 v\beta\alpha^2}\right) e\Big(\frac{ma\overline{v\beta\alpha^2\ell n}}{N_0}\Big)\tau_m(\ell n).
\end{aligned}
$$

In light of relation between $\tau_m(\ell n)$ and $G_m(\ell n)$, to handle the sum over n, we shall bound

$$
(4.5)\qquad \sum_{(n,\alpha v N_0)=1} \frac{\Lambda_E(n)}{n\sqrt{n}} H\left(\frac{\log n}{L}\right)\left(\frac{\pm\kappa v\beta N_0}{\ell n}\right) \check{\Phi}\left(\frac{Xm}{\ell n N_0 v\beta\alpha^2}\right) e\Big(\frac{ma\overline{v\beta\alpha^2\ell n}}{N_0}\Big) G_m(\ell n),
$$

and we therefore require the following explicit calculation for $G_m(\ell n)$ from [18, Lemma 2.3].

Lemma 4.1. *If ℓ and n are coprime and odd, then $G_m(\ell n) = G_m(\ell)G_m(n)$. Moreover, if p^α is the largest prime power of p that divides m (when $m = 0$, setting $\alpha = \infty$), then*

$$
G_m(p^\beta) = \begin{cases} \phi(p^\beta) & \text{if } \beta \le \alpha \text{ is even;} \\ -p^\alpha & \text{if } \beta = \alpha+1 \text{ is even;} \\ (\frac{mp^{-\alpha}}{p})p^{\alpha+\frac{1}{2}} & \text{if } \beta = \alpha+1 \text{ is odd;} \end{cases}
$$

and $G_m(p^\beta) = 0$ if $\beta \le \alpha$ is odd, or $\beta \ge \alpha+2$.

As discussed in [15, p. 12], if n is a prime power such that $(n,m) = 1$, then $G_m(\ell n) = 0$ unless n is a prime $p \nmid \ell$ (for such in instance, $G_m(\ell p) = (\frac{m}{p})p^{\frac{1}{2}}G_m(\ell)$). The contribution of these terms to (4.5) is

$$
G_m(\ell)\left(\frac{\pm\kappa v\beta N_0}{\ell}\right) \sum_{(p,\alpha\ell m v N_0)=1} \frac{\Lambda_E(p)}{p} \left(\frac{\pm\kappa m v\beta N_0}{p}\right) e\Big(\frac{ma\overline{v\beta\alpha^2\ell p}}{N_0}\Big) H\left(\frac{\log p}{L}\right) \check{\Phi}\left(\frac{Xm}{\ell p N_0 v\beta\alpha^2}\right).
$$

Also, as $\check{\Phi}(s)$ decays rapidly, we may further consider

$$
(4.6)\qquad p > \frac{X^{1-\varepsilon}|m|}{\ell N_0 v\beta\alpha^2}.
$$

(Otherwise, we have $\frac{X|m|}{\ell p N_0 v\beta\alpha^2} \ge X^\varepsilon$. This forces $\check{\Phi}\left(\frac{Xm}{\ell p N_0 v\beta\alpha^2}\right) \ll_K X^{-K\varepsilon}$ for any $K > 0$, which gives a negligible error.) By splitting p into arithmetic progressions modulo N_0, it suffices to estimate

$$
G_m(\ell) \sum_{\substack{(p,\alpha\ell m v N_0)=1\\ p\equiv c (\operatorname{mod} N_0)}} \frac{\Lambda_E(p)}{p} \left(\frac{p}{q}\right) H\left(\frac{\log p}{L}\right) \check{\Phi}\left(\frac{Xm}{\ell p N_0 v\beta\alpha^2}\right),
$$

for c coprime to N_0 and $q \mid mv\beta$, where the sum runs over p satisfying (4.6). Under GRH for $L(s,E)$ and its twists by quadratic characters and Dirichlet characters modulo N_0, Abel's summation then yields the bound

$$
\ll |G_m(\ell)| \left(\frac{\ell N_0 v\beta\alpha^2}{X^{1-\varepsilon}|m|}\right)^{\frac{1}{2}} \log^2 X \ll \frac{X^\varepsilon \ell^{\frac{3}{2}} (v\beta)^{\frac{1}{2}}\alpha}{\sqrt{X|m|}}.
$$

For the case that n is a power of a prime dividing m. As $G_m(\ell n) = 0$ for $n \nmid m^2$, we may assume $n \mid m^2$. Again, by the rapid decay of $\breve{\Phi}(s)$, we may assume

$$n > \frac{X^{1-\varepsilon}|m|}{\ell N_0 v\beta\alpha^2}.$$

Since $|G_m(\ell n)| \leq (|m|\ell n)^{\frac{1}{2}}$, the contribution of these terms to (4.5) is

$$\ll \sum_{n|m^2} \Lambda(n) \frac{(|m|\ell)^{\frac{1}{2}}}{(X^{1-\varepsilon}|m|)/(\ell N_0 v\beta\alpha^2)} \ll (\log m) X^{\varepsilon} \frac{\ell^{\frac{3}{2}} v\beta\alpha^2}{X|m|^{\frac{1}{2}}} \ll \frac{X^{2\varepsilon}\ell^{\frac{3}{2}}(v\beta)^{\frac{1}{2}}\alpha}{\sqrt{X|m|}},$$

as $\log m \ll \log X$, provided that $\alpha \leq A \leq \sqrt{X}$.

