ON THE EXISTENCE OF A SECOND POSITIVE SOLUTION TO MIXED LOCAL-NONLOCAL CONCAVE-CONVEX CRITICAL PROBLEMS

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ABSTRACT. We prove the existence of a second positive weak solution for mixed local-nonlocal critical semilinear elliptic problems with a sublinear perturbation in the spirit of [1].

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with smooth enough boundary $\partial\Omega$. We consider the following mixed local-nonlocal perturbed critical semilinear elliptic problem:

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} \mathcal{L}_\varepsilon u = \lambda\, u^p + u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $\lambda > 0$ is a positive real parameter, $p \in (0,1)$ and

$$\mathcal{L}_\varepsilon := -\Delta + \varepsilon\,(-\Delta)^s, \qquad s \in (0,1), \quad \varepsilon \in (0,1].$$

Along the paper it will sometimes be useful to denote the above problem as $(\mathrm{P}_\varepsilon)_\lambda$ to make it clear the choice of the parameter. Here, $(-\Delta)^s$ with $s \in (0,1)$ denotes the fractional Laplacian which acts in smooth enough functions as

$$(-\Delta)^s u(x) = 2\,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy = 2 \lim_{\delta\to 0^+} \int_{\{|x-y|\geq\delta\}} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy.$$

We neglect the normalization constant $C_{n,s} > 0$ usually appearing in front of the integral because we are not interested in asymptotics as $s \to 1^-$.

The above boundary value problem falls in the framework of the so called *concave-convex problems* whose model and most famous example is the one considered in the paper [1]:

$$(\mathrm{ABC}) \qquad \begin{cases} -\Delta u = \lambda\, u^p + u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Without any aim of completeness, we mention that (ABC) has been generalized to various directions, either allowing for different operators either considering different boundary conditions, see e.g. [32, 24, 22, 7].

Before entering into the details related to $(\mathrm{P}_\varepsilon)_\lambda$, let us spend a few more words on $\mathcal{L}_\varepsilon$. The operator $\mathcal{L}_1$ (i.e. with $\varepsilon = 1$) is a special instance of a wide more general class of operators whose study began in the '60s, see [14] and [21] for generalizations, in connection with the validity of a maximum principle. On the other hand, the operator $\mathcal{L}_1$ can be seen as the infinitesimal generator of a stochastic process obtained as a superposition of a Brownian motion and a Lévy flight, and hence there is a vast literature which establishes several regularity properties *adopting probabilistic techniques*, see e.g. [23] and the references therein.
More recently, the study of regularity properties related to this operator (and its quasilinear generalizations) has seen an increasing interest, mainly adopting more analytical and PDEs approaches, see, e.g., [8, 20, 27, 28, 30, 31, 37, 4, 38]. It is worth mentioning that the operator $\mathcal{L}_\varepsilon$ (usually with $\varepsilon = 1$) seems to be of interest in biological applications, see, e.g. [29] and the references therein.
The operator $\mathcal{L}_1$ has a also variational nature and it is associated to the energy

$$E(u) := \int_\Omega \frac{|\nabla u|^2}{2} \, dx + \frac{1}{2} \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy.$$

defined on a suitable space functions for which we refer to Section 2. It is clear that

$$E(\lambda u) = \lambda^2 E(u), \quad \text{for every } \lambda \in \mathbb{R},$$

but there is a lack of scaling invariance, namely,

$$E(u_t) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{t^{2s-2}}{2} \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \quad \text{for every } t > 0,$$

where $u_t(x) := t^{(n-2)/2} u(x)$, which typically preserves the L^2-gradient norm.

Among other aspects, this phenomenon is responsible of the following fact: the best Sobolev constant in the natural mixed Sobolev inequality is never achieved and it coincides with the one coming from the purely local one. We notice that something similar happens when dealing with mixed Hardy-type inequalities, see [11]. An interesting consequence of the absence of mixed Aubin-Talenti functions naturally affects PDEs with critical term. Indeed, when dealing with critical variational problems for $-\Delta$, it is well known that a major role is played by the best Sobolev constant, for example being a threshold of validity of Palais-Smale condition, and this is usually achieved by testing the critical equation with the Aubin-Talenti functions. An analogous procedure can be followed in the mixed setting as well, but the lack of invariance previously mentioned can create troubles, see e.g [10, 13, 25]. We stress that something similar may happen with non-homogeneous operators like the (p, q)-Laplacian, see [33].

Following the approach recently used in [13] for the case of mild singular and critical mixed problems, we are now interested in the following classical problem:

Find values of the parameter λ for which $(\mathrm{P}_\varepsilon)_\lambda$ admits one or more positive weak solutions.

We refer to Definition 2.2 for the precise definition of *weak solution* of $(\mathrm{P}_\varepsilon)_\lambda$. Differently from [10] and [13], here we are in presence of a sublinear perturbation $(\lambda \, u^p$ with $p \in (0, 1))$ of the critical term (u^{2^*-1}).

For $\varepsilon = 1$, a first step towards an answer has been recently made in [2, Theorem 1.2] and [36, Theorem 1.1] where the following has been proved:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ (with $n \geq 3$) be a bounded open set with smooth enough boundary, and let $p \in (0,1)$. Then, there exists $\Lambda > 0$ such that*

    a) *problem $(\mathrm{P}_1)_\lambda$ admits at least one weak solution for every $0 < \lambda \leq \Lambda$;*

    b) *problem $(\mathrm{P}_1)_\lambda$ does not admit weak solutions for every $\lambda > \Lambda$.*

Moreover, for $\lambda \in (0, \Lambda)$, the solution is minimal and increasing w.r.t. to λ.

