GLOBAL SOLUTION OF 2D HYPERBOLIC LIQUID CRYSTAL SYSTEM FOR SMALL INITIAL DATA

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Abstract

We prove the global stability of small perturbation near the the constant equilibrium for the two dimensional simplified Ericksen-Leslie's hyperbolic system for incompressible liquid crystal model, where the direction function of liquid crystal molecules satisfies a wave map equation with an acoustical metric. This improves the almost global existence result by Huang-Jiang-Zhao [14]. As byproducts, we obtain the sharp (same as the linear solution) L^∞_x-decay estimates for the nonlinear velocity and the nonlinear wave part. Moreover the nonlinear wave part scatters to a linear solution as time goes to infinity.

The main novelty of this paper is that we uncover a null structure inside the velocity equation on the Fourier side for the nonlinear interaction between nonlinear heat equation and nonlinear wave equation.

1. Introduction

This paper is devoted to studying the small data global regularity problem for the 2D simplified Ericksen-Leslie's hyperbolic liquid crystal model, which read as follows,

$$\left\{\begin{array}{l}\partial_t u + u\cdot\nabla u + \nabla p = \Delta u - \nabla\cdot\big(\nabla d\otimes\nabla d\big), \quad \nabla\cdot u = 0,\\ D_t^2 d - \Delta d = \big(-|D_t d|^2 + |\nabla_x d|^2\big)d,\end{array}\right. \tag{1.1}$$

where $D_t := \partial_t + u\cdot\nabla_x, d\in\mathbb{S}^1$. If we parametrize $d = (\cos\phi, \sin\phi)$, see Huang-Jiang-Zhao [14], then the following system of equation holds,

$$\left\{\begin{array}{ll}\partial_t u - \Delta u & = -u\cdot\nabla u - \nabla p - \partial_i(\nabla\phi\partial_i\phi), \quad \nabla\cdot u = 0\\ (\partial_t^2 - \Delta)\phi & = -u\cdot\nabla(\partial_t\phi + u\cdot\nabla\phi) - \partial_t(u\cdot\nabla\phi)\\ & = -2\partial_i(u_i\partial_t\phi) - \partial_i(\partial_t u_i\phi) - \partial_i(u_iu_j\partial_j\phi) =: \mathcal{N}_\phi\end{array}\right. \tag{1.2}$$

For the physical background of the system (1.2), we refer readers to [14] and references therein for more details. The system (1.2) came to the author's attention when one of the authors of [14], Prof. Ning Jiang, presented their small data almost global result in "Workshop on the recent progress of kinetic theory and related topics", Jan 15-19, 2024, Sanya, China. Comparing with the rich literature on nonlinear wave equation and nonlinear heat equation, it's fair to say that the picture of quadratic interaction between wave equation and heat equation, is far from complete.

Generally speaking, $2D$ nonlinear wave decays at most $t^{-1/2}$ rate over time. Due to the slow time decay issue, small data global regularity problem for 2D quadratic equations is very challenging. Very interestingly, in [14], the authors are able to identify an good unknown variable and successfully apply the ghost weight method of Alinhac [1]. Despite the ghost weight method is well-known in the study of wave equation, it's interesting to find its application in the study of heat equation. Remarkably, these observations allow them to prove almost global solution for small initial data.

A natural question is what happens after the almost global life-span of solution. Does the nonlinear solution continues to survive or blows up in finite time? If the nonlinear solution continues to survive, what's the asymptotic behavior of the solution?

The goal of this paper is to answer these questions affirmatively. The key observation is that we uncover a null structure inside the velocity equation on the Fourier side for the nonlinear interaction between nonlinear heat equation and nonlinear wave equation. The null structure comes from cancellation between the pressure term ∇p and $\partial_i(\nabla\phi\partial_i\phi)$. Due to its generality, we believe that this type of null structure will appear in other models of hyperbolic liquid crystal optics and the method presented in this paper will shade some light on future study. We will elaborate this observation in subsection 1.2 and explain how to make use of this null structure in subsection 1.3.

Different from the physical space approach employed in [14], we mainly analyze the system (1.2) from the Fourier side. Moreover, we use both the strength of the vector field method and the Fourier method. Using the combination of these two methods is not new, e.g., it works successfully in the seminal work of Ionescu-Pausader [18] for the study of Einstein-Klein-Gordon system, see also [27].

Lastly, for the sake of readers, we give a brief introduction on literature of these two methods. The celebrated Klainerman vector field method was introduced in the seminal work [22], in which the symmetries of the wave operator are exploited to prove decay estimate of the nonlinear wave equation. A great achievement of the vector field method lies in the monumental work, global stability of the Minkowski spacetime, by Christodoulou-Klainerman [3]. For the past ten years, the Fourier method, which studies nonlinear solutions on the Fourier side and controls the pull back of the nonlinear solution along the linear flow over time, also plays an important role in the study of small data global regularity problem for nonlinear dispersive PDEs and NLW. Since the literature is too vast to survey, for the purpose of giving a sense for readers, we only mention the spacetime resonance method and the Z-norm method. The spacetime resonance method was introduced by Germain-Masmoudi-Shatah [9] in the study of NLS. Now, it has very wide applications in the study of nonlinear dispersive equations, see e.g., [10, 11, 15, 16, 17, 18] and the nonlinear wave equations, see [4, 26]. The Z-norm method was firstly introduced by Ionescu-Pausader in [16]. This method is often used together with the spacetime resonance method. It depends essentially on identifying the "correct" Z-norm, depending on the problem, to prove sharp or almost sharp decay estimates for the nonlinear solution.

1.1. **Main result of this paper.**

The main result of this paper is stated as follows,

Theorem 1.1. *Let $N_0 = 10^5$. There exists absolute small constant ϵ_0 such that if the initial data (u_0, ϕ_0, ϕ_1) of the system (1.2) satisfies the following smallness condition*

$$\|u_0\|_{H^{N_0+1}} + \|U_0\|_{H^{N_0}} + \||x|\nabla_x u_0\|_{L^2} + \sum_{|\alpha|\leq 1} \||x|\nabla_x U_0\|_{L^2} + \sum_{|\beta|\leq 11} \|\nabla_x^\beta U_0\|_{L^1_x} + \|\nabla_x^\beta u_0\|_{L^1_x} \leq \epsilon_0, \quad (1.3)$$

where $U_0 := \phi_1 + i|\nabla|\phi_0$. Then the system (1.2) has unique global solution. The nonlinear solution $\phi(t)$ scatters to a linear solution $\phi_\infty(t)$. Moreover, the following sharp decay estimates hold for u and ϕ,

$$\sum_{|\alpha|\leq 3} \langle t\rangle^{1/2} \|\nabla_x^\alpha \nabla_{t,x}\phi\|_{L^\infty_x} + \langle t\rangle \|\nabla_x^\alpha u(t)\|_{L^\infty} \lesssim \epsilon_0. \quad (1.4)$$

Remark 1.1. The plausible goal of optimizing N_0 is not pursued here.

1.2. **Null structure.**

In this subsection, we show with details about the hidden null structure inside the velocity equation, c.f., (1.1). To this end, we first do some reduction for the nonlinear $\mathcal{N}_u$. Note that, by using the divergence free condition, we can solve the pressure as follows,

$$\Delta p = -\partial_j(u_i\partial_i u_j) - \partial_i\partial_j(\partial_i\phi\partial_j\phi), \quad \nabla p = -\nabla\Delta^{-1}\partial_j(u_i\partial_i u_j) - \nabla\Delta^{-1}\partial_i\partial_j(\partial_i\phi\partial_j\phi).$$

Therefore, we can formulate the equation satisfied by velocity u as follows,

$$\begin{aligned}
&\partial_t u - \Delta u = \mathcal{N}_u := Q(u,u) + \widetilde{Q}(\phi,\phi),\\
&\quad Q(u,u) = -u\cdot\nabla u + \nabla\Delta^{-1}\partial_j(u_i\partial_i u_j) = -\partial_i(u_i u) + \nabla\Delta^{-1}\partial_i\partial_j(u_i u_j),\\
&\quad \widetilde{Q}(\phi,\phi) = \nabla\Delta^{-1}\partial_i\partial_j(\partial_i\phi\partial_j\phi) - \partial_i(\partial_i\phi\nabla\phi).
\end{aligned} \quad (1.5)$$

Note that that terms in $\mathcal{N}_u$ always have one derivative outside, which contributes the smallness of output frequency. It motivates us to formulate $\mathcal{N}_u$ as follows,

$$\mathcal{N}_u = |\nabla|\widetilde{\mathcal{N}}_u, \quad \widetilde{\mathcal{N}}_u := -R_i(u_i u) - RR_iR_j(u_i u_j) - RR_iR_j(\partial_i\phi\partial_j\phi) - R_i(\partial_i\phi\nabla\phi). \quad (1.6)$$

Moreover, we observe that there is a null structure inside $\widetilde{Q}(\phi,\phi)$. More precisely, the symbol of $\widetilde{Q}(\phi,\phi)$ is given as follows,

$$\widetilde{q}(\xi-\eta,\eta) = -i(\xi\cdot(\xi-\eta))|\eta|\big[\frac{\xi(\xi\cdot\eta)}{|\xi|^2|\eta|} - \frac{\eta}{|\eta|}\big]. \tag{1.7}$$

Very interestingly, the null structure presented in $\widetilde{Q}(\phi,\phi)$ is not only helpful in the wave $\times$ wave $\to$ wave type interaction, which is more classic, but also helpful in the wave $\times$ wave $\to$ heat type interaction, which is new. More precisely, recall (1.7), note that,

$$\begin{gathered}\eta\cdot\xi = \frac{1}{2}\big[(|\eta|-|\eta-\xi|)(|\eta|+|\eta-\xi|)+|\xi|^2\big], \Longrightarrow \quad \widetilde{q}(\xi-\eta,\eta) = q^{wwh}_{null}(\xi-\eta,\eta)q^{www}_{null}(\xi-\eta,\eta),\\ q^{wwh}_{null}(\xi-\eta,\eta) := i\frac{|\eta|}{2}\underbrace{\big[(|\eta|-|\eta-\xi|)(|\eta|+|\eta-\xi|)-|\xi|^2\big]}_{\text{null structure for w}\times\text{w}\to\text{h}}, \quad q^{www}_{null}(\xi-\eta,\eta) := \underbrace{\big[\frac{\xi(\xi\cdot\eta)}{|\xi|^2|\eta|} - \frac{\eta}{|\eta|}\big]}_{\text{null structure for w}\times\text{w}\to\text{w}}.\end{gathered} \tag{1.8}$$

1.3. **Exploiting the null structure.** To see better in what sense what we mean by "null structure for w×w→h". We employ the following normal form transformation,

$$v := u + \sum_{\mu,\nu\in\{+,-\}} A_{\mu,\nu}(U^{\mu},U^{\nu}), \tag{1.9}$$

where the symbol $a_{\mu,\nu}(\xi-\eta,\eta)$ of the bilinear operator $A_{\mu,\nu}(U^{\mu},U^{\nu})$ is given as follows,

$$a_{\mu,\nu}(\xi-\eta,\eta) = \frac{-\mu\nu\widetilde{q}(\xi-\eta,\eta)}{4\big(-|\xi|^2+i(\mu|\xi-\eta|+\nu|\eta|)\big)}\frac{1}{|\xi-\eta||\eta|}. \tag{1.10}$$

Note that, from (1.15), we have

$$\begin{aligned}(\partial_t-\Delta)v &= Q(u,u)+\widetilde{Q}(\phi,\phi) - \sum_{\mu,\nu\in\{+,-\}} \Delta\big(A_{\mu,\nu}(U^{\mu},U^{\nu})\big) + A_{\mu,\nu}(\partial_t U^{\mu},U^{\nu}) + A_{\mu,\nu}(U^{\mu},\partial_t U^{\nu})\\ &= Q(u,u) + \sum_{\mu,\nu\in\{+,-\}} A_{\mu,\nu}\big(\big((\partial_t+i|\nabla|)U\big)^{\mu},U^{\nu}\big) + A_{\mu,\nu}\big(U^{\mu},\big((\partial_t+i|\nabla|)U\big)^{\nu}\big)\\ &= Q(u,u) + \sum_{\mu,\nu\in\{+,-\}} A_{\mu,\nu}\big((\mathcal{N}_\phi)^{\mu},U^{\nu}\big) + A_{\mu,\nu}\big(U^{\mu},(\mathcal{N}_\phi)^{\nu}\big) =: \mathcal{N}_v(t).\end{aligned} \tag{1.11}$$

Thanks to the "null structure for w×w→h", the symbol $a_{\mu,\nu}(\xi-\eta,\eta)$ of normal form transformation is non-singular. As a result of direct computation, $\forall k,k_1,k_2\in\mathbb{Z}$, we have

$$\|a_{\mu,\nu}(\xi-\eta)\|_{\mathcal{S}^{\infty}_{k,k_1,k_2}} + \|\frac{q^{wwh}_{null}(\xi-\eta,\eta)}{-|\xi|^2+i(\mu|\xi-\eta|+\nu|\eta|)}\|_{\mathcal{S}^{\infty}_{k,k_1,k_2}} \lesssim \begin{cases} 2^{-k_+} & \text{if } (k_1,k_2)\in\chi^2_k\cup\chi^3_k,\\ 2^{-k_{1,-}} & \text{if } (k_1,k_2)\in\chi^1_k.\end{cases} \tag{1.12}$$

Now, it's clear that the equation satisfied by v is much better than u. Naturally, from (1.9), we can also view that the velocity u consists of two parts: the heat part v, which is close to a free heat solution as time goes to infinity; the wave part $A_{\mu,\nu}(U^{\mu},U^{\nu})$, which is a higher order perturbation and also independent of heat flow.

Very importantly, and also very interestingly, since the w×w→w type null structure is not used in the normal form transformation process, c.f., (1.9), the bilinear form $A_{\mu,\nu}(\cdot,\cdot)$ still have wave type null structure. From (1.2) and (1.9), we have the following approximation equation after replacing u by v and $A_{\mu,\nu}(\cdot,\cdot)$ and neglecting the contribution of v,

$$\text{(Approximation equation)}: \qquad \Box\phi = \sum_{\mu,\nu\in\{+,-\}} Q(A_{\mu,\nu}(U^{\mu},U^{\nu}),\nabla_{t,x}\phi), \tag{1.13}$$

where Q is some bilinear form. If without wave type null structure, generally speaking, 2D cubic wave equation might blow up in almost global lifespan. If the approximation equation of (1.2) blows up, we don't expect the nonlinear solution of (1.2) exists globally in time. The fact that $A_{\mu,\nu}(\cdot,\cdot)$ still have wave type null structure is main reason why we expect global solution for the system (1.2).

1.4. **Bootstrap assumption.** Before stating our bootstrap assumption, we define the following normed space.

$$Z_u(t) := \sup_{k\in\mathbb{Z}} 2^{k+4k_+}\|P_k u\|_{L^2}, \qquad Z_\phi(t) := \sum_{k\in\mathbb{Z}} 2^{k/3+10k_+}\big(\|\partial_t\widehat{\phi}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi} + 2^k\|\widehat{\phi}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\big). \tag{1.14}$$

For any $\Gamma\in\{Id, S := t\partial_t + x\cdot\nabla_x, \Omega := x_1\partial_{x_2} - x_2\partial_{x_1}\}$, we define the half wave and its profile as follows,

$$\begin{aligned} &U^\Gamma(t) := (\partial_t - i|\nabla|)\Gamma\phi, \quad \Longrightarrow \quad \partial_t\Gamma\phi = \frac{U^\Gamma(t)+\overline{U^\Gamma(t)}}{2}, \quad \Gamma\phi = \sum_{\mu\in\{+,-\}}\frac{\mu}{2i|\nabla|}(U^\Gamma(t))^\mu, \\ &V^\Gamma(t) := e^{it|\nabla|}U^\Gamma(t), \quad \Longrightarrow \partial_t V^\Gamma(t) = e^{it|\nabla|}\mathcal{N}^\Gamma_\phi, \quad U^\Gamma(t) = e^{-it|\nabla|}U^\Gamma(0) + \int_0^t e^{-i(t-s)|\nabla|}\mathcal{N}^\Gamma_\phi(s)ds. \end{aligned} \tag{1.15}$$

For simplicity in notation, we abbreviate $U^{Id}(V^{Id})$ as $U(V)$.

Our bootstrap assumption is stated as follows,

$$\begin{aligned} &\sup_{t\in[0,T]} \langle t\rangle^{1/2-\delta}\|u(t)\|_{H^{N_0+1}} + \langle t\rangle^{1/2-2\delta}\big(\sum_{\Gamma\in\{S,\Omega\}}\|\Gamma u(t)\|_{H^1}\big) + \langle t\rangle Z_u(t) + \langle t\rangle^{-\delta}\|U(t)\|_{H^{N_0}} \\ &\quad + \langle t\rangle^{-2\delta}\big(\sum_{\Gamma\in\{S,\Omega\}}\|U^\Gamma(t)\|_{L^2} + \sum_{k\in\mathbb{Z}_-}2^{-\alpha k}\|P_kU^\Gamma(t)\|_{L^2}\big) + Z_\phi(t) \lesssim \epsilon_1 := \epsilon_0^{3/4}, \quad \alpha := 1/10. \end{aligned} \tag{1.16}$$

As a consequence of the above bootstrap assumption, we have the sharp decay estimate for the velocity field u as follows,

$$\sum_{0\le|\alpha|\le 3}\|\nabla_x^\alpha u(t)\|_{L^\infty} \lesssim \langle t\rangle^{-1}\epsilon_1. \tag{1.17}$$

1.5. **Outline of this paper.**

- In section 2, we introduce the notation used in this paper, a super-localized decay estimate, which plays an important role in later High $\times$ High type interaction.
- In section 3, we do energy estimates for both the velocity part and the wave part.
- In section 4, we estimate the Z-norms for the velocity part and the wave part, which give us the sharp decay estimates for nonlinear solutions.