Thus, (4.4) becomes

$$\ll \frac{X}{\ell L N_0} \sum_{0<|m|\leq \ell e^L A^2 X^{-1+3\varepsilon}} \sum_{\beta|v} \sum_{\alpha \leq A} \frac{1}{v\beta\alpha^2} \cdot \frac{X^{2\varepsilon}\ell^{\frac{3}{2}}(v\beta)^{\frac{1}{2}}\alpha}{\sqrt{X|m|}} \ll X^{\frac{1}{2}+3\varepsilon}\ell^{\frac{1}{2}} \sum_{0<|m|\leq \ell e^L A^2 X^{-1+3\varepsilon}} \frac{\log A}{\sqrt{|m|}},$$

which is

$$\ll \ell e^{\frac{L}{2}} A X^{5\varepsilon},$$

provided that $A \leq \sqrt{X}$. Balancing this with (4.1), we shall choose

$$A = X^{\frac{1}{2}-2\varepsilon}\ell^{-\frac{1}{2}}e^{-\frac{L}{4}}, \tag{4.7}$$

which gives the both error is at most $O(X^{\frac{1}{2}+3\varepsilon}\ell^{\frac{1}{2}}e^{\frac{L}{4}})$.

Finally, we focus on the main term of S_2 arising from terms with $m = 0$. Recall that $\tau_0(\ell n) = \phi(\ell n)$ if ℓn is a square, and $\tau_0(\ell n) = 0$ otherwise. From (4.2) and (4.3), combined with the above discussion for errors, it follows that the main term of S_2 is

$$\frac{X}{vLN_0}\breve{\Phi}(0) \sum_{\beta|v} \frac{\mu(\beta)}{\beta} \sum_{\substack{(n,vN_0)=1 \\ \ell n = \square}} \sum_{\substack{(\alpha,\ell n v N_0)=1 \\ \alpha \leq A}} \frac{\mu(\alpha)}{\alpha^2} \frac{\Lambda_E(n)}{\sqrt{n}} \frac{\phi(\ell n)}{\ell n} \left(\widehat{h}\left(\frac{\log n}{L}\right) + \widehat{h}\left(-\frac{\log n}{L}\right)\right), \tag{4.8}$$

where we used the condition $(\ell, v) = 1$.

Observe that as n in the sum of S_2 must be a prime power, to have $n\ell = \square$, ℓ can only be a square, or a prime times a square. Firstly, when ℓ is a square, writing $n = r^2$, by a direct calculation, we can express the inner triple sum in (4.8) as

$$\sum_{(r,vN_0)=1} \frac{\Lambda_E(r^2)}{r} \left(\prod_{p|r\ell v} \left(1 + \frac{1}{p}\right)^{-1} \prod_{p\nmid N_0} \left(1 - \frac{1}{p^2}\right) + O\left(\frac{v^{\varepsilon}}{A}\right)\right) \left(\widehat{h}\left(\frac{\log r^2}{L}\right) + \widehat{h}\left(-\frac{\log r^2}{L}\right)\right).$$

Recalling that the theory of Rankin-Selberg L-functions gives

$$\sum_{\substack{p \leq y \\ (p,N_0)=1}} \frac{\Lambda_E(p^2)}{p} = -\log y + O(1),$$

and applying Abel's summation, we then establish that (4.8) equals

$$
\begin{aligned}
&-\frac{X}{vLN_0}\check{\Phi}(0)\Bigg(\prod_{p\mid \ell v}\left(1+\frac{1}{p}\right)^{-1}\prod_{p\nmid N_0}\left(1-\frac{1}{p^2}\right)+O\left(\frac{v^{\varepsilon}}{A}\right)\Bigg)\\
&\times\left(\int_1^{\infty}\left(\widehat{h}\left(\frac{2\log y}{L}\right)+\widehat{h}\left(-\frac{2\log y}{L}\right)\right)\frac{dy}{y}+O(1)\right)\\
&=-\frac{X}{vN_0}\check{\Phi}(0)\prod_{p\mid \ell v}\left(1+\frac{1}{p}\right)^{-1}\prod_{p\nmid N_0}\left(1-\frac{1}{p^2}\right)\frac{h(0)}{2}+O\left(\frac{Xv^{\varepsilon}}{vA}+\frac{X}{vL}\right),
\end{aligned}
$$

(cf. [15, Eq. (30)]), where the error term involving A is acceptable by our choice of A in (4.7).