We stress that the the above results actually holds in the supercritical case as well. For sake of completeness, we recall that the existence of the first positive solution in [2] is obtained by means of a sub/supersolution scheme as in [1]. We also note that, if we fix $\varepsilon \in (0,1]$, the above result holds for a threshold Λ_ε now depending on ε.

We also mention that concave-convex problems in the mixed local-nonlocal setting has been previously studied in [26]: there, the leading operator is a combination of p-Laplacian and fractional p-Laplacian and the existence of weak solutions is proved under the assumption $p > n$, being the interest of the authors to study the limiting problem as $p \to +\infty$.

Our first result is somehow complementary to Theorem 1.1 stated above and it is the counterpart of [1, Theorem 2.2] in the mixed local-nonlocal setting.

Theorem 1.2. *Let $\varepsilon \in (0,1]$ be fixed. Then, there exists a constant $M_\varepsilon > 0$ such that, for every $\lambda \in (0, \Lambda_\varepsilon)$, problem $(\mathrm{P}_\varepsilon)_{\lambda, \varepsilon}$ has at most one solution u_λ with*

$$\|u_{\lambda, \varepsilon}\|_{L^\infty(\mathbb{R}^n)} \leq M_\varepsilon.$$

Note that the if the unique solution of Theorem 1.2 exists, it has to be the minimal one found in Theorem 1.1.

Following the seminal paper [1], it is natural to wonder whether a second positive solution exists or not for all $\lambda \in (0, \Lambda)$. A similar question has been answered in [36] for sublinear perturbations of a subcritical nonlinearity, so leaving the critical case unsolved.

To better understand the difficulties one has to face in the case of mixed operators like $\mathcal{L}_\varepsilon$, let us briefly recall the method employed in [1] to find a second positive solution of (ABC):

    i) show that the first solution found is a minimizer of the functional naturally associated with (ABC) in C^1-topology;

    ii) thanks to the famous result by Brezis and Nirenberg [18], one inherits that the first solution is actually a minimizer in the H^1-topology;

    iii) prove the validity of the Palais-Smale condition under a certain value proportional to the best Sobolev constant, here making use of the Aubin-Talenti functions;

    iv) find a second positive solution of mountain-pass-type.

Despite i) can still be proved in our case, the above scheme finds a first obstacle once a result like the one in [18] is needed. To this aim, and thanks to the recent regularity results proved in [3] and [38], we establish the following

Theorem 1.3. *Let* $\Phi : \mathcal{X}^{1,2}(\Omega) \to \mathbb{R}$ *defined as*

$$(1.1) \qquad \Phi(u) := \frac{1}{2}\,\rho(u)^2 - \int_\Omega F(x,u)\,dx,$$

where $F(x,u) := \int_0^u f(x,s)\,ds$ *and with* f *satisfying that*

$$(1.2) \qquad |f(x,u)| \leq C_f\,(1 + |u|^{2^*-1}).$$

Let $u_0 \in \mathcal{X}^{1,2}(\Omega)$ *be a local minimizer of* Φ *in the* C^1*-topology. Then,* u_0 *is a local minimizer in the* $\mathcal{X}^{1,2}$*-topology.*

Theorem 1.3 allows to achieve ii) as well, but a new difficulty arises once iii) has to be faced, the main reason being the lack of scaling invariance, and its consequences on the mixed Sobolev inequality, previously discussed. A similar issue occurred in the recent [13] when dealing with mild singular problems, and it is the main reason to consider $\mathcal{L}_\varepsilon$ instead of $\mathcal{L}_1$ because, roughly speaking, the presence of the parameter ε allows us to adjust the lack of scaling invariance, at least for small enough ε. In this way, we are able to fully follow the above scheme proving the following

Theorem 1.4. *Let* $\Omega \subset \mathbb{R}^n$ *(with* $n \geq 3$*) be a bounded open set with smooth enough boundary, and let* $p \in (0,1)$ *be fixed.*
Then, there exist $\lambda_\star > 0$ *and* $\varepsilon_0 \in (0,1)$ *such that problem* $(\mathrm{P}_\varepsilon)_\lambda$ *admits a second positive solutions for every* $\varepsilon \in (0, \varepsilon_0)$ *and for every* $\lambda \in (0, \lambda_\star)$*.*

As it clearly appears from the statement, there is a first price to pay once considering $\mathcal{L}_\varepsilon$, because the existence is not proved for every λ. Less evident is the second fee we have to pay. Let us briefly describe it: the first solution found in Theorem 1.1 can still be found once $\mathcal{L}_1$ is replaced with $\mathcal{L}_\varepsilon$ but now it implicitly depends on ε and this forces to carefully keep track of the dependence on ε of many ingredients like L^∞-bounds and so on.
As in [13], the key result to be proved is Lemma 5.2 and we want to explicit mention that a careful inspection of its proof shows that a better result can be obtained, this time with big restrictions on both the dimension n and the fractional parameter s:

Corollary 1.5. *Let* $n = 3$ *and* $s \in \left(0, \frac{1}{2}\right)$*. Then problem* $(\mathrm{P}_\varepsilon)_\lambda$ *admits at least two positive weak solutions for every* $\lambda \in (0, \Lambda)$*.*

Finally, once the existence of (at least) two solutions of problem $(\mathrm{P}_\varepsilon)_\lambda$ has been established (at least for ε small enough), by exploiting Theorem 1.2 and by proceeding essentially as in [1] we can prove the following qualitative result.

