ON THE BIRATIONAL GEOMETRY OF $\mathbb{Q}$-FANO THREEFOLDS OF LARGE FANO INDEX, I

YURI PROKHOROV

ABSTRACT. We investigate the rationality problem for $\mathbb{Q}$-Fano threefolds of Fano index ≥ 2.

1. INTRODUCTION

A three-dimensional algebraic projective variety X is called $\mathbb{Q}$-*Fano threefold* if it has only terminal $\mathbb{Q}$-factorial singularities, $\operatorname{Pic}(X) \simeq \mathbb{Z}$, and its anticanonical divisor $-K_X$ is ample. The class of these varieties is important in birational geometry because it is one of the possible outputs of the Minimal Model Program in dimension 3. It is known that $\mathbb{Q}$-Fano threefolds are bounded, i.e. they lie in a finite number of algebraic families. Moreover, the methods of [Kaw92] allow to obtain a (huge) list of numerical invariants of $\mathbb{Q}$-Fano threefolds [B$^+$]. At the moment there is no classification, but there are a lot of partial results.

This work is a sequel to our previous papers [Pro22b], [Pro22a]. We are interested in the birational geometry of $\mathbb{Q}$-Fano threefolds rather than biregular one. Mainly, we will discuss the rationality question.

The $\mathbb{Q}$-*Fano index* of a $\mathbb{Q}$-Fano threefold X is the maximal integer $\mathrm{q}_{\mathbb{Q}}(X)$ that divides the canonical class K_X in the Weil divisor class group modulo torsion (see (2.2.1)). A Weil divisor A such that $-K_X \sim_{\mathbb{Q}} \mathrm{q}_{\mathbb{Q}}(X)A$ we call the *fundamental divisor* and denote it by A_X. It turns out that the classification of $\mathbb{Q}$-Fano threefolds of large index $\mathrm{q}_{\mathbb{Q}}(X)$ is much simpler (see [Suz04] and [Pro10]). Moreover, $\mathbb{Q}$-Fano threefolds of large index $\mathrm{q}_{\mathbb{Q}}(X)$ are expected to be rational:

1.1. **Theorem** ([Pro22b]). *Let X be a $\mathbb{Q}$-Fano threefold. If $\mathrm{q}_{\mathbb{Q}} \geq 8$, then X is rational.*

On the other hand, there are nonrational $\mathbb{Q}$-Fano threefolds of large index. For example, T. Okada [Oka19] showed that there are $\mathbb{Q}$-Fano threefold hypersurfaces of index 2, 3, 5, and 7 that are not rational.

The following invariant will be very important in the sequel:

$$\mathrm{p}_n(X) := \max \left\{ \mathrm{h}^0(X, \mathscr{O}_X(D)) \mid D \sim_{\mathbb{Q}} nA_X \right\}.$$

If the Weil divisor class group $\operatorname{Cl}(X)$ is torsion free, then the above definition becomes simpler:

$$\mathrm{p}_n(X) = \mathrm{h}^0(X, \mathscr{O}_X(nA_X)).$$

Our main result is as follows:

1.2. **Theorem.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) \geq 2$. If one of the following conditions hold, then X is rational*

(i) $\mathrm{p}_1(X) \geq 4$,
(ii) $\mathrm{q}_{\mathbb{Q}}(X) \geq 3$ *and* $\mathrm{p}_1(X) \geq 3$,
(iii) $\mathrm{q}_{\mathbb{Q}}(X) \geq 4$ *and* $\mathrm{p}_1(X) \geq 2$,
(iv) $\mathrm{q}_{\mathbb{Q}}(X) \geq 5$ *and* $\mathrm{p}_2(X) \geq 2$ *and X is not of type* [B$^+$, # 41422] (*see Proposition* 7.4),

This work is supported by the Russian Science Foundation under grant no. 23-11-00033, https://rscf.ru/project/23-11-00033/.

(v) $\mathrm{q}_{\mathbb{Q}}(X) \ge 6$ *and* $\mathrm{p}_3(X) \ge 2$.

We also study birational geometry of $\mathbb{Q}$-Fano threefolds with $\mathrm{p}_1(X) \ge 2$ and $\mathrm{q}_{\mathbb{Q}}(X) = 2$ or 3 (see Propositions 6.4 and 8.1).

2. Preliminaries

2.1. **Notation.** We employ the following notation.

- $\sim$ and $\sim_{\mathbb{Q}}$ denote the linear and $\mathbb{Q}$-linear equivalences of divisors, respectively;
- $\mathrm{Cl}(X)$ denotes the Weil divisor class group of a normal variety X;
- $\mathrm{Cl}(X)_{\mathrm{t}}$ is the torsion subgroup of $\mathrm{Cl}(X)$;
- $\mathrm{g}(X)$ is the genus of the $\mathbb{Q}$-Fano threefold X, that is, $\mathrm{g}(X) := \mathrm{h}^0(X, \mathscr{O}_X(-K_X)) - 2$;
- $\mathbb{P}(w_1, \dots, w_n)$ is the weighted projective space with weights $w_1, \dots, w_n$; coinciding weights we group together as follows $\mathbb{P}(\underbrace{w_1, \dots, w_1}_{k}, w_2, \dots) = \mathbb{P}(w_1^k, w_2, \dots)$;
- $\mathbf{B}(X)$ is the basket of singularities of a terminal threefold X (see [Rei87]); this is a collection of virtual cyclic quotient singularities $\frac{1}{r_P}(1, -1, b_P)$ associated to each actual singular point of X. For short, in $\mathbf{B}(X)$ we list only indices of these virtual points, i.e. $\mathbf{B}(X) = (\{r_P\})$.
- $\boldsymbol{\mu}_N$ denotes the multiplicative cyclic group of order N. If $\boldsymbol{\mu}_N$ acts on $\mathbb{A}^n$ by
$$(x_1, \dots, x_n) \longmapsto (\zeta_n^{w_1} x_1, \dots, \zeta_n^{w_n} x_n),$$
where ζ_n is a primitive N-th root of 1, then we say that $(w_1, \dots, w_n)$ are the weights of the action an we write $\boldsymbol{\mu}_N(w_1, \dots, w_n)$ to specify the action.

2.2. **$\mathbb{Q}$-Fano threefolds.** For a Fano variety with at worst log terminal singularities we define *Fano-Weil* and *$\mathbb{Q}$-Fano indices* as follows:

$$\begin{aligned} \mathrm{q_W}(X) &:= \max\{q \in \mathbb{Z} \mid -K_X \sim qA,\ A \text{ is a Weil divisor}\}, \\ \mathrm{q}_{\mathbb{Q}}(X) &:= \max\{q \in \mathbb{Z} \mid -K_X \sim_{\mathbb{Q}} qA,\ A \text{ is a Weil divisor}\}. \end{aligned} \tag{2.2.1}$$

The *fundamental divisor* of X is a Weil divisor A_X such that

$$-K_X \sim_{\mathbb{Q}} \mathrm{q}_{\mathbb{Q}}(X) A_X. \tag{2.2.2}$$

Note that if $\mathrm{Cl}(X)_{\mathrm{t}} \ne 0$, then the class of A_X is not uniquely defined modulo linear equivalence. However, in the case $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q_W}(X)$ we always accept the following.

2.2.3. **Convention.** If $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q_W}(X)$ we take A_X so that

$$-K_X \sim \mathrm{q_W}(X) A_X.$$

The *Hilbert series* of a $\mathbb{Q}$-Fano threefold X is the formal power series [ABR02]

$$\mathrm{h}_X(t) = \sum_{m \ge 0} \mathrm{h}^0(X, mA_X) \cdot t^m.$$

It is computed by using the orbifold Riemann-Roch formula [Rei87]. If the group $\mathrm{Cl}(X)$ contains an element T of N-torsion, we define *T-Hilbert series* $\mathrm{h}_X(t, \sigma) \in \mathbb{Z}[[t, \sigma]]/(\sigma^N - 1)$ as follows:

$$\mathrm{h}_X(t, \sigma) = \sum_{m \ge 0} \sum_{j=0}^{N-1} \mathrm{h}^0(X, mA_X + jT) \cdot t^m \sigma^j.$$

Obviously, the above definition depends on the choice the class of A_X in $\mathrm{Cl}(X)$. Typically calculating $\mathrm{h}_X(t)$ or $\mathrm{h}_X(t, \sigma)$ for our purposes we need only a few initial terms of the series.

Recall that the *Gorenstein index* of a normal $\mathbb{Q}$-Gorenstein variety X is a minimal positive integer r such that the Weil divisor rK_X is Cartier.

2.2.4. **Theorem** ([Suz04], [Pro10]). *Let X be a Fano threefold with terminal singularities and let r be its global Gorenstein index. Then the following assertions hold:*

(i) $\mathrm{q_W}(X)$ *divides* $\mathrm{q_{\mathbb{Q}}}(X)$*;*
(ii) $\mathrm{q_W}(X) = \mathrm{q_{\mathbb{Q}}}(X)$ *if and only if r and $\mathrm{q_W}(X)$ are coprime;*
(iii) rA_X^3 *is an integer;*
(iv) $\mathrm{q_W}(X),\ \mathrm{q_{\mathbb{Q}}}(X) \in \{1, \dots, 9, 11, 13, 17, 19\}$.

2.2.5. **Theorem** ([CF93], [San96]). *Let X be a Fano threefold with terminal singularities. Assume that there exists a Cartier divisor H on X such that $H \sim_{\mathbb{Q}} mA_X$, where $m < \mathrm{q_{\mathbb{Q}}}(X)$. Then the general member $S \in |H|$ is a smooth del Pezzo surface, the group $\mathrm{Cl}(X)$ is torsion free, and (X, H) is described by the following table. The general member X of each family is a $\mathbb{Q}$-Fano threefold.*

	$\mathrm{q_{\mathbb{Q}}}$	$\mathbf{B}(X)$	A_X^3	X	$\mathrm{g}(X)$	Rat?	m	K_S^2
1^o	4	$\varnothing$	1	$\mathbb{P}^3$	33	R	1, 2, 3	9, 8, 3
2^o	3	$\varnothing$	2	$X_2 \subset \mathbb{P}^4$	28	R	1, 2	8, 4
3^o	2	$\varnothing$	d	del Pezzo threefold of degree $d \le 5$	$4d+1$		1	d
4^o	5	(2)	1/2	$\mathbb{P}(1^3, 2)$	32	R	2, 4	9, 2
5^o	7	(2, 3)	1/6	$\mathbb{P}(1^2, 2, 3)$	29	R	6	1
6^o	4	(3^2)	1/3	$X_6 \subset \mathbb{P}(1^2, 2, 3^2)$	11	R	3	1
7^o	5	(2, 4)	1/4	$X_6 \subset \mathbb{P}(1^2, 2, 3, 4)$	16	R	4	1
8^o	6	(5)	1/5	$X_6 \subset \mathbb{P}(1^2, 2, 3, 5)$	22	R	5	1
9^o	3	(2^3)	1/2	$X_6 \subset \mathbb{P}(1^2, 2^2, 3)$	7		2	1
10^o	3	(2^2)	1	$X_4 \subset \mathbb{P}(1^3, 2^2)$	14	R	2	2
11^o	4	(3)	2/3	$X_4 \subset \mathbb{P}(1^3, 2, 3)$	22	R	3	2
12^o	3	(2)	3/2	$X_3 \subset \mathbb{P}(1^4, 2)$	21	R	2	3

The column "Rat" indicates the rationality of X. One can see that almost all the $\mathbb{Q}$-Fano threefolds in the table are rational. In the case 9^o it is known that a very general variety in the family is not rational (and even not stably rational) [Oka19]. Rationality question of del Pezzo threefolds (case 3^o) has a long story. We refer to the book [IP99] for references and to [CPS19] for a detailed discussion of the case $d = 2$.

2.3. **Singularities.** For the classification of terminal threefold singularities we refer to [Rei87].

2.3.1. **Definition.** Let X be a threefold with terminal $\mathbb{Q}$-factorial singularities. An *extremal blowup* of X is a birational morphism $f : \tilde{X} \to X$ such that $\tilde{X}$ has only terminal $\mathbb{Q}$-factorial singularities and $\rho(\tilde{X}/X) = 1$.

Note that in this situation the divisor $-K_{\tilde{X}}$ is f-ample.

2.3.2. **Theorem** ([Kaw96]). *Let $X \ni P$ a cyclic quotient terminal threefold singularity of type $\frac{1}{r}(a, r-a, 1)$, $r \ge 2$ and let $f : \tilde{X} \to X$ be an extremal blowup $f : \tilde{X} \to X \ni P$ whose center contains P. Then f is the weighted blowup with the system of weights $\frac{1}{r}(a, r-a, 1)$, in particular, the discrepancy of the f-exceptional divisor equals $1/r$.*

2.3.3. **Theorem** ([Kaw93]). *Let $X \ni P$ a terminal threefold singularity of index r. Then there exists an exceptional divisor E over P whose discrepancy equals $1/r$.*

The following useful fact is a consequence of the classification of extremal blowups [Kaw05] (see [Pro13, Lemma 2.6] for explanations).

2.3.4. **Lemma.** *Let $X \ni P$ be a threefold terminal point of index $r > 1$ with basket $\mathbf{B}(X, P)$, let $f : \tilde{X} \to X$ be an extremal blowup with $f(E) = P$, where E is the exceptional divisor, and let α be the discrepancy of E.*

(i) *If $X \ni P$ is a point of type other than cA/r and $r > 2$, then $\alpha = 1/r$.*
(ii) *If $X \ni P$ is of type cA/r and $\mathbf{B}(X, P)$ consists of n points of index r, then $\alpha = a/r$, where $n \equiv 0 \mod a$.*

2.4. **Du Val del Pezzo surfaces.** Let S be a del Pezzo surface with only Du Val singularities. We assume that $\rho(S) = 1$. The definitions of Fano indices and the fundamental divisor A_X are applicable to S (see (2.2.1) and (2.2.2)). Recall also our Convention 2.2.3. Let $\mu : \tilde{S} \to S$ be the minimal resolution. We say that a curve $L \subset S$ is a *line* if there exists a (-1)-curve $E \subset \tilde{S}$ such that $L = \mu(E)$.