Lastly, if ℓ is q times a square, with a (unique) prime q, we must have $n = qr^2$ for some r. However, as n has to be a prime power in S_2, it can only be an odd power of q. Therefore, we can bound (4.8), the main term of S_2, by

$$
\ll \frac{X}{vLN_0}\frac{\log q}{\sqrt{q}}\prod_{p\mid \ell v}\left(1+\frac{1}{p}\right)^{-1}\prod_{p\nmid N_0}\left(1-\frac{1}{p^2}\right)
$$

(cf. [15, Eq. (31)]), which concludes the proof.

5. Proof of Proposition 2.2

In this section, we shall prove Proposition 2.2. To begin, opening the sum, we see that the left of (2.6) is

(5.1)

$$
\sum_{j=0}^{k}\binom{k}{j}(-c)^j(b+c)^{k-j}\sum_{d\in\mathcal{E}(\kappa,a)}\sum_{\log X\le p_1,\dots,p_j\le x}C_{p_1}(d)\cdots C_{p_j}(d)\mathcal{P}(d;x)^{k-j}\sum_{\gamma_d}h\left(\frac{\gamma_d L}{2\pi}\right)\Phi\left(\frac{\kappa d}{X}\right).
$$

As noted in [14, pp. 1048-1049], if $q_1 < q_2 < \cdots < q_u$ are the distinct primes appearing in $p_1, \dots, p_j$, denoting the multiplicity of q_i by m_i, one has

$$
C_{p_1}(d)\cdots C_{p_j}(d)=\prod_{i=1}^{u}C_{q_i}(d)^{m_i}=\sum_{v\mid (d,q_1\cdots q_u)}\sum_{rs=v}\mu(r)\prod_{i=1}^{u}C_{q_i}(s)^{m_i}.
$$

Hence, the inner sum (over d) of (5.1) can be written as

$$
\sum_{\log X\le p_1,\dots,p_j\le x}\sum_{v\mid q_1\cdots q_u}\sum_{rs=v}\mu(r)\prod_{i=1}^{u}C_{q_i}(s)^{m_i}\sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ v\mid d}}\mathcal{P}(d;x)^{k-j}\sum_{\gamma_d}h\left(\frac{\gamma_d L}{2\pi}\right)\Phi\left(\frac{\kappa d}{X}\right). \tag{5.2}
$$

(Note that $v \le x^k = X^{k/\log\log\log X}$ for v appearing in (5.2); we shall use this repeatedly in the remaining discussion.) Expanding out $\mathcal{P}(d;x)^{k-j}$, we see that the last double sum becomes

$$
\sum_{\substack{p_{j+1},\dots,p_k\le x\\ p_i\nmid vN_0}}\frac{\lambda_E(p_{j+1})\cdots\lambda_E(p_k)}{\sqrt{p_{j+1}\cdots p_k}}\sum_{\substack{d\in\mathcal{E}(\kappa,a)\\ v\mid d}}\sum_{\gamma_d}h\left(\frac{\gamma_d L}{2\pi}\right)\chi_d(p_{j+1}\cdots p_k)\Phi\left(\frac{\kappa d}{X}\right), \tag{5.3}
$$

where we used the fact that $\chi_d(p_{j+1}\cdots p_k)=0$ if $p_i \mid v$ for some i, as $v \mid d$.

From the first part of Proposition 2.1, it follows that the contribution of the terms, arising from $p_{j+1}\cdots p_k$ that is not a square nor a prime times a square, to (5.3) is

$$\ll \sum_{p_{j+1},\dots,p_k\le x} \frac{|\lambda_E(p_{j+1})\cdots\lambda_E(p_k)|}{\sqrt{p_{j+1}\cdots p_k}} X^{\frac12+\varepsilon}(p_{j+1}\cdots p_k)^{\frac12} e^{\frac{L}{4}} \ll_k x^k X^{\frac12+\varepsilon} e^{\frac{L}{4}} \tag{5.4}$$

as $|\lambda_E(p)|\le 2$.