Theorem 1.6. *Let* $\varepsilon \in (0,1]$ *be fixed, and assume that* Ω *is* star-shaped*. Then,*

$$\|v_{\lambda,\varepsilon}\|_{L^\infty(\Omega)} \to +\infty \ as \ \lambda \to 0^+,$$

where $v_{\lambda,\varepsilon} \in \mathcal{X}^{1,2}(\Omega)$ *is* any weak solution of *problem* $(\mathrm{P}_\varepsilon)_\lambda$ *distinct from its minimal solution* $u_{\lambda,\varepsilon}$ *(see Theorem 1.1 and Lemma 2.6).*

Plan of the paper: The paper is organized as follows:

- In Section 2 we collect all the relevant notation, definitions and preliminary results needed for the proof of our main results.

- In Section 3 we briefly study the *purely sublinear* counterpart of problem $(P_\varepsilon)_\lambda$ (starting from the results proved in [12]); in particular, we establish some *uniform bounds* of the unique solution $w_{\lambda,\varepsilon}$ of this problem which will be used subsequent sections, and we establish Theorem 1.2.
- In Section 4 we prove Theorem 1.3.
- In Section 5 we prove Theorem 1.4.
- Finally, in Section 6 we give the proof of Theorem 1.6.

2. Preliminaries

In this section we collect some preliminary definitions and results which will be used throughout the rest of the paper. First of all, we review some basic properties of the fractional Sobolev spaces, and we properly introduce the adequate functional setting for the study of mixed local-nonlocal operators; we then give the precise definition of *weak sub/supersolution* of problem $(P_\varepsilon)_\lambda$, and we establish some qualitative properties of the solutions of this problem (provided they exist). Finally, we spend a few words the applicability of Theorem 1.1 (established in the case when $\varepsilon = 1$) to our ε-dependent operator $\mathcal{L}_\varepsilon$.

i) Sobolev spaces of fractional order. We begin this section by collecting a few basic facts fractional Sobolev spaces, which are naturally related to the fractional Laplacian $(-\Delta)^s$; we refer to [34] for a thorough introduction to this topic.

Let $\varnothing \neq \mathcal{O} \subseteq \mathbb{R}^n$ be an arbitrary open set. The *fractional Sobolev space* $H^s(\mathcal{O})$ (of order $s \in (0,1)$) is the subset of $L^2(\mathcal{O})$ defined as follows

$$H^s(\mathcal{O}) := \Big\{ u \in L^2(\mathcal{O}) : [u]_{s,\mathcal{O}}^2 = \iint_{\mathcal{O}\times\mathcal{O}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy < +\infty \Big\}.$$

We then list the few basic properties of $H^s(\mathcal{O})$ we will exploit in this paper.

a) $H^s(\mathcal{O})$ is a real Hilbert space, with the scalar product

$$\langle u, v \rangle_{s,\mathcal{O}} := \iint_{\mathcal{O}\times\mathcal{O}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dx\, dy \qquad (u, v \in H^s(\mathcal{O})).$$

b) $C_0^\infty(\mathcal{O})$ is a *linear subspace of* $H^s(\mathcal{O})$; in addition, in the particular case when $\mathcal{O} = \mathbb{R}^n$, we have that $C_0^\infty(\mathbb{R}^n)$ is *dense* in $H^s(\mathbb{R}^n)$.

c) If $\mathcal{O} = \mathbb{R}^n$ or if $\mathcal{O}$ has *bounded boundary* $\partial\mathcal{O} \in C^{0,1}$, we have the *continuous embedding* $H^1(\mathcal{O}) \hookrightarrow H^s(\mathcal{O})$, that is, there exists $\mathbf{c} = \mathbf{c}(n,s) > 0$ s.t.

$$(2.1) \qquad \iint_{\mathcal{O}\times\mathcal{O}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \leq \mathbf{c}\, \|u\|_{H^1(\mathcal{O})}^2 \quad \text{for every } u \in H^1(\mathcal{O}).$$

In particular, if $\mathcal{O} \subseteq \mathbb{R}^n$ is a *bounded open set* (with no regularity assumptions on $\partial\mathcal{O}$) and if $u \in H_0^1(\mathcal{O})$, setting $\hat{u} = u \cdot \mathbf{1}_\mathcal{O} \in H^1(\mathbb{R}^n)$ we have

$$(2.2) \qquad \iint_{\mathbb{R}^{2n}} \frac{|\hat{u}(x)-\hat{u}(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \leq \beta \int_\mathcal{O} |\nabla u|^2\, dx,$$

where $\beta > 0$ is a suitable constant depending on n, s and on $|\Omega|$. Here and throughout, $|\cdot|$ denotes the n-dimensional Lebesgue measure.

ii) The space $\mathcal{X}^{1,2}(\Omega)$**.** Now we have briefly recalled some basic facts regarding fractional Sobolev spaces, we are in a position to introduce the adequate functional setting for the study of mixed local-nonlocal operators.