2.4.1. **Lemma.** *In the above notation assume that $d := K_S^2 < 8$. One has:*

(i) *The set of lines on S is finite and non-empty.*
(ii) *The group $\mathrm{Cl}(S)$ is generated by the classes of lines.*
(iii) *For every effective Weil divisor D on S there is a presentation*
$$D \sim a_0(-K_S) + \sum a_i L_i,$$
where the L_i are lines in S, the a_i are non-negative integers, and $a_0 = 0$ if $d > 1$.
(iv) *For any line L on S we have $L \sim_{\mathbb{Q}} A_S$, hence $\mathrm{q}_{\mathbb{Q}}(S) = d$.*
(v) *If D is a divisor on S such that $D \sim_{\mathbb{Q}} A_S$, then either*
  (a) *$\dim |D| = 0$ and $D \sim L$, where L is a line, or*
  (b) *$\dim |D| = 1$, $d = 1$, and $D \sim -K_S$.*
(vi) *If D is an ample divisor, then $|D| \neq \varnothing$.*
(vii) *If D is an effective divisor such that $\dim |D| = 0$ and $D \neq 0$, then D is a line.*
(viii) *Assume that $d > 1$. Then for any two lines L_1 and L_2 on S the divisor $L_1 - L_2$ is a non-trivial torsion element in $\mathrm{Cl}(S)$. In particular, $\mathrm{Cl}(S) \simeq \mathbb{Z}$ if and only if S contains exactly one line.*

Sketch of the proof. The assertion (i) follows from the cone theorem applied to $\tilde{S}$. For (ii) and (iii) we refer to [CP21, Lemma 2.9]. The assertion (iv) follows from the equality $d = -dK_S \cdot L = \mathrm{q}_{\mathbb{Q}}(S)(-K_S \cdot A_S)$. To prove (v) assume that $D \not\sim -K_S$ and $D \sim_{\mathbb{Q}} A_S$. By the orbifold Riemann-Roch formula [Rei87] applied to D and $-D$ we have:

$$\begin{aligned} \chi(S, \mathscr{O}_S(D)) &= \frac{1}{2} D \cdot (D - K_S) + 1 + \sum c_P(D), \\ \chi(S, \mathscr{O}_S(-D)) &= \frac{1}{2} D \cdot (D + K_S) + 1 + \sum c_P(-D). \end{aligned}$$

By the Kawamata-Viehweg vanishing and Serre duality $\mathrm{h}^i(S, \mathscr{O}_S(-D)) = 0$ for $i = 0, 1, 2$ and $\mathrm{h}^i(S, \mathscr{O}_S(D)) = 0$ for $i = 1, 2$. Since $c_P(-D) = c_P(D)$, we obtain $\mathrm{h}^0(S, \mathscr{O}_S(D)) = 1$. Hence we may assume that D is effective. In this case (v) is a consequence of (iii) The assertion (vi) follows from (v). For (vii), in view of (iii) it is sufficient to show that $-K_S \cdot D = 1$. Let

$C \in |-K_S|$ be a general element. Then C is a smooth elliptic curve lying in the smooth part of S. Since $H^1(S, \mathscr{O}_S(D+K_S)) = 0$, from the exact sequence

$$0 \longrightarrow \mathscr{O}_S(D+K_S) \longrightarrow \mathscr{O}_S(D) \longrightarrow \mathscr{O}_C(D) \longrightarrow 0$$

we obtain $\mathrm{h}^0(C, \mathscr{O}_C(D)) = 1$, hence $D \cdot C = \deg \mathscr{O}_C(D) = 1$.

Finally, (viii) follows from (i), (ii) and (v). □

2.4.2. **Lemma.** *Let S be a del Pezzo surface with only Du Val singularities of type $\mathrm{A_n}$ and $\rho(S) = 1$. If the group $\mathrm{Cl}(S)$ is torsion free, then there are only the following possibilities:*

K_S^2	S	$\mathrm{h}_S(t)$
9	$\mathbb{P}^2$	$1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + \cdots$
8	$\mathbb{P}(1^2, 2)$	$1 + 2t + 4t^2 + 6t^3 + 9t^4 + 12t^5 + \cdots$
6	$\mathbb{P}(1, 2, 3)$	$1 + t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + \cdots$
5	$S_6 \subset \mathbb{P}(1, 2, 3, 5)$	$1 + t + 2t^2 + 3t^3 + 4t^4 + 6t^5 + \cdots$

where $S_6 \subset \mathbb{P}(1, 2, 3, 5)$ is a hypersurface of degree 6 in $\mathbb{P}(1, 2, 3, 5)$ having a unique singular point which point of type A_4; this surface is unique up to isomorphism.

Proof. The classification can be found in [MZ88] and computation of $\mathrm{h}_S(t)$ follows from the orbifold Riemann-Roch [Rei87]. □

3. $\mathbb{Q}$-Fano threefolds with torsion in the divisor class group

In this section we discuss $\mathbb{Q}$-Fano threefolds whose class group $\mathrm{Cl}(X)$ contains non-trivial torsion.

3.1. **Proposition** (see [Pro22b, § 3])**.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) \geq 5$ and $|\mathrm{Cl}(X)_{\mathrm{t}}| = N > 1$. Then $\mathrm{q}_{\mathbb{Q}}(X) = 5$ or 7 and X belongs to one of the following classes:*

	A_X^3	$\mathbf{B}(X)$	$\mathrm{g}(X)$	$\mathrm{h}_X(t, \sigma)$
				$\mathrm{q}_{\mathbb{Q}}(X) = 7,\ N = 2$
1^o	1/24	$(2^2, 3, 4, 8)$	6	$1 + t\sigma + t^2 + t^2\sigma + 2t^3 + 2t^3\sigma + 3t^4 + 3t^4\sigma + 4t^5 + 4t^5\sigma + \cdots$
2^o	1/30	$(2, 6, 10)$	5	$1 + t + t^2 + t^2\sigma + t^3 + 2t^3\sigma + 2t^4 + 3t^4\sigma + 3t^5 + 4t^5\sigma + \cdots$
				$\mathrm{q}_{\mathbb{Q}}(X) = 5,\ N = 3$
3^o	1/18	$(2, 9^2)$	2	$1 + t + t^2 + t^2\sigma + t^2\sigma^2 + t^3 + 2t^3\sigma + 2t^3\sigma^2 + \cdots$
				$\mathrm{q}_{\mathbb{Q}}(X) = 5,\ N = 2$
4^o	1/6	$(2, 4^2, 6)$	10	$1+t+t\sigma+2t^2+3t^2\sigma+4t^3+5t^3\sigma+8t^4+7t^4\sigma+12t^5+11t^5\sigma+\cdots$
5^o	1/8	$(2^2, 4, 8)$	7	$1+t+t\sigma+2t^2+2t^2\sigma+3t^3+4t^3\sigma+6t^4+6t^4\sigma+9t^5+9t^5\sigma+\cdots$
6^o	1/12	$(4^2, 12)$	4	$1 + t + t^2 + t^2\sigma + 2t^3 + 2t^3\sigma + 4t^4 + 4t^4\sigma + 6t^5 + 6t^5\sigma + \cdots$
7^o	1/28	$(2, 4, 14)$	1	$1 + t + t^2 + t^3 + t^3\sigma + 2t^4 + 2t^4\sigma + 3t^5 + 3t^5\sigma + \cdots$

In particular, $\mathrm{p}_1(X) = 1$.

The following fact is a consequence of computer calculations as explained in Appendix A. In principle, one can perform them by hand but since they are not conceptual it is more reasonable to use a computer.

3.2. **Proposition.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) \geq 3$ and $\mathrm{q}_{\mathrm{W}}(X) \neq \mathrm{q}_{\mathbb{Q}}(X)$. Then $\mathrm{q}_{\mathbb{Q}}(X) = 3$ or 4 and X belongs to one of the following classes:*

	A_X^3	$\mathbf{B}(X)$	$\mathrm{g}(X)$	$\mathrm{p}_1(X)$	$\mathrm{h}_X(t,\sigma)$
$\mathrm{q}_{\mathbb{Q}}(X)=3$, $\mathrm{Cl}(X)_{\mathrm{t}} \simeq \mathbb{Z}/3\mathbb{Z}$					
1^o	$1/2$	$(3^4,6)$	6	2	$1+(1+\sigma+2\sigma^2)t+(3+4\sigma+4\sigma^2)t^2+(8+8\sigma+7\sigma^2)t^3+\cdots$
2^o	$1/10$	$(3^4,5,6)$	0	1	$1+\sigma^2 t+(\sigma+\sigma^2)t^2+(2+2\sigma+\sigma^2)t^3+\cdots$
3^o	$1/4$	$(2,3^2,12)$	2	1	$1+(\sigma+\sigma^2)t+2(1+\sigma+\sigma^2)t^2+4(1+\sigma+\sigma^2)t^3+\cdots$
$\mathrm{q}_{\mathbb{Q}}(X)=4$, $\mathrm{Cl}(X)_{\mathrm{t}} \simeq \mathbb{Z}/2\mathbb{Z}$					
4^o	$1/3$	$(2^5,6)$	10	2	$1+(1+2\sigma)t+(4+3\sigma)t^2+7(1+\sigma)t^3+12(1+\sigma)t^4+\cdots$
5^o	$2/15$	$(2^5,5,6)$	3	1	$1+\sigma t+(2+\sigma)t^2+3(1+\sigma)t^3+5(1+\sigma)t^4+\cdots$
6^o	$1/21$	$(2^5,6,7)$	0	1	$1+\sigma t+t^2+(1+\sigma)t^3+2(1+\sigma)t^4+\cdots$
7^o	$2/5$	$(2^3,10)$	12	2	$1+(1+2\sigma)t+4(1+\sigma)t^2+8(1+\sigma)t^3+14(1+\sigma)t^4+\cdots$
8^o	$1/15$	$(2^3,3,10)$	1	1	$1+\sigma t+(1+\sigma)t^2+2(1+\sigma)t^3+3(1+\sigma)t^4+\cdots$
9^o	$1/5$	$(2^3,5,10)$	5	1	$1+\sigma t+2(1+\sigma)t^2+4(1+\sigma)t^3+7(1+\sigma)t^4+\cdots$
10^o	$4/35$	$(2^3,7,10)$	2	1	$1+\sigma t+(1+\sigma)t^2+2(1+\sigma)t^3+4(1+\sigma)t^4+\cdots$

3.3. **Proposition.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) \geq 3$, $\mathrm{q}_{\mathrm{W}}(X) \neq \mathrm{q}_{\mathbb{Q}}(X)$, and $\mathrm{p}_1(X) \geq 2$. Then one of the following holds*

(i) *X is of type 1^o of Proposition 3.2 and then X is the quotient $X'/\boldsymbol{\mu}_3$, where X' is a hypersurface of degree 3 in $\mathbb{P}(1^4,2)$:*

$$\left\{x_2x_1^{(1)} + \phi_3\bigl(x_1^{(1)},\dots,x_1^{(4)}\bigr) = 0\right\}/\boldsymbol{\mu}_3(0,0,1,1,-1).$$

(ii) *X is of type 4^o of Proposition 3.2 and then X is the quotient $X'/\boldsymbol{\mu}_2$, where X' is a hypersurface of degree 4 in $\mathbb{P}(1^3,2,3)$:*

$$\left\{x_3x_1 + x_2^2 + \phi_4(x_1',x_1'') + \phi_2(x_1',x_1'')x_1^2 = 0\right\}/\boldsymbol{\mu}_2(0,1,1,1,0).$$

Here the subscript is the degree of the corresponding variable or polynomial.

In both cases X is rational.

Proof. By Proposition 3.2 we have either $\mathrm{q}_{\mathbb{Q}}(X) = 3$ or $\mathrm{q}_{\mathbb{Q}}(X) = 4$. First, consider the case $\mathrm{q}_{\mathbb{Q}}(X) = 3$. Then the variety X is of type 1^o of Proposition 3.2. The group $\mathrm{Cl}(X)_{\mathrm{t}} \simeq \mathbb{Z}/3\mathbb{Z}$ defines a triple cyclic cover $\pi : X' \to X$ which is étale outside $\mathrm{Sing}(X)$. Thus $X = X'/\boldsymbol{\mu}_3$, where X' is a Fano threefold with terminal singularities (not necessarily $\mathbb{Q}$-Fano) such that $\mathrm{q}_{\mathrm{W}}(X')$ is divisible by 3, $A_{X'}^3 = 3/2$, and $\mathbf{B}(X') = (2)$. By Theorem 2.2.5 the variety X' is a hypersurface of degree 3 in $\mathbb{P}(1^4,2)$. In these settings, the unique point $P' \in X'$ of index 2 has coordinates $(0,0,0,0,1)$. Since $P' \in X'$ is a terminal cyclic quotient singularity, the variable x_2 of degree 2 appears in the equation of X'.

The embedding $X' \subset \mathbb{P}(1^4,2)$ is canonical, so it is $\boldsymbol{\mu}_3$-equivariant and the action of $\boldsymbol{\mu}_3$ on X' is induced by a liner action on the ambient space $\mathbb{P}(1^4,2)$. Moreover, the homogeneous coordinates $x_1^{(k)}$ and x_2 can be taken to be semi-invariant. Modulo a linear coordinate change we may assume that the equation of X' is as follows:

$$x_2x_1^{(1)} + \phi(x_1^{(1)},\dots,x_1^{(4)}) = 0.$$

Consider the affine chart $U_2 := \{x_2 \neq 0\} \subset X'$. Then

$$U_2 = \{y^{(1)} + \phi'(y^{(1)}, \dots, y^{(4)}) = 0\}/\boldsymbol{\mu}_2(1,1,1,1).$$

The quotient $U_2/\boldsymbol{\mu}_3$ is a terminal cyclic quotient singularity. Hence $\boldsymbol{\mu}_3$-action on U_2 has weights $(a,1,1,-1)$ for some $a \in \{0,1,2\}$ modulo permutations of $y^{(2)}, y^{(3)}, y^{(4)}$ and changing the generator of $\boldsymbol{\mu}_3$. Thus $\boldsymbol{\mu}_3$-action on $\mathbb{P}(1^4,2)$ has weights $(0,a,1,1,-1)$. If $a=1$, then the fixed point locus contains the curve $\{x_2 = x_1^{(4)} = 0\} \cap X'$, a contradiction. If $a=-1$, then, by the same reason, the line $x_2 = x_1^{(2)} = x_1^{(3)} = 0$ is not contained in X' and so $\phi(x_1^{(1)}, 0, 0, x_1^{(4)}) \neq 0$. But in this case ϕ and $x_2 x_1^{(1)}$ must be $\boldsymbol{\mu}_3$-invariant, a contradiction. Hence $a=0$. This proves (i).

Now consider the case $\mathrm{q}_{\mathbb{Q}}(X) = 4$. Then again by Proposition 3.2 the variety X is of type 4^o or 7^o. The group $\mathrm{Cl}(X)_{\mathrm{t}} \simeq \mathbb{Z}/2\mathbb{Z}$ defines a double cover $\pi : X' \to X$ which is étale outside $\mathrm{Sing}(X)$, where X' is a Fano threefold with terminal singularities. Moreover, in our two cases we have

4^o. $\mathrm{q}_{\mathrm{W}}(X') = 4$, $A_{X'}^3 = 2/3$, $\mathbf{B}(X') = (3)$, $\dim |A_{X'}| = 2$, $\mathrm{g}(X') = 22$;
7^o. $\mathrm{q}_{\mathrm{W}}(X') = 4$, $A_{X'}^3 = 4/5$, $\mathbf{B}(X') = (5)$, $\dim |A_{X'}| = 2$, $\mathrm{g}(X') = 26$.