For the case that $p_{j+1}\cdots p_k$ is a square (when this happens, $k-j$ must be even), by the second part of Proposition 2.1, we see that the main term of (5.3) is

$$\begin{aligned} &\frac{X}{vN_0}\prod_{p\nmid N_0}\left(1-\frac{1}{p^2}\right)\prod_{p|v}\left(1+\frac1p\right)^{-1}\check{\Phi}(0)\left(\frac{2\log X}{L}\widehat{h}(0)+\frac{h(0)}{2}+O(L^{-1})\right)\\ &\times \sum_{\substack{p_{j+1},\dots,p_k\le x\\ p_i\nmid vN_0\\ p_{j+1}\cdots p_k=\square}} \frac{\lambda_E(p_{j+1})\cdots\lambda_E(p_k)}{\sqrt{p_{j+1}\cdots p_k}} \prod_{p|p_{j+1}\cdots p_k}\left(1+\frac1p\right)^{-1} \end{aligned} \tag{5.5}$$

(while the error is still $\ll_k x^k X^{\frac12+\varepsilon}e^{\frac{L}{4}}$ as argued above). Observe that the terms above with some p_i dividing v contribute at most

$$\ll \frac{X(\log X)(\log\log X)^k}{vL\log X}$$

as every prime factor of v is larger than $\log X$. Therefore, removing the condition $p_i\nmid v$ for $j+1\le i\le k$ results in an error $\ll X(\log X)(\log\log X)^k/(vL\log X)$. Now, we further denote the distinct primes appearing in $p_{j+1}\cdots p_k$ by $q_1',\dots,q_{u'}'$ with multiplicity $m_1',\dots,m_{u'}'$, respectively. Note that as each $m_i'\ge 2$ and $\sum_{i=1}^{u'} m_i' = k-j$, we know $u\le\frac{k-j}{2}$. The contribution of $q_1',\dots,q_{u'}'$ with $u'<\frac{k-j}{2}$ (which forces $u'\le\frac{k-j}{2}-1$) to (5.5) is

$$\ll \frac{X}{v}\prod_{p|v}\left(1+\frac1p\right)^{-1}\frac{\log X}{L}\left(\sum_{\substack{p\le x\\ p\nmid N_0}}\frac{\lambda_E(p)^2}{p}\right)^{\frac{k-j}{2}-1} \ll \frac{X\log X}{vL}(\log\log X)^{\frac{k-j}{2}-1},$$

which is negligible. Lastly, for $u'=\frac{k-j}{2}$, we know that each m_i must equal 2, and thus the contribution of $q_1',\dots,q_{(k-j)/2}'$ (to the sum over p_i in (5.5)) is

$$\frac{(k-j)!}{2^{(k-j)/2}((k-j)/2)!}\sum_{\substack{q_1',\dots,q_{(k-j)/2}'\le x\\ q_i'\nmid N_0}}\frac{\prod_{i=1}^{(k-j)/2}\lambda_E(q_i')^2}{(q_1'+1)\cdots(q_{(k-j)/2}'+1)}.$$

(A justification of the factor $\frac{(k-j)!}{2^{(k-j)/2}((k-j)/2)!}$ can be found in [6, p. 5, especially Eq. (4.2)].) As

$$\sum_{\substack{q\le y\\ q\nmid N_0}}\frac{\lambda_E(q)^2}{q+1}=\sum_{\substack{q\le y\\ q\nmid N_0}}\frac{\lambda_E(q)^2}{q}+O(1)=\log\log y+O(1),$$

we then conclude that the main term of (5.3) is equal to

$$\begin{aligned}(5.6)\qquad &\frac{X}{vN_0}\prod_{p\nmid N_0}\left(1-\frac{1}{p^2}\right)\prod_{p|v}\left(1+\frac{1}{p}\right)^{-1}\breve{\Phi}(0)\left(\frac{2\log X}{L}\widehat{h}(0)+\frac{h(0)}{2}+O(L^{-1})\right)\\ &\times\left(\frac{(k-j)!}{2^{(k-j)/2}((k-j)/2)!}+o(1)\right)(\log\log X)^{\frac{k-j}{2}}\end{aligned}$$

in this case (namely, the contribution of terms with $p_{j+1}\cdots p_k=\square$).

Finally, by the third part of Proposition 2.1, we can bound the contribution of the terms arising from $p_{j+1}\cdots p_k$ that is a prime times a square (which forces $k-j$ to be odd) by

$$\begin{aligned}(5.7)\qquad &\ll\frac{X}{vLN_0}\sum_{q\le x}\frac{\log q}{q}\prod_{p|v}\left(1+\frac{1}{p}\right)^{-1}\sum_{\substack{p_{j+1},\dots,p_{k-1}\le x\\ p_i\nmid N_0\\ p_{j+1}\cdots p_{k-1}=\square}}\frac{\prod_{i=j+1}^{k-1}\lambda_E(p_i)}{\sqrt{p_{j+1}\cdots p_{k-1}}}+x^kX^{\frac12+\varepsilon}e^{\frac{L}{4}}\\ &\ll\frac{X\log x}{vLN_0}\prod_{p|v}\left(1+\frac{1}{p}\right)^{-1}(\log\log X)^{\frac{k-j-1}{2}}+x^kX^{\frac12+\varepsilon}e^{\frac{L}{4}}.\end{aligned}$$