Let then $\Omega \subseteq \mathbb{R}^n$ be a bounded open set Lipschitz boundary $\partial\Omega$. We define the space $\mathcal{X}^{1,2}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the *global norm*

$$\rho(u) := \left(\||\nabla u|\|_{L^2(\mathbb{R}^n)}^2 + [u]_{s,\mathbb{R}^n}^2 \right)^{1/2}, \qquad u \in C_0^\infty(\Omega).$$

Due to its relevance in the sequel, we also introduce a distinguished notation for the *cone of the non-negative functions* in $\mathcal{X}^{1,2}(\Omega)$: we set

$$\mathcal{X}_+^{1,2}(\Omega) := \{u \in \mathcal{X}^{1,2}(\Omega) : u \geq 0 \text{ a.e. in } \Omega\}.$$

Since this norm ρ is induced by the scalar product

$$\mathcal{B}_\rho(u,v) := \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx + \langle u, v \rangle_{s,\mathbb{R}^n}$$

(where $\cdot$ denotes the usual scalar product in $\mathbb{R}^n$), the space $\mathcal{X}^{1,2}(\Omega)$ is a real *Hilbert space*; most importantly, since Ω is bounded and $\partial\Omega$ is Lipschitz, by combining the above (2.1) with the classical Poincaré inequality we infer that

$$\vartheta^{-1}\|u\|_{H^1(\mathbb{R}^n)} \leq \rho(u) \leq \vartheta\|u\|_{H^1(\mathbb{R}^n)} \qquad \text{for every } u \in C_0^\infty(\Omega),$$

where $\vartheta > 1$ is a suitable constant depending on n, s and on $|\Omega|$. Thus, $\rho(\cdot)$ and the full H^1-norm in $\mathbb{R}^n$ are *actually equivalent* on the space $C_0^\infty(\Omega)$, so that

$$(2.3) \qquad \begin{aligned} \mathcal{X}^{1,2}(\Omega) &= \overline{C_0^\infty(\Omega)}^{\,\|\cdot\|_{H^1(\mathbb{R}^n)}} \\ &= \{u \in H^1(\mathbb{R}^n) : u|_\Omega \in H_0^1(\Omega) \text{ and } u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}. \end{aligned}$$

We explicitly observe that, on account of (2.3), the functions in $\mathcal{X}^{1,2}(\Omega)$ naturally satisfy the nonlocal Dirichlet condition prescribed in problem $(P_\varepsilon)_\lambda$, that is,

$$u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \text{ for every } u \in \mathcal{X}^{1,2}(\Omega).$$

Remark 2.1 (Properties of the space $\mathcal{X}^{1,2}(\Omega)$). For a future reference, we list in this remark some properties of the function space $\mathcal{X}^{1,2}(\Omega)$ which will be repeatedly exploited in the rest of the paper.

1) Since both $H^1(\mathbb{R}^n)$ and $H_0^1(\Omega)$ are *closed* under the maximum/minimum operation, it is readily seen that

$$u_\pm \in \mathcal{X}^{1,2}(\Omega) \quad \text{for every } u \in \mathcal{X}^{1,2}(\Omega),$$

where $u_+ = \max\{u,0\}$ and $u_- = \max\{-u,0\}$.

2) Since we are assuming that $\partial\Omega$ is smooth, from (2.3) we see that a function $u \in H^1(\mathbb{R}^n) \cap C(\overline{\Omega})$ belongs to the space $\mathcal{X}^{1,2}(\Omega)$ *if and only if*

$$u \equiv 0 \text{ pointwise on } \mathbb{R}^n \setminus \Omega.$$

3) On account of (2.2), for every $u \in \mathcal{X}^{1,2}(\Omega)$ we have

$$(2.4) \qquad [u]_{s,\mathbb{R}^n}^2 = \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \leq \beta \int_\Omega |\nabla u|^2 \, dx.$$

As a consequence, the norm ρ is *globally equivalent* on $\mathcal{X}^{1,2}(\Omega)$ to the H_0^1-norm: in fact, by (2.4) there exists a constant $\Theta > 0$ such that

$$(2.5) \qquad \|\nabla u\|_{L^2(\Omega)} \leq \rho(u) \leq \Theta \|\nabla u\|_{L^2(\Omega)} \quad \text{for every } u \in \mathcal{X}^{1,2}(\Omega).$$

4) By the (local) Sobolev inequality, for every $u \in \mathcal{X}^{1,2}(\Omega)$ we have

$$S_n \|u\|_{L^{2^*}(\Omega)}^2 = \|u\|_{L^{2^*}(\mathbb{R}^n)}^2 \leq \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \leq \rho(u)^2.$$

This, together with Hölder's inequality (recall that Ω is *bounded*), proves the *continuous embedding* $\mathcal{X}^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \leq q \leq 2^*$.

5) By combining (2.5) with the *compact embedding* of $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ (holding true for every $1 \leq q < 2^*$), we derive that also the embedding

$$\mathcal{X}^{1,2}(\Omega) \hookrightarrow L^q(\Omega) \text{ is compact for every } 1 \leq q < 2^*.$$

As a consequence, if $\{u_k\}_k$ is a bounded sequence in $\mathcal{X}^{1,2}(\Omega)$, it is possible to find a (unique) function $u \in \mathcal{X}^{1,2}(\Omega)$ such that (up to a sub-sequence)
   a) $u_n \to u$ weakly in $\mathcal{X}^{1,2}(\Omega)$;
   b) $u_n \to u$ *strongly* in $L^q(\Omega)$ for every $1 \leq q < 2^*$;
   c) $u_n \to u$ pointwise a.e. in Ω.
   Clearly, since both u_n (for all $n \geq 1$) and u *identically vanish* out of Ω, see (2.3), we can replace Ω with $\mathbb{R}^n$ in the above assertions b)-c).
We will exploit these properties without any further comment.