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2. Preliminary

2.1. **Notation.** We define a class of symbol and its associated norms as follows,

$$\begin{aligned} &\mathcal{S}^\infty := \{m : m\in C(\mathbb{R}^4)\cup C(\mathbb{R}^6), \quad \|m\|_{\mathcal{S}^\infty} := \sum_{|\alpha|+|\beta|\le 20}\|\xi^\alpha\eta^\beta\nabla_\xi^\alpha\nabla_\eta^\beta m(\xi,\eta)\|_{L^\infty_{\xi,\eta}} < \infty\}, \\ &\|m(\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} := \|m(\xi,\eta)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty}, \\ &\|m(\xi,\eta,\sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}} := \|m(\xi,\eta,\sigma)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta-\sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty}. \end{aligned} \tag{2.1}$$

For any $k\in\mathbb{Z}$, we use k_+ to denote $\max\{k,0\}$ and use k_- to denote $\min\{k,0\}$. Moreover, for any $k\in\mathbb{Z}$, we use P_k, $P_{\le k}$ and $P_{\ge k}$ to denote the projection operators by the Fourier multipliers $\psi_k(\cdot)$, $\psi_{\le k}(\cdot)$ and $\psi_{\ge k}(\cdot)$ respectively. We use $P_{[k_1,k_2]}$ to denotes $\sum_{k\in[k_1,k_2]}P_k$. For convenience in notation, we also use $f_k(x)$ to abbreviate $P_kf(x)$.

For any chosen threshold $\bar{l}\in\mathbb{Z}_-, l\in[\bar{l},2]\cap\mathbb{Z}$, we define the cutoff function with the chosen threshold $\varphi_{l;\bar{l}}(x):\mathbb{R}^3\to\mathbb{R}$ as follows,

$$\varphi_{l;\bar{l}}(x):=\begin{cases}\psi_{\leq\bar{l}}(x) & \text{if}\, l=\bar{l},\\ \psi_{\leq\bar{l}}(x) & \text{if}\, l\in(\bar{l},2),\\ \psi_{\geq 2}(x) & \text{if}\, l=2.\end{cases} \tag{2.2}$$

We define the following index sets, which correspond to High $\times$ High type interaction, Low $\times$ High type interaction, and High $\times$ Low type interaction respectively,

$$\begin{aligned}\chi_k^1 &:= \{(k_1,k_2): k_1,k_2\in\mathbb{Z}, |k_1-k_2|\leq 10, k_1\geq k-10\},\\ \chi_k^2 &:= \{(k_1,k_2): k_1,k_2\in\mathbb{Z}, |k_2-k|\leq 10, k_1\leq k-10\},\\ \chi_k^3 &:= \{(k_1,k_2): k_1,k_2\in\mathbb{Z}, |k_1-k|\leq 10, k_2\leq k-10\}.\end{aligned} \tag{2.3}$$

To study the High$\times$High $\longrightarrow$ Low type interaction in the energy estimate, we exploit the following decay estimate with the super-localized cutoff function. Comparing with the usual dyadic decomposition, the width of the annulus, which is the support of super-localized cutoff function, is not comparable with the radius of annulus. The idea of using super-localized decay estimate to get around the summation issue of the High$\times$High $\longrightarrow$ Low type interaction goes back to the work of Ionescu-Pausader [18] for the study of Einstein-Klein-Gordon system.

Lemma 2.1. *For any $t\in[2^{m-1},2^m], m\in\mathbb{Z}_+, k,\tilde{k},n\in\mathbb{Z}, x\in\mathbb{R}^3,\mu\in\{+,-\}$, s.t., $\tilde{k}\leq k$, $n\in[2^{k-2},2^{k+2}]\cap\mathbb{Z}$, $m\gg 1$, $k\geq -m$, we have*

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2} e^{ix\cdot\xi-i\mu t|\xi|}m(\xi)\widehat{f}(\xi)\psi_{\tilde{k}}(|\xi|-n)d\xi\Big|\lesssim 2^{-m/2}\|m(\xi)\|_{\mathcal{S}^\infty_k}\big[2^{\tilde{k}+k/2}\|\widehat{f}(\xi)\psi_{[\tilde{k}-2,\tilde{k}+2]}(|\xi|-n)\|_{L^\infty_\xi}\\ &+2^{-m/4+3k/4+\tilde{k}/2}\|\nabla_\xi\widehat{f}(\xi)\psi_{\tilde{k}}(|\xi|-n)\|_{L^2_\xi}\big].\end{aligned} \tag{2.4}$$

Proof. • If $|x|\leq 2^{m-10}$.

Note that, for this case, we have $\big|\nabla_\xi(x\cdot\xi-\mu t|\xi|)\big|\sim t\sim 2^m$. To exploit the high oscillation in ξ, we do integration by parts in ξ once. As a result, we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2} e^{ix\cdot\xi-i\mu t|\xi|}m(\xi)\widehat{f}(\xi)\psi_{\tilde{k}}(|\xi|-n)d\xi\Big|\lesssim\Big|\int_{\mathbb{R}^2} e^{ix\cdot\xi-i\mu t|\xi|}\nabla_\xi\cdot\big[\frac{(x|\xi|-\mu t\xi)}{|x|\xi|-\mu t\xi|^2}|\xi|m(\xi)\widehat{f}(\xi)\psi_{\tilde{k}}(|\xi|-n)\big]d\xi\Big|\\ &\lesssim 2^{-m}\|m(\xi)\|_{\mathcal{S}^\infty_k}\big[2^k\|\widehat{f}(\xi)\psi_{[\tilde{k}-1,\tilde{k}+1]}(|\xi|-n)\|_{L^\infty_\xi}+2^{(k+\tilde{k})/2}\|\nabla_\xi\widehat{f}(\xi)\psi_{\tilde{k}}(|\xi|-n)\|_{L^2_\xi}\big].\end{aligned} \tag{2.5}$$

• If $|x|\geq 2^{m-10}$.

Note that $\xi\times\nabla_\xi(x\cdot\xi-\mu t|\xi|)=0$ if and only if $\xi/|\xi|=\mu x/|x|:=\xi_0$. For this case, we do dyadic decomposition for the angle between ξ and ξ_0 with the threshold $\bar{l}:=-m/2-k/2$. From the volume of support of ξ, the following estimate holds for the small angle case,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2} e^{ix\cdot\xi-i\mu t|\xi|}\widehat{f}(\xi)m(\xi)\psi_{\tilde{k}}(|\xi|-n)\psi_{\leq\bar{l}}(\frac{\xi}{|\xi|}-\xi_0)d\xi\Big|\\ &\lesssim\|m(\xi)\|_{\mathcal{S}^\infty_k}2^{k+\tilde{k}+\bar{l}}\|\widehat{f}(\xi)\psi_{\tilde{k}}(|\xi|-n)\|_{L^\infty_\xi}\lesssim 2^{-m/2+k/2+\tilde{k}}\|m(\xi)\|_{\mathcal{S}^\infty_k}\|\widehat{f}(\xi)\psi_{\tilde{k}}(|\xi|-n)\|_{L^\infty_\xi}.\end{aligned} \tag{2.6}$$

For the large angle case, we first do dyadic decomposition for the size of angle between ξ and ξ_0 and then do integration by parts in ξ once. As a result, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2} e^{ix\cdot\xi - i\mu t|\xi|}\widehat{f}(\xi)m(\xi)\psi_{\tilde{k}}(|\xi|-n)\psi_{>\bar{l}}(\frac{\xi}{|\xi|}-\xi_0)d\xi\Big| \\
&\lesssim \sum_{l>\bar{l}}\Big|\int_{\mathbb{R}^2} e^{ix\cdot\xi - i\mu t|\xi|}(\xi_2\partial_{\xi_1}-\xi_1\partial_{\xi_2})\big[\frac{m(\xi)\widehat{f}(\xi)\psi_{\tilde{k}}(|\xi|-n)}{\xi_2 x_1-\xi_1 x_2}\varphi_{l;\bar{l}}(\frac{\xi}{|\xi|}-\xi_0)\big]d\xi\Big| \\
&\lesssim \sum_{l>\bar{l}} 2^{-m-l}\|m(\xi)\|_{\mathcal{S}^\infty_k}\big[2^{-k-l}2^{k+\tilde{k}+l}\|\widehat{f}(\xi)\psi_k(\xi)\|_{L^\infty_\xi} + 2^{(k+\tilde{k}+l)/2}\|\nabla_\xi\widehat{f}(\xi)\psi_k(\xi)\|_{L^2_\xi}\big] \\
&\lesssim 2^{-m/2}\|m(\xi)\|_{\mathcal{S}^\infty_k}\big[2^{\tilde{k}+k/2}\|\widehat{f}(\xi)\psi_k(\xi)\|_{L^\infty_\xi} + 2^{-m/4+3k/4+\tilde{k}/2}\|\nabla_\xi\widehat{f}(\xi)\psi_k(\xi)\|_{L^2_\xi}\big].
\end{aligned}\tag{2.7}$$

To sum up, our desired estimate (2.4) holds from the above estimate and the obtained estimate (2.5). □

By using the above linear decay estimate of wave equation, in the following Lemma, we show that the nonlinear wave part decays sharply over time under the bootstrap assumption (1.16).

Lemma 2.2. *Under the bootstrap assumption* (1.16)*, the following L^∞_x-decay estimate holds for the wave part,*

$$\sum_{k\in\mathbb{Z}}\langle t\rangle^{1/2}2^{-k/2+8k_+}\|P_k U\|_{L^\infty_x}\lesssim \epsilon_1. \tag{2.8}$$

Proof. Let $t\in[2^{m-1},2^m]\subset[0,T], m\gg 1$. By using the volume of support of ξ, we first rule out the very low and relatively high frequency cases as follows,

$$\begin{aligned}
&\sum_{k\notin[-m/2,2m/(N_0-20)]} 2^{-k/2+8k_+}\|\nabla_{t,x}P_k\phi\|_{L^\infty_x}\lesssim \sum_{k\notin[-m/2,2m/(N_0-20)]} 2^{-k/2+3k_+}\min\big\{2^{2k}\|\widehat{U}(t,\xi)\psi_k(\xi)\|_{L^\infty_\xi}, \\
&2^k\|\widehat{U}(t,\xi)\psi_k(\xi)\|_{L^2_\xi}\big\}\lesssim \sum_{k\notin[-m/2,2m/(N_0-20)]} 2^{-k/2+8k_+}\min\{2^{5k/3},2^{-(N_0-1)k+\delta m}\}\epsilon_1\lesssim 2^{-m/2}\epsilon_1.
\end{aligned}$$

Now, we focus on the case $k\in[-m/2,2m/(N_0-20)]\cap\mathbb{Z}$. Note that

$$\big|\nabla_\xi\widehat{V}(t,\xi)\big|\lesssim t|\xi|^{-1}\big|\partial_t\widehat{V}(t,\xi)\big| + |\xi|^{-1}\big[|\widehat{U^S}(t,\xi)| + |\widehat{U^\Omega}(t,\xi)| + |\widehat{U}(t,\xi)|\big].$$

Recall (1.2). Note that, from (1.17), we have

$$\|\partial_t\widehat{V}(t,\xi)\psi_k(\xi)\|_{L^2}\lesssim\|\nabla_{t,x}\phi\|_{H^3}\|u\|_{W^{3,\infty}}\lesssim\langle t\rangle^{-1+\delta}\epsilon_1^2.$$

Therefore, from the above two estimates and the bootstrap assumption (1.16), we have

$$\sup_{k\in\mathbb{Z}}2^{k-\alpha k_-}\big\|\nabla_\xi\widehat{V}(t,\xi)\psi_k(\xi)\big\|_{L^2}\lesssim\langle t\rangle^\delta\epsilon_1. \tag{2.9}$$

For the rest of cases, we use the linear decay estimate (2.4) in Lemma 2.1. From (2.9), we have

$$\begin{aligned}
&\|P_k U(t)\|_{L^\infty_x}\lesssim 2^{-m/2}\big(2^{k-10k_+} + 2^{-m/4+(1/4+\alpha)k+\delta m}\big)\epsilon_1, \\
\Longrightarrow &\sum_{k\in[-m/2,2m/(N_0-20)]} 2^{-k/2+8k_+}\|P_k U\|_{L^\infty_x}\lesssim 2^{-m/2}\epsilon_1.
\end{aligned}\tag{2.10}$$

Hence finishing the proof of the desired estimate (2.8). □

3. Energy estimate of the velocity part and the wave part

3.1. Energy estimate of the velocity field.

Lemma 3.1. *Let $t \in [2^{m-1}, 2^m] \subset [0,T], m \in \mathbb{Z}_+$, s.t., $m \gg 1$. Under the bootstrap assumption* (1.16), *the following improved energy estimate for the velocity field u holds,*

$$\|u(t)\|_{H^{N_0+1}} \lesssim 2^{-m/2+\delta m}\epsilon_0. \tag{3.1}$$

Proof. Recall the normal form transformation we did in (1.11). We first estimate the energy of the normal form part. Thanks to the "null structure for w×w→h", the symbol $a_{\mu,\nu}(\xi-\eta,\eta)$ of normal form transformation is non-singular. For any $\mu,\nu \in \{+,-\}$, from the estimate of symbol in (1.12), we have

$$\sum_{|\alpha|\leq N_0+1} \|\nabla_x^\alpha\big(A_{\mu,\nu}(U^\mu,U^\nu)\big)\|_{L^2} \lesssim \sum_{\substack{|k_1-k_2|\leq 10\\ k_1\geq k+20}} \|P_k\big(A_{\mu,\nu}(P_{k_1}(U^\mu),P_{k_2}(U^\mu))\big)\|_{L^2} + \|U\|_{H^{N_0}}\|U\|_{W^{3,\infty}}. \tag{3.2}$$

In the above estimate we single out the High × High type interaction because of two reasons. Firstly, the High × Low and Low × High type interactions are standard. Lastly and most importantly, the High × High type interaction is non-trivial in the sense that there is summability issue with respect to the output frequency k if without paying any price of decay rate. Note that this case only happens when $\alpha = 0$, which explains why only L^2-norm appears in (3.2).