In light of the above discussion (especially, (5.3), (5.4), (5.6), and (5.7)), to handle (5.2), we shall estimate

$$(5.8)\qquad \sum_{\log X\le p_1,\dots,p_j\le x}\ \sum_{v|q_1\cdots q_u}\ \sum_{rs=v}\mu(r)\prod_{i=1}^{u}C_{q_i}(s)^{m_i}\frac{1}{v}\prod_{p|v}\left(1+\frac{1}{p}\right)^{-1}$$

(for $j\ge1$). Noting that $\frac{1}{v}\prod_{p|v}(1+\frac{1}{p})^{-1}=\prod_{p|v}\frac{1}{p+1}$ (as v is square-free in our consideration), we follow [14, p. 1050] to set

$$G(q_1^{m_1}\cdots q_u^{m_u})=\sum_{v|q_1\cdots q_u}\ \sum_{rs=v}\mu(r)\prod_{i=1}^{u}C_{q_i}(s)^{m_i}\prod_{p|v}\frac{1}{p+1},$$

which is multiplicative. In addition, we recall that

$$G(p^\alpha)=(\log c(p))^\alpha\left(\frac{1}{p+1}\left(1-\frac{1}{p+1}\right)^\alpha+\frac{p}{p+1}\left(\frac{-1}{p+1}\right)^\alpha\right),$$

which implies that $G(q_1^{m_1}\cdots q_u^{m_u})$ is non-vanishing only if $m_i\ge2$ for all i, and also

$$G(p^\alpha)\ll(\log c(p))^\alpha/p$$

for all $\alpha\ge2$.

Given $q_1<\cdots<q_u$ and $m_i\ge2$ with $\sum_{i=1}^u m_i=j$, the number of choices for $p_1,\dots,p_j$ is $\frac{j!}{m_1!\cdots m_u!}$. Hence, the first double sum in (5.8) can be written as

$$(5.9)\qquad \sum_{\substack{u\\ m_i\ge2\,\forall i\\ \sum_{i=1}^u m_i=j}}\frac{j!}{m_1!\cdots m_u!}\sum_{\log X\le q_1<\cdots<q_u<x}G(q_1^{m_1}\cdots q_u^{m_u}).$$

If $m_i\ge3$ for some i, then the above inner sum over q_i is at most

$$\ll(\log\log X)^{(j-1)/2}.$$

Otherwise, if $m_i = 2$ for all i, we must have $j = 2u$ (which is even). As a direct calculation shows

$$G(p^2) = \frac{(\log c(p))^2}{p} + O\left(\frac{1}{p^2}\right),$$

it then follows from (2.2) that the terms with all $m_i = 2$ contribute

$$\frac{j!}{2^{j/2}(j/2)!}((\sigma(E)^2 - 1)\log\log X + O(\log\log\log X))^{j/2} \tag{5.10}$$

to (5.9). Therefore, we conclude that the main term must arise from terms with even j. Consequently, when k is even (which forces $k - j$ to be even), it follows from (5.2), (5.3), (5.4), (5.6), (5.8), and (5.10) that (5.1) (and thus the left of (2.6)) becomes

$$\sum_{\substack{j=0 \\ j \text{ even}}}^{k} \binom{k}{j}(-1)^j \frac{X}{N_0}\prod_{p\nmid N_0}\left(1 - \frac{1}{p^2}\right)\check{\Phi}(0)\left(\frac{2\log X}{L}\widehat{h}(0) + \frac{h(0)}{2} + O(L^{-1})\right)$$
$$\times \frac{j!}{2^{j/2}(j/2)!}(c^2(\sigma(E)^2 - 1 + o(1))\log\log X)^{j/2}\frac{(k-j)!}{2^{(k-j)/2}((k-j)/2)!}(((b+c)^2 + o(1))\log\log X)^{\frac{k-j}{2}},$$

which yields the claimed estimate for even k. In the case that k is odd, the "main term" of (5.1) does not manifest since by (5.2), (5.3), (5.4), (5.7), (5.8), and (5.10), it is

$$\ll_{b,c} \sum_{\substack{j=0 \\ j \text{ even}}}^{k} \binom{k}{j}\frac{X\log x}{LN_0}(\log\log X)^{j/2}(\log\log X)^{\frac{k-j-1}{2}} \ll \frac{X\log X}{LN_0}(\log\log X)^{\frac{k-1}{2}},$$

which completes the proof.