We now observe that, since the leading operator of $(\mathrm{P}_\varepsilon)_\lambda$ is given by the ε-dependent operator $\mathcal{L}_\varepsilon = -\Delta + \varepsilon(-\Delta)^s$, it follows that the bilinear form naturally associated with $\mathcal{L}_\varepsilon$ is the following

$$\mathcal{B}_\varepsilon(u, v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx + \varepsilon \langle u, v \rangle_{s, \mathbb{R}^n};$$

in its turn, this form $\mathcal{B}_\varepsilon$ induces the ε-dependent quadratic form

$$\rho_\varepsilon(u) = \|\nabla u\|_{L^2(\mathbb{R}^n)}^2 + \varepsilon \, [u]_s^2 \qquad (\text{for } u \in \mathcal{X}^{1,2}(\Omega)).$$

While in this perspective it should seem more *natural* to use the norm ρ_ε in place of ρ on the space $\mathcal{X}^{1,2}(\Omega)$, it is readily seen that these two norms are indeed *equivalent* on $\mathcal{X}^{1,2}(\Omega)$ (and equivalent to the H_0^1-norm), *uniformly with respect to ε*: in fact, taking into account (2.5) (and since $0 < \varepsilon \leq 1$), we have

$$(2.6) \qquad \|\nabla u\|_{H_0^1(\Omega)} \leq \rho_\varepsilon(u) \leq \rho(u) \leq \Theta \|\nabla u\|_{H_0^1(\Omega)}$$

for some $\Theta > 0$ only depending on n, s. On account of (2.6), we can indifferently use $\rho_\varepsilon(\cdot)$, $\rho(\cdot)$ and the H_0^1-norm to define the topology of the space $\mathcal{X}^{1,2}(\Omega)$, and this choice does not produce any dependence on $\varepsilon \in (0, 1]$.

Notation. We conclude this second part of the section with a short list of notation, which will be used in the sequel; here, as usual, $\varepsilon \in (0, 1]$ is a fixed parameter.

1) Given any open set $\mathcal{O} \subseteq \mathbb{R}^n$ (not necessarily bounded), we set

   a) $\mathcal{B}_{\varepsilon, \mathcal{O}}(u, v) = \int_{\mathcal{O}} \nabla u \cdot \nabla v \, dx + \varepsilon \langle u, v \rangle_{s, \mathbb{R}^n}$ (for $u, v \in \mathcal{X}^{1,2}(\Omega)$);

   b) $\mathcal{Q}_{\varepsilon, \mathcal{O}}(u) = \mathcal{B}_{\rho_\varepsilon, \mathcal{O}}(u, u)$ (for $u \in \mathcal{X}^{1,2}(\Omega)$).

Since $\mathcal{X}^{1,2}(\Omega) \subseteq H^1(\mathbb{R}^n)$ (see (2.3)), the above forms $\mathcal{B}_{\varepsilon, \mathcal{O}}$ and $\mathcal{Q}_{\varepsilon, \mathcal{O}}$ are well-defined; moreover, again by taking into account (2.3) we have

- $\mathcal{B}_{\varepsilon,\Omega}(u,v) = \mathcal{B}_{\varepsilon,\mathbb{R}^n}(u,v) \equiv \mathcal{B}_\varepsilon(u,v)$ for all $u,v \in \mathcal{X}^{1,2}(\Omega)$;
- $\mathcal{Q}_{\varepsilon,\Omega}(u) = \mathcal{Q}_{\varepsilon,\mathbb{R}^n}(u) \equiv \rho_\varepsilon(u)$ for all $u \in \mathcal{X}^{1,2}(\Omega)$.

2) Given any *bounded* open set $\mathcal{O} \subseteq \mathbb{R}^n$, we set

$$\|u\|_{H_0^1(\mathcal{O})} := \||\nabla u|\|_{L^2(\mathcal{O})} = \Big(\int_{\mathcal{O}} |\nabla u|^2 \, dx \Big)^{1/2}.$$

iii) Weak sub/supersolutions of problem $(\mathrm{P}_\varepsilon)_\lambda$. Thanks to all the preliminaries reviewed so far, we are ready to provide the precise definition of *weak sub/supersolutions* of problem $(\mathrm{P}_\varepsilon)_\lambda$. Actually, for a reason which will be clear later on, we consider the more general problem

$$(2.7) \qquad \begin{cases} \mathcal{L}_\varepsilon u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an arbitrary Carathéodory function satisfying the growth condition (1.2), that is, *there exists a constant $C > 0$ such that*

$$|f(x,t)| \leq C(1 + |t|^{2^*-1}) \quad \text{for a.e. } x \in \Omega \text{ and every } t \in \mathbb{R}.$$

Clearly, problem $(\mathrm{P}_\varepsilon)_\lambda$ is of the form (2.7), with the choice $f(x,t) = \lambda t^p + t^{2^*-1}$.

Definition 2.2. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition (1.2). We say that a function $u \in \mathcal{X}^{1,2}(\Omega)$ is

    a) a *weak subsolution* (resp. *supersolution*) of problem (2.7) if we have

$$(2.8) \qquad \mathcal{B}_\rho(u,\varphi) \leq [\text{resp. } \geq] \int_\Omega f(x,u)\varphi \, dx$$

        for every test function $\varphi \in C_0^\infty(\Omega), \varphi \geq 0$ in Ω.

    b) a *weak solution* of problem (2.7) if u is both a weak subsolution and a weak supersolution of the same problem;

    c) a *weak subsolution* (resp. *supersolution, solution*) of problem

$$(2.9) \qquad \begin{cases} \mathcal{L}_\varepsilon u = f(x,u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

        if u is a weak subsolution (resp. supersolution, solution) of problem (2.7) in the sense already specified, further satisfying

$$u > 0 \text{ a.e. in } \Omega.$$

Now we have introduced Definition 2.2, we prove a regularity result for solutions of problem (2.7) which will be used in the sequel.