By Theorem 2.2.5 in the case 4^o the variety X' is a hypersurface of degree 4 in $\mathbb{P}(1^3,2,3)$ and by Proposition 3.3.1 below the case 7^o does not occur. As above the embedding $X' \subset \mathbb{P}(1^3,2,3)$ is $\boldsymbol{\mu}_2$-equivariant, where the action on X' is induced by a liner action on the ambient space $\mathbb{P}(1^3,2,3)$. Moreover, we may assume that the coordinates $x_1, x_1', x_1'', x_2, x_3$ are semi-invariants with eigenvalues ± 1 and x_1 is an invariant. Since $\mathbf{B}(X') = (3)$, the terms x_2^2 and some of $x_3 x_1$, $x_3 x_1'$ or $x_3 x_1''$ appear in the equation. Modulo an obvious coordinate change we may assume that the equation of X' is as follows:

$$x_3 x_1 + x_2^2 + \phi(x_1, x_1', x_1'') = 0,$$

where ϕ is a semi-invariant of degree 4. Then we see that ϕ is in fact an invariant and x_3 must be an invariant as well. Since the set of fixed points is finite, the variables x_1', x_1'', x_2 cannot be invariant. Hence the action have the desired form. The rest is obvious. □

3.3.1. **Proposition.** *Let Y be a weak Fano threefold with terminal singularities with $\mathrm{g}(Y) > 22$. Assume that there exists a Weil divisor B on Y such that $-K_Y \sim 4B$ and $\dim |B| \geq 2$. Then $Y \simeq \mathbb{P}^3$.*

Proof. If Y is a $\mathbb{Q}$-Fano threefold, then the assertion follows from [Pro13, Theorem 1.2]. Thus we assume that Y is not $\mathbb{Q}$-Fano. Replacing Y with its $\mathbb{Q}$-factorialization, we may assume that Y is $\mathbb{Q}$-factorial. Then $\uprho(Y) > 1$. Run the MMP on Y:

$$Y = Y^{(0)} \dashrightarrow Y^{(1)} \dashrightarrow \cdots \dashrightarrow Y^{(n-1)} \dashrightarrow Y^{(n)}. \tag{3.3.2}$$

Let $\mathscr{B}_k$ be the proper transform of the linear system $\mathscr{B}_0 := |B|$ on $Y^{(k)}$. On each step the relation $-K_{Y^{(k)}} \sim 4\mathscr{B}_k$ is kept. Moreover, $\dim \mathscr{B}_k = \dim |B| \geq 2$. Thus $Y^{(n)}$ has a structure of Mori-Fano fiber space $\varphi : Y^{(n)} \to Z$ such that for a general fiber F we have $-K_F \sim 4\mathscr{B}_n|_F$. This is not possible if φ is a del Pezzo or a rational curve fibration. Hence Z is a point and $Y^{(n)}$ is a $\mathbb{Q}$-Fano threefold such that $\mathrm{q}_{\mathrm{W}}(Y^{(n)})$ is divisible by 4 and $\mathrm{g}(Y^{(n)}) \geq \mathrm{g}(Y) > 22$. By [Pro13, Theorem 1.2] we have $Y^{(n)} \simeq \mathbb{P}^3$. Let us consider the last step $\psi : Y^{(n-1)} \dashrightarrow Y^{(n)}$ of the MMP. Since $\uprho(Y^{(n)}) = 1$, the map $\psi : Y^{(n-1)} \dashrightarrow Y^{(n)}$ must be a divisorial contraction and since $-K_{Y^{(n-1)}}$ is divisible, ψ cannot be a contraction of a divisor to a curve. Thus ψ contracts a divisor $E \subset Y^{(n-1)}$ to a point $P \in Y^{(n)}$. Since $-K_{Y^{(n-1)}} \sim 4\mathscr{B}_{n-1}$ is ψ-ample, we see that $P \in \mathrm{Bs}\, \mathscr{B}_n$. Then $\mathscr{B}_n$ is subsystem in $|\mathscr{O}_{\mathbb{P}^3}(1)|$ of codimension 1 and so $\mathscr{B}_n$ has a single base point, say P. Therefore, $\mathscr{B}_{n-1}$ has no base points outside E. Since $-K_{Y^{(n-1)}}$ is ample on E, we see that $-K_{Y^{(n-1)}}$ is nef. Since $-K_{Y^{(n-1)}}$ is the proper transform of $-K_Y$, it is also big. Thus $Y^{(n-1)}$ is a weak Fano threefold with terminal singularities and $-K_{Y^{(n-1)}} \sim 4\mathscr{B}_{n-1}$. According to [Kaw01] the contraction ψ is the weighted blowup with weights $(1, w_1, w_2)$, where $\gcd(w_1, w_2) = 1$. Then we have

$$\mathbf{B}(Y^{(n-1)}) = (w_1, w_2), \qquad K_{Y^{(n-1)}} = \psi^* K_{\mathbb{P}^3} + (w_1 + w_2)E.$$

Here $w_1 + w_2$ is divisible by 4 because so $K_{Y^{(n-1)}}$ is. Then

$$0 < (-K_{Y^{(n-1)}})^3 = 64 - \frac{(w_1 + w_2)^3}{w_1 w_2}.$$

Up to permutation of w_1 and w_2 we obtain the following possibilities for $\big(w_1, w_2, A^3_{Y^{(n-1)}}\big)$:

$$(1, 3, 2/3), \qquad (3, 5, 7/15), \qquad (5, 7, 8/35).$$

Then from the Kawamata-Viehweg vanishing and the orbifold Riemann-Roch formula (see (A.1.2)) we obtain $\mathrm{g}(Y^{(n-1)}) = 22, 15, 7$ in these cases, respectively. This contradicts our assumption $\mathrm{g}(Y) > 22$. □

3.3.3. **Corollary.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) \geq 6$. If either $\mathrm{p}_1(X) \geq 2$ or $\mathrm{q}_{\mathbb{Q}}(X) \geq 7$ and $\mathrm{p}_2(X) \geq 2$, then X is rational.*

Proof. By Theorem 1.1 we may assume that $\mathrm{q}_{\mathbb{Q}}(X) = 6$ or 7. By Proposition 3.1 the group $\mathrm{Cl}(X)$ is torsion free. If $\mathrm{q}_{\mathbb{Q}}(X) = 6$ and $\mathrm{p}_1(X) \geq 2$, then computer search (see Sect. A) gives only one possibility: $A_X^3 = 1/5$, $\mathbf{B}(X) = (5)$, [B$^+$, # 41469]. In this case X is rational by Theorem 2.2.5. Similarly in the case $\mathrm{q}_{\mathbb{Q}}(X) = 7$ and $\mathrm{p}_2(X) \geq 2$ there are four possibilities:

- $A_X^3 = 1/6$, $\mathbf{B}(X) = (2, 3)$, $\mathrm{g}(X) = 29$, [B$^+$, # 41492];
- $A_X^3 = 1/12$, $\mathbf{B}(X) = (2, 3^2, 4)$, $\mathrm{g}(X) = 14$, [B$^+$, # 41484];
- $A_X^3 = 1/10$, $\mathbf{B}(X) = (2^3, 5)$, $\mathrm{g}(X) = 17$, [B$^+$, # 41489];
- $A_X^3 = 1/15$, $\mathbf{B}(X) = (2^2, 3, 5)$, $\mathrm{g}(X) = 11$, [B$^+$, # 41481].

In the case $A_X^3 = 1/6$ we have $X \simeq \mathbb{P}(1^2, 2^2)$ by Theorem 2.2.5 and in the cases $A_X^3 = 1/12$ and $A_X^3 = 1/10$ there is explicit descriptions of X as a weighted hypersurface $X_6 \subset \mathbb{P}(1, 2, 3^2, 4)$ and $X_6 \subset \mathbb{P}(1, 2^2, 3, 5)$, respectively (see [Pro13, Theorem 1.4] and [Pro16, Theorem 1.2]). It is easy to see that these varieties are rational. Rationality of X in the case $A_X^3 = 1/15$ was proved in [Pro22b, Proposition 5.1]. □

3.4. **Proposition.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) \geq 3$ and $N := |\operatorname{Cl}(X)_{\mathrm{t}}| > 1$. Assume either $N \geq 4$ or $\mathrm{q}_{\mathbb{Q}}(X) = 4$ and $N = 3$. Then $\operatorname{Cl}(X)_{\mathrm{t}} \simeq \mathbb{Z}/N\mathbb{Z}$ and X is the quotient $X'/\boldsymbol{\mu}_N$, where X' is a Fano threefold with terminal singularities and the action of $\boldsymbol{\mu}_N$ on X' is free outside a finite number of points. Moreover, X and X' are described by the following table:*

	$\mathrm{q}_{\mathbb{Q}}(X)$	$\mathbf{B}(X)$	N	A_X^3	$\mathrm{g}(X)$	$\mathrm{p}_1(X)$	$\mathbf{B}(X')$	X'
1^o	3	$(2^2, 8^2)$	4	1/4	2	1	(2^2)	$X_4' \subset \mathbb{P}(1^3, 2^2)$
2^o	3	(5^4)	5	2/5	4	1	$\varnothing$	$Q \subset \mathbb{P}^4$
3^o	4	(5^4)	5	1/5	5	1	$\varnothing$	$\mathbb{P}^3$
4^o	4	(9^2)	3	1/9	3	1	(3^2)	$X_6' \subset \mathbb{P}(1^2, 2, 3^2)$

where $X_4' \subset \mathbb{P}(1^3, 2^2)$ is a hypersurface of degree 4 and $Q \subset \mathbb{P}^4$ is a smooth quadric.

In particular, X is rational.

Proof. The computer search by the algorithm outlined in Appendix A produces exactly three possibilities with numerical invariants as in the table. Then $\mathrm{Cl}(X)_{\mathrm{t}} \simeq \mathbb{Z}/N\mathbb{Z}$, where $N = 4$ or 5 and, as in the proof of Proposition 3.3, we see that the generator of this group defines a global cyclic cover $X' \to X$ of degree N. Thus $X = X'/\boldsymbol{\mu}_N$. Here the Gorenstein index of X' is strictly less than $\mathrm{q}_{\mathbb{Q}}(X')$. Hence X' by Theorem 2.2.5 X' is either a hypersurface of degree 4 in $\mathbb{P}(1^3, 2^2)$, of degree 6 in $\mathbb{P}(1^2, 2, 3^2)$, the projective space $\mathbb{P}^3$, or a quadric $Q \subset \mathbb{P}^4$.

It remains to show that X is rational. In the case 3^o the variety X is toric, so rationality is obvious. In the case 2^o the action of $\boldsymbol{\mu}_5$ on Q is induced by a linear action on $\mathbb{P}^4$. The projection $Q \dashrightarrow \mathbb{P}^3$ from a fixed point $P \in Q$ is equivariant, hence $Q/\boldsymbol{\mu}_5$ is birational to the toric variety $\mathbb{P}^3/\boldsymbol{\mu}_5$, so it is rational.

Consider the case 1^o. As above, action of $\boldsymbol{\mu}_4$ on X' is induced by a liner action on the ambient space $\mathbb{P}(1^3, 2^2)$. Moreover, we may assume that the coordinates x_1, x_2, x_3, y_1, y_2 are semi-invariants with $\deg x_i = 1$, $\deg y_j = 2$. The equation of X' can be written as follows

$$q(y_1, y_2) + y_1\phi_2'(x_1, x_2, x_3) + y_2\phi_2''(x_1, x_2, x_3) + \phi_4(x_1, x_2, x_3) = 0,$$

where q, ϕ_2', ϕ_2'', ϕ_4 are homogeneous polynomials of degree 2, 2, 2, 4, respectively. Since the line $\mathrm{Sing}(\mathbb{P}(1^3, 2^2)) = \{x_1 = x_2 = x_3 = 0\}$ is not contained in X', we have $q(y_1, y_2) \neq 0$. The projection $\Psi : X' \dashrightarrow \mathbb{P}(1^3) = \mathbb{P}^2$ is an equivariant rational map whose fibers are conics in $\mathbb{P}(1, 2^2) = \mathbb{P}^2$ and whose indeterminacy locus $\mathrm{Ind}(\Psi) = \{q(y_1, y_2) = x_1 = x_2 = x_3 = 0\}$ consists of one or two points of index 2. These points cannot be switched by the action of $\boldsymbol{\mu}_4$ because their images on X are exactly the points of index 8. Hence points in $\mathrm{Ind}(\Psi) = \{q(y_1, y_2) = x_1 = x_2 = x_3 = 0\}$ give invariant sections of Ψ. This implies that the rational curve fibration $X = X'/\boldsymbol{\mu}_4 \dashrightarrow \mathbb{P}^2/\boldsymbol{\mu}_4$ has a section, hence X is rational.

Finally, in the case 4^o, as above, the equation of X' can be reduced to one of the following forms

$$\begin{aligned} x_3x_3' &+ ax_2^3 + x_2\phi_4(x_1, x_1') + \phi_6(x_1, x_1') = 0, \\ x_3^2 &+ ax_2^3 + x_2\phi_4(x_1, x_1') + \phi_6(x_1, x_1') = 0. \end{aligned}$$

In the former case the projection to $\mathbb{P}(1^2, 2, 3)$ establishes the rationality of X. In the latter case the point $(0, 0, 0, 0, 1)$ on X' is a unique non-Gorenstein point and it is not of type cA/3 [Rei87]. Hence its quotient by $\boldsymbol{\mu}_3$ cannot be terminal, a contradiction. □

3.5. **Proposition.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{Cl}(X)_{\mathrm{t}} \neq 0$. Assume either*

(i) $\mathrm{q}_{\mathbb{Q}}(X) = 5$, *and* $\mathrm{p}_2(X) \geq 2$, *or*

(ii) $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q_W}(X) = 3$, $\mathrm{p}_1(X) \geq 2$, *and* $\dim |A_X| \leq 0$.

Then $\mathrm{Cl}(X)_{\mathrm{t}} \simeq \mathbb{Z}/2\mathbb{Z}$ and X is the quotient $X'/\boldsymbol{\mu}_2$, where X' is a Fano threefold with terminal singularities and the action of $\boldsymbol{\mu}_2$ on X' is free outside a finite number of points. Moreover, X and X' are described by the following table:

	$\mathrm{q}_{\mathbb{Q}}(X)$	$\mathbf{B}(X)$	A_X^3	$\mathrm{g}(X)$	$\mathbf{B}(X')$	$\mathrm{g}(X')$	X'
1^o	5	$(2^2, 4, 8)$	$1/8$	7	$(2, 4)$	16	$X_6' \subset \mathbb{P}(1^2, 2, 3, 4)$
2^o	5	$(2, 4^2, 6)$	$1/6$	10	$(2^2, 3)$	21	$X_4' \subset \mathbb{P}(1^2, 2^2, 3)$
3^o	3	$(2^4, 4^2)$	$1/2$	6	(2^2)	14	$X_4' \subset \mathbb{P}(1^3, 2^2)$

In particular, X is rational.

Proof. The numerical invariants of X and its global cover X' are obtained from Proposition 3.1 in the case $\mathrm{q}_{\mathbb{Q}}(X) = 5$ and by computer search in the case $\mathrm{q}_{\mathbb{Q}}(X) = 3$. Then in the case 1^o we see that X' is a hypersurface $X'_6 \subset \mathbb{P}(1^2, 2, 3, 4)$ by Theorem 2.2.5. As above, we may assume that the coordinates x_1, x'_1, x_2, x_3, x_4 are semi-invariants with $\deg x_i = i$, $\deg x'_1 = 1$. The set of non-Gorenstein points of X' consists of either two cyclic quotient singularities P_1 and P_2 of types $\frac{1}{2}(1, 1, 1)$ and $\frac{1}{4}(1, 1-1)$ or one point P of type cAx/4. The latter case does not occur because the quotient $(X' \ni P)/\boldsymbol{\mu}_2$ must be a terminal singularity. Hence the equation of X' must contain the term x_4x_2. Then the projection $X' \to \mathbb{P}(1^2, 2, 3)$ is equivariant and birational. Therefore the variety $X = X'/\boldsymbol{\mu}_2$ is birational to $\mathbb{P}(1^2, 2, 3)/\boldsymbol{\mu}_2$, so it is rational.