6. Derivation of Theorem 1.4

In this section, we shall take $x = X^{1/(\log\log X)^2}$ and let V_1, V_2 be real. Recall that in [14, pp. 1045-1048], Radziwiłł and Soundararajan proved the following:

(1) If $d \in \mathcal{E}$, with $\frac{X}{\log X} < |d| \le |X|$, satisfies

$$\log L(\tfrac{1}{2}, E_d) + \log\log X \ge V_1\sqrt{\log\log X}, \tag{6.1}$$

then one of the following cases must happen:
   (a) $\mathcal{P}(d;x) \ge (V_1 - \varepsilon)\sqrt{\log\log X}$;
   (b) $|\mathcal{P}(d;x)| \ge \log\log X$;
   (c) $|\mathcal{P}(d;x)| \le \log\log X$ but $L(\frac{1}{2}, E_d)(\log X)^{\frac{1}{2}}\exp(-\mathcal{P}(d;x)) \ge \exp(\varepsilon\sqrt{\log\log X})$.

(2) If $d \in \mathcal{E}$, with $\frac{X}{\log X} < |d| \le |X|$, satisfies

$$\frac{\log L(\frac{1}{2}, E_d) - \sum_{p|d}\log c(p) - \mu(E)\log\log X}{\sqrt{\sigma(E)^2\log\log X}} \ge V_2, \tag{6.2}$$

then either (b), (c), or one of the following cases must happen:
   (d) $\mathcal{P}(d;x) - \mathcal{C}(d;x) \ge (V_2 - \varepsilon)\sqrt{\sigma(E)^2\log\log X}$;
   (e) $|\log \mathrm{Tam}(E_d) + (\mu(E) + \frac{1}{2})\log\log X - \mathcal{C}(d;x)| \ge \frac{\varepsilon}{10}\sqrt{\log\log X}$.

In addition, cases (b), (c), and (e) occur for at most $o(X)$ discriminants $d \in \mathcal{E}$. Consequently, if $d \in \mathcal{E}$, with $\frac{X}{\log X} < |d| \le |X|$, satisfies both (6.1) and (6.2), then cases (a) and (d) happen, simultaneously, for all but at most $o(X)$ discriminants $d \in \mathcal{E}$.

On the other hand, it follows from (3.1) that

$$\begin{aligned}\log(S(E_d)/\sqrt{|d|}) &= \log L(\tfrac{1}{2}, E_d) - \log \operatorname{Tam}(E_d) + O(1) \\ &= \log L(\tfrac{1}{2}, E_d) + (\mu(E) + \tfrac{1}{2}) \log\log X - \mathcal{C}(d;x) + O(\varepsilon\sqrt{\log\log X} + 1),\end{aligned}$$

except for at most $o(X)$ discriminants $d \in \mathcal{E}$, where the implied constant is independent of ε, as case (e) appears with frequency $o(X)$. By the definition of $\mathcal{C}(d;x)$ and (2.1), one has

$$\begin{aligned}\log(S(E_d)/\sqrt{|d|}) - \mu(E)\log\log X = \log L(\tfrac{1}{2}, E_d) - \mu(E)\log\log X - \sum_{\substack{\log X \le p \le x \\ p \mid d}} \log c(p) \\ + O(\varepsilon\sqrt{\log\log X} + \log\log\log X + 1).\end{aligned}$$

In addition, by the argument leading to the first estimate in [14, p. 1048], one knows

$$\sum_{\substack{d\in\mathcal{E} \\ X/\log X \le |d| \le X}} \Bigg(\sum_{\substack{p<\log X \\ p\mid d}} \log c(p) + \sum_{\substack{p>x \\ p\mid d}} \log c(p) \Bigg) \ll X \log\log\log X,$$

and thus the number of $d \in \mathcal{E}$, with $\frac{X}{\log X} < |d| \le |X|$, satisfying $\sum_{\substack{p<\log X \\ p\mid d}} \log c(p) + \sum_{\substack{p>x \\ p\mid d}} \log c(p) \ge (\log\log\log X)^2$ is $O(X/\log\log\log X) = o(X)$. Hence, $\log(S(E_d)/\sqrt{|d|}) - \mu(E)\log\log X$ is

$$\log L(\tfrac{1}{2}, E_d) - \mu(E)\log\log X - \sum_{p\mid d} \log c(p) + O(\varepsilon\sqrt{\log\log X} + (\log\log\log X)^2)$$

for all but $o(X)$ $d \in \mathcal{E}$, with $\frac{X}{\log X} < |d| \le |X|$. Thus, we conclude that