To get around this summability issue, we first do super localization for two inputs and then use the super localized decay estimate (2.4) in Lemma 2.1 and the orthogonality in L^2. Let $P_{k_1;k,n}$ denotes the Fourier multiplier operator with Fourier symbol $\psi_{k_1}(\xi)\psi_k(|\xi|-2^k n)$. From the estimate (2.9) and the estimate of symbol in (1.12), we have

$$\begin{aligned}
&\|P_k\big(A_{\mu,\nu}(P_{k_1}(U^\mu),P_{k_2}(U^\nu))\big)\|_{L^2} \lesssim \sum_{\substack{n,m\in[2^{k_1-k-10},2^{k_1-k+10}]\cap\mathbb{Z}\\ |n-m|\leq 2^{10}}} \|P_k\big(A_{\mu,\nu}(P_{k_1;k,n}(U^\mu),P_{k_2;k,m}(U^\nu))\big)\|_{L^2}\\
&\lesssim \sum_{\substack{n,m\in[2^{k_1-k-5},2^{k_1-k+5}]\cap\mathbb{Z}\\ |n-m|\leq 2^{10}}} 2^{-m/2-k_{1,-}}\big[2^{k-k_{2,+}}\epsilon_1 + 2^{-m/4+3k_1/4+k/2}\|\nabla_\xi \widehat{V}(\xi)\psi_{k_2}(\xi)\psi_k(|\xi|-2^k n)\|_{L^2}\big]\\
&\times \|P_{k_1;k,n}(U^\mu)\|_{L^2} \lesssim 2^{-m/2-k_{1,-}}\min\{2^{-N_0k_{1,+}+\delta m}, 2^{k_1}\}(2^{k/2+k_1/2}+2^{-m/4+k/4+\delta m})\epsilon_1^2.
\end{aligned} \tag{3.3}$$

After combining the obtained estimates (3.2) and (3.3), we have

$$\sum_{|\alpha|\leq N_0+1} \|\nabla_x^\alpha\big(A_{\mu,\nu}(U^\mu,U^\nu)\big)\|_{L^2} \lesssim 2^{-m/2+\delta m}\epsilon_1^2. \tag{3.4}$$

Now, we focus on the estimate of H^{N_0}-norm of v. Recall the equation satisfied by "v" in (1.11). Note that the following Duhamel's formula holds,

$$v(t) = e^{t\Delta}v(0) + \int_0^t e^{(t-s)\Delta}\mathcal{N}_v(s)ds. \tag{3.5}$$

Note that, for any fixed $k\in\mathbb{Z}$, the following estimate holds for the frequency localized heat kernel,

$$\Big|\int_{\mathbb{R}^3} e^{iy\cdot\xi-(t-s)|\xi|^2}\psi_k(\xi)dy\Big| \lesssim 2^{3k}(1+2^{2k}|t-s|)^{-100}(1+2^k|y|)^{-100}. \tag{3.6}$$

By using the precise form of the heat kernel and Hölder inequality, we have

$$\|v(t)\|_{H^{N_0+1}} \lesssim 2^{-m/2}\epsilon_0 + \sum_{k\in\mathbb{Z}}\int_0^t (1+2^{2k}|t-s|)^{-10}\big[2^k\|P_k\mathcal{N}_v(s)\|_{L^1_x} + 2^{(N_0+1)k}\|P_k\mathcal{N}_v(s)\|_{L^2}\big]ds. \tag{3.7}$$

Recall (1.11) and (1.5). We first estimate the $H_x^{N_0-1/2}$-norm of $\mathcal{N}_v(s)$. From the estimate of symbol in (1.12), the $L^2 - L^\infty$-type bilinear estimate and the decay estimates in (1.17) and (2.8), we have

$$\|\mathcal{N}_v(s)\|_{H_x^{N_0-1/2}} \lesssim \langle s\rangle^{-3/2+2\delta}\epsilon_1^2 + \epsilon_1\langle s\rangle^{-1/2}\|\mathcal{N}_\phi(t)\|_{H_x^{N_0-3/2}} + \epsilon_1\langle s\rangle^{\delta}\|\mathcal{N}_\phi(t)\|_{L_x^\infty} \lesssim \langle s\rangle^{-3/2+2\delta}\epsilon_1^2. \tag{3.8}$$

Recall (1.2) and (1.6). Note that $\mathcal{N}_\phi$ has one space derivative outside, which contributes the smallness of output frequency. More precisely, we decompose $\mathcal{N}_\phi(s)$ as follows,

$$\mathcal{N}_\phi(s) = -2\partial_i(u_i\partial_t\phi) - \partial_i(\Delta u_i\phi) - \partial_i(u_iu_j\partial_j\phi) - \partial_i(|\nabla|\widetilde{\mathcal{N}}_u(s)\phi(s)) \tag{3.9}$$

Note that, after using the $L^2 - L^\infty$-type bilinear estimate and using $Z_\phi(t)$-norm to control ϕ, we have

$$\begin{aligned}\sum_{(k_1,k_2)\in\chi_k^2\cup\chi_k^3} &\|P_k(P_{k_1}(u_i)P_{k_2}(\partial_t\phi))\|_{L_x^2} + \|P_k(P_{k_1}(\Delta u_i)P_{k_2}(\phi))\|_{L_x^2} \\ &+ \|P_k(P_{k_1}(|\nabla|\widetilde{\mathcal{N}}_u(s))P_{k_2}(\phi))\|_{L_x^2} \lesssim 2^{k-3k_+}\langle s\rangle^{-1}\epsilon_1^2.\end{aligned} \tag{3.10}$$

For the High $\times$ High type interaction, we use the super-localized decay estimate for ϕ-part. After employing the same strategy used in (3.3), we have

$$\begin{aligned}\sum_{(k_1,k_2)\in\chi_k^1} &\|P_k(P_{k_1}(u_i)P_{k_2}(\partial_t\phi))\|_{L_x^2} + \|P_k(P_{k_1}(\Delta u_i)P_{k_2}(\phi))\|_{L_x^2} \\ &+ \|P_k(P_{k_1}(|\nabla|\widetilde{\mathcal{N}}_u(s))P_{k_2}(\phi))\|_{L_x^2} \lesssim 2^{k/4-3k_+}\langle s\rangle^{-1}\epsilon_1^2.\end{aligned} \tag{3.11}$$

Lastly, from $L^2 \to L^1$-type Sobolev embedding, we have

$$\|P_k(u_iu_j\partial_j\phi)\|_{L_x^2} \lesssim 2^k\|P_k(u_iu_j\partial_j\phi)\|_{L_x^1} \lesssim 2^{k-3k_+}\|u\|_{H^3}^2\|U\|_{W_x^{3,\infty}} \lesssim 2^k\langle s\rangle^{-5/4}\epsilon_1^3. \tag{3.12}$$

To sum up, after combining (3.9)–(3.12), we have

$$\|P_k\big(\mathcal{N}_\phi(s)\big)\|_{L_x^2} \lesssim 2^{5k/4-3k_+}\langle s\rangle^{-1}\epsilon_1. \tag{3.13}$$

For any fixed $k\in\mathbb{Z}$, from the obtained estimate (3.14), $L^2 - L^2$-type bilinear estimate, the following estimate holds if we put u in L^2 and ϕ-part in L_x^∞ for the L_x^2-estimate of $\mathcal{N}_\phi$,

$$\|P_k\mathcal{N}_v(s)\|_{L^1} \lesssim \|u(s)\|_{L^2}\|\nabla u(s)\|_{L^2} + \sum_{(k_1,k_2)\in\cup_{i=1,2,3}\chi_k^i}\|P_{k_2}U\|_{L^2}\|P_{k_1}\mathcal{N}_\phi\|_{L^2} \lesssim \langle s\rangle^{-1}\epsilon_1^2. \tag{3.14}$$

Moreover, After combining the above obtained estimates (3.4), (3.7), (3.8), and (3.14), we have

$$\begin{aligned}\|v(t)\|_{H^{N_0+1}} + \|u(t)\|_{H^{N_0+1}} &\lesssim 2^{-m/2+\delta m}\epsilon_0 + \int_0^t (t-s)^{-1/2}\langle s\rangle^{-1}\epsilon_1^2 + (t-s)^{-3/4}\langle s\rangle^{-3/2+2\delta}\epsilon_1^2 ds \\ &\lesssim 2^{-m/2+\delta m}\epsilon_0.\end{aligned} \tag{3.15}$$

Hence finishing the proof of our desired estimate (3.1). □

In the following Lemma, we control the energy of velocity with vector fields.

Lemma 3.2. *Let $t\in[2^{m-1},2^m]\subset[0,T], m\in\mathbb{Z}_+$, s.t., $m\gg 1$. Under the bootstrap assumption, the following improved energy estimate for the velocity field u holds,*

$$\sum_{\Gamma\in\{S,\Omega\}}\|\Gamma u(t)\|_{H^1} \lesssim 2^{-m/2+2\delta m}\epsilon_0. \tag{3.16}$$

Proof. Recall the normal form we did in (1.9). Again, we first estimate the energy of the normal form part. Similar to the obtained estimate (3.3), for any $\mu,\nu\in\{+,-\},\Gamma\in\{S,\Omega\}$, after putting U^Γ in L^2 and the other piece in L^∞, from the sharp decay estimate in (2.8) and the estimate of symbol in (1.12), we have

$$\|\Gamma\big(A_{\mu,\nu}(U^\mu(t),U^\nu(t))\big)\|_{H^1} \lesssim 2^{-m/2+2\delta m}\epsilon_1^2.$$

Now, we focus on the estimate of v-part. Note that, S doesn't commute with the heat equation. As a result of direct computation, we have

$$[\partial_t - \Delta, \Omega] = 0, \quad [\partial_t - \Delta, S] = [\partial_t - \Delta, t\partial_t + x \cdot \nabla_x] = \partial_t - 2\Delta$$

Therefore,

$$(\partial_t - \Delta)Sv = S\mathcal{N}_v + \mathcal{N}_v - \Delta v, \quad (\partial_t - \Delta)\Omega v = \Omega \mathcal{N}_v.$$

We rewrite the above two equations uniformly as follows,

$$(\partial_t - \Delta)\Gamma v = \Gamma \mathcal{N}_v + c_\Gamma^1 \mathcal{N}_v + c_\Gamma^2 \Delta v, \quad c_S^1 = 1, c_S^2 = -1, c_\Omega^1 = c_\Omega^2 = 0.$$

We also have the following Duhamel's formula for $\Gamma v(t)$,

$$\Gamma v(t) = e^{t\Delta}\Gamma v(0) + \int_0^t e^{(t-s)\Delta}\big(\Gamma \mathcal{N}_v + c_\Gamma^1 \mathcal{N}_v + c_\Gamma^2 \Delta v\big) ds. \tag{3.17}$$

From the explicit formula of heat kernel and the obtained estimate (3.14), we have

$$\begin{aligned} \|\int_0^t e^{(t-s)\Delta}\mathcal{N}_v ds\|_{H_x^1} &\lesssim \sum_{k\in\mathbb{Z}} \int_0^t (1 + 2^{2k}|t-s|)^{-10} 2^k \big(\|P_k \mathcal{N}_v(s)\|_{L_x^1} + \|P_k \mathcal{N}_v(s)\|_{L_x^2}\big) ds \\ &\lesssim \int_0^t (t-s)^{-1/2}\langle s\rangle^{-1}\epsilon_1^2 ds \lesssim 2^{-m/2+0.1\delta m}\epsilon_0. \end{aligned} \tag{3.18}$$

Again, from the explicit formula of heat kernel, the following estimate holds from the obtained H^{N_0}-norm estimate of v in (3.15),

$$\|\int_0^t e^{(t-s)\Delta}\Delta v ds\|_{H_x^1} \lesssim \int_0^{t-1} (t-s)^{-1}\|v(s)\|_{H_x^1} ds + \int_{t-1}^t \|v(s)\|_{H_x^3} ds \lesssim 2^{-m/2+1.1\delta m}\epsilon_0. \tag{3.19}$$

Now, we focus on the contribution of the main term $\Gamma\mathcal{N}_v$. Recall (1.11) and (1.5). Since $Q(u,u)$ has one space derivative outside, which contributes the smallness of output frequency, from $L^2 - L^\infty$-type bilinear estimate, we have

$$\|P_k\big(\Gamma\big(Q(u(s),u(s))\big)\big)\|_{L_x^2} \lesssim 2^{k-2k_+}\langle s\rangle^{-3/2+3\delta}\epsilon_1^2.$$

After doing dyadic decomposition for the output frequency and using the pointwise estimate of the heat kernel, the following estimate holds from the above obtained estimate,

$$\begin{aligned} \|\int_0^t e^{(t-s)\Delta}\big(\Gamma\big(Q(u(s),u(s))\big)\big) ds\|_{H_x^1} &\lesssim \sum_{k\in\mathbb{Z}} \int_0^t (1+2^{2k}|t-s|)^{-10} 2^k \langle s\rangle^{-3/2+3\delta}\epsilon_1^2 ds \\ &\lesssim \int_0^t (t-s)^{-1/2}\langle s\rangle^{-3/2+3\delta}\epsilon_0 ds \lesssim 2^{-m/2}\epsilon_0. \end{aligned} \tag{3.20}$$

Now, we consider the case Γ hits $A_{\mu,\nu}\big((\mathcal{N}_\phi)^\mu, U^\nu\big)$ and $A_{\mu,\nu}((U)^\mu, (\mathcal{N}_\phi)^\nu)$. Due to symmetry, it would be sufficient to consider the case Γ hits $A_{\mu,\nu}\big((\mathcal{N}_\phi)^\mu, U^\nu\big)$. As a result of computation, we have

$$\begin{aligned} \Gamma\big(A_{\mu,\nu}\big((\mathcal{N}_\phi)^\mu, U^\nu\big)\big) &= A_{\mu,\nu}\big((\Gamma\mathcal{N}_\phi)^\mu, U^\nu\big) + A_{\mu,\nu}\big((\mathcal{N}_\phi)^\mu, (U^\Gamma)^\nu\big) + Comm_\Gamma(\phi) \\ Comm_\Gamma(\phi) &:= \Gamma\big(A_{\mu,\nu}\big((\mathcal{N}_\phi)^\mu, U^\nu\big)\big) - A_{\mu,\nu}\big((\Gamma\mathcal{N}_\phi)^\mu, U^\nu\big) - A_{\mu,\nu}\big((\mathcal{N}_\phi)^\mu, (U^\Gamma)^\nu\big) \end{aligned} \tag{3.21}$$

For the commutators, we use the same strategy of estimating $\mathcal{N}_v$ in (3.14) and (3.18). As a result, we have

$$\|\int_0^t e^{(t-s)\Delta}\big(Comm_\Gamma(\phi)\big) ds\|_{H_x^1} \lesssim 2^{-m/2+0.1\delta m}\epsilon_1^2. \tag{3.22}$$

Now, it would be sufficient to focus on the estimate of $A_{\mu,\nu}\big((\Gamma\mathcal{N}_\phi)^\mu, U^\nu\big)$ and $A_{\mu,\nu}\big((\mathcal{N}_\phi)^\mu, (U^\Gamma)^\nu\big)$. From the estimate of symbol in (1.12), the obtained estimate (3.13), the $L^2 - L^\infty$-type bilinear estimate and the $L^\infty \to L^2$ type Sobolev embedding for the High $\times$ Low and Low $\times$ High type interactions, the

$L^2 - L^1$-type Sobolev embedding and the $L^\infty \to L^2$ type bilinear estimate for the High $\times$ High type interaction, we have

$$\begin{aligned}\sum_{(k_1,k_2)\in\chi_k^2\cup\chi_k^3} \|P_k A_{\mu,\nu}(P_{k_1}\big(\mathcal{N}_\phi(s)\big)^\mu, P_{k_2}(U^\Gamma)^\nu(s))\|_{L^2_x} &\lesssim 2^{9k/4-3k_+}\langle s\rangle^{-1+2\delta}\epsilon_1^2,\\ \sum_{(k_1,k_2)\in\chi_k^1} \|P_k A_{\mu,\nu}(P_{k_1}\big(\mathcal{N}_\phi(s)\big)^\mu, P_{k_2}(U^\Gamma)^\nu(s))\|_{L^2_x} &\lesssim 2^{k-2k_+}\langle s\rangle^{-1+2\delta}\epsilon_1^2.\end{aligned} \tag{3.23}$$

Similar to the obtained estimate (3.13), we have

$$\|P_k\big(\Gamma\mathcal{N}_\phi(s)\big)\|_{L^2_x} \lesssim 2^{k/2+k_+/2}\langle s\rangle^{-1+2\delta}\epsilon_1^2.$$

From the above estimate, the estimate of symbol in (1.12), the decay estimate (2.8), the $L^2 - L^\infty$-type bilinear estimate and the $L^\infty \to L^2$ type Sobolev embedding for the High $\times$ Low and Low $\times$ High type interactions, the $L^2 - L^1$-type Sobolev embedding and the $L^\infty \to L^2$ type bilinear estimate for the High $\times$ High type interaction, we have

$$\begin{aligned}\sum_{(k_1,k_2)\in\chi_k^2\cup\chi_k^3} \|P_k A_{\mu,\nu}(P_{k_1}\big(\Gamma\mathcal{N}_\phi(s)\big)^\mu, P_{k_2}(U)^\nu(s))\|_{L^2_x} &\lesssim 2^{k/2-k_+/2}\min\{2^k\langle s\rangle^{-1+2\delta}, \langle s\rangle^{-3/2+2\delta}\}\epsilon_1^2,\\ \sum_{(k_1,k_2)\in\chi_k^1} \|P_k A_{\mu,\nu}(P_{k_1}\big(\Gamma\mathcal{N}_\phi(s)\big)^\mu, P_{k_2}(U)^\nu(s))\|_{L^2_x} &\lesssim 2^{k-2k_+}\langle s\rangle^{-1+2\delta}\epsilon_1^2.\end{aligned} \tag{3.24}$$

Recall the decomposition (3.21). To sum up, after combining the above obtained estimates (3.22)–(3.24), we have

$$\begin{aligned}&\|\int_0^t e^{(t-s)\Delta}\Gamma\big(A_{\mu,\nu}\big((\mathcal{N}_\phi)^\mu, U^\nu\big)\big)\|_{H^1_x} \lesssim 2^{-m/2+0.1\delta m}\epsilon_1^2 + \sum_{k\in\mathbb{Z}}\int_0^t (1+2^{2k}|t-s|)^{-10}2^k\langle s\rangle^{-1+2\delta}\epsilon_1^2 ds\\ &\lesssim 2^{-m/2+0.1\delta m}\epsilon_1^2 + \int_0^t (t-s)^{-1/2}\langle s\rangle^{-1+2\delta}\epsilon_1^2 ds \lesssim 2^{-m/2+2\delta m}\epsilon_1^2.\end{aligned} \tag{3.25}$$

Therefore, our desired estimate (3.16) holds after combining the obtained estimates (3.17), (3.18), (3.19), (3.20), and (3.25). □

In the following Lemma, we control the energy of the wave part.