Consider the case 2^o. We claim that X' is a $\mathbb{Q}$-Fano threefold. Indeed, otherwise, as in the proof of Proposition 3.3.1 we can take $\mathbb{Q}$-factorialization $Y \to X'$ and run the MMP (3.3.2). At the end we obtain a $\mathbb{Q}$-Fano threefold $Y^{(n)}$ such that $\mathrm{g}(Y^{(n)}) \geq \mathrm{g}(X) \geq 21$. Hence $Y^{(n)} \simeq \mathbb{P}(1^3, 2)$ or $Y^{(n)} \simeq Y_4 \subset \mathbb{P}(1^2, 2^2, 3)$ by [Pro13]. In both cases the group $\mathrm{Cl}(Y^{(n)})$ is torsion free. Therefore, the same is true for $\mathrm{Cl}(Y^{(k)})$, $0 \leq k \leq n$. Consider the first step $\psi : Y \dashrightarrow Y^{(1)}$. Assume that ψ is a flip and let C be a component of the flipping locus. Then $-K_Y \cdot C = 5A_Y \cdot C \geq 5/\operatorname{Index}(Y) = 5/6$. On the other hand, by [Mor88, (2.3.2)] we have $-K_Y \cdot C = 1 - \sum_P w_P(0)$, where the sum runs through the set of points lying on C and w_P is a local invariant defined in [Mor88, (2.2.1)]. By definition w_P takes values in $\frac{1}{r}\mathbb{Z}_{>0}$, where r is the index of P. Since $r \in \{2, 3\}$ in our case we obtain $-K_Y \cdot C \leq 2/3$, a contradiction. Therefore, ψ is a divisorial contraction. Let S be the exceptional divisor. Since K_Y is divisible, $\psi(S)$ is not a curve. Thus $Q := \psi(S)$ is a point. Let m be its index. Assume that $m > 1$. Since $-K_{Y^{(1)}} \sim 5\psi_*A_Y$ and $-K_{Y^{(1)}}$ is a (local) generator of the group $\mathrm{Cl}(Y^{(1)}, Q)_{\mathrm{t}} \simeq \mathbb{Z}/m\mathbb{Z}$, the numbers m and 5 must be coprime. Write $mK_Y \sim \psi^*(mK_{Y^{(1)}}) + m\alpha S$, where $\alpha \in \frac{1}{m}\mathbb{Z}_{>0}$ is the discrepancy of S. Since the group $\mathrm{Cl}(Y^{(1)})$ is torsion free, this implies that $m\alpha$ is divisible by 5. In particular, $\alpha > 1/m$. On the other hand, by Theorem 2.3.3 there exists an exceptional over $Q \in Y^{(1)}$ divisor S' whose discrepancy equals $1/m$. Then the discrepancy of S' over Y must be strictly less than $1/m$. Since $\mathbf{B}(Y) = (2^2, 3)$, the only possibility is $m = 2$. In this situation $-K_{Y^{(1)}}$ is nef and big, and does not contract any divisors. Therefore, $Y^{(1)}$ is an almost Fano threefold with terminal singularities of Gorenstein index ≤ 2. The anticanonical model $Y^{(1)}_{\mathrm{can}}$ of $Y^{(1)}$ a Fano threefold whose singularities are also of index ≤ 2. By Theorem 2.2.5 we have $Y^{(1)} \simeq \mathbb{P}(1^3, 2)$ because $\mathrm{g}(Y^{(1)}_{\mathrm{can}}) \geq \mathrm{g}(X') \geq 21$ and $\mathbb{P}(1^3, 2)$ is $\mathbb{Q}$-factorial. But in this case $Y^{(1)}$ has a unique singular point which is of type $\frac{1}{2}(1, 1, 1)$ and its extraction ψ produces a smooth variety, a contradiction. Therefore, $m = 1$, i.e. $Q = \psi(S)$ is a Gorenstein point. Then we can apply the above arguments replacing Y with $Y^{(1)}$ and get a contradiction again. Thus X' is a $\mathbb{Q}$-Fano threefold. Then X' is a hypersurface of degree 4 in $\mathbb{P}(1^2, 2^2, 3)$ by [Pro13]. As in the case 1^o we see that the projection $X' \to \mathbb{P}(1^2, 2^2)$ is $\boldsymbol{\mu}_2$-equivariant and birational. Hence $X = X'/\boldsymbol{\mu}_2$ is rational.

Finally in the case 3^o we see that X' is a hypersurface $X'_4 \subset \mathbb{P}(1^3, 2^2)$ by Theorem 2.2.5. As above, we may assume that the coordinates $x_1, x'_1, x''_1, x_2, x'_2$ are semi-invariants with eigenvalues ± 1 and x_1 is an invariant. Since the fixed point locus if zero-dimensional, the $\boldsymbol{\mu}_2$-action on $\mathbb{P}(1^3, 2^2)$ has weights $(0, 0, 1; 1, 1)$. Then $X = X'/\boldsymbol{\mu}_2$ is rational by the arguments similar to that in the case 1^o. □

4. Sarkisov link

The following construction will be systematically used throughout the paper. From now on we adopt the following notation.

4.1. Let X be a non-Gorenstein $\mathbb{Q}$-Fano threefold of $\mathbb{Q}$-Fano index $q = \mathrm{q}_{\mathbb{Q}}(X) > 1$. Let $\mathscr{M}$ be a linear system on X such that $\mathscr{M} \sim_{\mathbb{Q}} nA_X$ with $n < q$, $\dim \mathscr{M} > 0$, and $\mathscr{M}$ has no fixed components. This $\mathscr{M}$ will be chosen at the beginning and fixed throughout this section. We usually take $\mathscr{M} = |nA_X|$ if $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{W}}(X)$. Let $c := \mathrm{ct}(X, \mathscr{M})$ be the canonical threshold of the pair $(X, \mathscr{M})$. We assume that $c \le 1$ (see Lemma 4.2.3 below). According to [Cor95, Proposition 2.10] (see also [Pro21, Claim 4.5.1]) there exists an extremal blowup $f : \tilde{X} \to X$. that is crepant with respect to $K_X + c\mathscr{M}$. By our construction $\rho(\tilde{X}) = 2$ and $-(K_{\tilde{X}} + c\tilde{\mathscr{M}})$ is nef and big. As in [Ale94], run the log minimal model program on $\tilde{X}$ with respect to $K_{\tilde{X}} + c\tilde{\mathscr{M}}$ (see e.g. [Ale94, 4.2] or [Pro21, 12.2.1]). We obtain the following Sarkisov link:

(4.1.1)
$$\begin{tikzcd} & \tilde{X} \arrow[rr, dashed, "\chi"] \arrow[dl, "f"'] & & \bar{X} \arrow[dr, "\bar{f}"] & \\ X & & & & \hat{X} \end{tikzcd}$$

where χ is an isomorphism in codimension 1, the variety $\bar{X}$ also has only terminal $\mathbb{Q}$-factorial singularities, $\rho(\bar{X}) = 2$, and $\bar{f} : \bar{X} \to \hat{X}$ is an extremal $K_{\bar{X}}$-negative Mori contraction which can be either divisorial or fiber type.

4.1.2. **Remark.** The proof of [Cor95, Proposition 2.10] shows that for any zero-dimensional canonical center P of the pair $(X, c\mathscr{M})$ there exists an extremal blowup $f : \tilde{X} \to X$ as in 4.1 with center P. Thus in general the link (4.1.1) is not determined by the choice of $\mathscr{M}$.

4.2. In what follows, for a divisor (or a linear system) D on X by $\tilde{D}$ and $\bar{D}$ we denote proper transforms of D on $\tilde{X}$ and $\bar{X}$, respectively. By E we denote the f-exceptional divisor. For $1 \le k < q$, let $\mathscr{M}_k$ be a linear system such that $\mathscr{M}_k \sim_{\mathbb{Q}} kA_X$. As in the case $k = n$, we usually take $\mathscr{M}_k = |kA_X|$ if $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{W}}(X)$. By M_k we denote a general member of $\mathscr{M}_k$. We can write

(4.2.1)
$$\begin{aligned} K_{\tilde{X}} &\sim_{\mathbb{Q}} f^*K_X + \alpha E, & \alpha \in \mathbb{Q},\ \alpha > 0, \\ \tilde{\mathscr{M}}_k &\sim_{\mathbb{Q}} f^*\mathscr{M}_k - \beta_k E, & \beta_k \in \mathbb{Q},\ \beta_k \ge 0. \end{aligned}$$

Then by taking proper transforms on $\bar{X}$ we obtain

$$kK_{\bar{X}} + q\bar{\mathscr{M}}_k \sim_{\mathbb{Q}} (k\alpha - q\beta_k)\bar{E}.$$

Moreover, if $kK_X + q\mathscr{M}_k \sim 0$ near $f(E)$, then $k\alpha - q\beta_k$ is an integer and we have linear equivalence

(4.2.2)
$$kK_{\bar{X}} + q\bar{\mathscr{M}}_k \sim (k\alpha - q\beta_k)E.$$

In particular, this holds if $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{W}}(X)$.

4.2.3. **Lemma** ([Pro10, Lemma 4.2])**.** *Let $P \in X$ be a point of index $r > 1$. Assume that in a neighborhood of P we have $\mathscr{M} \sim -mK_X$, where $0 < m < r$. Then $\mathrm{ct}(X, \mathscr{M}) \le 1/m$. Therefore,*

$$\beta_n \ge m\alpha \quad \textit{and} \quad q\beta_n - n\alpha \ge \alpha > 0.$$

4.3. Assume that the contraction $\bar{f}$ is birational. Then $\hat{X}$ is a $\mathbb{Q}$-Fano threefold. In this case, denote by $\bar{F}$ the $\bar{f}$-exceptional divisor, by $\tilde{F} \subset \tilde{X}$ its proper transform, $F := f(\tilde{F})$, and $\hat{q} := \mathrm{q}_{\mathbb{Q}}(\hat{X})$. The divisor $\bar{E}$ is not contracted by $\bar{f}$, i.e. $\bar{E} \ne \bar{F}$ (see e.g. [Pro10, Claim 4.6]). Let $A_{\hat{X}}$ be a fundamental divisor on $\hat{X}$. Write

$$F \sim_{\mathbb{Q}} dA_X, \qquad \hat{E} \sim_{\mathbb{Q}} eA_{\hat{X}}, \qquad \hat{\mathscr{M}}_k \sim_{\mathbb{Q}} s_kA_{\hat{X}},$$

where d, $e \in \mathbb{Z}_{>0}$, $s_k \in \mathbb{Z}_{\ge 0}$.

Note that $s_k = 0$ if and only if $\dim \mathscr{M}_k = 0$ and the unique element $M_k \in \mathscr{M}_k$ coincides with the $\bar{f}$ -exceptional divisor $\bar{F}$.

4.3.1. **Lemma.** *If* $\mathrm{Cl}(X)_{\mathrm{t}} = 0$*, then* $\mathrm{Cl}(\hat{X})_{\mathrm{t}} \simeq \mathbb{Z}/\mathbb{Z}_{d/e}$.

Proof. Follows from obvious isomorphisms

$$\mathbb{Z}/d\mathbb{Z} \simeq \mathrm{Cl}(X)/(F\cdot\mathbb{Z}) \simeq \mathrm{Cl}(\bar{X})/(\bar{F}\cdot\mathbb{Z}\oplus\bar{E}\cdot\mathbb{Z}) \simeq \mathrm{Cl}(\hat{X})/(\hat{E}\cdot\mathbb{Z})$$

and $\mathrm{Cl}(\hat{X})/(\mathrm{Cl}(\hat{X})_{\mathrm{t}}\oplus\hat{E}\cdot\mathbb{Z}) \simeq \mathbb{Z}/e\mathbb{Z}$. □

4.4. Assume that $\bar{f}$ is a fibration. Then we denote by $\bar{F}$ a general geometric fiber. Then $\bar{F}$ is either a smooth rational curve or a del Pezzo surface contained in the smooth part of $\bar{X}$. The image of the restriction map $\mathrm{Cl}(\bar{X}) \to \mathrm{Pic}(\bar{F})$ is isomorphic to $\mathbb{Z}$. Let Ξ be its ample generator. As above, we can write

$$-K_{\bar{X}}|_{\bar{F}} = -K_{\bar{F}} \sim \hat{q}\Xi, \qquad \bar{E}|_{\bar{F}} \sim e\Xi, \qquad \bar{\mathscr{M}}_k|_{\bar{F}} \sim s_k\Xi,$$

where $\hat{q}$, $e \in \mathbb{Z}_{>0}$, and $s_k \in \mathbb{Z}_{\geq 0}$.

4.4.1. **Lemma.** *Assume that* $\dim \hat{X} = 2$*. Then* $\hat{X}$ *is a del Pezzo surface with Du Val singularities of type* A*,* $\uprho(\hat{X}) = 1$*, and* $\mathrm{Cl}(\hat{X})_{\mathrm{t}} \simeq \mathrm{Cl}(\bar{X})_{\mathrm{t}}$*. Furthermore, there is an embedding*

$$\mathrm{Cl}(\hat{X})_{\mathrm{t}} \subset \mathrm{Cl}(X)_{\mathrm{t}}.$$

In particular, if $\mathrm{Cl}(X)$ *is torsion free, then so* $\mathrm{Cl}(\hat{X})$ *is and so* $\hat{X}$ *is one of the four surfaces described in Lemma* 2.4.2*.*

Proof. By [MP08, Theorem 1.2.7] the surface $\hat{X}$ has only Du Val singularities of type $\mathrm{A_n}$. Since $\uprho(\hat{X}) = \uprho(\bar{X}) - 1 = 1$ and $\hat{X}$ is uniruled, $-K_{\hat{X}}$ is ample. Further, since both $\bar{X}$ and $\hat{X}$ have only isolated singularities and $\mathrm{Pic}(\bar{X}/\hat{X}) \simeq \mathbb{Z}$, there is a well-defined injective map

$$\bar{f}^*\colon\ \mathrm{Cl}(\hat{X}) \longrightarrow \mathrm{Cl}(\bar{X}).$$

Hence $\mathrm{Cl}(\hat{X})_{\mathrm{t}} \simeq \mathrm{Cl}(\bar{X})_{\mathrm{t}} \simeq \mathrm{Cl}(\tilde{X})_{\mathrm{t}}$. On the other hand, the push-forward map $f_* : \mathrm{Cl}(\tilde{X}) \to \mathrm{Cl}(X)$ is the quotient by the subgroup $\mathbb{Z}\cdot E$, hence f_* is injective on $\mathrm{Cl}(\tilde{X})_{\mathrm{t}}$. □

Regardless of whether $\bar{f}$ is birational or not, from (4.2.2) we obtain

4.5. **Corollary.** *In the notation of* 4.3 *and* 4.4 *one has*

$$k\hat{q} = qs_k + (q\beta_k - k\alpha)e, \tag{4.5.1}$$

where $q\beta_n - n\alpha > 0$*. If furthermore* $\mathrm{q_{\mathbb{Q}}}(X) = \mathrm{q_W}(X)$*, then* $q\beta_n - n\alpha$ *is a positive integer.*

Also from our construction we obtain the following easy corollary. It shows that most of $\mathbb{Q}$-Fano threefolds of $\mathbb{Q}$-Fano index at least two are not birationally rigid. In the case where X is a weighted hypersurface much stronger result was proved in [ACP21].