Theorem 2.3. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition (1.2), and suppose that there exists a weak solution $u_0 \in \mathcal{X}^{1,2}(\Omega)$ of problem (2.7) (in the sense of Definition 2.2). Then, we have*

$$(2.10) \qquad u \in C^{1,\alpha}(\overline{\Omega}) \text{ for some } \alpha \in (0,1).$$

Proof. First of all we observe that, since the function f satisfies the growth condition (1.2), we can apply [38, Theorem 1.1], ensuring that $u_0 \in L^\infty(\Omega)$; from this, again by exploiting condition (1.2), we derive that

$$(2.11) \qquad g(x) := f(x,u(x)) \in L^\infty(\Omega).$$

On account of (2.11), and since Ω has smooth boundary, we can then invoke the *global regularity result* for the weak solutions of the $\mathcal{L}_\varepsilon$ - *Dirichlet problem*

$$\begin{cases} \mathcal{L}_\varepsilon u = g & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

proved in [3, Theorem 1.1], from which we derive (2.10). This ends the proof. $\square$

To make the paper self-contained, we finally state a strong maximum principle and Hopf Lemma which is a mere combination of results already appearing in the literature, see [3, 8, 12].

Theorem 2.4. *Let $u \in \mathcal{X}^{1,2}(\Omega) \cap C^1(\overline{\Omega})$ be a weak supersolution of*

$$(2.12) \qquad \begin{cases} \mathcal{L}_\varepsilon u = 0 & in \ \Omega, \\ u = 0 & in \ \mathbb{R}^n \setminus \Omega. \end{cases}$$

We also assume that $u \not\equiv 0$ in Ω. Then

  i) *$u > 0$ pointwise in Ω and $u = 0$ pointwise in $\partial\Omega$;*
  ii) *$\partial_\nu < 0$ pointwise on $\partial\Omega$ (where ν is the outer exterior normal at $\partial\Omega$).*

Proof. We first observe that, since $u \in \mathcal{X}^{1,2}(\Omega) \cap C^1(\overline{\Omega})$, by Remark 2.1 we have

$$(2.13) \qquad u \in H^1(\mathbb{R}^n) \text{ and } u \equiv 0 \text{ pointwise in } \mathbb{R}^n \setminus \Omega;$$

moreover, since u is a weak supersolution of (2.12), we also have

$$(2.14) \qquad \mathcal{B}_\varepsilon(u, \varphi) \geq 0 \quad \forall \ \varphi \in C_0^\infty(\Omega), \ \varphi \geq 0 \text{ in } \Omega.$$

Gathering (2.13)-(2.14), we are then entitled to apply the Weak Maximum Principle for $\mathcal{L}_\varepsilon$ in [8, Theorem 1.2], ensuring that $u \geq 0$ pointwise in $\mathbb{R}^n$; as a consequence, since $u \geq 0$, $u \not\equiv 0$ in Ω and since $u \equiv 0$ on $\mathbb{R}^n \setminus \Omega$ (see (2.13)), by combining the Strong Maximum Principle in [12, Theorem 3.1] (applied here with $f \equiv 0$) and the Hopf lemma in [3, Theorem 1.2] (remind that $u \in C^1(\overline{\Omega})$), we conclude that

$$u > 0 \text{ pointwise in } \Omega \quad \text{and} \quad \partial_\nu u < 0 \text{ on } \partial\Omega.$$

This ends the proof. $\square$

Lemma 2.5. *Let $n > 3$ and $h = h(x) \in L^p(\Omega)$ with $p > \frac{n}{2}$. Assume that there exists a weak solution $u \in \mathcal{X}^{1,2}(\Omega)$ of the Dirichlet problem*

$$\begin{cases} \mathcal{L}_\varepsilon u = h & in \ \Omega, \\ u = 0 & in \ \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then $u \in L^\infty(\mathbb{R}^n)$ and

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C \, \|h\|_{L^p(\Omega)},$$

for some positive constant $C > 0$ independent of ε.

Proof. It is enough to closely follow the proof of [8, Theorem 4.7] in the case $\varepsilon = 1$. Take $\delta > 0$ to be chosen later on and define the functions

$$\tilde{u} := \frac{\delta \, u}{K} \quad \text{and} \quad \tilde{v} := \frac{\delta \, h}{K},$$

where $K = K(u, h, \Omega, p, n) := \|u\|_{L^{2^*}(\Omega)} + \|h\|_{L^p(\Omega)}$. One can now perform the same computations made in [8, Theorem 4.7], getting rid of the nonlocal part thanks to [8, Equation (4.47)], reaching the following:

$$(2.15) \qquad u(x) \leq \frac{K}{\delta} = \frac{\|u\|_{L^{2^*}(\Omega)} + \|h\|_{L^p(\Omega)}}{\delta}, \quad \text{for every } x \in \Omega,$$

where $\delta > 0$ has been chosen conveniently small but independently of ε. Now, by Sobolev and Hölder inequalities, and testing the equation with u itself, we get

$$\mathcal{S}_n \|u\|^2_{L^{2^*}(\Omega)} \leq |\Omega|^{1/q} \|u\|_{L^{2^*}(\Omega)} \|h\|_{L^p(\Omega)},$$

where $\frac{1}{2^*} + \frac{1}{p} + \frac{1}{q} = 1$ are the Hölder exponents. Therefore,

$$\|u\|_{L^{2^*}(\Omega)} \leq \frac{|\Omega|^{1/q}}{\mathcal{S}_n} \|h\|_{L^p(\Omega)},$$

which combined with (2.15) gives the desired conclusion. $\qquad \square$

iv) First positive solution to $(\mathrm{P}_\varepsilon)_\lambda$. We conclude this section of preliminaries by spending a few words on the applicability of Theorem 1.1 (which is proved in the case when $\varepsilon = 1$) to our operator $\mathcal{L}_\varepsilon$.