Lemma 3.3. *Let $t \in [2^{m-1}, 2^m] \subset [0,T], m \in \mathbb{Z}_+$, s.t., $m \gg 1$. Under the bootstrap assumption* (1.16)*, the following improved energy estimate for the wave part,*

$$2^{-\delta m}\|U(t)\|_{H^{N_0}} + 2^{-2\delta m}\big(\sum_{\Gamma\in\{S,\Omega\}}\|U^\Gamma(t)\|_{L^2} + \sum_{k\in\mathbb{Z}_-} 2^{-\alpha k}\|P_k U^\Gamma(t)\|_{L^2}\big) \lesssim \epsilon_0. \tag{3.26}$$

Proof. Since the vector field $\Gamma \in \{S,\Omega\}$ plays the same role as $\nabla_x^\alpha, |\alpha| = N_0$, the estimate of H^{N_0}-norm of U and the energy estimate of $U^\Gamma(t)$ follows in the same argument. We first give a detailed argument for the H^{N_0}-estimate of U by using a modified energy method and then we define the corresponding modified energy for the vector field part.

For the H^{N_0}-estimate of U, we define the following modified energy

$$E_{modi}(t) = \sum_{|\alpha|\le N_0}\int_{\mathbb{R}^3} |\nabla_x^\alpha\nabla_x\phi|^2 + |\nabla_x^\alpha\partial_t\phi|^2 - u_iu_j\big(\partial_i\nabla_x^\alpha\phi\partial_j\nabla_x^\alpha\phi\big) + 2\partial_t\nabla_x^\alpha\phi\nabla_x^\alpha u_i\partial_i\phi dx. \tag{3.27}$$

As a result of direct computations, we have

$$\begin{aligned}\frac{d}{dt}E_{modi}(t) = &\sum_{|\alpha|\le N_0}\int_{\mathbb{R}^3} 2\nabla_x^\alpha\partial_t\phi\big(\partial_t^2-\Delta\big)\nabla_x^\alpha\phi - 2u_iu_j\big(\partial_i\nabla_x^\alpha\partial_t\phi\partial_j\nabla_x^\alpha\phi\big) + 2\partial_t\nabla_x^\alpha\phi\partial_t\nabla_x^\alpha u_i\partial_i\phi dx\\ &+ 2\partial_t\nabla_x^\alpha\phi\nabla_x^\alpha u_i\partial_i\partial_t\phi - 2\partial_t u_iu_j\big(\partial_i\nabla_x^\alpha\phi\partial_j\nabla_x^\alpha\phi\big) + 2\partial_t^2\nabla_x^\alpha\phi\nabla_x^\alpha u_i\partial_i\phi dx.\end{aligned} \tag{3.28}$$

Recall (1.2). Note that

$$
\begin{aligned}
&(\partial_t^2-\Delta)\nabla_x^\alpha\phi = Semi_\alpha(\phi)+\sum_{i=1,2}Quas^i_\alpha(\phi),\quad Semi_\alpha(\phi):=\nabla_x^\alpha\mathcal{N}_\phi-\sum_{i=1,2}Quas^i_\alpha(\phi),\\
&Quas^1_\alpha(\phi):=-2u_i\partial_t\partial_{x_i}\nabla_x^\alpha\phi+u_iu_j\partial_{x_i}\partial_{x_j}\nabla_x^\alpha\phi,\quad Quas^2_\alpha(\phi):=-\partial_t\nabla_x^\alpha u_i\partial_i\phi.
\end{aligned}
\tag{3.29}
$$

For the semilinear part, we use the L^2-L^∞-type bilinear estimate. As a result, we have

$$
\begin{aligned}
&\big|\int_{\mathbb{R}^3}\partial_t\nabla_x^\alpha\phi Semi_\alpha(\phi)dx\big|+\big|\int_{\mathbb{R}^3}\partial_tu_iu_j\big(\partial_i\nabla_x^\alpha\phi\partial_j\nabla_x^\alpha\phi\big)dx\big|+\big|\int_{\mathbb{R}^3}\partial_t\nabla_x^\alpha\phi\nabla_x^\alpha u_i\partial_i\partial_t\phi dx\big|\\
&\lesssim\|U(t)\|_{H^{N_0}}\big[\|U(t)\|_{H^{N_0}}\|u(t)\|_{W^{2,\infty}}+\|U(t)\|_{W^{2,\infty}}\|u(t)\|_{H^{N_0+1}}\big]\lesssim\langle t\rangle^{-1+2\delta}\epsilon_1^3.
\end{aligned}
\tag{3.30}
$$

Thanks to the symmetric structure and the divergence free condition of u, for the quasilinear part $Quas^1_\alpha(\phi)$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}2\partial_t\nabla_x^\alpha\phi Quas^1_\alpha(\phi)dx-\int_{\mathbb{R}^3}2u_iu_j\partial_i\partial_t\nabla_x^\alpha\phi\partial_j\nabla_x^\alpha\phi dx\\
&=-\int_{\mathbb{R}^3}2\partial_t\nabla_x^\alpha\phi\big[2u_i\partial_t\partial_{x_i}\nabla_x^\alpha\phi+u_iu_j\partial_{x_i}\partial_{x_j}\nabla_x^\alpha\phi\big]dx-\int_{\mathbb{R}^3}2u_iu_j\partial_i\partial_t\nabla_x^\alpha\phi\partial_j\nabla_x^\alpha\phi dx\\
&=\int_{\mathbb{R}^3}-2\partial_{x_i}\big(\partial_t\nabla_x^\alpha\phi\partial_j\nabla_x^\alpha\phi\big)u_iu_jdx=\int_{\mathbb{R}^3}2\big(\partial_t\nabla_x^\alpha\phi\partial_j\nabla_x^\alpha\phi\big)u_i\partial_{x_i}u_jdx.
\end{aligned}
$$

Therefore, by using the L^2-L^∞-type bilinear estimate, we have

$$
\begin{aligned}
&\big|\int_{\mathbb{R}^3}2\partial_t\nabla_x^\alpha\phi Quas^1_\alpha(\phi)dx-\int_{\mathbb{R}^3}2u_iu_j\partial_i\partial_t\nabla_x^\alpha\phi\partial_j\nabla_x^\alpha\phi dx\big|\\
&\lesssim\|U(t)\|_{H^{N_0}}\big[\|U(t)\|_{H^{N_0}}\|u(t)\|_{W^{2,\infty}}+\|U(t)\|_{W^{2,\infty}}\|u(t)\|_{H^{N_0+1}}\big]\lesssim\langle t\rangle^{-1+2\delta}\epsilon_1^3.
\end{aligned}
\tag{3.31}
$$

In particular, we add the cubic correction term $2\partial_t\nabla_x^\alpha\phi\nabla_x^\alpha u_i\partial_i\phi$ in (3.27) to cancel the Quasilinear part $Quas^2_\alpha(\phi)$. More precisely, we have

$$
\int_{\mathbb{R}^3}2\partial_t\nabla_x^\alpha\phi Quas^2_\alpha(\phi)dx+\int_{\mathbb{R}^3}2\partial_t\nabla_x^\alpha\phi\partial_t\nabla_x^\alpha u_i\partial_i\phi dx=0.
$$

Recall (3.28). Now, it remains to estimate the last piece, which is $\partial_t^2\nabla_x^\alpha\phi\nabla_x^\alpha u_i\partial_i\phi$. By using the equation satisfied by $\nabla_x^\alpha\phi$ in (3.29), we make the following decomposition,

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\partial_t^2\nabla_x^\alpha\phi\nabla_x^\alpha u_i\partial_i\phi dx=I_1(t)+I_2(t),\\
&I_1(t):=\int_{\mathbb{R}^3}\Delta\nabla_x^\alpha\phi\nabla_x^\alpha u_i\partial_i\phi dx+\int_{\mathbb{R}^3}\big[Semi_\alpha(\phi)+Quasi^1_\alpha(\phi)\big]\nabla_x^\alpha u_i\partial_i\phi dx,\\
&I_2(t):=\int_{\mathbb{R}^3}Quasi^2_\alpha(\phi)\nabla_x^\alpha u_i\partial_i\phi dx=-\int_{\mathbb{R}^3}\partial_t\nabla_x^\alpha u_i\partial_i\phi\nabla_x^\alpha u_i\partial_i\phi dx.
\end{aligned}
$$

For $I_2(t)$, we use the equation satisfied by u in (1.2). As a result, we have

$$
I_2(t)=I_2^1(t)+I_2^2(t),\quad I_2^1(t):=-\int_{\mathbb{R}^3}\Delta\nabla_x^\alpha u_i\partial_i\phi\nabla_x^\alpha u_i\partial_i\phi dx,\quad I_2^2(t)=-\int_{\mathbb{R}^3}\nabla_x^\alpha\mathcal{N}_u\cdot\nabla\phi\nabla_x^\alpha u_i\partial_i\phi dx.
$$

Since $u\in H^{N_0+1}$, which is one regularity better than $\nabla_{t,x}\phi$. By using this fact, now it's easy to see that there is no losing derivative issue for $I_1(t), I_2^1(t)$, and $I_2^2(t)$. From the L^2-L^∞-type bilinear estimate, we have

$$
\begin{aligned}
&\big|\int_{\mathbb{R}^3}\partial_t^2\nabla_x^\alpha\phi\nabla_x^\alpha u_i\partial_i\phi dx\big|\lesssim\big|I_1(t)\big|+\big|I_2^1(t)\big|+\big|I_2^1(t)\big|\lesssim\|U(t)\|^2_{H^{N_0}}\|U(t)\|^2_{W^{3,\infty}}\\
&+\|U(t)\|_{H^{N_0}}\|u(t)\|_{H^{N_0+1}}\|U(t)\|_{W^{3,\infty}}+\|u(t)\|^2_{H^{N_0+1}}\|U(t)\|_{W^{3,\infty}}\lesssim\langle t\rangle^{-1+2\delta}\epsilon_1^3.
\end{aligned}
\tag{3.32}
$$

To sum up, from the above obtained estimates (3.30), (3.31),and (3.32), we have

$$\begin{aligned} &\|U(t)\|_{H^{N_0}}^2 \lesssim E_{modi}(t) + \|U(t)\|_{H^{N_0}}^2 \|u(t)\|_{W^{2,\infty}}^2 + \|U(t)\|_{H^{N_0}} \|u(t)\|_{H^{N_0}} \|U(t)\|_{W^{2,\infty}} \\ &\lesssim \int_0^t \langle s\rangle^{-1+2\delta} \epsilon_1^3 ds + \epsilon_1^4 \lesssim \langle t\rangle^{2\delta}\epsilon_0^2. \end{aligned} \tag{3.33}$$

Similar to what we did in the H^{N_0}-norm estimate, to avoid the losing derivative issue for the energy estimate of $U^{\Gamma}(t)$, we define the following modified energy,

$$E_{modi}^{vec}(t) = \sum_{\Gamma\in\{S,\Omega\}} \int_{\mathbb{R}^3} |\partial_t \Gamma\phi|^2 + |\nabla_x \Gamma\phi|^2 - \int_{\mathbb{R}^3} u_i u_j \big(\partial_i \Gamma\phi \partial_j \Gamma\phi\big) + 2\partial_t \Gamma\phi \Gamma u_i \partial_i \phi dx. \tag{3.34}$$

Again, due to the symmetric structure of $\mathcal{N}_\phi$ and the fact that $\Gamma u \in H^1$, which is one regularity better than $\nabla_{t,x}\Gamma\phi$. With the minor modification in the arguments for the estimate of $E_{modi}(t)$, we have

$$\begin{aligned} &\big|\frac{d}{dt} E_{modi}^{vec}(t)\big| \lesssim \sum_{\Gamma\in\{S,\Omega\}} \|U^{\Gamma}(t)\|_{L^2}^2 \|U(t)\|_{W^{3,\infty}}^2 + \|U^{\Gamma}(t)\|_{L^2} \|\Gamma u(t)\|_{H^1} \|U(t)\|_{W^{3,\infty}} \\ &+ \|\Gamma u(t)\|_{H^1}^2 \|U(t)\|_{W^{3,\infty}} \lesssim \langle t\rangle^{-1+4\delta}\epsilon_1^3. \\ \Longrightarrow\quad &\|U^{\Gamma}(t)\|_{L^2}^2 \lesssim E_{modi}(t) + \|U^{\Gamma}(t)\|_{L^2}^2 \|u(t)\|_{W^{2,\infty}}^2 \\ &+ \|U^{\Gamma}(t)\|_{L^2} \|\Gamma u(t)\|_{H^1} \|U(t)\|_{W^{2,\infty}} \lesssim \int_0^t \langle s\rangle^{-1+4\delta}\epsilon_1^3 ds + \epsilon_1^4 \lesssim \langle t\rangle^{4\delta}\epsilon_0^2. \end{aligned} \tag{3.35}$$

Lastly, we consider the low frequency part, for which there is no losing derivative issue. More precisely, for the the High $\times$ High type interaction, there is no losing derivative issue since derivatives can be distributed between two inputs. Meanwhile, for the High $\times$ Low type interaction and the Low $\times$ High type interaction, the frequency of the High frequency part is comparable with the output frequency, hence the frequency of the High frequency part is less one.

Recall (1.2). Since the nonlinearity $\mathcal{N}_\phi$ has a derivative outside, which contributions the smallness of 2^k. Therefore, from the $L^2 - L^\infty_x$ type bilinear estimate, for any $\Gamma \in \{S,\Omega\}$, we have

$$\begin{aligned} &\sum_{k\in\mathbb{Z}_-} 2^{-\alpha k} \|P_k U^{\Gamma}(t)\|_{L^2} \lesssim \epsilon_0 + \int_0^t \sum_{k\in\mathbb{Z}_-} 2^{-\alpha k} \big(\|P_k \Gamma \mathcal{N}_\phi(s)\|_{L^2} + \|P_k \mathcal{N}_\phi(s)\|_{L^2}\big) ds \\ &\lesssim \epsilon_0 + \int_0^t \sum_{k\in\mathbb{Z}_-} 2^{(1/2-\alpha)k} \langle s\rangle^{-1+2\delta}\epsilon_1^2 ds \lesssim \langle t\rangle^{2\delta}\epsilon_1^2. \end{aligned} \tag{3.36}$$

Hence finishing the proof of our desired estimate (3.26). $\square$

4. Z-norm estimate of the velocity part and the wave part

In this section, we improve the estimates of $Z_u(t)$ and $Z_\phi(t)$ within the time interval $[0,T]$. Hence finishing the bootstrap argument. Before proceeding to the the estimate of $Z_u(t)$ and $Z_\phi(t)$, we first prove a technical bilinear estimate, which will be used to study the bilinear interaction of wave part with angular localization on the Fourier side.

Let $\mu,\nu \in \{+,-\}$, $m(\xi,\eta) \in S^\infty$, $\phi \in C_0^\infty(\mathbb{R}^2)$, $l, k, k_1, k_2 \in \mathbb{Z}$, $\alpha \in \mathbb{R}_+$, $l \leq 0$, and $f, g \in L^2 \cap L^1$. We define the rescaled Schwartz function $\phi_l(x) = \phi(2^{-l}x)$ and a bilinear form as follows,

$$T^l_{k;k_1,k_2}(f,g) = \int_{\mathbb{R}^2} e^{it(|\xi|-\mu|\xi-\eta|-\nu|\eta|)} \psi_k(\xi) m(\xi,\eta) \big(\frac{\xi}{|\xi|} \times \frac{\eta}{|\eta|}\big)^{\alpha} \phi_l\big(\frac{\xi}{|\xi|} - \mu\frac{\eta}{|\eta|}\big) \widehat{f_{k_1}}(\xi-\eta)\widehat{g_{k_2}}(\eta) d\eta, \tag{4.1}$$

Lemma 4.1. *For the bilinear form defined in* (4.1)*, the following estimate holds,*

$$\begin{aligned}\|T^l_{k;k_1,k_2}(f,g)\|_{L^2} \lesssim & 2^{\alpha l}\|m\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \min\Big\{\|\hat{f}(\xi)\psi_{[k_1-2,k_1+2]}(\xi)\|_{L^2_\xi}\|\mathcal{F}^{-1}[e^{-it\nu|\xi|}\widehat{g}(\xi)\psi_{k_2}(\xi)]\|_{L^\infty_x},\\ & \|\hat{g}(\xi)\psi_{[k_2-2,k_2+2]}(\xi)\|_{L^2}\|\mathcal{F}^{-1}[e^{-it\mu|\xi|}\widehat{f}(\xi)\psi_{k_1}(\xi)]\|_{L^\infty_x},\\ & 2^{2k_2+l}\|f\|_{L^2}\|\widehat{g}\|_{L^\infty_\xi}, \min\big\{2^{k+(k_1+k_2)/2+l}, 2^{k_2+k_1+l}\big\}\|\widehat{f}\|_{L^\infty_\xi}\|g\|_{L^2}\Big\}.\end{aligned} \tag{4.2}$$

Proof. For any $l \in (-\infty,-100)\cap\mathbb{Z}_-$, we define $K_l := \lfloor \pi 2^{5-l}\rfloor$. Moreover, $\forall i\in\{1,\cdots,K_l\}$, denote $\phi^l_i := i2^{l-4}$, and $\theta^l_i := (\cos\phi^l_i, \sin\phi^l_i)$. Let

$$\varphi_{l,i}(\xi) := \frac{\psi_{\le l-4}(\frac{\xi}{|\xi|}-\theta^l_i)}{\sum_{i=1,\cdots,K}\psi_{\le l-4}(\frac{\xi}{|\xi|}-\theta^l_i)}, \quad \Longrightarrow \sum_{i\in\{1,\cdots,K_l\}}\varphi_{l,i}(\xi)=1. \tag{4.3}$$