4.6. **Corollary.** *Let* X *be a* $\mathbb{Q}$*-Fano threefold with* $\mathrm{q_{\mathbb{Q}}}(X) > 1$*. If* $\mathrm{p}_n(X) \geq 2$ *for some* $n < \mathrm{q_{\mathbb{Q}}}(X)$*, then* X *is not birationally rigid.*

Proof. Assume the converse. Let n be the minimal positive integer such that $\mathrm{p}_n(X) \geq 2$. Thus there is a (complete) linear system $\mathscr{M}$ such that $\dim \mathscr{M} > 0$ and $\mathscr{M} \sim_{\mathbb{Q}} nA_X$. Apply the construction (4.1.1). Since X is birationally rigid, $\hat{X} \simeq X$ and so $\hat{q} = q$. By (4.5.1) we have $n > s_n$. But then $\dim \hat{\mathscr{M}} > 0$ and $\hat{\mathscr{M}} \sim_{\mathbb{Q}} s_nA_{\hat{X}}$. This contradicts our assumption on minimality of n. □

5. Sarkisov link and rationality

5.1. **Lemma.** *In the notation of* 4.1 *assume that* X *is not rational.*

(i) *If the contraction* $\bar{f}$ *is birational, then either* $\hat{q} \le 6$ *or* $\hat{q} = 7$ *and* $s_n \ge 2$.

(ii) *If* $\bar{f}$ *is a fibration, then* $\hat{q} = 1$.

Proof. The assertion (i) follows from Theorem 1.1 and Corollary 3.3.3. To prove (ii) assume that $\hat{q} > 1$. If $\hat{X}$ is a curve, then $\hat{X} \simeq \mathbb{P}^1$ and $\bar{F}$ is a smooth del Pezzo surface with divisible canonical class. Thus $\bar{f}$ is either a generically $\mathbb{P}^2$-bundle or quadric bundle. Then $\bar{f}$ must be locally trivial in Zariski topology, hence $\bar{X}$ is rational in this case. Now assume that $\dim \hat{X} = 2$. Let L be an effective Weil divisor on $\hat{X}$ such that $L \sim A_{\hat{X}}$ and let B be a Weil divisor on $\bar{X}$ whose image is Ξ (see 4.4). Then B is a section of $\bar{f}$ over a Zariski open subset in $\hat{X}$. Since the general fiber is a smooth rational curve and $\hat{X}$ is rational, the variety $\bar{X}$ is rational as well. $\square$

5.2. **Lemma.** *In the notation of* 4.1 *assume that* $\mathrm{q_W}(X) = \mathrm{q}_{\mathbb{Q}}(X)$, $\mathrm{p}_1(X) \ge 2$, $s_1 = 0$, *and* X *is not rational. Then* $\hat{q} = 1$ *and there is an embedding* $\mathrm{Cl}(\hat{X})_{\mathrm{t}} \subset \mathrm{Cl}(X)_{\mathrm{t}}$.

(i) *If* $\mathrm{Cl}(\hat{X})_{\mathrm{t}} = 0$, *then* $\hat{X} \simeq \mathbb{P}^1$, $\mathbb{P}^2$, *or* $\mathbb{P}(1,1,2)$.

(ii) *If* $\mathrm{Cl}(\hat{X})_{\mathrm{t}} \ne 0$, *then* $\hat{X}$ *is a del Pezzo surface of degree* 1 *and* $|\mathrm{Cl}(\hat{X})_{\mathrm{t}}| \ge 3$.

The numbers $\mathrm{p}_i(X)$ *satisfy the following conditions:*

	$\hat{X}$	$\mathrm{p}_1(X)$	$\mathrm{p}_2(X)$	$\mathrm{p}_3(X)$
1^o	$\hat{X} \simeq \mathbb{P}^1$	2	≥ 3	≥ 4
2^o	$\hat{X} \simeq \mathbb{P}^2$	3	≥ 6	≥ 10
3^o	$\hat{X} \simeq \mathbb{P}(1,1,2)$	2	≥ 4	≥ 6
4^o	$\hat{X}$ is a del Pezzo surface of degree 1	2	≥ 4	≥ 7

Furthermore, assume that $\mathrm{Cl}(\hat{X})_{\mathrm{t}} = 0$. *If* $s_2 = 0$ (*resp.,* $s_3 = 0$), *then equalities hold for* $\mathrm{p}_2(X)$ (*resp., for* $\mathrm{p}_3(X)$).

Proof. We have $\hat{q} = 1$ by Lemma 5.1. For short, we consider only the case where $\hat{X}$ is a surface. The case where $\hat{X}$ is a curve is much easier. By Lemma 4.4.1 we have $\mathrm{Cl}(\hat{X})_{\mathrm{t}} \subset \mathrm{Cl}(X)_{\mathrm{t}}$ and $\hat{X}$ is a del Pezzo surface with Du Val singularities of type A and $\rho(\hat{X}) = 1$. The pull-back map $\bar{f}^*$ of Weil divisors is well-defined and injective (see the proof of Lemma 4.4.1). Hence $\bar{\mathscr{M}} = \bar{f}^*|A_{\hat{X}}|$ for a primitive element $A_{\hat{X}} \in \mathrm{Cl}(\hat{X})$ and $\dim |A_{\hat{X}}| = \dim \mathscr{M} = \dim |A_X| \ge 1$. If $K_{\hat{X}}^2 > 6$, then $\hat{X}$ is either $\mathbb{P}^2$ or $\mathbb{P}(1^2, 2)$ (see e.g. [HW81, Theorem 3.4]) and $\mathrm{p}_1(X) = 3$ or 2 in these cases, respectively. Let $K_{\hat{X}}^2 \le 6$. Then $K_{\hat{X}}^2 = 1$ and $A_{\hat{X}} \sim -K_{\hat{X}}$ by Lemma 2.4.1. Hence $\mathrm{p}_1(X) = \dim |\bar{\mathscr{M}}| + 1 = 2$. Finally, $|\mathrm{Cl}(\hat{X})_{\mathrm{t}}| \ge 3$ by the classification [MZ88].

Note that $k\bar{\mathscr{M}} \sim \bar{f}^*(kA_{\hat{X}})$ for any k, hence the linear system $f_*\chi_*^{-1}|\bar{f}^*(kA_{\hat{X}})|$ is contained in $|kA_X| = \mathscr{M}_k$. This implies that $\dim |kA_{\hat{X}}| \le \dim |kA_X|$. Then the inequalities in the table follow from Lemma 2.4.2 and the Riemann-Roch for $-kK_{\hat{X}}$ in the case $K_{\hat{X}}^2 = 1$. Finally, if $s_k = 0$ and $\mathrm{Cl}(\hat{X})_{\mathrm{t}} = 0$, then the linear system $\bar{\mathscr{M}}_k$ is $\bar{f}$-vertical, that is, $\bar{\mathscr{M}}_k = \bar{f}^*(kA_{\hat{X}})$ and so $\dim |kA_{\hat{X}}| = \dim |kA_X|$. Hence, the inequalities in the table are in fact equalities in this case. $\square$

6. Case $\mathrm{p}_1(X) \ge 2$.

6.1. **Set-up.** Let X be a non-rational $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q_W}(X) > 1$ and $\mathrm{p}_1(X) \ge 2$. We assume that X is has at least one non-Gorenstein point. This holds automatically if

$\mathrm{q}_{\mathbb{Q}}(X) \geq 3$ because X is not rational. The linear system $|A_X|$ has no fixed components. Apply the construction (4.1.1) with $n = 1$, i.e. $\mathscr{M} = |A_X|$. The relation (4.5.1) for $k = 1$ has the form

$$\hat{q} = qs_1 + (q\beta_1 - \alpha)e, \tag{6.1.1}$$

where $q\beta_1 - \alpha$ is a positive integer by Corollary 4.5. Taking Lemma 5.1 into account we obtain two possibilities:

6.1.1. *Case* $s_1 > 0$. Then $\bar{f}$ is birational and $\hat{q} \geq q + 1$.

6.1.2. *Case* $s_1 = 0$. Then $\bar{f}$ is a fibration and $\hat{q} = e = q\beta_1 - \alpha = 1$.

6.2. **Proposition.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) \geq 5$. If $\mathrm{p}_1(X) \geq 2$, then X is rational.*

Proof. By Corollary 3.3.3 we may assume that $\mathrm{q}_{\mathbb{Q}}(X) = 5$. By Proposition 3.1 the group $\mathrm{Cl}(X)$ is torsion free. Apply the construction (4.1.1) with $\mathscr{M} = |A_X|$ (see 6.1). Assume that X is not rational.

If $s_1 > 0$, then $\hat{q} \geq 6$ (see 6.1.1). By Theorem 1.1 and Corollary 3.3.3 we have $s_1 \geq 2$. Hence $\hat{q} \geq 11$ and X is rational by Theorem 1.1, a contradiction.

Therefore, $s_1 = 0$, $\bar{f}$ is fibration, $\hat{q} = e = 1$, and $5\beta_1 = \alpha + 1$ (see 6.1.2). Let $P \in X$ be any point of index $r > 1$. Since $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{W}}(X)$, the numbers r and 5 are coprime. Take $m \in \mathbb{Z}_{>0}$ so that $5m \equiv 1 \mod r$. Then $m(-K_X) \sim \mathscr{M}$ near P and so $\beta_1 \geq m\alpha$ by Lemma 4.2.3. We obtain $\alpha + 1 = 5\beta_1 \geq 5m\alpha$ and $\alpha \leq \frac{1}{5m-1}$. Since X is not rational, we can take P so that $r \notin \{2, 4\}$ by Theorem 2.2.5. Then $5 \not\equiv 1 \mod r$, hence $m \geq 2$ and so $\alpha \leq 1/9$. This implies that $f(E)$ is a point of index ≥ 9. In this case computer search (see Sect. A) gives the only possibility [B$^+$, # 41446]:

$$\mathbf{B}(X) = (2, 9), \quad A_X^3 = 5/18, \quad \mathrm{p}_1(X) = 2, \quad \mathrm{p}_2(X) = 4, \quad \mathrm{p}_3(X) = 7.$$

Then $\alpha = \beta_1 = 1/9$ because $\mathbf{B}(X) = (2, 9)$. Recall that $e = 1$. Then one can show that (4.5.1) implies $s_2 = s_3 = 0$. This contradicts Lemma 5.2. □

6.3. **Proposition.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) \geq 4$. If $\mathrm{p}_1(X) \geq 2$, then X is rational.*

Proof. By Proposition 6.2 we may assume that $\mathrm{q}_{\mathbb{Q}}(X) = 4$. By Proposition 3.3 the group $\mathrm{Cl}(X)$ is torsion free. Apply the construction (4.1.1) with $\mathscr{M} = |A_X|$. Assume that X is not rational.

If $s_1 > 0$, then $\hat{q} \geq 5$ (see 6.1.1), hence $s_1 \geq 2$ by Theorem 1.1, Corollary 3.3.3, and Proposition 6.2. In this case (6.1.1) implies $\hat{q} \geq 9$, hence X is rational, a contradiction.

Therefore, $s_1 = 0$, $\bar{f}$ is a fibration, $\hat{q} = 1$, and $e = 4\beta_1 - \alpha = 1$. Let $P \in X$ be a point of maximal index r. Recall that r must be odd because $\mathrm{q}_{\mathrm{W}}(X) = 4$. Take m so that $4m \equiv 1 \mod r$ and $0 < m < r$, that is,

$$m = \begin{cases} (3r+1)/4 & \text{if } r \equiv 1 \mod 4, \\ (r+1)/4 & \text{if } r \equiv -1 \mod 4. \end{cases}$$

In both cases, $4m - 1 \geq r$. By Lemma 4.2.3 we have $\beta_1 \geq m\alpha$, hence

$$\alpha + 1 = 4\beta_1 \geq 4m\alpha, \qquad \frac{1}{4m-1} \geq \alpha \geq \frac{1}{r}, \qquad r \geq 4m - 1.$$

On the other hand, $4m - 1 \geq r$. Therefore, $r = 4m - 1$ and $\alpha = 1/r$. We may assume that $f(E) = P$. If X has a point P' of index r' with $r' \equiv 1 \mod 4$, then similar computations show that $1/r + 1 = \alpha + 1 \geq 4m'\alpha = (3r' + 1)/r$ and so $3r' \leq r$.

Thus the variety X and the point $f(E)$ satisfy the following properties:

(1) $\mathrm{p}_1(X) \geq 2$ and $\mathrm{g}(X) \leq 21$ [Pro13, Theorem 1.2];
(2) $f(E)$ is a point whose index r is maximal, $r \equiv -1 \mod 4$ and $r \geq 7$;
(3) if $P \in X$ is a point of index r' with $r' \equiv 1 \mod 4$, then $3r' \leq r$.

Applying computer search (see Sect. A) under these conditions we obtain the following possibilities:

	A_X^3	$\mathbf{B}(X)$	$\mathrm{g}(X)$	$\mathrm{p}_1(X)$	$\mathrm{p}_2(X)$	$\mathrm{p}_3(X)$	[B$^+$]
1^o	$3/7$	(7)	14	2	5	9	# 41372
2^o	$8/21$	$(3,7)$	12	2	4	8	# 41367
3^o	$3/7$	(7^2)	13	2	4	8	# 41370
4^o	$5/9$	(9)	18	2	6	11	# 41381
5^o	$13/33$	$(3,11)$	12	2	4	8	# 41368

The relation (4.5.1) for $k = 2$ and $k = 3$ has the form

$$2 = 2\hat{q} = 4s_2 + 4\beta_2 - 2\alpha, \qquad 3 = 3\hat{q} = 4s_3 + 4\beta_3 - 3\alpha.$$

Since X has no points of index 2, we have $\beta_2 > 0$ and so $\beta_2 \geq 1/r = \alpha$. We obtain $s_2 = 0$. Then the cases 1^o and 4^o $\mathbf{B}(X) = (7)$ and (9) are impossible by Lemma 5.2 and our assumptions. In the remaining cases 2^o, 3^o and 5^o again by Lemma 5.2 we have $s_3 \neq 0$, hence $\beta_3 = 0$ and $\alpha = 1/3$. Thus $f(E)$ is a point of index 3. This contradicts the property (2). $\square$

6.4. **Proposition.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) = 3$.*

(i) *If $\mathrm{p}_1(X) \geq 3$, then X is rational.*
(ii) *If $\mathrm{p}_1(X) = 2$ and X is not rational, then $\mathrm{Cl}(X)$ is torsion free and one of the following holds:*
 (a) $A_X^3 = 1/2$, $\mathbf{B}(X) = (2,2,2)$, $\mathrm{g}(X) = 7$, [B$^+$, # 41198];
 (b) $A_X^3 = 2/5$, $\mathbf{B}(X) = (5)$, $\mathrm{g}(X) = 6$, [B$^+$, # 41195];
 (c) $A_X^3 = 6/11$, $\mathbf{B}(X) = (11)$, $\mathrm{g}(X) = 7$, [B$^+$, # 41196];
 (d) $A_X^3 = 10/17$, $\mathbf{B}(X) = (17)$, $\mathrm{g}(X) = 7$, [B$^+$, # 41197].
 In these cases X is unirational and has a conic bundle structure.

6.4.1. **Remark.** By Theorem 2.2.5 a variety of type (ii)(a) is a hypersurface $X_6 \subset \mathbb{P}(1^2, 2^2, 3)$. It is known that a very general variety in the family is not rational [Oka19]. We do not know if the varieties satisfying (ii)(b), (ii)(c), and (ii)(d) are rational or not. Moreover, we do not know if the varieties satisfying (ii)(c) and (ii)(d) really exist. The general complete intersection $X_{6,6} \subset \mathbb{P}(1^2, 2, 3^2, 5)$ satisfies (ii)(b) but still we do not know if this is the only example.