In the case $\varepsilon = 1$, the existence of a first positive solution is proved in [2] by means of a classical sub and supersolution scheme that can be performed in the same way even replacing $\mathcal{L}_1$ with $\mathcal{L}_\varepsilon$. In particular, one can define

$$(2.16) \qquad \Lambda_\varepsilon := \sup \{\lambda > 0 : (\mathrm{P}_\varepsilon)_\lambda \text{ admits a weak solution }\}.$$

One then has to show what follows:

    a) Λ_ε is well defined and $\Lambda_\varepsilon < +\infty$;
    b) problem $(\mathrm{P}_\varepsilon)_\lambda$ admits a weak solution for every $0 < \lambda \leq \Lambda_\varepsilon$.

For every *fixed* $0 < \varepsilon \leq 1$, assertions a)-b) can be proved exactly as in [2]. In particular, following the proof of [2, Theorem 1.1], where the authors used a recent result in critical point theory proved in [35], there exists $\mu_{\sharp,\varepsilon} > 0$ (a priori depending on ε in a not explicit way) such that a first solution of $(\mathrm{P}_\varepsilon)_\lambda$ (for $0 < \lambda < \mu_{\sharp,\varepsilon}$) is obtained as a critical point over the set

$$M = \left\{v \in \mathcal{X}^{1,2}(\Omega) : \|v\|_{L^\infty(\Omega)} \leq r, u \geq 0 \text{ a.e. in } \Omega\right\}.$$

This is the reason why we need the following

Lemma 2.6. *There exists $\mu_\sharp > 0$ independent of ε such that $(\mathrm{P}_\varepsilon)_\lambda$ possesses a minimal weak solution $\bar{u}_{\lambda,\varepsilon} \in \mathcal{X}^{1,2}(\Omega) \cap L^\infty(\Omega)$ for every $0 < \lambda \leq \mu_\sharp$ and every $\varepsilon \in (0,1)$, which is increasing w.r.t. λ and further satisfying that*

$$(2.17) \qquad \|\bar{u}_{\lambda,\varepsilon}\|_{L^\infty(\Omega)} \leq C,$$

where $C > 0$ is independent of ε.

Proof. As already discussed, Theorem 1.1 applies to our problem $(\mathrm{P}_\varepsilon)_\lambda$, the role of ε being immaterial; as a consequence, there exists $\Lambda_\varepsilon > 0$ such that problem $(\mathrm{P}_\varepsilon)_\lambda$ possesses a *minimal weak solution*

$$\bar{u}_{\lambda,\varepsilon} \in \mathcal{X}^{1,2}(\Omega)$$

for every $0 < \lambda \leq \Lambda_\varepsilon$; moreover, if $0 < \lambda_1 < \lambda_2 < \Lambda_\varepsilon$, we have

$$\bar{u}_{\lambda_1,\varepsilon} \leq \bar{u}_{\lambda_2,\varepsilon} \text{ a.e. in } \Omega.$$

We claim that there exists a constant $\lambda_\sharp > 0$, independent of $\varepsilon \in (0,1)$, such that

$$(2.18) \qquad\qquad \Lambda_\varepsilon \geq \lambda_\sharp.$$

Indeed, let $\varepsilon \in (0,1)$ be fixed. By arguing as in the proof of [2, Theorem 1.2] we see that there exists $\Lambda_\varepsilon \geq \mu_{\sharp,\varepsilon}$, where $\mu_{\sharp,\varepsilon} > 0$ satisfies the following property (see, precisely, the proof of [2, Lemma 3.9]):

$(\star)$ for every $0 < \lambda < \lambda_{\sharp,\varepsilon}$ there exist $0 < r_1 < r_2$ such that
$$C_\varepsilon(r^{2^*-2} + \lambda r^{p-1}) \leq 1 \text{ for all } r \in [r_1, r_2];$$

here, $C_\varepsilon > 0$ is a positive constant such that, for every $h \in L^p(\Omega)$ (with $p > n/2$) and every weak solution $u \in \mathcal{X}^{1,2}(\Omega)$ of the problem

$$(2.19) \qquad\qquad \begin{cases} \mathcal{L}_\varepsilon u = h & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

we have the *a-priori estimate*

$$(2.20) \qquad\qquad \|u\|_{L^\infty(\mathbb{R}^n)} \leq C_\varepsilon \|h\|_{L^p(\Omega)}$$

(the existence of such a constant follows from [8, Theorem 4.7]). Hence, the lower bound $\lambda_{\sharp,\varepsilon}$ depends on ε *only through the constant C_ε in estimate* (2.20). On the other hand, by Lemma 2.5, the constant C_ε can be chosen *independently of ε*: this means, precisely, that there exists $C_0 > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C_0 \|h\|_{L^p(\Omega)}$$

for every $h \in L^p(\Omega)$ (with $p > n/2$), every weak solution $u \in \mathcal{X}^{1,2}(\Omega)$ of (2.19) *and every $\varepsilon \in (0,1]$*. Hence, if we choose $\mu_\sharp > 0$ in such a way that $(\star)$ holds with C_0 in place of C_ε (that is, if we choose the *very same constant considered in* [2] and corresponding to the case $\varepsilon = 1$), we conclude that (2.18) is satisfied.