$$\begin{aligned}\#\Bigg\{ d \in \mathcal{E}, \frac{X}{\log X} < |d| \le X : \frac{\log L(\frac{1}{2}, E_d) + \frac{1}{2}\log\log X}{\sqrt{\log\log X}} \ge V_1, \\ \frac{\log(S(E_d)/\sqrt{|d|}) - \mu(E)\log\log X}{\sqrt{\sigma(E)^2 \log\log X}} \ge V_2 \Bigg\}\end{aligned}$$

is less than

$$\#\Bigg\{ d \in \mathcal{E}, \frac{X}{\log X} < |d| \le X : \frac{\mathcal{P}(d;x)}{\sqrt{\log\log X}} \ge V_1 - \varepsilon, \frac{\mathcal{P}(d;x) - \mathcal{C}(d;x)}{\sqrt{\sigma(E)^2 \log\log X}} \ge V_2 - \varepsilon \Bigg\} + o(X),$$

which is $\sim (\Xi_E((V_1 - \varepsilon, V_2 - \varepsilon), (\infty, \infty)) + o(1)) \#\{d \in \mathcal{E} : |d| \le X\}$, and so the desired result follows.

7. Concluding remark

It is known that assuming GRH and the 1-level density conjectures of Katz and Sarnak [10], then Keating-Snaith's conjecture, as mentioned in the introduction, would be true (see [19, p. 992] and also [15, p. 1033]). Indeed, as remarked by Soundararajan [19], if most of $L(s, E_d)$ do not have a zero very close to $s = \frac{1}{2}$ (which is implied by the 1-level density conjectures), then one could prove Keating-Snaith's conjecture. In addition, similarly, Conjecture 1, proposed by Radziwiłł-Soundararajan, and Conjecture 2 would follow from the same assumption.

To end this article, we shall discuss how Radziwiłł-Soundararajan's argument leads to a quantified formulation of Soundararajan's remark above as follows. From the proof of [15, Proposition 1], under GRH, one has

$$\log L(\tfrac{1}{2}, E_d) = \mathcal{P}(d;x) - \tfrac{1}{2}\log\log x + \frac{1}{\log x}\sum_{\gamma_d} \mathfrak{Re} \int_{\frac{1}{2}}^{\infty} \frac{x^{\rho_d-\sigma}}{(\rho_d-\sigma)^2} d\sigma + O\left(\frac{\log|d|}{\log x}\right).$$

In addition, it was shown that for $|\gamma_d \log x| \ge 1$,

$$\int_{\frac{1}{2}}^{\infty} \frac{x^{\rho_d-\sigma}}{(\rho_d-\sigma)^2} d\sigma \ll \frac{1}{\gamma_d^2}\int_{\frac{1}{2}}^{\infty} x^{\frac{1}{2}-\sigma} d\sigma \ll \frac{1}{\gamma_d^2 \log x},$$

which is $\le \log x$ for this range of γ_d. Also, in the two estimates above [15, Eq. (15)], for $|\gamma_d \log x| \le 1$, one has

$$\mathfrak{Re}\int_{\frac{1}{2}+\frac{1}{\log x}}^{\infty} \frac{x^{\rho_d-\sigma}}{(\rho_d-\sigma)^2} d\sigma \ll \int_{\frac{1}{2}+\frac{1}{\log x}}^{\infty} \frac{x^{\frac{1}{2}-\sigma}}{(\frac{1}{2}-\sigma)^2} d\sigma \ll \log x$$

and

$$\begin{aligned}\mathfrak{Re}\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{\log x}} \frac{x^{\rho_d-\sigma}}{(\rho_d-\sigma)^2} d\sigma &= \mathfrak{Re}\left(\frac{-1}{i\gamma_d} - \frac{1}{1/\log x - i\gamma_d} + (\log x)\log\frac{-i\gamma_d}{1/\log x - i\gamma_d} + O(\log x)\right)\\ &= -\frac{1/\log x}{(1/\log x)^2+\gamma_d^2} + (\log x)\log\left|\frac{(1/\log x + i\gamma_d)(i\gamma_d)}{(1/\log x)^2+\gamma_d^2}\right| + O(\log x)\\ &= (\log x)\log\frac{|\gamma_d|\log x}{(1+(\gamma_d^2\log x)^2)^{1/2}} + O(\log x).\end{aligned}$$