Proof. Assume that X is not rational and $\mathrm{p}_1(X) \geq 2$. By Proposition 3.3 we have $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{W}}(X)$ and $\dim |A_X| \geq 1$ by Proposition 3.5. Apply the construction (4.1.1) with $\mathscr{M} = |A_X|$ (see 6.1).

If $s_1 > 0$, then $\hat{q} \geq 4$ (see 6.1.1), hence $s_1 \geq 2$ by Proposition 6.3. Then $\hat{q} = 7$ and by Corollary 3.3.3 we have $s_1 \geq 3$ and so $\hat{q} > 7$, a contradiction.

Therefore, $s_1 = 0$, $\hat{q} = e = 1$, and $3\beta_1 = \alpha + 1$, $\bar{f}$ is a fibration, and $\bar{\mathscr{M}} = \bar{f}^*|A_{\hat{X}}|$. Since $\beta_1 \geq \alpha$ by Lemma 4.2.3, we have $\alpha \leq 1/2$. In particular, this implies that $f(E)$ is non-Gorenstein point.

Assume that $\mathrm{Cl}(X)_{\mathrm{t}} \neq 0$. Then $|\mathrm{Cl}(X)_{\mathrm{t}}| \geq 3$ by Lemma 5.2. Since $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{W}}(X)$, we have $|\mathrm{Cl}(X)_{\mathrm{t}}| \neq 3$ by Theorem 2.2.4(ii). Then X is rational by Proposition 3.4, a contradiction. Thus we may assume that $\mathrm{Cl}(X)$ is torsion free.

Let P be a point of maximal index r. Then r is not divisible by 3 by Theorem 2.2.4(ii).

Consider the case where $r = 2$. Then X is as in (ii)(a) by Theorem 2.2.5, 9^o and we may assume that $f(E) = P$. Moreover, $\alpha = \beta_1 = 1/2$ and $s_2 = 1$ by (4.5.1). Note that in this case the linear system $\mathscr{M}_2 = |2A_X|$ is base point free and $\dim \mathscr{M}_2 = 4$. Let $\mathscr{M}'_2 \subset \mathscr{M}_2$ be the subsystem consisting of all divisors passing through P. Then $\dim \mathscr{M}'_2 = 3$. Similar to (4.5.1) we have

$$2 = k\hat{q} = qs'_k + (q\beta'_k - k\alpha)e = 3s'_2 + 3\beta'_2 - 1$$

where β'_2 is a positive integer. Hence, $s'_2 = 0$, that is, $\bar{\mathscr{M}}'_2 \subset \bar{f}^*|2A_{\hat{X}}|$. Thus $\dim |A_{\hat{X}}| = 1$ and $\dim |2A_{\hat{X}}| \geq 3$. Then $\hat{X} \not\simeq \mathbb{P}^1$, hence $\hat{X}$ is a surface and $\bar{f}$ is a $\mathbb{Q}$-conic bundle. A general member $S \in \mathscr{M}_2$ is a smooth del Pezzo surface of degree 1 and its proper transform $\bar{S} \subset \bar{X}$ is a rational multisection of $\bar{f}$ (because $s_2 = 1$). This implies that $\bar{X}$ is unirational. Unirationality of X in this case is also proved in [CF93].

From now on we assume that $r \geq 4$. Put

$$m =: \begin{cases} (2r+1)/3 & \text{if } r \equiv 1 \mod 3, \\ (r+1)/3 & \text{if } r \equiv -1 \mod 3. \end{cases}$$

Note that in both cases $r \leq 3m - 1$. Then $\beta_1 \geq m\alpha$ by Lemma 4.2.3. Hence,

$$\alpha + 1 = 3\beta_1 \geq 3m\alpha, \qquad \frac{1}{3m-1} \geq \alpha \geq \frac{1}{r}, \qquad r \geq 3m - 1.$$

Since $r \leq 3m - 1$, we obtain $r = 3m - 1$ and $\alpha = 1/r$. Thus we may assume that $f(E) = P$. If X has a point P' of index r' with $r' \equiv 1 \mod 3$, then similar computations show that $1/r + 1 = \alpha + 1 \geq 3m'\alpha = (2r' + 1)/r$ and so $2r' \leq r$. Thus X satisfies the following properties:

(1) $\mathrm{p}_1(X) \geq 2$ and $\mathrm{g}(X) \leq 20$ [Pro13, Theorem 1.2];
(2) if r is the maximal index r of points on X, then $r \equiv -1 \mod 3$ and $r \geq 5$;
(3) if X has a point of index r' with $r' \equiv 1 \mod 3$, then $2r' \leq r$.

Then computer search (see Sect. A) under these conditions gives the following possibilities:

	A_X^3	$\mathbf{B}(X)$	$\mathrm{g}(X)$	$\mathrm{p}_1(X)$	$\mathrm{p}_2(X)$		A_X^3	$\mathbf{B}(X)$	$\mathrm{g}(X)$	$\mathrm{p}_1(X)$	$\mathrm{p}_2(X)$
1^o	2/5	(5)	6	2	4	2^o	11/10	(2, 5)	15	3	8
3^o	3/5	$(2^2, 5)$	8	2	5	4^o	6/5	(5^2)	16	3	8
5^o	7/10	$(2, 5^2)$	9	2	5	6^o	9/8	(2, 8)	15	3	8
7^o	4/5	(5^3)	10	2	5	8^o	7/8	$(2^2, 4, 8)$	11	2	6
9^o	5/8	$(2^2, 8)$	8	2	5	10^o	29/40	(2, 5, 8)	9	2	5
11^o	49/40	(5, 8)	16	3	8	12^o	25/22	(2, 11)	15	3	8
13^o	6/11	(11)	7	2	4	14^o	35/44	(4, 11)	10	2	5
15^o	7/11	$(2^2, 11)$	8	2	5	16^o	11/14	(14)	10	2	5
17^o	81/110	(2, 5, 11)	9	2	5	18^o	10/17	(17)	7	2	4
19^o	9/14	$(2^2, 14)$	8	2	5						

Note that $|2A_X|$ has no fixed components in our case. Then one can show that (4.5.1) implies $s_2 = 0$. Hence by Lemma 5.2 the only cases 1^o, 13^o, and 18^o are possible. We get (ii)(b), (ii)(c), and (ii)(d). Note that in all cases X has only cyclic quotient singularities. By Theorem 2.3.2 the f-exceptional divisor E is toric, in particular, rational. Its proper transform $\bar{E} \subset \bar{X}$ is a multisection of $\bar{f}$. Hence $\bar{X}$ is unirational. This finishes the proof of Proposition 6.4. $\square$

7. $\mathbb{Q}$-Fano threefolds with $\mathrm{q}_{\mathbb{Q}}(X) \geq 5$

7.1. **Proposition.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) \geq 6$. If $\mathrm{p}_2(X) \geq 2$, then X is rational.*

Proof. The group $\mathrm{Cl}(X)$ is torsion free by Proposition 3.1 and by Corollary 3.3.3 we may assume that $\mathrm{q}_{\mathbb{Q}}(X) = 6$. Assume that X is not rational. By Theorem 2.2.5 the global Gorenstein index of X is at least 6. Apply the computer search (see Sect. A) or [B$^+$] under the assumption $\mathrm{p}_2(X) \geq 2$ and $\mathrm{p}_1(X) \leq 0$. We obtain the only possibility [B$^+$, # 41466]:

$$A_X^3 = 3/35, \quad \mathbf{B}(X) = (5, 7), \quad \mathrm{g}(X) = 9, \quad \mathrm{h}_X(t) = 1 + t + 2t^2 + 3t^3 + \cdots .$$

Since $\dim |A_X| = 0$, the linear system $|2A_X|$ has no fixed components. Hence we can apply the construction (4.1.1) with $\mathscr{M} = |2A_X|$. In a neighborhood of the point of index 7 we have $\mathscr{M} \sim 5(-K_X)$ and so $\beta_2 \geq 5\alpha$ by Lemma 4.2.3. The relation (4.5.1) for $k = 2$ has the form

$$\hat{q} = 3s_2 + (3\beta_2 - \alpha)e \geq 3s_2 + 14\alpha e.$$

Since $\alpha \geq 1/7$, we see that $\hat{q} \geq 2$. Then the contraction $\bar{f}$ is birational by Lemma 5.1. Since $\dim \mathscr{M}_2 > 0$, we have $s_2 > 0$ and so $\hat{q} \geq 5$. Then $s_2 \geq 2$ by Proposition 6.3. Hence $\hat{q} \geq 8$ and X is rational by Theorem 1.1, a contradiction. □

7.2. **Proposition.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) = 6$. If $\mathrm{p}_3(X) \geq 2$, then X is rational.*

Proof. Assume that X is not rational. Assume also that $\mathrm{p}_2(X) \leq 1$ and $\mathrm{p}_3(X) \geq 2$. Applying computer search (see Sect. A or [B$^+$]) we obtain the only possibility [B$^+$, # 41462]:

$$A_X^3 = 2/35, \quad \mathbf{B}(X) = (5, 7^2), \quad \mathrm{g}(X) = 5, \quad \mathrm{h}_X(t) = 1 + t^2 + 2t^3 + 3t^4 + \cdots .$$

Apply the construction (4.1.1) with $\mathscr{M} = |3A_X|$. In a neighborhood of the point of index 7 we have $\mathscr{M} \sim 4(-K_X)$ and so $\beta_3 \geq 4\alpha$ by Lemma 4.2.3. The relation (4.5.1) for $k = 3$ has the form

$$\hat{q} = 2s_3 + (2\beta_3 - \alpha)e \geq 2s_3 + 7\alpha e.$$

We claim that $\alpha = 1/7$ and $e = 1$. Indeed, otherwise $\hat{q} \geq 7\alpha e > 1$. Hence $\bar{f}$ is birational by Lemma 5.1. In this case, $s_3 > 0$ and so $\hat{q} \geq 4$. Then $s_3 \geq 2$ by Proposition 6.3 and so $\hat{q} \geq 6$. From Proposition 7.1 we obtain $s_3 \geq 3$ and so $\hat{q} \geq 8$. This contradicts Theorem 1.1.

Thus $\alpha = 1/7$ and $e = 1$. Then we consider (4.5.1) for $k = 2$. Since $f(E)$ is a point of index 7, the number $7\beta_2$ is an integer and $7\hat{q} = 3(7s_2 + 7\beta_2) - 1$, hence $\hat{q} \equiv -1 \mod 3$. In particular, $\hat{q} \neq 1$, hence $\bar{f}$ is birational and s_2, $s_3 > 0$. Then the only possibility is $\hat{q} = 5$ and $s_3 = 2$ by Proposition 6.3. Hence $\mathrm{p}_2(\hat{X}) \geq 2$. Since $e = 1$ and $|A_X| = \varnothing$, we have $\mathrm{Cl}(X)_{\mathrm{t}} \neq 0$ by Lemma 4.3.1. Then $\hat{X}$ is rational by Proposition 3.5. □

7.3. **Proposition.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) = 7$. If $\mathrm{p}_3(X) \geq 2$, then X is rational.*

Proof. Assume that X is not rational. By Proposition 3.2 we have $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{W}}(X)$ and $\mathrm{p}_2(X) \leq 1$ by Corollary 3.3.3. Computer search (see Sect. A or [B$^+$]) produces four numerical possibilities which will be considered below. We will use the construction (4.1.1) with $\mathscr{M} = |3A_X|$ or $\mathscr{M} = |4A_X|$. The relation (4.5.1) for $k = 3$ and 4 has the form

$$3\hat{q} = 7s_3 + (7\beta_3 - 3\alpha)e, \tag{7.3.1}$$

$$4\hat{q} = 7s_4 + (7\beta_4 - 4\alpha)e. \tag{7.3.2}$$

Case $A_X^3 = 1/24$**,** [B$^+$, # 41477]**.** Then $\mathbf{B}(X) = (2^2, 3, 4, 8)$ and $\mathrm{h}_X(t) = 1 + t^2 + 2t^3 + 3t^4 + \cdots$. Thus $\dim |3A_X| = 1$. Note that in this case X can have a 2-torsion in $\mathrm{Cl}(X)$ (see Proposition 3.1). Apply the construction (4.1.1) with $\mathscr{M} = |3A_X|$. In a neighborhood of the point of index 8 we have $\mathscr{M} \sim 5(-K_X)$ and so $\beta_3 \ge 5\alpha$ by Lemma 4.2.3. By (7.3.1)

$$3\hat{q} = 7s_3 + (7\beta_3 - 3\alpha)e \ge 7s_3 + 32\alpha e \ge 7s_3 + 4e.$$

Hence, $\hat{q} \ge 2$ and $\bar{f}$ is birational. Then $s_3 > 0$ and $\hat{q} \ge 4$. By Proposition 6.3 we have $s_3 \ge 2$ and $\hat{q} \ge 6$. By Proposition 7.1 we have $s_3 \ge 3$ and $\hat{q} \ge 9$. This contradicts Theorem 1.1.

Case $A_X^3 = 1/18$**,** [B$^+$, # 41480]**.** Then $\mathbf{B}(X) = (3, 6, 9)$, the group $\mathrm{Cl}(X)$ is torsion free, and and $\mathrm{h}_X(t) = 1 + t + t^2 + 2t^3 + 3t^4 + \cdots$. Thus $\dim |4A_X| = 2$. Apply the construction (4.1.1) with $\mathscr{M} = |4A_X|$. In a neighborhood of the point of index 9 we have $\mathscr{M} \sim 7(-K_X)$ and so $\beta_4 \ge 7\alpha$ by Lemma 4.2.3. From (7.3.2) we have

$$4\hat{q} = 7s_4 + (7\beta_4 - 4\alpha)e \ge 7s_4 + 45\alpha e \ge 7s_4 + 5e.$$

Hence $\hat{q} \ge 2$, $\bar{f}$ is birational, $s_4 \ge 1$, and $\hat{q} \ge 3$. Then $\alpha < 1$, so $f(E)$ is a point of index $r = 3$, 6 or 9 and $\alpha = 1/r$ (see Theorem 2.3.2). The relation (4.5.1) for $k = 1$ has the form $\hat{q} = 7s_1 + (7\beta_1 - \alpha)e$, where $\beta_1 \ge \frac{1}{4}\beta_4 \ge \frac{7}{4}\alpha$ because $4M_1 \in \mathscr{M}$. This gives us $s_1 = 0$, $e = 1$ by Lemma 4.3.1, and $\hat{q} = 7\beta_1 - \alpha$. If $\hat{q} \ge 6$, then $s_4 \ge 3$ by Proposition 7.1, and $\hat{q} > 7$, a contradiction.

Thus $3 \le \hat{q} \le 5$. If $\hat{q} = 3$, then $s_4 = e = 1$ and $\alpha = 1/9$. Thus $\hat{\mathscr{M}}_4 \subset |A_{\hat{X}}|$. We obtain $\dim |A_{\hat{X}}| \ge 2$. This contradicts Proposition 6.4. Therefore, $\hat{q} \ge 4$, then $s_3 \ge 2$ by Proposition 6.3, hence $\hat{q} = 5$ by (7.3.1). Then $\beta_1 = (5 + \alpha)/7 = (5r + 1)/7r$, so $5r + 1 \equiv 0 \mod 7$. This is contradicts $r \in \{3, 6, 9\}$.