Therefore $\{\varphi_{l,i}(\xi)\}_{i=1}^{K^l}$ is a partition of unity with finite overlap. Moreover, $supp(\varphi_{l,i}(\cdot))$ lies in an angular sector centered at $\theta^l_i\in\mathbb{S}^1$. Note that, for any $l,\tilde{l}\in\mathbb{Z}_-$, after doing integration by parts in ξ along $\{\theta^{\tilde{l}}_{i_1},(\theta^{\tilde{l}}_{i_1})^\perp\}$ directions for and doing integration by parts η along $\{\theta^l_{i_2},(\theta^l_{i_2})^\perp\}$ directions many times, the following estimate holds for the kernel of the angle localized symbol,

$$\begin{aligned}&\big|\mathcal{F}^{-1}_{(\xi,\eta)\to(x,y)}\big[m(\xi,\eta)\psi_k(\xi)\psi_{k_2}(\eta)\big(\frac{\xi}{|\xi|}-\mu\frac{\eta}{|\eta|}\big)^\alpha\phi_l\big(\frac{\xi}{|\xi|}-\mu\frac{\eta}{|\eta|}\big)\varphi_{\tilde{l},i_1}(\xi)\varphi_{l,i_2}(\eta)\big]\big|\\ &\lesssim 2^{\alpha l}\|m(\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}2^{2k+\min\{l,\tilde{l}\}}(1+2^k|\theta^{\tilde{l}}_{i_1}\cdot x|)^{-4}(1+2^{k+\min\{l,\tilde{l}\}}|\theta^{l_1}_{i_1}\times x|)^{-4}\\ &\times 2^{2k_2+l}(1+2^{k_2}|\theta^l_{i_2}\cdot x|)^{-4}(1+2^{k_2+l}|\theta^l_{i_2}\times y|)^{-4}.\end{aligned} \tag{4.4}$$

Applying the above partition of unity, we decompose the bilinear form $T^l_{k;k_1,k_1}(f,g)$ as follows,

$$\begin{aligned}T^l_{k;k_1,k_2}(f,g) &= \sum_{\substack{i_1\in\{1,\cdots,K^{(k_2-k_1)+l}\}, i_2\in\{1,\cdots,K^l\}\\ |\theta^{(k_2-k_1)+l}_{i_1}-\theta^l_{i_2}|\le 2^{l+10}}} T^{l;i_0,i_1,i_2}_{k;k_1,k_2}(f,g)\\ T^{l;i_0,i_1,i_2}_{k;k_1,k_2}(f,g) &= \int_{\mathbb{R}^2} e^{it(|\xi|-\mu|\xi-\eta|-\nu|\eta|)}\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)\big(\frac{\xi}{|\xi|}-\mu\frac{\eta}{|\eta|}\big)^\alpha\phi_l\big(\frac{\xi}{|\xi|}-\mu\frac{\eta}{|\eta|}\big)\\ &\times\varphi_{k_2-k_1+l,i_1}(\xi)\varphi_{l,i_2}(\eta)\varphi_{k_2-k_1+l,i_1}(\pm(\xi-\eta))m(\xi,\eta)\widehat{f^\mu}(\xi-\eta)\widehat{g^\nu}(\eta)d\eta,\end{aligned} \tag{4.5}$$

In the above decomposition, we used the fact that $|\frac{\xi}{|\xi|}\times\frac{\xi-\eta}{|\xi-\eta|}|\sim 2^{k_2-k_1+l}$. Due to the orthogonality in L^2, from the above estimate of kernel, we have

$$\begin{aligned}&\|T^l_{k;k_1,k_2}(f,g)\|^2_{L^2_\xi}\lesssim \sum_{\substack{i_1\in\{1,\cdots,K^{(k_2-k_1)+l}\}, i_2\in\{1,\cdots,K^l\}\\ |\theta^{(k_2-k_1)+l}_{i_1}-\theta^l_{i_2}|\le 2^{l+10}}}\|T^{l;i_0,i_1,i_2}_{k;k_1,k_2}(f,g)\|^2_{L^2_\xi}\lesssim \sum_{\substack{i_1\in\{1,\cdots,K^{(k_2-k_1)+l}\}, i_2\in\{1,\cdots,K^l\}\\ |\theta^{(k_2-k_1)+l}_{i_1}-\theta^l_{i_2}|\le 2^{l+10}}} 2^{2\alpha l}\\ &\|m(\xi,\eta)\|^2_{\mathcal{S}^\infty_{k,k_1,k_2}}\min\big\{\|\mathcal{F}^{-1}_{\xi\to x}[e^{-it\mu|\xi|}\varphi_{k_2-k_1+l,i_1}(\pm\xi)\hat{f}(\xi)\psi_{k_1}(\xi)]\|^2_{L^\infty_x}\|\hat{g}(\xi)\varphi_{l,i_2}(\xi)\psi_{[k_2-2,k_2+2]}(\xi)\|^2_{L^2_\xi},\\ &\|\varphi_{k_2-k_1+l,i_1}(\pm\xi)\hat{f}(\xi)\psi_{k_1}(\xi)\|^2_{L^2_\xi}\|\mathcal{F}^{-1}_{\xi\to x}[e^{-it\nu|\xi|}\hat{g}(\xi)\varphi_{l,i_2}(\xi)\psi_{[k_2-2,k_2+2]}(\xi)\|^2_{L^\infty_x}\big\}\\ &\lesssim 2^{2\alpha l}\|m(\xi,\eta)\|^2_{\mathcal{S}^\infty_{k,k_1,k_2}}\big(\min\big\{2^{k_2+k_1+l}\|\hat{f}(\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\|\hat{g}(\xi)\psi_{[k_2-2,k_2+2]}(\xi)\|_{L^2_\xi},\\ &2^{2k_2+l}\|\hat{g}(\xi)\psi_{[k_2-2,k_2+2]}(\xi)\|_{L^\infty_\xi}\|\hat{f}(\xi)\psi_{k_1}(\xi)\|_{L^2_\xi}\big\}\big)^2.\end{aligned} \tag{4.6}$$

Note that if $|k_1-k_2|\ge 10$, we have $|k-\max\{k_1,k_2\}|\le 10$. To prove the final estimate of (4.2), it would be sufficient to consider the case $|k_1-k_2|\le 10$. As in (4.6), if instead of using the L^2-L^∞-type bilinear estimate, we first use the $L^2 \to L^1$-type Sobolev embedding and then use the $L^2 - L^2$-type bilinear estimate, then we have

$$\|T^l_{k;k_1,k_2}(f,g)\|_{L^2_\xi} \lesssim 2^{\alpha l}\|m(\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} 2^{k+l/2}2^{(k_1+k_2+l)/2}\|\hat{f}(\xi)\psi_{k_1}(\xi)\|_{L^\infty_\xi}\|\hat{g}(\xi)\psi_{[k_2-2,k_2+2]}(\xi)\|_{L^2_\xi}. \quad (4.7)$$

Moreover, from the obtained estimate (4.6), we have

$$\|T^l_{k;k_1,k_2}(f,g)\|_{L^2_\xi} \lesssim 2^{\alpha l}\|m(\xi,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\|\hat{f}(\xi)\psi_{[k_1-2,k_1+2]}(\xi)\|_{L^2_\xi}\|\mathcal{F}^{-1}[e^{-it\nu|\xi|}\widehat{g}(\xi)\psi_{k_2}(\xi)]\|_{L^\infty_x}. \quad (4.8)$$

Alternatively, if we apply the partition of unity with $i_1, i_2 \in \{1, \cdots, K^l\}$, for ξ, η, we have

$$\begin{aligned}\|T^l_{k;k_1,k_2}(f,g)\|^2_{L^2_\xi} &\lesssim \sum_{i_1,i_2\in\{1,\cdots,K^l\},|i_1-i_2|\le 2^{10}} 2^{2\alpha l}\|\hat{g}(\xi)\varphi_{l,i_2}(\xi)\psi_{[k_2-2,k_2+2]}(\xi)\|^2_{L^2_\xi}\|\mathcal{F}^{-1}_{\xi\to x}[e^{-it\mu|\xi|}\hat{f}(\xi)\psi_{k_1}(\xi)]\|^2_{L^\infty_x} \\ &\times\|m(\xi,\eta)\|^2_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim 2^{2\alpha l}\|m(\xi,\eta)\|^2_{\mathcal{S}^\infty_{k,k_1,k_2}}\|\hat{g}(\xi)\psi_{[k_2-2,k_2+2]}(\xi)\|^2_{L^2_\xi}\|\mathcal{F}^{-1}_{\xi\to x}[e^{-it\mu|\xi|}\hat{f}(\xi)\psi_{k_1}(\xi)]\|^2_{L^\infty_x}\end{aligned} \quad (4.9)$$

To sum up, our desired estimates in (4.2) hold from the above obtained estimates (4.6)–(4.9). □

4.1. The estimate of $Z_u(t)$.

Lemma 4.2. *Let $t \in [2^{m-1}, 2^m] \subset [0,T], m \in \mathbb{Z}_+$, s.t., $m \gg 1$. Under the bootstrap assumption* (1.16), *the following improved energy estimate for the velocity field u holds,*

$$Z_u(t) \lesssim 2^{-m}\epsilon_0. \quad (4.10)$$

Proof. Recall the normal form we did in (1.9) and the equation satisfied by v in (1.11).

- The estimate of v-part.

Note that, since $Q(u,u)$ has one derivative outside, which contributes one degree of smallness, from the $L^2 - L^\infty$-type bilinear estimate, we have

$$\forall k \in Z, \quad \|P_kQ(u(s),u(s))\|_{L^2} \lesssim 2^{k-10k_+}\min\{2^k\langle s\rangle^{-1+2\delta}, \langle s\rangle^{-3/2+\delta}\}\epsilon_1^2.$$

From the above estimate and the estimate of heat kernel, we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} 2^{k+4k_+}\Big\|\int_0^t e^{(t-s)\Delta}P_kQ(u(s),u(s))ds\Big\|_{L^2_x} \\ &\lesssim \sum_{k\in\mathbb{Z}}\int_0^t 2^{2k-5k_+}(1+2^k|t-s|)^{-10}\min\{2^k\langle s\rangle^{-1+2\delta}, \langle s\rangle^{-3/2+\delta}\}\epsilon_1^2 ds \lesssim \langle t\rangle^{-3/2+3\delta}\epsilon_0.\end{aligned} \quad (4.11)$$

Now we estimate the contribution of $A_{\mu,\nu}((\mathcal{N}_\phi)^\mu, U^\nu)$. Note that, the rough estimate obtained for $\mathcal{N}_\phi$ in (3.13) is almost sufficient. We first rule out most of cases and then focus on the bulk scenario.

By using the $L^2_x - L^\infty_x$-type bilinear estimate, and the L^∞_ξ-estimate of (u, U), we have

$$\|P_kA_{\mu,\nu}((P_{k_1}\mathcal{N}_\phi)^\mu, P_{k_2}U^\nu)\|_{L^2_x} \lesssim 2^{-(N_0-5)k_+}\langle s\rangle^{-3/2+3\delta}\epsilon_1^2.$$

The above estimate allows us to rule out the case $k \ge 0$ as follows,

$$\begin{aligned}&\sum_{k\in\mathbb{Z}_+} 2^{k+4k_+}\Big\|\int_0^t e^{(t-s)\Delta}P_kA_{\mu,\nu}((P_{k_1}\mathcal{N}_\phi)^\mu, P_{k_2}U^\nu)ds\Big\|_{L^2_x} \\ &\lesssim \sum_{k\in\mathbb{Z}_+}\int_0^t 2^{-(N_0-10)k_+}(1+2^k|t-s|)^{-10}\langle s\rangle^{-3/2+3\delta}\epsilon_1^2 ds \lesssim \langle t\rangle^{-3/2+3\delta}\epsilon_0.\end{aligned} \quad (4.12)$$

Now, we focus on the case $k \le 0$. From the $L_x^2 - L_x^2$-type bilinear estimate, the obtained estimate (3.13) and the L_ξ^∞-estimate of U, we first rule out the High $\times$ Low and Low $\times$ High type interaction as follows,

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\chi_k^2\cup\chi_k^3} \|P_k A_{\mu,\nu}(\big(P_{k_1}\mathcal{N}_\phi\big)^\mu, P_{k_2}U^\nu)\|_{L_x^1} \lesssim 2^{5k/4}\langle s\rangle^{-1},\\
\Longrightarrow &\sum_{k\in\mathbb{Z}_-}\sum_{(k_1,k_2)\in\chi_k^2\cup\chi_k^3} 2^{k+4k_+}\|\int_0^t e^{(t-s)\Delta}P_k A_{\mu,\nu}(\big(P_{k_1}\mathcal{N}_\phi(s)\big)^\mu, P_{k_2}U^\nu(s))ds\|_{L_x^2}\\
&\lesssim \sum_{k\in\mathbb{Z}_-}\int_0^t 2^{9k/4}(1+2^{2k}|t-s|)^{-10}\langle s\rangle^{-1}\epsilon_1^2 ds \lesssim \langle t\rangle^{-1}\epsilon_0.
\end{aligned} \tag{4.13}$$

Lastly, we consider the High $\times$ High type interaction. For this case, we use different strategies for different pieces of $\mathcal{N}_\phi$. Recall (1.2). We decompose $\mathcal{N}_\phi$ into two pieces as follows,

$$\mathcal{N}_\phi = \mathcal{N}_\phi^1 + \mathcal{N}_\phi^2, \quad \mathcal{N}_\phi^1 = \sum_{k\in\mathbb{Z}} -2P_{\le k-10}(u_i)\partial_i P_k(\partial_t\phi), \quad \mathcal{N}_\phi^2 = \mathcal{N}_\phi - \mathcal{N}_\phi^1. \tag{4.14}$$

For $\mathcal{N}_\phi^2$ part, the following improved estimate holds since the L_x^2 of ∇u decay faster than u,

$$\|P_k\mathcal{N}_\phi^2\|_{L^2} \lesssim 2^{k-3k_+}\langle s\rangle^{-3/2+2\delta}\epsilon. \tag{4.15}$$

From the above estimate and the estimate of symbol in (1.12), we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}_-}\sum_{(k_1,k_2)\in\chi_k^1} 2^{k+4k_+}\|\int_0^t e^{(t-s)\Delta}P_k A_{\mu,\nu}(\big(P_{k_1}\mathcal{N}_\phi^2(s)\big)^\mu, P_{k_2}U^\nu(s))ds\|_{L_x^2}\\
&\lesssim \sum_{k\in\mathbb{Z}_-}\int_0^t 2^{2k}(1+2^{2k}|t-s|)^{-10}\langle s\rangle^{-3/2+2\delta}\epsilon_1^2 ds \lesssim \langle t\rangle^{-1}\epsilon_0.
\end{aligned} \tag{4.16}$$

It remains to consider the contribution of $\mathcal{N}_\phi^1$. For this case, as in (3.3), we use super localized decay estimate for the half wave. More precisely, we first consider the case when the frequency of u in $\mathcal{N}_\phi^1$ is less than 2^k. For this case, we decompose the high frequency inputs into fine pieces as follows,

$$\begin{aligned}
&\|P_k A_{\mu,\nu}(P_{k_1}\big(P_{\le k}(u_i)P_{k_1'}(\partial_t\partial_i\phi)\big)^\mu, P_{k_2}U^\nu(s))\|_{L_x^1}\\
&\lesssim \sum_{\substack{n_1,n_2\in[2^{k_1-k-5},2^{k_1-k+5}]\cap\mathbb{Z}\\ |n_1-n_2|\le 2^{10}}} \|P_k A_{\mu,\nu}(P_{k_1}\big(P_{k'}(u_i)P_{k_1';k,n_1}(\partial_t\partial_i\phi)\big)^\mu, P_{k_2;k,n_2}U^\nu(s))\|_{L_x^1}
\end{aligned} \tag{4.17}$$

where $|k_1' - k_1| \le 5$ and $P_{k_2;k,n_1}$ denotes the Fourier multiplier operator with the Fourier symbol $\psi_{k_2}(\xi)\psi_k(|\xi| - 2^k m)$.