To prove [15, Proposition 1], Radziwiłł and Soundararajan bounded the above three integrals all by $\ll (\log x)\log(1+\frac{1}{(\gamma_d \log x)^2}))$. It is worth noting that by the above discussion (especially the last estimate), one may further establish

$$\begin{aligned}\log L(\tfrac{1}{2}, E_d) = \mathcal{P}(d;x) - \tfrac{1}{2}\log\log x + \sum_{|\gamma_d|\le((\log X)(\log\log X))^{-1}} \log(|\gamma_d|\log x)\\ + O\left(\frac{\log|d|}{\log x} + \sum_{|\gamma_d|\ge((\log X)(\log\log X))^{-1}} \log\left(1+\frac{1}{(\gamma_d\log x)^2}\right)\right),\end{aligned}$$

where $x = X^{1/\log\log\log X}$, and the last sum is at most $O((\log\log\log X)^3)$ for all but $o(X)$ $d \in \mathcal{E}$ by Lemma 3.3. Hence, from this and the proof of Theorem 1.2, we then deduce the following:

Theorem 7.1. *Assume GRH for the family of twisted L-functions $L(s, E\otimes\chi)$ with all Dirichlet characters χ. Suppose, further, that for all but $o(X)$ $d \in \mathcal{E}$, with $X < |d| \le 2X$, one has*

$$\sum_{|\gamma_d|\le((\log X)(\log\log X))^{-1}} \log(|\gamma_d|\log x) = o(\sqrt{\log\log X}),$$

as $X \to \infty$, where $x = X^{1/\log\log\log X}$. Then Conjecture 2 (and thus both conjectures of Keating-Snaith and Radziwiłł-Soundararajan) is true.

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References

[1] P. Billingsley, *Probability and Measure*, 3rd ed., Wiley Series in Probability and Mathematical Statistics, John Wiley and Sons, New York, 1995.

[2] H.M. Bui, N. Evans, S. Lester, and K. Pratt, *Weighted central limit theorems for central values of L-functions*, preprint, `https://arxiv.org/abs/2109.06829`.

[3] E.J. Dudewicz and S.N. Mishra, *Modern Mathematical Statistics*, John Wiley and Sons, New York, 1988.

[4] M. Fréchet and J. Shohat, *A proof of the generalized second-limit theorem in the theory of probability.* Trans. Amer. Math. Soc. 33 (1931), 533-543.

[5] G.T. Gilbert, *Positive definite matrices and Sylvester's criterion*, Amer. Math. Monthly 98 (1991), no. 1, 44-46.

[6] A. Harper, *Sharp conditional bounds for moments of the Riemann zeta function*, preprint, `https://arxiv.org/abs/1305.4618`.

[7] D.R. Heath-Brown, *The average analytic rank of elliptic curves*, Duke Math. J. 122(3) (2004), 591-623.

[8] T. Hida and M. Hitsuda, *Gaussian Processes*, Translated from the 1976 Japanese original by the authors, Translations of Mathematical Monographs 120, American Mathematical Society, Providence, RI, 1993.

[9] P.-H. Hsu and P.-J. Wong, *On Selberg's central limit theorem for Dirichlet L-functions*, J. Théor. Nombres Bordeaux 32 (2020), no. 3, 685-710.

[10] N.M. Katz and P. Sarnak, *Random matrices, Frobenius eigenvalues, and monodromy*, American Mathematical Society Colloquium Publications 45, American Mathematical Society, Providence, RI, 1999.

[11] J.P. Keating and N.C. Snaith, *Random matrix theory and L-functions at $s = 1/2$*, Comm. Math. Phys. 214(1) (2000), 91-110.

[12] X. Li, *Moments of quadratic twists of modular L-functions*, preprint, `https://arxiv.org/abs/2208.07343`.

[13] L. Mai and M.R. Murty, *A note on quadratic twists of an elliptic curve*, CRM Proceedings and Lecture Notes, 4 (1994), 121-124.

[14] M. Radziwiłł and K. Soundararajan, *Moments and distribution of central L-values of quadratic twists of elliptic curves*, Invent. math. 202(3) (2015), 1029-1068.

[15] M. Radziwiłł and K. Soundararajan, *Conditional lower bounds on the distribution of central values in families of L-functions*, preprint, `https://arxiv.org/abs/2308.00169`.

[16] S.I. Resnick, *A Probability Path*, Birkhäuser/Springer, New York, 2014.

[17] A. Smith, *2^∞-Selmer groups, 2^∞-class groups, and Goldfeld's conjecture*, preprint, `https://arxiv.org/abs/1702.02325`.

[18] K. Soundararajan. *Nonvanishing of quadratic Dirichlet L-functions at $s = \frac{1}{2}$*, Ann. of Math. (2) 152 (2000), 447-488.

[19] K. Soundararajan, *Moments of the Riemann zeta function*, Ann. of Math. (2) 170 (2009), 981-993.

[20] K. Soundararajan and M.P. Young, *The second moment of quadratic twists of modular L-functions*, J. Eur. Math. Soc. (JEMS) 12 (2010), no. 5, 1097-1116.

(Peng-Jie Wong) Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung City, Taiwan

Email address: `pjwong@math.nsysu.edu.tw`