Case $A_X^3 = 1/33$**,** [B$^+$, # 41476]**.** Then $\mathbf{B}(X) = (2^2, 3, 11)$, the group $\mathrm{Cl}(X)$ is torsion free, and $\mathrm{h}_X(t) = 1 + t^2 + 2t^3 + 2t^4 + \cdots$. Apply the construction (4.1.1) with $\mathscr{M} = |4A_X|$. In a neighborhood of the point of index 11 we have $\mathscr{M} \sim 10(-K_X)$ and so $\beta_4 \ge 10\alpha$ by Lemma 4.2.3. By (7.3.2)

$$4\hat{q} = 7s_4 + (7\beta_4 - 4\alpha)e \ge 7s_4 + 66\alpha e \ge 7s_4 + 6e.$$

Hence $\hat{q} \ge 2$, $\bar{f}$ is birational, and $s_4 \ge 1$. Then $\hat{q} \ge 5$, $s_4 \ge 2$ by Proposition 6.3, and $\alpha \le 7/33$, so $f(E)$ is a point of index 11 and $\alpha = 1/11$ (see Theorem 2.3.2). If $\hat{q} \ge 6$, then s_3, $s_4 \ge 3$ by Proposition 7.1. Since $\beta_3 \ge \alpha = 1/11$, from (7.3.1) we obtain $\hat{q} > 7$, a contradiction. Therefore, $\hat{q} = 5$, $s_3 = s_4 = 2$, and $e = 1$. Hence $\mathrm{p}_1(\hat{X}) \ge 1$ and $\mathrm{p}_2(\hat{X}) \ge 2$. On the other hand, $\mathrm{Cl}(\hat{X})_{\mathrm{t}} \ne 0$ by Lemma 4.3.1 because $|A_X| = \varnothing$. Then by Proposition 3.5 the variety $\hat{X}$ is rational.

Case $A_X^3 = 1/30$**,** [B$^+$, # 41479]**.** Then $\mathbf{B}(X) = (2, 6, 10)$ and

$$\mathrm{h}_X(t) = 1 + t + t^2 + t^3 + 2t^4 + 3t^5 + \cdots.$$

In particular, $\dim |3A_X| = 0$ and $\dim |4A_X| = 1$. By our assumption $\mathrm{p}_3(X) \ge 2$, hence X has to have a 2-torsion $T \in \mathrm{Cl}(X)$ (see Proposition 3.1). Hence X is of type 3.12^o. Apply the construction (4.1.1) with $\mathscr{M} = |4A_X|$. In a neighborhood of the point of index 6 we have $\mathscr{M} \sim 4(-K_X)$ and so $\beta_4 \ge 4\alpha$ by Lemma 4.2.3. By (7.3.2)

$$4\hat{q} = 7s_4 + (7\beta_4 - 4\alpha)e \ge 7s_4 + 24\alpha e.$$

If $\alpha = 1$, then $\hat{q} \ge 6$ and so $s_4 \ge 2$ by Proposition 6.3. But in this case $\hat{q} > 7$, a contradiction. Thus $\alpha < 1$ and so $f(E)$ is a cyclic quotient singularity of index $r = 2$, 6 or 10. In particular,

$\alpha = 1/r$ by Theorem 2.3.2. The relation (4.5.1) for $k = 1$ has the form

$$\hat{q} = 7s_1 + (7\beta_1 - \alpha)e.$$

Here $\beta_1 \ge \alpha$, hence $\hat{q} = 1$ because $\hat{q} < 8$ by Theorem 1.1. Then $s_1 = s_4 = 0$, $r = 6$, $\alpha = 1/6$, and $e = 1$. This means that $\bar{f}$ is a fibration, $\bar{M}_1 \sim \bar{f}^* A_{\hat{X}}$ and $\bar{\mathscr{M}}_4 = \bar{f}^*|4A_{\hat{X}}|$ for a primitive element $A_{\hat{X}} \in \mathrm{Cl}(\hat{X})$. Hence $\dim|A_{\hat{X}}| = 0$ and $\dim|4A_{\hat{X}}| = 1$. This is impossible if $\hat{X} \simeq \mathbb{P}^1$. Therefore, $\hat{X}$ is a Du Val del Pezzo surface. Thus $A_{\hat{X}}$ is a line on $\hat{S}$ by Lemma 2.4.1(vii). Since $\dim|2A_{\hat{X}}| = \dim|3A_{\hat{X}}| = 0$ we see that $\mathrm{Cl}(\hat{X})_{\mathrm{t}} \ne 0$ by Lemma 2.4.2. Therefore, $\hat{X}$ contains a line $\hat{L}$ other than $A_{\hat{X}}$ (see Lemma 2.4.1(viii)). Let $\bar{D} := \bar{f}^*\hat{L}$. Then $\bar{D}$ is an irreducible effective divisor and $\bar{D} \ne \bar{M}_1$. Hence $\tilde{D} := \chi_*^{-1}\bar{D} \ne \tilde{M}_1$ and $D := f_*\tilde{D} \ne M_1$. So, D is an effective divisor on X such that $D \not\sim A_X$ but $D \sim_{\mathbb{Q}} A_X$. This contradicts Proposition 3.12^o. □

7.3.3. **Corollary.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) \ge 6$. If either $\mathrm{Cl}(X)_{\mathrm{t}} \ne 0$ or $\mathrm{p}_3(X) \ge 2$, then X is rational.*

7.4. **Proposition.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) = 5$ and $\mathrm{p}_2(X) \ge 2$. Assume that X is not rational. Then X belongs to the following class:*

(*) $\mathbf{B}(X) = (2^2, 3, 4)$, $A_X^3 = 1/12$, $\mathrm{Cl}(X)_{\mathrm{t}} = 0$, $\mathrm{p}_1(X) = 1$, $\mathrm{p}_2(X) = 2$, [B$^+$, # 41422].

Moreover, X is birational to a conic bundle, and if the point of index 4 is a cyclic quotient singularity, then X is unirational.

7.4.1. **Remark.** A general hypersurface $X_{10} \subset \mathbb{P}(1, 2, 3, 4, 5)$ belongs to the class (*) and according to [Oka19] a very general such a hypersurface is not rational. However we do not know that the family of such hypersurfaces exhaust (*).

Proof. We have $\mathrm{q}_{\mathbb{Q}}(X) = \mathrm{q}_{\mathrm{W}}(X) = 5$ by Proposition 3.2.

7.4.2. **Claim.** $|A_X| \ne \varnothing$ *and* $\dim|2A_X| \ge 1$.

Proof. If $\mathrm{Cl}(X)_{\mathrm{t}} \ne 0$, the assertions follow from Proposition 3.1. Thus we may assume that $\mathrm{Cl}(X) \simeq \mathbb{Z} \cdot A_X$. Then $\dim|2A_X| \ge 1$ by our assumption $\mathrm{p}_2(X) \ge 2$. Computer search shows that there are 35 Hilbert series of $\mathbb{Q}$-Fano threefolds with $\mathrm{q}_{\mathrm{W}}(X) = 5$ and $\mathrm{Cl}(X)_{\mathrm{t}} = 0$, and in all cases $\dim|2A_X| \ge 1$ implies $|A_X| \ne \varnothing$ (see also [B$^+$]). □

7.4.3. **Claim.** *We have $s_1 = 0$, $e = 1$, and one of the following holds:*

(i) $s_2 = 0$, $\hat{q} = 1$, $5\beta_1 = \alpha + 2s_2 + 1$, $5\beta_2 = 2\alpha + 2$, *$\bar{f}$ is a fibration, or*

(ii) $s_2 = 1$, $\hat{q} = 3$, $5\beta_1 = \alpha + 3$, $5\beta_2 = 2\alpha + 1$, *and $\bar{f}$ is birational.*

Moreover, in the case (i) *we have $s_3 = 0$ if $\beta_3 > 0$ and $s_4 = 0$ if $\beta_4 > 0$, and in the case* (ii) *we have $s_3 = 1$ if $\beta_3 > 0$.*

Proof. The relation (4.5.1) for $k = 1$ and 2 has the form

$$\begin{aligned} \hat{q} &= 5s_1 + (5\beta_1 - \alpha)e, \\ 2\hat{q} &= 5s_2 + (5\beta_2 - 2\alpha)e. \end{aligned}$$

Note that $5\beta_1 \ge \frac{5}{2}\beta_2 > \alpha$. Hence $\hat{q} > 5s_1$. If $s_1 > 0$, then $\hat{q} \ge 6$ and $s_2 \ge 3$ by Proposition 7.1. But then $\hat{q} > 15$, a contradiction. Therefore, $s_1 = 0$ and $\hat{q} = (5\beta_1 - \alpha)e$.

Consider the case $s_2 = 0$. Then $\hat{q} = 1$. Since $5\beta_1 - \alpha$ is an integer, we have $5\beta_1 = \alpha + 1$, $e = 1$, and so $5\beta_2 = 2\alpha + 2$.

Consider the case $s_2 > 0$, then $\hat{q} \ge 3$. If moreover $\hat{q} > 3$, then $s_2 \ge 2$ by Proposition 6.3. In this case $\hat{q} > 5$ and $s_2 \ge 3$ by Proposition 7.1. But then $\hat{q} > 7$, a contradiction. Thus $\hat{q} = 3$, then $s_2 = 5\beta_2 - 2\alpha = e = 1$ and $5\beta_1 = \alpha + 3$.

The last statement follows from (4.5.1). □

Proof of Proposition 7.4 (*continued*). Thus we have

$$\hat{q} = 2s_2 + 1, \qquad 5\beta_1 = \alpha + 2s_2 + 1, \qquad 5\beta_2 = 2\alpha + 2 - s_2.$$

Let $P \in X$ be a point of index $r > 1$. Take m so that $5m \equiv 2 \mod r$ and $0 < m < r$. Then $\beta_2 \geq m\alpha$ by Lemma 4.2.3. This gives us $2 \geq 2 - s_2 \geq 3m\alpha$. By Theorem 2.2.5 we may assume that $r > 2$ and then $m \geq 1$ and $\alpha \leq 2/3$. Hence $f(E)$ is a non-Gorenstein point. Now take $P = f(E)$. Then $b_i := \beta_i r$ and $a := \alpha r$ are integers such that

$$5b_1 = a + (2s_2 + 1)r, \qquad 5b_2 = 2a + (2 - s_2)r \geq 5ma.$$

Since $r \leq 24$ by (A.1.1), we have $(5m-2)a \leq 48$. Now it is easy to enumerate all the possibilities for the point $f(E)$ and numbers β_i, s_2, and m:

r	s_2	$\hat{q}$	α	β_1	β_2	m	$\operatorname{ct}(X, \mathscr{M})$
3	1	3	1/3	2/3	1/3	1	1
3	0	1	2/3	1/3	2/3	1	1
4	0	1	1/4	1/4	1/2	2	1/2
8	1	3	1/8	5/8	1/4	2	1/2
8	0	1	1/4	1/4	1/2	2	1/2
9	0	1	1/9	2/9	4/9	4	1/4

Let $P' \in X$ be a point P' of index $r' > 1$ and let m' is an integer such that $5m' \equiv 2 \mod r'$ and $0 < m' < r'$. Then $\operatorname{ct}(X, \mathscr{M}) \leq 1/m'$ by Lemma 4.2.3 and so $m' \leq m \leq 4$. This shows that X can contain only points of indices $r' = 2, 3, 4, 6, 8, 9, 13, 18$. Computer search shows that under the assumptions $\mathrm{p}_2(X) \geq 2$ and $\mathrm{p}_1(X) \leq 1$ we have $\mathbf{B}(X) = (2, 4^2, 6)$, $(2^2, 3, 9)$, $(2^3, 3, 8)$,$(2^3, 3^2)$, $(2^2, 4, 8)$, or $(2^2, 3, 4)$. Consider these cases separately.

Case $\mathbf{B}(X) = (2, 4^2, 6)$, [B⁺, # 41434]. In this case $r = 4$ and for $r' = 6$ we have $m' = 4 > m = 2$, a contradiction.

Case $\mathbf{B}(X) = (2^2, 3, 9)$, [B⁺, # 41423]. Then the group $\operatorname{Cl}(X)$ is torsion free,

(7.4.4) $$\dim |A_X| = 0, \quad \dim |2A_X| = 1, \quad \dim |3A_X| = 2, \quad \text{and} \quad \dim |4A_X| = 4.$$

In this case $r = 9$ and $\hat{q} = 1$. Then $s_3 = s_4 = 0$ by Claim 7.4.3 because $3A_X$ and $4A_X$ are not Cartier at $P = f(E)$. We see that $\bar{\mathscr{M}}_k = \bar{f}^*|\hat{M}_k|$ for $k = 2, 3, 4$. If $\hat{X} \simeq \mathbb{P}^1$, then $\bar{\mathscr{M}}_2 = \bar{f}^*|\mathscr{O}_{\mathbb{P}^1}(1)|$ and $\bar{\mathscr{M}}_4 = \bar{f}^*|\mathscr{O}_{\mathbb{P}^1}(2)|$. This contradicts (7.4.4). Hence by Lemma 4.4.1 $\hat{X}$ is a Du Val del Pezzo surface with only type A-singularities and $\operatorname{Cl}(\hat{X}) \simeq \mathbb{Z}$. This contradicts Lemma 2.4.2.

Case $\mathbf{B}(X) = (2^3, 3, 8)$, [B⁺, # 41440]. Then the group $\operatorname{Cl}(X)$ is torsion free, $r = 8$, $\dim |A_X| = 0$, $\dim |2A_X| = 2$, and $\dim |3A_X| = 4$. If $s_2 = 1$, then $\hat{q} = 3$, $\hat{X}$ is a $\mathbb{Q}$-Fano with $\mathrm{p}_1(\hat{X}) \geq 3$. In this case $\hat{X}$ is rational by Proposition 6.4. Let $s_2 = 0$. Then $s_3 = 0$ by Claim 7.4.3 because $3A_X$ is not Cartier at $P = f(E)$. Hence $\bar{\mathscr{M}}_k = \bar{f}^*|\hat{M}_k|$ for $k = 2$ and 3, where $\dim |\hat{M}_2| = \dim |2A_X| = 2$ and $\dim |\hat{M}_3| = \dim |3A_X| = 4$. As above, $\hat{X} \not\simeq \mathbb{P}^1$ and we get a contradiction by Lemma 2.4.2.

Case $\mathbf{B}(X) = (2^3, 3^2)$, [B$^+$, # 41439]. In this case $r = 3$, the group $\mathrm{Cl}(X)$ is torsion free, $\dim |A_X| = 0$, and $\dim |2A_X| = 2$. If $\hat{q} = 3$, then $\hat{X}$ ia a $\mathbb{Q}$-Fano threefold with $\mathrm{p}_1(\hat{X}) \geq 3$ because $s_2 = 1$. Then $\hat{X}$ is rational by Proposition 6.4. Let $\hat{q} = 1$. Then $\bar{f}$ is a fibration such that $\bar{\mathscr{M}}_2 = \bar{f}^*|\hat{M}_2|$ with $\dim |\hat{M}_2| = 2$. Since $\dim |\bar{M}_1| = 0$, we have $\hat{X} \not\simeq \mathbb{P}^1$. Hence $\hat{X}$ is a Du Val del Pezzo surface such that $\mathrm{Cl}(\hat{X}) \simeq \mathbb{Z}$, $\dim |A_{\hat{X}}| = 0$ and $\dim |2A_{\hat{X}}| = 2$. This contradicts Lemma 2.4.2.