From the $L^2 - L^2 - L_x^\infty$ type multilinear estimate, the estimate (2.9), the estimate of symbol in (1.12), and the super localized decay estimate (2.4) in Lemma 2.1, we have

$$\begin{aligned}
(4.17) \lesssim &\sum_{\substack{n_1,n_2\in[2^{k_1-k-5},2^{k_1-k+5}]\cap\mathbb{Z}\\ |n_1-n_2|\le 2^{10}}} \langle s\rangle^{-1/2}2^{-k_{1,-}}\|P_{k_1';k,n_1}(\partial_t\partial_i\phi)\|_{L_x^2}\|P_{\le k}(u)\|_{L^2}\\
&\times\big[2^k\epsilon_1 + 2^{-m/4+3k_1/4+k/2}\|\nabla_\xi\widehat{V}(\xi)\psi_{k_2}(\xi)\psi_k(|\xi|-2^k n_2)\|_{L^2}\big]\\
&\lesssim 2^{k/2+k_1-4k_{1,+}}\epsilon_1^2\langle s\rangle^{-1+\delta}.
\end{aligned} \tag{4.18}$$

Similarly, for any $k' \in [k, k_1 - 10]$, we have

$$\|P_k A_{\mu,\nu}(P_{k_1}\big(P_{k'}(u_i)P_{k_1'}(\partial_t\partial_i\phi)\big)^\mu, P_{k_2}U^\nu(s))\|_{L_x^1} \lesssim \langle s\rangle^{-1/2}2^{k'/2+k_1-4k_{1,+}}\|P_{k'}(u)\|_{L^2}$$

Moreover, from $L^2-L^\infty_x-L^\infty_x$ type multilinear estimate, we have

$$\sum_{k'\leq k_1-10}\|P_kA_{\mu,\nu}(P_{k_1}\big(P_{k'}(u_i)P_{k_1'}(\partial_t\partial_i\phi)\big)^\mu,P_{k_2}U^\nu(s))\|_{L^1_x}\lesssim\langle s\rangle^{-3/2}2^{k_1-3k_{1,+}}\epsilon_1^2.$$

From the above obtained estimates, we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}_-}\sum_{(k_1,k_2)\in\chi_k^1}2^{k+4k_+}\|\int_0^t e^{(t-s)\Delta}P_kA_{\mu,\nu}((P_{k_1}\mathcal{N}^1_\phi(s))^\mu,P_{k_2}U^\nu(s))ds\|_{L^2_x}\\&\lesssim\sum_{k\in\mathbb{Z}_-}\int_0^t\epsilon_1^2 2^k(1+2^{2k}|t-s|)^{-10}\big[\min\big\{2^{3k/2}\langle s\rangle^{-1+\delta},\langle s\rangle^{-3/2}\big\}+2^k\langle s\rangle^{-5/4+2\delta}\big]ds\lesssim\langle t\rangle^{-1}\epsilon_1^2.\end{aligned}\tag{4.19}$$

To sum up, after combining the obtained estimates (4.12), (4.13), and (4.19), we have

$$\sum_{k\in\mathbb{Z}}2^{k+4k_+}\|\int_0^t e^{(t-s)\Delta}P_kA_{\mu,\nu}((\mathcal{N}^1_\phi(s))^\mu,U^\nu(s))ds\|_{L^2_x}\lesssim\langle t\rangle^{-1}\epsilon_1^2.\tag{4.20}$$

Due to symmetry of the bilinear form $A_{\mu,\nu}(\cdot,\cdot)$, from the above estimate and the obtained estimate (4.11), we have

$$\sum_{k\in\mathbb{Z}}2^{k+4k_+}\|\int_0^t e^{(t-s)\Delta}P_k\mathcal{N}_v(s)ds\|_{L^2_x}\lesssim\langle t\rangle^{-1}\epsilon_1^2.\tag{4.21}$$

- The estimate of the normal form part.

Recall (1.10) and (1.8). We first use trivial estimates to rule out the very low frequency case and the relatively high frequency case. More precisely, from the estimate of symbol in (1.12), the $L^2_x\to L^1_x$ type Sobolev embedding and the $L^2_x-L^2_x$-type bilinear estimate, we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z},k\notin[-m/2-10\delta m,2m/N_0]}2^{k+4k_+}\|P_kA_{\mu,\nu}(U^\mu(t),U^\nu(t))\|_{L^2}\\&\lesssim\sum_{k\in\mathbb{Z},k\notin[-m/2-10\delta m,2m/N_0]}2^{2k-(N_0-10)k_++\delta m}\epsilon_1^2\lesssim2^{-m}\epsilon_0.\end{aligned}\tag{4.22}$$

Now, we focus on the case $k\in[-m/2-10\delta m,2m/N_0]\cap\mathbb{Z}$. To exploit the wave-wave-wave-type null structure, we localize the angle between ξ and $\pm\eta$ as follows,

$$\begin{aligned}&P_kA_{\mu,\nu}(U^\mu(t),U^\nu(t))=\sum_{(k_1,k_2)\in\chi_k^1\cup\chi_k^2\cup\chi_k^3}\sum_{l\in[\bar{l},2]}T^l_{k;k_1,k_2}(t,x),\quad\bar{l}:=-m/2-\beta(k,k_1,k_2)/2\\&T^l_{k;k_1,k_2}(t,x):=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{ix\cdot\xi+it\Phi^{\mu,\nu}(\xi,\eta)}a^l_{\mu,\nu}(\xi-\eta,\eta)\widehat{V^\mu}(t,\xi-\eta)\widehat{V^\nu}(t,\eta)d\eta d\xi,\\&a^l_{\mu,\nu}(\xi-\eta,\eta):=a_{\mu,\nu}(\xi-\eta,\eta)\varphi_{l;\bar{l}}(\frac{\xi}{|\xi|}\times\frac{\eta}{|\eta|}),\\&\beta(k,k_1,k_2):=k\mathbf{1}_{(k_1,k_2)\in\chi_k^1}+(2k-k_1)\mathbf{1}_{(k_1,k_2)\in\chi_k^2}+k_2\mathbf{1}_{(k_1,k_2)\in\chi_k^3}.\end{aligned}\tag{4.23}$$

- If $(k_1,k_2)\in\chi_k^1$, i.e., $|k_1-k_2|\leq10$. From the bilinear estimate (4.2) in Lemma 4.1, and the estimate of symbol in (1.12), we have

$$\|T^{\bar{l}}_{k;k_1,k_2}(t,x)\|_{L^2_x}\lesssim2^{-k_{1,-}+2\bar{l}}2^{k+4k_1/3-10k_{1,+}}\epsilon_1^2\lesssim2^{-m+k_1/3-8k_{1,+}}\epsilon_1^2.\tag{4.24}$$

For $T^{l}_{k;k_1,k_2}(t,x)$, we do integration by parts in η once. As a result, we have

$$T^{l}_{k;k_1,k_2}(t,x) := T^{l;1}_{k;k_1,k_2}(t,x) + T^{l;2}_{k;k_1,k_2}(t,x)$$

$$T^{l;1}_{k;k_1,k_2}(t,x) := \sum_{\tilde{\mu}\in\{+,-\}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi+it\Phi^{\mu,\nu}(\xi,\eta)}\nabla_\eta\cdot\big[\frac{i\nabla_\eta\Phi^{\mu,\nu}(\xi,\eta)}{t|\nabla_\eta\Phi^{\mu,\nu}(\xi,\eta)|^2}a^{l}_{\mu,\nu}(\xi-\eta,\eta)\big]\widehat{V^{\mu}}(t,\xi-\eta)\widehat{V^{\nu}}(t,\eta)d\eta d\xi$$

$$T^{l;2}_{k;k_1,k_2}(t,x) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi+it\Phi^{\mu,\nu}(\xi,\eta)}\nabla_\eta\cdot\big[\widehat{V^{\mu}}(t,\xi-\eta)\widehat{V^{\nu}}(t,\eta)\big]\frac{i\nabla_\eta\Phi^{\mu,\nu}(\xi,\eta)}{t|\nabla_\eta\Phi^{\mu,\nu}(\xi,\eta)|^2}a^{l}_{\mu,\nu}(\xi-\eta,\eta)d\eta d\xi,$$

For $T^{l;1}_{k;k_1,k_2}(t,x)$, we do integration by parts in η again. As a result, we have

$$\begin{aligned}T^{l;1}_{k;k_1,k_2}(t,x) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi+it\Phi^{\mu,\nu}(\xi,\eta)}\nabla_\eta\cdot\big[\frac{i\nabla_\eta\Phi^{\mu,\nu}(\xi,\eta)}{t|\nabla_\eta\Phi^{\mu,\nu}(\xi,\eta)|^2}\nabla_\eta\cdot\big[\frac{i\nabla_\eta\Phi^{\mu,\nu}(\xi,\eta)}{t|\nabla_\eta\Phi^{\mu,\nu}(\xi,\eta)|^2}\\ \times a^{l}_{\mu,\nu}(\xi-\eta,\eta)\big]\widehat{V^{\mu}}(t,\xi-\eta)\widehat{V^{\nu}}(t,\eta)\big]d\eta d\xi\end{aligned}$$

Note that $\big((\xi-\eta)/|\xi-\eta|\big)\times\big(\eta/|\eta|\big) = (\xi\times\eta)/\big(|\xi-\eta||\eta|\big)$. From the estimate of symbol in (1.12), and the bilinear estimate (4.2) in Lemma 4.1, we have

$$\sum_{l\in(\bar{l},2]\cap\mathbb{Z}} \|T^{l;1}_{k;k_1,k_2}(t,x)\|_{L^2_x} \lesssim 2^{-\max\{k_1,k_2\}_- -2m-2(k+2l)}2^{k+4k_1/3+2l-10k_{1,+}}\epsilon_1^2 \lesssim 2^{-m+k_1/3-8k_{1,+}}\epsilon_1^2, \tag{4.25}$$

Moreover, from the estimate of symbol in (1.12), the bilinear estimate (4.2) in Lemma 4.1, the estimate (2.9), and the decay estimate (2.4) in Lemma 2.1, we have

$$\begin{aligned}\sum_{l\in(\bar{l},2]\cap\mathbb{Z}} \|T^{l;2}_{k;k_1,k_2}(t,x)\|_{L^2_x} &\lesssim 2^{-\max\{k_1,k_2\}_- -m-(k-k_1+l)+l}2^{-(1-\alpha)k_1+2\delta m}2^{-m/2+k_1/2-8k_{1,+}}\\ &\lesssim 2^{-3m/2+3\delta m-k+(3/2+\alpha)k_1-8k_{1,+}}\epsilon_1^2.\end{aligned} \tag{4.26}$$

- If $(k_1,k_2)\in\chi_k^2$, i.e., $k_1\leq k-10$. By using the same strategy, from the estimate of symbol in (1.12), the bilinear estimate (4.2) in Lemma 4.1, and the decay estimate (2.4), we have

$$\begin{aligned}\|T^{\bar{l}}_{k;k_1,k_2}(t,x)\|_{L^2_x} &\lesssim 2^{-k_+ +2l}2^{k_1/3+2k-10k_+}\epsilon_1^2 \lesssim 2^{-m+k_1/3-8k_+}\epsilon_1^2,\\ \sum_{l\in(\bar{l},2]\cap\mathbb{Z}} \|T^{l;1}_{k;k_1,k_2}(t,x)\|_{L^2_x} &\lesssim \sum_{l\in(\bar{l},2]\cap\mathbb{Z}} 2^{-k_+ -2m-2(2k-k_1+2l)+2l}2^{k_1/3+2k-10k_+} \lesssim 2^{-m+k_1/3-8k_+}\epsilon_1^2,\\ \sum_{l\in(\bar{l},2]\cap\mathbb{Z}} \|T^{l;2}_{k;k_1,k_2}(t,x)\|_{L^2_x} &\lesssim 2^{-k_+ -m-(k-k_1+l)+l}2^{-(1-\alpha)k_1+2\delta m}2^{-m/2+k_2/2-8k_{2,+}}\\ &\lesssim 2^{-5m/4+\alpha k_1-8k_+}\epsilon_1^2.\end{aligned} \tag{4.27}$$

- If $(k_1,k_2)\in\chi_k^2\cup\chi_k^3$, i.e., $k_2\leq k-10$. By using the same strategy, from the estimate of symbol in (1.12), the bilinear estimate (4.2) in Lemma 4.1, and the decay estimate (2.4), we have

$$\begin{aligned}\|T^{\bar{l}}_{k;k_1,k_2}(t,x)\|_{L^2_x} &\lesssim 2^{-k_+ +2l}2^{4k_2/3+k-10k_+}\epsilon_1^2 \lesssim 2^{-m+k_2/3-8k_+}\epsilon_1^2,\\ \sum_{l\in(\bar{l},2]\cap\mathbb{Z}} \|T^{l;1}_{k;k_1,k_2}(t,x)\|_{L^2_x} &\lesssim \sum_{l\in(\bar{l},2]\cap\mathbb{Z}} 2^{-k_+ -2m-2(k_2+2l)+2l}2^{4k_2/3+k-10k_+} \lesssim 2^{-m+k_2/3-8k_+}\epsilon_1^2,\\ \sum_{l\in(\bar{l},2]\cap\mathbb{Z}} \|T^{l;2}_{k;k_1,k_2}(t,x)\|_{L^2_x} &\lesssim \sum_{l\in(\bar{l},2]\cap\mathbb{Z}} 2^{-k_+ -m-l+l}2^{-(1-\alpha)k_2+2\delta m-8k_+}\min\{2^{-m/2+k_1/2},2^{k_2+l+2k_1/3}\}\\ &\lesssim 2^{-m/2-\alpha m/4+3\delta m+\alpha k_2/2-8k_+}\epsilon_1^2,\end{aligned} \tag{4.28}$$

Recall the decomposition (4.23). After combining the obtained estimates (4.24–4.28), we have

$$\sum_{k\in[-m/2-10\delta m,2m/N_0]\cap\mathbb{Z}} 2^{k+4k_+}\|P_k A_{\mu,\nu}(U^{\mu}(t),U^{\nu}(t))\|_{L^2} \lesssim 2^{-m}\epsilon_1^2. \tag{4.29}$$

To sum up, our desired estimate (4.10) holds from the above estimate and the obtained estimate (4.21). $\square$

4.2. **The estimate of $Z_\phi(t)$.**

Lemma 4.3. *Let $t_1, t_2 \in [2^{m-1}, 2^m] \subset [0,T], m \in \mathbb{Z}_+$, s.t., $m \gg 1$. Under the bootstrap assumption* (1.16)*, we have*

$$\sup_{k\in\mathbb{Z}} 2^{k/3+10k_+} \| \int_{t_1}^{t_2} e^{is|\xi|}\widehat{\mathcal{N}_\phi}(s,\xi)\psi_k(\xi)ds\|_{L^\infty_\xi} \lesssim 2^{-0.04m+10\delta m}\epsilon_0. \tag{4.30}$$

Proof. Recall (1.14) and (1.2). We decompose $\mathcal{N}_\phi$ into two parts as follows,

$$\mathcal{N}_\phi = M_\phi + G_\phi, \quad M_\phi := -2u\cdot\nabla_x\partial_t\phi - \Delta u\cdot\nabla_x\phi, \quad G_\phi = -(\partial_t-\Delta)u\cdot\nabla_x\phi - u\cdot\nabla(u\cdot\nabla_x\phi). \tag{4.31}$$

More precisely, from the $L^2_x - L^2_x - L^\infty_x$ type multilinear estimate, we have

$$\sum_{|\alpha|\le 10} \|\nabla_x^\alpha \partial_i(u_i(s)u_j(s)\partial_j\phi(s))\|_{L^1_\xi} \lesssim \langle s\rangle^{-3/2+2\delta}\epsilon_1^3. \tag{4.32}$$

Moreover, from the estimate (4.44) in Lemma 4.4 and the $L^2_x - L^2_x - L^\infty_x$ type multilinear estimate, we have

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} 2^{k/3+10k_+}\|\mathcal{F}_{x\to\xi}\big[Q(u,u)\cdot\nabla_x\phi\big](\xi)\psi_k(\xi)\|_{L^\infty_\xi} \lesssim \langle s\rangle^{-3/2+5\delta}\epsilon_1^3,\\
&\sup_{k\in\mathbb{Z}} 2^{k/3+10k_+}\|\mathcal{F}_{x\to\xi}\big[\widetilde{Q}(\phi,\phi)\cdot\nabla_x\phi\big](\xi)\psi_k(\xi)\|_{L^\infty_\xi} \lesssim \langle s\rangle^{-1.04+5\delta}\epsilon_1^3,\\
\Longrightarrow\quad &\sup_{k\in\mathbb{Z}} 2^{k/3+10k_+}\|\mathcal{F}_{x\to\xi}\big[(\partial_t-\Delta)u_i\partial_i\phi\big](\xi)\psi_k(\xi)\|_{L^\infty_\xi} \lesssim \langle s\rangle^{-5/4+5\delta}\epsilon_1^3.
\end{aligned} \tag{4.33}$$

From the above estimate and the obtained estimate (4.32), we have

$$\sup_{k\in\mathbb{Z}} 2^{k/3+10k_+}\|\mathcal{F}_{x\to\xi}\big[\int_{t_1}^{t_2} e^{is|\nabla|}G_\phi ds\big](\xi)\psi_k(\xi)\|_{L^\infty_\xi} \lesssim 2^{-0.04m+10\delta m}\epsilon_1^2. \tag{4.34}$$

Now, we focus on the contribution of the main part M_ϕ. Note that, from the Duhamel's formula, on the Fourier side, for any $\xi \in supp(\psi_k(\cdot))$, we have

$$\begin{aligned}
&\mathcal{F}_{x\to\xi}\big[\int_{t_1}^{t_2} e^{is|\nabla|}M_\phi ds\big](\xi)\psi_k(\xi) = \sum_{\mu\in\{+,-\}}\sum_{(k_1,k_2)\in\chi_k^1\cup\chi_k^2\cup\chi_k^3}\int_{t_1}^{t_2} I^\mu_{k;k_1,k_2}(s,\xi)ds,\\
&I^\mu_{k;k_1,k_2}(s,\xi) := \int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)} m^{\mu;j}_{k,k_1,k_2}(\xi,\eta)\cdot\widehat{u_j}(s,\xi-\eta)\widehat{V^\mu}(s,\eta)d\eta,
\end{aligned} \tag{4.35}$$

where $m^{\mu;j}_{k,k_1,k_2}(\xi,\eta) := i\eta_j\big(1+c_\mu|\xi-\eta|^2|\eta|^{-1}\big)\psi_k(\xi)\psi_{k_1}(\xi-\eta)\psi_{k_2}(\eta)$. By using the energy norm and the volume of support, we can rule out the very low and relative high frequency as follows,

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\mathbb{Z}^2/[-2m/3,2m/N_0]^2} 2^{k/3+10k_+}\|I^\mu_{k;k_1,k_2}(s,\xi)\|_{L^\infty_\xi} \lesssim \sum_{(k_1,k_2)\in\mathbb{Z}^2/[-2m/3,2m/N_0]^2} 2^{k/3+10k_+-m/2+\delta m}\\
&\times\min\{2^{2\min\{k_1,k_2\}/3}, 2^{-(N_0-5)\max\{k_1,k_2\}}\}\epsilon_1^2 \lesssim 2^{-1.1m}\epsilon_1^2.
\end{aligned} \tag{4.36}$$

Now, we focus on the case $k_1,k_2\in[-2m/3,2m/N_0]$. For this case, we do integration by parts in η once for $I^\mu_{k;k_1,k_2}(t,\xi)$. As a result, we have

$$\begin{aligned}
&I^\mu_{k;k_1,k_2}(s,\xi) := I^{\mu;1}_{k;k_1,k_2}(s,\xi) + I^{\mu;2}_{k;k_1,k_2}(s,\xi),\\
&I^{\mu;1}_{k;k_1,k_2}(s,\xi) := \int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)}\widehat{u_j}(s,\xi-\eta)\nabla_\eta\cdot\big[\frac{i\mu\eta}{s|\eta|}m^{\mu;j}_{k,k_1,k_2}(\xi,\eta)\widehat{V^\mu}(s,\eta)\big]d\eta,\\
&I^{\mu;2}_{k;k_1,k_2}(s,\xi) := \int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)}\nabla_\eta\big(\widehat{u_j}(t,\xi-\eta)\big)\cdot\big[\frac{i\mu\eta}{s|\eta|}m^{\mu;j}_{k,k_1,k_2}(\xi,\eta)\widehat{V^\mu}(s,\eta)\big]d\eta.
\end{aligned} \tag{4.37}$$

Note that,

$$\begin{aligned}\nabla_\xi \widehat{u}(s,\xi) &= -\frac{\xi}{|\xi|^2}\widehat{Su}(s,\xi) + \frac{(\xi_2,-\xi_1)}{|\xi|^2}(\xi_2\partial_{\xi_1}-\xi_1\partial_{\xi_2})\widehat{u}(s,\xi) + s\frac{\xi}{|\xi|^2}\widehat{\partial_s u}(s,\xi)\\ &= -\frac{\xi}{|\xi|^2}\widehat{Su}(s,\xi) + \frac{(\xi_2,-\xi_1)}{|\xi|^2}\widehat{\Omega u}(s,\xi) + s\frac{\xi}{|\xi|^2}\widehat{\mathcal{N}}_u(s,\xi) - s\xi\widehat{u}(s,\xi).\end{aligned}$$

From the above equality, we decompose $I^{\mu;2}_{k;k_1,k_2}(s,\xi)$ into three parts as follows,

$$I^{\mu;2}_{k;k_1,k_2}(s,\xi) := I^{\mu;2;1}_{k;k_1,k_2}(s,\xi) + I^{\mu;2;2}_{k;k_1,k_2}(s,\xi) + I^{\mu;2;3}_{k;k_1,k_2}(s,\xi), \tag{4.38}$$

$$\begin{aligned}I^{\mu;2;1}_{k;k_1,k_2}(s,\xi) &:= \int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)} m^{\mu;j}_{k,k_1,k_2}(\xi,\eta)\frac{i\mu\eta}{s|\eta|}\cdot\big[-\frac{\xi-\eta}{|\xi-\eta|^2}\widehat{Su_j}(s,\xi-\eta) + \frac{(\xi_2-\eta_2,-\xi_1+\eta_1)}{|\xi-\eta|^2}\\ &\quad\times\widehat{\Omega u_j}(s,\xi-\eta) + \frac{s(\xi-\eta)}{|\xi-\eta|^2}\mathcal{F}[Q_j(u,u)](s,\xi-\eta)\big]\widehat{V^\mu}(s,\eta)d\eta,\\ I^{\mu;2;2}_{k;k_1,k_2}(s,\xi) &:= \int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)}\big[\frac{i\mu\eta\cdot(\xi-\eta)}{|\eta||\xi-\eta|^2} m^{\mu;j}_{k,k_1,k_2}(\xi,\eta)\widehat{V^\mu}(s,\eta)\mathcal{F}[\widetilde{Q}_j(\phi,\phi)](s,\xi-\eta)d\eta,\\ I^{\mu;2;3}_{k;k_1,k_2}(s,\xi) &:= \int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)}\frac{i\mu\eta\cdot(\xi-\eta)}{|\eta|} m^{\mu;j}_{k,k_1,k_2}(\xi,\eta)\widehat{V^\mu}(s,\eta)\widehat{u_j}(s,\xi-\eta)d\eta,\end{aligned}$$

where we use $Q_j(\cdot,\cdot)$ and $\widetilde{Q}_j$ to denote the j-th component of the vector $Q(\cdot,\cdot)$ and $\widetilde{Q}(\cdot,\cdot)$.

For $I^{\mu;2;3}_{k;k_1,k_2}(s,\xi)$, we repeat the procedure we did for $I_\mu(t,\xi)$. As a result, we have

$$I^{\mu;2;3}_{k;k_1,k_2}(s,\xi) = \widetilde{I}^{\mu;1}_{k;k_1,k_2}(s,\xi) + \widetilde{I}^{\mu;2}_{k;k_1,k_2}(s,\xi) + \widetilde{I}^{\mu;3}_{k;k_1,k_2}(s,\xi) + \widetilde{I}^{\mu;4}_{k;k_1,k_2}(s,\xi), \tag{4.39}$$

$$\begin{aligned}\widetilde{I}^{\mu;1}_{k;k_1,k_2}(s,\xi) &= \int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)}\widehat{u_j}(s,\xi-\eta)\nabla_\eta\cdot\big[\frac{\eta(\xi-\eta)\cdot\eta}{s|\eta|^2} m^j_\mu(\xi,\eta)\widehat{V^\mu}(s,\eta)\big]d\eta,\\ \widetilde{I}^{\mu;2}_{k;k_1,k_2}(s,\xi) &= -\int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)} m^{\mu;j}_{k,k_1,k_2}(\xi,\eta)\frac{\eta(\xi-\eta)\cdot\eta}{s|\eta|^2}\cdot\big[\frac{\xi-\eta}{|\xi-\eta|^2}\widehat{Su_j}(s,\xi-\eta) - \frac{(\xi_2-\eta_2,-\xi_1+\eta_1)}{|\xi-\eta|^2}\\ &\quad\times((\xi_2-\eta_2)\partial_{\xi_1}-(\xi_1-\eta_1)\partial_{\xi_2})\widehat{\Omega u_j}(s,\xi-\eta) - \frac{s(\xi-\eta)}{|\xi-\eta|^2}\mathcal{F}[Q_j(u,u)](s,\xi-\eta)\big]\widehat{V^\mu}(s,\eta)d\eta,\\ \widetilde{I}^{\mu;3}_{k;k_1,k_2}(s,\xi) &= -\int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)}\frac{(\eta\cdot(\xi-\eta))^2}{|\eta|^2|\xi-\eta|^2} m^{\mu;j}_{k,k_1,k_2}(\xi,\eta)\widehat{V^\mu}(s,\eta)\mathcal{F}[\widetilde{Q}_j(\phi,\phi)](s,\xi-\eta)d\eta,\\ \widetilde{I}^{\mu;4}_{k;k_1,k_2}(s,\xi) &= -\int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)}\frac{\big(\eta\cdot(\xi-\eta)\big)^2}{|\eta|^2} m^{\mu;j}_{k,k_1,k_2}(\xi,\eta)\widehat{V^\mu}(s,\eta)\widehat{u_j}(s,\xi-\eta)d\eta.\end{aligned}$$

For $\widetilde{I}^{\mu;4}_{k;k_1,k_2}(s,\xi)$, we do integration by parts in times s once. As a result, we have

$$\begin{aligned}&\int_{t_1}^{t_2}\widetilde{I}^{\mu;4}_{k;k_1,k_2}(s,\xi)ds = End^\mu_{k;k_1,k_2}(t_1,t_2,\xi) + \mathcal{I}^{\mu;1}_{k;k_1,k_2}(t_1,t_2,\xi) + \mathcal{I}^{\mu;2}_{k;k_1,k_2}(t_1,t_2,\xi),\\ &End^\mu_{k;k_1,k_2}(t_1,t_2,\xi) = \sum_{j=1,2}(-1)^j\int_{\mathbb{R}^3} e^{it_j(|\xi|-\mu|\eta|)}\frac{m^j_\mu(\xi,\eta)\widehat{V^\mu}(t_j,\eta)\widehat{u_j}(t_j,\xi-\eta)}{|\xi-\eta|^2+i(|\xi|-\mu|\eta|)}\frac{\big(\eta\cdot(\xi-\eta)\big)^2}{|\eta|^2}d\eta,\\ &\mathcal{I}^{\mu;1}_{k;k_1,k_2}(t_1,t_2,\xi) = \int_{t_1}^{t_2}\int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)}\frac{m^j_\mu(\xi,\eta)\partial_s\widehat{V^\mu}(s,\eta)\widehat{u_j}(s,\xi-\eta)}{|\xi-\eta|^2+i(|\xi|-\mu|\eta|)}\frac{\big(\eta\cdot(\xi-\eta)\big)^2}{|\eta|^2}d\eta dt,\\ &\mathcal{I}^{\mu;2}_{k;k_1,k_2}(t_1,t_2,\xi) = \int_{t_1}^{t_2}\int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)}\frac{m^j_\mu(\xi,\eta)\widehat{V^\mu}(s,\eta)\mathcal{F}[Q_j(u,u)](s,\xi-\eta)}{|\xi-\eta|^2+i(|\xi|-\mu|\eta|)}\frac{\big(\eta\cdot(\xi-\eta)\big)^2}{|\eta|^2}d\eta dt,\\ &\mathcal{I}^{\mu;3}_{k;k_1,k_2}(t_1,t_2,\xi) = \int_{t_1}^{t_2}\int_{\mathbb{R}^3} e^{is(|\xi|-\mu|\eta|)}\frac{m^j_\mu(\xi,\eta)\widehat{V^\mu}(s,\eta)\mathcal{F}[\widetilde{Q}_j(\phi,\phi)](s,\xi-\eta)}{|\xi-\eta|^2+i(|\xi|-\mu|\eta|)}\frac{\big(\eta\cdot(\xi-\eta)\big)^2}{|\eta|^2}d\eta dt.\end{aligned} \tag{4.40}$$

From the L^2-L^2 type bilinear estimate, the estimate (2.9), we have

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} 2^{k/3+10k_+}\big(\|I^{\mu;1}_{k;k_1,k_2}(s,\xi)\psi_k(\xi)\|_{L^\infty_\xi}+\|I^{\mu;2;1}_{k;k_1,k_2}(s,\xi)\psi_k(\xi)\|_{L^\infty_\xi}+\|\widetilde{I}^{\mu;1}_{k;k_1,k_2}(s,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\\
&\qquad +\|\widetilde{I}^{\mu;2}_{k;k_1,k_2}(s,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\big)\lesssim \langle s\rangle^{-3/2+2\delta}\epsilon_1^2,\\
&\sup_{k\in\mathbb{Z}} 2^{k/3+10k_+}\big(\|End^{\mu}_{k;k_1,k_2}(t_1,t_2,\xi)\psi_k(\xi)\|_{L^\infty_\xi}+\|\mathcal{I}^{\mu;2}_{k;k_1,k_2}(t_1,t_2,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\\
&\qquad +\mathcal{I}^{\mu;2}_{k;k_1,k_2}(t_1,t_2,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\big)\lesssim 2^{-0.04m+10\delta}\epsilon_1^2.
\end{aligned}\tag{4.41}$$

From the estimate (4.44) in Lemma 4.4, we have

$$\begin{aligned}
\sup_{k\in\mathbb{Z}} 2^{k/3+10k_+}\big(2^m\|I^{\mu;2;2}_{k;k_1,k_2}(s,\xi)\psi_k(\xi)\|_{L^\infty_\xi}+2^m\|\widetilde{I}^{\mu;3}_{k;k_1,k_2}(s,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\\
+\|\mathcal{I}^{\mu;3}_{k;k_1,k_2}(t_1,t_2,\xi)\psi_k(\xi)\|_{L^\infty_\xi}\big)\lesssim 2^{-0.04m+10\delta}\epsilon_1^2.
\end{aligned}\tag{4.42}$$

Recall the decomposition in (4.37), (4.38), (4.39), and (4.40). To sum up, after combining the obtained estimates (4.41) and (4.42), for any $k_1,k_2\in[-m/2-10\delta m, 2m/N_0]\cap\mathbb{Z}$, we have

$$\sup_{k\in\mathbb{Z}} 2^{k/3+10k_+}\|\int_{t_1}^{t_2} I^{\mu}_{k;k_1,k_2}(s,\xi)ds\|_{L^\infty_\xi}\lesssim 2^{-m/2+10\delta m}\epsilon_1^2.\tag{4.43}$$

Recall decompositions in (4.13) and (1.5). Hence our desired estimate (4.30) holds from the above estimate, the obtained estimates (4.34) and (4.36). □

Lemma 4.4. *Let $t\in[2^{m-1},2^m], m\in\mathbb{Z}_+$. For any $\mu,\nu,\iota\in\{+,-\}$, and bilinear operator $T(\cdot,\cdot)$ with symbol $m\in\mathcal{S}^\infty_{k,k_1,k_2}$, s.t., $\|m\|_{\mathcal{S}^\infty_{k,k_1,k_2}}\lesssim 2^{-k_{1,-}+2\max\{k_1,k_2\}_+}$, under the bootstrap assumption* (1.16), *we have*

$$\sup_{k\in\mathbb{Z}}\sum_{\substack{(k_1,k_2)\in\cup_{i=1,2,3}\chi^i_k\\ (k_1',k_2')\in\cup_{i=1,2,3}\chi^i_{k_1}}} 2^{k/3+10k_+}\|\mathcal{F}\big[T\big(P_{k_1}\big(\widetilde{Q}(P_{k_1'}U^\mu(t),P_{k_2'}U^\nu(t))\big),P_{k_2}U^\iota(t)\big)\big](\xi)\|_{L^\infty_\xi}\lesssim 2^{-1.04m}\epsilon_1^3.\tag{4.44}$$

where the bilinear operator $\widetilde{Q}(\cdot,\cdot)$ is defined by the symbol $\widetilde{q}(\xi-\eta,\eta)$ in (1.7).

Proof. Recall the symbol (1.7). By using the volume of support of frequency variable and the H^{N_0}-norm of the wave part, we have

$$\begin{aligned}
&\sum_{\substack{(k_1,k_2)\in\mathbb{Z}^2/[-3m/4,2N_0/m]^2\\ (k_1',k_2')\in\mathbb{Z}^2}} 2^{k/3+10k_+}\|\mathcal{F}\big[T\big(P_{k_1}\big(\widetilde{Q}(P_{k_1'}U^\mu(t),P_{k_2'}U^\nu(t))\big),P_{k_2}U^\iota(t)\big)\big](\xi)\|_{L^\infty_\xi}\\
&\lesssim \sum_{(k_1,k_2)\in\mathbb{Z}^2/[-3m/4,2N_0/m]^2}\min\{2^{5\min\{k_1,k_2\}/3},2^{-(N-15)\max\{k_1,k_2\}_+}\}\epsilon_1^3\lesssim 2^{-5m/4}\epsilon_1^3.
\end{aligned}\tag{4.45}$$

Moreover, for any fixed $(k_1,k_2)\in[-3m/4,2N_0/m]^2$, we can rule out the very small k_1',k_2' case and the relatively big k_1',k_2' case as follows,

$$\begin{aligned}
&\sum_{(k_1',k_2')\in\mathbb{Z}^2/[-3m/4,2N_0/m]^2} 2^{k/3+10k_+}\|\mathcal{F}\big[T\big(P_{k_1}\big(\widetilde{Q}(P_{k_1'}U^\mu(t),P_{k_2'}U^\nu(t))\big),P_{k_2}U^\iota(t)\big)\big](\xi)\|_{L^\infty_\xi}\\
&\lesssim \sum_{(k_1,k_2)\in\mathbb{Z}^2/[-3m/4,2N_0/m]^2}\min\{2^{5\min\{k_1,k_2\}/3},2^{-(N-15)\max\{k_1,k_2\}_+}\}\epsilon_1^3\lesssim 2^{-5m/4}\epsilon_1^3.
\end{aligned}\tag{4.46}$$

Since there are at most m^4 cases left, which only cause logarithmic loss, it would be sufficient to let $k_1,k_2,k_1',k_2'\in[-3m/4,2N_0/m]\cap\mathbb{Z}$ be fixed. As in the proof of the obtained estimate (4.29), we also exploit the null structure of the bilinear operator $\widetilde{Q}(\cdot,\cdot)$. Motivated from the proof of (4.29), we decompose into two parts as follows,

$$\widetilde{Q}(P_{k_1'}U^{\mu}(t), P_{k_2'}U^{\nu}(t))) = \widetilde{Q}^{\leq}(P_{k_1'}U^{\mu}(t), P_{k_2'}U^{\nu}(t))) + \widetilde{Q}^{>}(P_{k_1'}U^{\mu}(t), P_{k_2'}U^{\nu}(t))), \tag{4.47}$$

where $\forall \star \in \{\leq, >\}$,

$$\widetilde{Q}^{\star}(P_{k_1'}U^{\mu}(t), P_{k_2'}U^{\nu}(t))) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\widetilde{q}(\xi-\eta,\eta)\psi_{\star\bar{a}}(\frac{\xi}{|\xi|}\times\frac{\eta}{|\eta|})\widehat{U^{\mu}}(t,\xi-\eta)\widehat{U^{\nu}}(t,\eta)d\eta d\xi. \tag{4.48}$$

where $\bar{a} := -0.45m - \beta(k_1,k_1',k_2')/2$, and $\beta(k_1,k_1',k_2')$ is defined in (4.23).