Case $\mathbf{B}(X) = (2^2, 4, 8)$, [B$^+$, # 41425]. Then $\dim |kA_X| = k - 1$ for $k = 1, 2, 3$. If $\mathrm{Cl}(X)_{\mathrm{t}} \neq 0$, then X is rational by Proposition 3.5. Thus we may assume that $\mathrm{Cl}(X)$ is torsion free. Apply the construction (4.1.1) with $\mathscr{M} = |3A_X|$. In a neighborhood of the point of index 8 we have $\mathscr{M} \sim 7(-K_X)$ and so $\beta_3 \geq 7\alpha$ by Lemma 4.2.3. The relation (4.5.1) for $k = 3$ has the form

$$3\hat{q} = 5s_3 + (5\beta_3 - 3\alpha)e \geq 5s_3 + 32\alpha e \geq 5s_3 + 4e.$$

Hence $\hat{q} > 1$, $\bar{f}$ is birational, $s_3 > 0$, and $\hat{q} \geq 3$. Then $s_3 \geq 2$ by Proposition 6.4 because $\dim |3A_X| = 2$. Hence $\hat{q} \geq 5$. If $\hat{q} \geq 6$, then $s_3 \geq 4$ by Proposition 7.3.3 and $\hat{q} \geq 8$. This contradicts Theorem 1.1. Thus $\hat{q} = 5$, $e = 1$, $s_3 = 2$, and $\alpha < 1/4$. Hence $\alpha = 1/8$ and $\beta_3 = 43/40 \notin \frac{1}{8}\mathbb{Z}$, a contradiction.

Case $\mathbf{B}(X) = (2^2, 3, 4)$, [B$^+$, # 41422]. Then the group $\mathrm{Cl}(X)$ is torsion free, $\dim |kA_X| = k - 1$ for $k = 1, 2, 3$, $r = 4$, and $\hat{q} = 1$. As above, we obtain $\hat{X}$ is a Du Val del Pezzo surface and $\bar{f}$ is a $\mathbb{Q}$-conic bundle, i.e. X is in situation (*) of 7.4. If $f(E)$ is a cyclic quotient singularity, then $E \simeq \mathbb{P}(1, 1, 3)$ by Theorem 2.3.2 and as in the proof of Proposition 6.4 we conclude that $\bar{X}$ is unirational. This finishes the proof of Proposition 7.4. □

8. $\mathbb{Q}$-Fano threefolds with $\mathrm{q}_{\mathbb{Q}}(X) = 2$

The following proposition slightly improves the corresponding result in [Pro22a].

8.1. **Proposition.** *Let X be a $\mathbb{Q}$-Fano threefold with $\mathrm{q}_{\mathbb{Q}}(X) = 2$. Assume that X is not Gorenstein.*

- (i) *If $\mathrm{p}_1(X) \geq 2$, then X is not solid, i.e. it is birational to a strict Mori fiber space.*
- (ii) *If $\mathrm{p}_1(X) \geq 3$, then X is birational to a conic bundle. If furthermore non-Gorenstein points of X are of types $\mathrm{cA/r}$, then X is unirational.*
- (iii) *If $\mathrm{p}_1(X) \geq 4$, then X is rational.*

Proof. Assume that X is not rational and $\mathrm{p}_1(X) \geq 2$. Let A' be a divisor such that $A' \sim_{\mathbb{Q}} A_X$ and $\dim |A'| = \mathrm{p}_1(X) - 1$. Apply the construction (4.1.1) with $\mathscr{M} := |A'|$ (cf. 6.1). We have

$$\hat{q} = 2s_1 + (2\beta_1 - \alpha)e, \tag{8.1.1}$$

where $2\beta_1 - \alpha > 0$ by Lemma 4.2.3.

First, consider the case $s_1 \geq 2$. Then $\hat{q} \geq 5$. If $\hat{q} \geq 6$, then $s_1 \geq 4$ by Corollary 7.3.3. But in this case $\hat{q} > 7$, a contradiction. Therefore, $\hat{q} = 5$. Then $s_1 = 2$, $\mathrm{p}_2(\hat{X}) \geq 2$, and so X is birational to a conic bundle by Proposition 7.4. If furthermore $\mathrm{p}_1(X) \geq 3$, then $\mathrm{p}_2(\hat{X}) \geq 3$ and X is rational again by Proposition 7.4.

Consider the case $s_1 = 1$. Then $\hat{q} \geq 3$, $\mathrm{p}_1(\hat{X}) \geq 2$, and X is unirational and has a conic bundle structure by Propositions 6.3 and 6.4. Moreover, if $\mathrm{p}_1(X) \geq 3$, then X is rational.

Finally, consider the case $s_1 = 0$. Then, $\hat{q} = 1$ and $\bar{f}$ is a fibration. This proves (i). Now, assume that $\mathrm{p}_1(X) \geq 3$. Then $\hat{X} \simeq \mathbb{P}^2$ (see Lemma 5.2) and $\bar{f}$ is a $\mathbb{Q}$-conic bundle. Moreover, in this case we have $\mathrm{p}_1(X) = 3$. This proves (iii). To prove (ii) we note that $2\beta_1 \leq \alpha + 1$ by (8.1.1). On the other hand, $\beta_1 \geq \alpha$ by Lemma 4.2.3. Hence $\alpha \leq 1$. If $\alpha < 1$, then $f(E)$

is a point of index > 1. If $\alpha = 1$, then $f(E)$ there is a canonical center of $(X, \mathscr{M})$ which is a point of index > 1 again by Lemma 4.2.3. Thus replacing f with another extremal blowup if necessary (see Remark 4.1.2) we may assume that $f(E)$ is a non-Gorenstein point. By our assumption $f(E)$ must be a point of type cA/r. In this case the divisor E must be a rational surface [Pro02]. Its proper transform $\bar{E} \subset \bar{X}$ is a multisection of $\bar{f}$. Hence $\bar{X}$ is unirational. This proves (ii). □

Appendix A.

In this section we present a computer algorithm (see [Pro22b, § 3] or [Car08, § 3]) that alow to list all the numerical possibilities for $\mathbb{Q}$-Fano threefolds of index at least 3. Let X be a $\mathbb{Q}$-Fano threefold with $q := \mathrm{q}_{\mathbb{Q}}(X) \geq 3$ and let $T \in \mathrm{Cl}(X)_{\mathrm{t}}$ be an element of order N.

Step 1. By [Kaw92] we have the inequality

$$0 < -K_X \cdot c_2(X) = 24 - \sum_{P \in \mathbf{B}} \frac{r_P - 1}{r_P}. \tag{A.1.1}$$

This produces a finite (but huge) number of possibilities for the basket $\mathbf{B}(X)$ and the number $-K_X \cdot c_2(X)$.

Step 2. Theorem 2.2.4 implies that $q \in \{3, \dots, 11, 13, 17, 19\}$. In each case we compute A_X^3 by the formula

$$A_X^3 = \frac{12}{(q-1)(q-2)} \Big(1 - \frac{A_X \cdot c_2(X)}{12} + \sum_{P \in B} c_P(-A_X) \Big)$$

(see [Suz04]), where c_P is the correction term in the orbifold Riemann-Roch formula [Rei87]. The number rA_X^3 must be a positive integer by Theorem 2.2.4(iii).

Step 3. On this step we can use an improved version of Bogomolov–Miyaoka inequality [LL23] instead of the one used in [Kaw92] and [Suz04]. Thus we have

$$(-K_X)^3 \begin{cases} < 3(-K_X) \cdot c_2(X) & \text{if } \mathrm{q}_{\mathbb{Q}}(X) \neq 4,\ 5, \\ \leq \frac{25}{8}(-K_X) \cdot c_2(X) & \text{otherwise.} \end{cases}$$

This removes a lot of possibilities.

Step 4. In a neighborhood of each point $P \in X$ we can write $A_X \sim l_P K_X$, where $0 \leq l_P < r_P$. There is a finite number of possibilities for the collection $\{(l_P)\}$. If $\mathrm{q_W}(X) = \mathrm{q}_{\mathbb{Q}}(X)$, then $\gcd(q, r) = 1$ by Theorem 2.2.4. In this case the numbers l_P are uniquely determined by $1 + q l_P \equiv 0 \mod r_P$ because $K_X + qA_X \sim 0$.

Step 5. Similarly, a neighborhood of each point $P \in X$ we can write $T \sim l'_P K_X$, where $0 \leq l'_P < r_P$. The collection $\{(l'_P)\}$ and the number N satisfy the following properties:

$$\chi(X,\ \mathscr{O}_X(NT)) = 1 \qquad \text{and} \qquad \chi(X,\ \mathscr{O}_X(jT)) = 0 \quad \text{for } j = 1, \dots, N-1$$

(by the Kawamata–Viehweg vanishing). Thus we obtain a finite number of possibilities for $\{(l'_P)\}$ and N.

Step 6. Finally, applying Kawamata–Viehweg vanishing we obtain

$$\chi(X,\ \mathscr{O}_X(mA_X + jT)) = \mathrm{h}^0(X,\ \mathscr{O}_X(mA_X + jT)) = 0.$$

for $-q < m < 0$ and $0 \leq j < n$. Again, we check this condition using orbifold Riemann-Roch and remove a lot of possibilities.

Step 7. We obtain a list of collections $(q, \mathbf{B}(X), A_X^3, \{(l_P)\}, n\{(l'_P)\})$. In each case we compute $\mathrm{g}(X)$ and $\mathrm{h}_X(t,\sigma)$ by using the orbifold Riemann-Roch theorem. For example,

$$\mathrm{g}(X) = -\frac{1}{2}K_X^3 + 1 - \sum_{P\in\mathbf{B}} \frac{b_P(r_P - b_P)}{2r_P}. \tag{A.1.2}$$

References

[ABR02] Selma Altınok, Gavin Brown, and Miles Reid. Fano 3-folds, $K3$ surfaces and graded rings. In *Topology and geometry: commemorating SISTAG*, volume 314 of *Contemp. Math.*, pages 25–53. Amer. Math. Soc., Providence, RI, 2002.

[ACP21] Hamid Abban, Ivan Cheltsov, and Jihun Park. On geometry of Fano threefold hypersurfaces. In Gavril Farkas, Gerard van der Geer, Mingmin Shen, and Lenny Taelman, editors, *Rationality of Varieties*, pages 1–14, Cham, 2021. Springer International Publishing.

[Ale94] Valery Alexeev. General elephants of **Q**-Fano 3-folds. *Compositio Math.*, 91(1):91–116, 1994.

[B$^+$] Gavin Brown et al. Graded Ring Database.

[Car08] Jorge Caravantes. Low codimension Fano-Enriques threefolds. *Note Mat.*, 28(2):117–147, 2008.

[CF93] F. Campana and H. Flenner. Projective threefolds containing a smooth rational surface with ample normal bundle. *J. Reine Angew. Math.*, 440:77–98, 1993.

[Cor95] Alessio Corti. Factoring birational maps of threefolds after Sarkisov. *J. Algebraic Geom.*, 4(2):223–254, 1995.

[CP21] Ivan Cheltsov and Yuri Prokhorov. Del Pezzo surfaces with infinite automorphism groups. *Algebraic Geometry*, 8(3):319–357, 2021.

[CPS19] Ivan Cheltsov, Victor Przyjalkowski, and Constantin Shramov. Which quartic double solids are rational? *J. Algebr. Geom.*, 28(2):201–243, 2019.

[HW81] Fumio Hidaka and Keiichi Watanabe. Normal Gorenstein surfaces with ample anti-canonical divisor. *Tokyo J. Math.*, 4(2):319–330, 1981.

[IP99] V. A. Iskovskikh and Yu. Prokhorov. *Fano varieties. Algebraic geometry V*, volume 47 of *Encyclopaedia Math. Sci.* Springer, Berlin, 1999.

[Kaw92] Yujiro Kawamata. Boundedness of **Q**-Fano threefolds. In *Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989)*, volume 131 of *Contemp. Math.*, pages 439–445. Amer. Math. Soc., Providence, RI, 1992.

[Kaw93] Yujiro Kawamata. The minimal discrepancy coefficients of terminal singularities in dimension three. Appendix to V. V. Shokurov's paper "3-fold log flips". *Russ. Acad. Sci., Izv., Math.*, 40(1):193–195, 1993.

[Kaw96] Yujiro Kawamata. Divisorial contractions to 3-dimensional terminal quotient singularities. In *Higher-dimensional complex varieties (Trento, 1994)*, pages 241–246. de Gruyter, Berlin, 1996.

[Kaw01] Masayuki Kawakita. Divisorial contractions in dimension three which contract divisors to smooth points. *Invent. Math.*, 145(1):105–119, 2001.

[Kaw05] Masayuki Kawakita. Three-fold divisorial contractions to singularities of higher indices. *Duke Math. J.*, 130(1):57–126, 2005.

[LL23] Haidong Liu and Jie Liu. Kawamata–Miyaoka type inequality for canonical **Q**-Fano varieties. *ArXiv e-print*, 2023.

[Mor88] Shigefumi Mori. Flip theorem and the existence of minimal models for 3-folds. *J. Amer. Math. Soc.*, 1(1):117–253, 1988.

[MP08] Shigefumi Mori and Yuri Prokhorov. On **Q**-conic bundles. *Publ. Res. Inst. Math. Sci.*, 44(2):315–369, 2008.

[MZ88] M. Miyanishi and D.-Q. Zhang. Gorenstein log del Pezzo surfaces of rank one. *J. Algebra*, 118(1):63–84, 1988.

[Oka19] Takuzo Okada. Stable rationality of orbifold Fano 3-fold hypersurfaces. *J. Algebr. Geom.*, 28(1):99–138, 2019.

[Pro02] Yu. Prokhorov. A remark on the resolution of three-dimensional terminal singularities. *Russian Math. Surveys*, 57(4):815–816, 2002.

[Pro10] Yuri Prokhorov. **Q**-Fano threefolds of large Fano index, I. *Doc. Math.*, 15:843–872, 2010.

[Pro13] Yu. Prokhorov. Fano threefolds of large Fano index and large degree. *Sbornik: Math.*, 204(3):347–382, 2013.

[Pro16] Yu. Prokhorov. Q-Fano threefolds of index 7. *Proc. Steklov Inst. Math.*, 294:139–153, 2016.

[Pro21] Yuri Prokhorov. Equivariant minimal model program. *Russian Math. Surv.*, 76(3):461–542, 2021.

[Pro22a] Yuri Prokhorov. Conic bundle structures on $\mathbf{Q}$-Fano threefolds. *Electron. Res. Arch.*, 30(5):1881–1897, 2022. Special issue on birational geometry and moduli of projective varieties.

[Pro22b] Yuri Prokhorov. *Rationality of* $\mathbf{Q}$*-Fano threefolds of large Fano index*, volume 478 of *London Mathematical Society Lecture Note Series*, page 253–274. Cambridge University Press, 2022.

[Rei87] Miles Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

[San96] Takeshi Sano. Classification of non-Gorenstein $\mathbf{Q}$-Fano d-folds of Fano index greater than $d-2$. *Nagoya Math. J.*, 142:133–143, 1996.

[Suz04] Kaori Suzuki. On Fano indices of $\mathbf{Q}$-Fano 3-folds. *Manuscripta Math.*, 114(2):229–246, 2004.

Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russian Federation

Email address: prokhoro@mi-ras.ru