Since the threshold we choose is $\bar{a}$ instead of $-m/2-\min\{k_1',k_2',k_1\}/2$, and the symbol $\widetilde{q}(\xi-\eta,\eta)$ of the bilinear operator $\widetilde{Q}(\cdot,\cdot)$ is better than the symbol of the bilinear operator $A_{\mu,\nu}(U^{\mu}(t),U^{\nu}(t))$ in the sense that it provides the smallness of the output frequency, which plays the small role of 2^k in (4.29). After rerunning the argument used for obtaining (4.29), we have

$$\begin{aligned} &\|P_{k_1}\widetilde{Q}(P_{k_1'}U^{\mu}(t), P_{k_2'}U^{\nu}(t)))\|_{L^2} \lesssim 2^{-m}\epsilon_1^2,\\ &\|P_{k_1}\widetilde{Q}^{>}(P_{k_1'}U^{\mu}(t), P_{k_2'}U^{\nu}(t)))\|_{L^2} \lesssim 2^{-1.1m}\epsilon_1^2, \end{aligned} \tag{4.49}$$

From the first estimate in (4.49), we can rule out very low output frequency as follows,

$$\sup_{k\in(-\infty,-0.15m)\cap Z} 2^{k/3+10k_+}\|\mathcal{F}\big[T\big(P_{k_1}\big(\widetilde{Q}(P_{k_1'}U^{\mu}(t), P_{k_2'}U^{\nu}(t))\big), P_{k_2}U^{\iota}(t)\big)\big](\xi)\|_{L^\infty_\xi} \lesssim 2^{-1.05m}\epsilon_1^3.$$

- Now, we focus on the case $k\in[-0.15m, 2m/N_0]\cap Z$ and $k_1,k_2,k_1',k_2'\in[-3m/4, 2N_0/m]\cap\mathbb{Z}$.

From the second estimate in (4.49) and the L^2-L^2-type bilinear estimate, we rule out further the large angle case as follows,

$$2^{k/3+10k_+}\|\mathcal{F}\big[T\big(P_{k_1}\big(\widetilde{Q}^{>}(P_{k_1'}U^{\mu}(t), P_{k_2'}U^{\nu}(t))\big), P_{k_2}U^{\iota}(t)\big)\big](\xi)\|_{L^\infty_\xi} \lesssim 2^{-1.05m}\epsilon_1^3. \tag{4.50}$$

It remains to estimate the contribution of $\widetilde{Q}^{\leq}(P_{k_1'}U^{\mu}(t), P_{k_2'}U^{\nu}(t))\big)$. Note that, on the Fourier side, after doing dyadic decomposition for the angle between $\xi-\eta$ and $\pm(\eta-\sigma)$, we have

$$\begin{aligned} &\mathcal{F}\big[T\big(P_{k_1}\big(\widetilde{Q}^{\leq}(P_{k_1'}U^{\mu}(t), P_{k_2'}U^{\nu}(t))\big), P_{k_2}U^{\iota}(t)\big)\big](\xi) = \sum_{l_1\in[\bar{l},2]\cap\mathbb{Z}} \mathcal{H}^{\mu,\nu,\iota;l}_{k;k_1,k_2;k_1',k_2'}(t,\xi),\\ &\mathcal{H}^{\mu,\nu,\iota;l}_{k;k_1,k_2;k_1',k_2'}(t,\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-it\left(\mu|\eta-\sigma|+\nu|\sigma|+\iota|\xi-\eta|\right)}\widehat{V^{\mu}_{k_1'}}(t,\eta-\sigma)\widehat{V^{\nu}_{k_2'}}(t,\sigma)\widehat{V^{\iota}_{k_2}}(t,\xi-\eta)\\ &\qquad\times\widetilde{q}(\eta-\sigma,\sigma)m(\eta,\xi-\eta)\psi_{k_1}(\eta)\varphi_{l;\bar{l}}(\frac{\xi-\eta}{|\xi-\eta|}\times\frac{\eta-\sigma}{|\eta-\sigma|})\psi_{\leq\bar{a}}(\frac{\eta}{|\eta|}\times\frac{\sigma}{|\sigma|})d\eta d\sigma, \end{aligned} \tag{4.51}$$

where $\bar{l} := -m/2-\min\{k_1',k_1,k_2\}/2$. Note that

$$\begin{aligned} |\frac{\eta-\sigma}{|\eta-\sigma|}\times\frac{\eta}{|\eta|}| \lesssim 2^{k_2'-k_1'+\bar{a}}, &\Longrightarrow |\frac{\xi-\eta}{|\xi-\eta|}\times\frac{\eta}{|\eta|}| \lesssim 2^{k_2'-k_1'+\bar{a}}+2^l,\\ &\Longrightarrow |\frac{\xi}{|\xi|}\times\frac{\eta}{|\eta|}| \lesssim 2^{k_1-k}\big(2^{k_2'-k_1'+\bar{a}}+2^l\big). \end{aligned} \tag{4.52}$$

For the threshold case, i.e., $l=\bar{l}$, we use the volume of support of η,σ. As a result, from the above estimate (4.52), we have

$$\begin{aligned} &\big|\mathcal{H}^{\mu,\nu,\iota;\bar{l}}_{k;k_1,k_2;k_1',k_2'}(t,\xi)\big| \lesssim 2^{k_1-k}\big(2^{k_2'-k_1'+\bar{a}}+2^l\big)2^{2\bar{a}}2^{4\min\{k_1,k_2\}/3+5\min\{k_1',k_2'\}/3-10\max\{k_1',k_2'\}_+-10k_{2,+}}\epsilon_1^3\\ &\Longrightarrow 2^{k/3+10k_+}\big|\mathcal{H}^{\mu,\nu,\iota;\bar{l}}_{k;k_1,k_2;k_1',k_2'}(t,\xi)\big| \lesssim 2^{-1.1m}\epsilon_1^3. \end{aligned} \tag{4.53}$$

For the threshold case, i.e., $l>\bar{l}$, we do integration by parts in η once. As a result, we have

$$\mathcal{H}^{\mu,\nu,\iota;l}_{k;k_1,k_2;k_1',k_2'}(t,\xi) = \mathcal{H}^{\mu,\nu,\iota;l;1}_{k;k_1,k_2;k_1',k_2'}(t,\xi) + \mathcal{H}^{\mu,\nu,\iota;l;2}_{k;k_1,k_2;k_1',k_2'}(t,\xi), \tag{4.54}$$

$$\mathcal{H}^{\mu,\nu,\iota;l;1}_{k;k_1,k_2;k_1',k_2'}(t,\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-it\left(\mu|\eta-\sigma|+\nu|\sigma|+\iota|\xi-\eta|\right)}\nabla_\eta\cdot\Big[\widehat{V^{\mu}_{k_1'}}(t,\eta-\sigma)\widehat{V^{\iota}_{k_2}}(t,\xi-\eta)\widehat{V^{\nu}_{k_2'}}(t,\sigma)\Big]$$
$$\times\frac{-i\left(\mu\frac{\eta-\sigma}{|\eta-\sigma|}-\iota\frac{\xi-\eta}{|\xi-\eta|}\right)}{t\left|\mu\frac{\eta-\sigma}{|\eta-\sigma|}-\iota\frac{\xi-\eta}{|\xi-\eta|}\right|^2}\widetilde{q}(\eta-\sigma,\sigma)m(\eta,\xi-\eta)\psi_{k_1}(\eta)\varphi_{l;\bar{l}}\Big(\frac{\xi-\eta}{|\xi-\eta|}\times\frac{\eta-\sigma}{|\eta-\sigma|}\Big)\psi_{\leq\bar{a}}\Big(\frac{\eta}{|\eta|}\times\frac{\sigma}{|\sigma|}\Big)d\eta d\sigma,$$
$$\mathcal{H}^{\mu,\nu,\iota;l;2}_{k;k_1,k_2;k_1',k_2'}(t,\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-it\left(\mu|\eta-\sigma|+\nu|\sigma|+\iota|\xi-\eta|\right)}\widehat{V^{\mu}_{k_1'}}(t,\eta-\sigma)\widehat{V^{\iota}_{k_2}}(t,\xi-\eta)\widehat{V^{\nu}_{k_2'}}(t,\sigma)$$
$$\times\nabla_\eta\cdot\Big[\frac{-i\left(\mu\frac{\eta-\sigma}{|\eta-\sigma|}-\iota\frac{\xi-\eta}{|\xi-\eta|}\right)}{t\left|\mu\frac{\eta-\sigma}{|\eta-\sigma|}-\iota\frac{\xi-\eta}{|\xi-\eta|}\right|^2}\widetilde{q}(\eta-\sigma,\sigma)m(\eta,\xi-\eta)\psi_{k_1}(\eta)\varphi_{l;\bar{l}}\Big(\frac{\xi-\eta}{|\xi-\eta|}\times\frac{\eta-\sigma}{|\eta-\sigma|}\Big)\psi_{\leq\bar{a}}\Big(\frac{\eta}{|\eta|}\times\frac{\sigma}{|\sigma|}\Big)\Big]d\eta d\sigma.$$

For $\mathcal{H}^{\mu,\nu,\iota;l;1}_{k;k_1,k_2;k_1',k_2'}(t,\xi)$, we use the L^2-L^2 type bilinear estimate and use the volume of support of η,σ. As a result, from the estimate (4.52), we have

$$\begin{aligned}
&2^{k/3+10k_+}\big|\mathcal{H}^{\mu,\nu,\iota;l;1}_{k;k_1,k_2;k_1',k_2'}(t,\xi)\big| \lesssim 2^{k/3+10k_+}2^{-m-l+\bar{a}}\big[2^{\min\{k_1,k_2\}-(1-\alpha)k_2}2^{(k_1-k)/2}\big(2^{k_2'-k_1'+\bar{a}}+2^l\big)^{1/2}\\
&\times 2^{4\min\{k_1',k_2'\}/3+\bar{a}-10\min\{k_1',k_2'\}_+}+2^{4\min\{k_1,k_2\}/3}2^{k_1-k}\big(2^{k_2'-k_1'+\bar{a}}+2^l\big)2^{\bar{a}/2}2^{\min\{k_1',k_2'\}-\alpha k_1'}\big]\epsilon_1^3\\
&\lesssim 2^{-2k/3+(1+\alpha)k_1+10k_+}\big(2^{-m-l}2^{5\bar{a}/2+4\min\{k_1',k_2'\}/3-k_1'/2}+2^{-m-l/2}2^{2\bar{a}+4\min\{k_1',k_2'\}/3}\big)\epsilon_1^3\\
&\quad+2^{-2k/3+7k_1/3+10k_+}\big(2^{-m+3\bar{a}/2+\min\{k_1',k_2'\}-\alpha k_1'}+2^{-m-l+5\bar{a}/2+k_2'+\min\{k_1',k_2'\}-(1+\alpha)k_1'}\big)\epsilon_1^3\\
&\lesssim 2^{-1.3m}\epsilon_1^3.
\end{aligned}\tag{4.55}$$

For $\mathcal{H}^{\mu,\nu,\iota;l;2}_{k;k_1,k_2;k_1',k_2'}(t,\xi)$, we use the volume of support of η,σ. As a result, from the estimate (4.52), we have

$$\begin{aligned}
&2^{k/3+10k_+}\big|\mathcal{H}^{\mu,\nu,\iota;l;2}_{k;k_1,k_2;k_1',k_2'}(t,\xi)\big| \lesssim 2^{k/3+10k_+}2^{-m-l+2\bar{a}}2^{k_1-k}\big(2^{k_2'-k_1'+\bar{a}}+2^l\big)\\
&\times 2^{2\min\{k_1,k_2\}+2\min\{k_1',k_2'\}-(k_2+k_1'+k_2')/3}\big[2^{-l-\min\{k_2,k_1'\}}+2^{-\bar{a}-k_1}\big]\\
&\lesssim 2^{-m-2k/3+5\min\{k_1,k_2\}/3+4\min\{k_1',k_2'\}/3+10k_+}\big[2^{-2l-\min\{k_2,k_1'\}+3\bar{a}+k_1+k_2'-k_1'}+2^{-l+2\bar{a}+k_2'-k_1'}\\
&\quad+2^{-l+2\bar{a}+k_1-\min\{k_2,k_1'\}}+2^{\bar{a}}\big]\lesssim 2^{-1.2m}\epsilon_1^3.
\end{aligned}\tag{4.56}$$

Recall the decompositions in (4.47), (4.51), and (4.54). After combining the above two estimates and the obtained estimates (4.50) and (4.53), we have

$$\sup_{k\in[-0.15m,2m/N_0]\cap Z}2^{k/3+10k_+}\|\mathcal{F}\big[T\big(P_{k_1}\big(\widetilde{Q}(P_{k_1'}U^{\mu}(t),P_{k_2'}U^{\nu}(t))\big),P_{k_2}U^{\iota}(t)\big)\big](\xi)\|_{L^\infty_\xi}\lesssim 2^{-1.05m}\epsilon_1^3.$$

Hence finishing the proof of our desired estimate (4.44). □

4.3. **Proof of the theorem 1.1.** In this subsection, we summarize the results we obtained so far and prove our main theorem 1.1. From the obtained estimate (3.1) in Lemma 3.1, the obtained estimate (3.16) in Lemma 3.2, the obtained estimate (3.26) in Lemma 3.3, the obtained estimate (4.10) in Lemma 4.2, the obtained estimate (4.30) in Lemma 4.3, the following improved estimate holds in the time interval $[0,T]$,

$$\begin{aligned}
&\sup_{t\in[0,T]}\langle t\rangle^{1/2-\delta}\|u(t)\|_{H^{N_0+1}}+\langle t\rangle^{1/2-2\delta}\Big(\sum_{\Gamma\in\{S,\Omega\}}\|\Gamma u(t)\|_{H^1}\Big)+\langle t\rangle Z_u(t)+\langle t\rangle^{-\delta}\|U(t)\|_{H^{N_0}}\\
&\quad+\langle t\rangle^{-2\delta}\Big(\sum_{\Gamma\in\{S,\Omega\}}\|U^{\Gamma}(t)\|_{L^2}+\sum_{k\in\mathbb{Z}_-}2^{-\alpha k}\|P_kU^{\Gamma}(t)\|_{L^2}\Big)+Z_\phi(t)\lesssim\epsilon_0.
\end{aligned}\tag{4.57}$$

Hence finishing the bootstrap argument, i.e., T can be extended to infinity. Moreover, as a by product of the $Z_\phi(t)$-estimate, from the obtained estimate (4.30) in Lemma 4.3, we have

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} 2^{k/3+10k_+}\|\int_t^\infty e^{is|\xi|}\widehat{\mathcal{N}_\phi}(s,\xi)\psi_k(\xi)ds\|_{L^\infty_\xi} \lesssim \langle t\rangle^{-0.04+10\delta}\epsilon_0,\\
&\quad\Longrightarrow \sup_{k\in\mathbb{Z}} 2^{k/3+10k_+}\|\big[\widehat{U}(t,\xi)-e^{-it|\xi|}V_\infty(\xi)\big]\psi_k(\xi)ds\|_{L^\infty_\xi} \lesssim \langle t\rangle^{-0.04+10\delta}\epsilon_0, \qquad (4.58)\\
&\quad V_\infty(\xi) := \widehat{U}_0(\xi) + \int_0^\infty e^{is|\xi|}\widehat{\mathcal{N}_\phi}(s,\xi)ds.
\end{aligned}$$

That's to say, the nonlinear solution scatters to a linear solution.

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