We now turn to prove (2.17). To this end we notice that, by proceeding exactly as in the proof of [2, Theorem 1.1] (and by recalling that our constant $\mu_\sharp$ is precisely the one considered in [2] and corresponding to the case $\varepsilon = 1$), there exists a solution $w_{\lambda,\varepsilon} \in \mathcal{X}^{1,2}(\Omega)$ of problem $(\mathrm{P}_\varepsilon)_\lambda$ satisfying the a-priori estimate

$$\|w_{\lambda,\varepsilon}\|_{L^\infty(\Omega)} \leq r,$$

where $r \in [r_1, r_2]$ is arbitrarily chosen, and $r_1 < r_2$ are as in $(\star)$ (and they depend on the fixed λ); thus, by the minimality property of $\bar{u}_{\lambda,\varepsilon}$ we infer that

$$0 \leq \bar{u}_{\lambda,\varepsilon} \leq w_{\lambda,\varepsilon} \leq r \quad \text{a.e. in } \Omega,$$

and this readily gives the desired (2.17). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.7. Arguing exactly as in [2, Proof of Theorem 1.2] one can also realize that there exists $\Lambda_\star > 0$ (indicated there as $\tilde{\lambda}$), depending on the first Dirichlet eigenvalue of $\mathcal{L}_1$ (see e.g. [9]) but independent of ε, such that

$$\Lambda_\varepsilon \leq \Lambda_\star.$$

3. Purely sublinear problems and the proof of Theorem 1.2

In this short section we investigate the *purely sublinear* problem

$$(3.1) \qquad \begin{cases} \mathcal{L}_\varepsilon u = \lambda\, u^p & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

To be more precise, we establish/review some results which will be used (as fundamental tools) in the proof of Theorem 1.4, and we demonstrate our first main result, namely Theorem 1.2.

To begin with, we prove the following proposition.

Proposition 3.1. *Let $\varepsilon \in (0,1]$ be fixed. Moreover, let $\lambda > 0$ and $p \in (0,1)$. Then, the following assertions hold.*

   i) *There exists a* unique weak solution $w_{\lambda,\varepsilon} \in \mathcal{X}^{1,2}(\Omega)$ *of problem* (3.1), *which is the unique* global minimizer *of the functional*

$$(3.2) \qquad J_{\lambda,\varepsilon}(u) = \frac{1}{2}\rho_\varepsilon(u)^2 - \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}\, dx.$$

   Moreover, $J_{\lambda,\varepsilon}(w_{\lambda,\varepsilon}) < 0$.

   ii) $w_{\lambda,\varepsilon} \in C^{1,\alpha}(\overline{\Omega})$ *for every $\alpha \in (0,1)$, and $\partial_\nu w_{\lambda,\varepsilon} < 0$ on $\partial\Omega$.*

   iii) *There exists $\varepsilon_0 > 0$ with the following property:* given any ball $B_R(x_0) \subseteq \Omega$ *and any $0 < r \leq \min\{1, R\}$, for every $0 < \varepsilon \leq \varepsilon_0$ we have*

$$w_{\lambda,\varepsilon} \geq c_2 \quad \text{a.e. on } B_r(x_0),$$

   for some constant $c_2 > 0$ independent of ε.

We explicitly stress that, since problem (3.1) is of the form (2.7) (with $f = \lambda t^p$), the definition of weak solution of (3.1) is given in Definition 2.2.

Proof. We prove separately the three assertions.

i) To begin with, by using [12, Theorem 1.2] with the choice $f(t) = \lambda t^p$ (notice that this theorem certainly applies to the operator $\mathcal{L}_\varepsilon$, the role of ε being immaterial) we immediately get the existence of a *unique weak solution*

$$w_{\lambda,\varepsilon} \in \mathcal{X}^{1,2}(\Omega) \cap L^\infty(\Omega)$$

of problem (3.1); moreover, owing to [12, Proposition 6.2] (which also can be applied to the ε-dependent operator $\mathcal{L}_\varepsilon$), we know that this $w_{\lambda,\varepsilon}$ satisfies

$$(3.3) \qquad J_{\lambda,\varepsilon}(w_{\lambda,\varepsilon}) = \min_{\varphi \in \mathcal{X}^{1,2}(\Omega)} J_{\lambda,\varepsilon}(\varphi) \quad \text{and} \quad J_{\lambda,\varepsilon}(w_{\lambda,\varepsilon}) < 0.$$

ii) Since $f(t) = \lambda t^p$ clearly satisfies the growth condition (1.2) (with $C_f = 1$), the regularity up the boundary of $w_{\lambda,\varepsilon}$ follows from Theorem 2.3; this, jointly with the fact that $w_{\lambda,\varepsilon} > 0$ in Ω, allows us to apply 2.4, obtaining

$$\partial_\nu w_{\lambda,\varepsilon} < 0 \text{ pointwise on } \partial\Omega.$$

iii) The proof of this assertion is split into three steps, and it is very similar to that of [13, Proposition 3.7]; we present it in detail for the sake of completeness.

   Step 1). In this first step, we prove that

$$(3.4) \qquad \|w_{\lambda,\varepsilon}\|_{H_0^1(\Omega)} \leq \lambda^{\frac{1}{1-p}}\, c \quad \text{for all } \varepsilon \in (0,1].$$

for some constant $c > 0$ only depending on Ω and on p.

Indeed, since $w_{\lambda,\varepsilon} \in \mathcal{X}^{1,2}(\Omega)$ is a weak solution of problem (3.1) (in the sense of Definition 2.2), by choosing $w_{\lambda,\varepsilon}$ as a test function in (2.8), we get

$$\|w_{\lambda,\varepsilon}\|_{H_0^1(\Omega)}^2 \leq \rho_\varepsilon(w_{\lambda,\varepsilon})^2 = \frac{\lambda}{p+1}\int_\Omega w_{\lambda,\varepsilon}^{p+1}\,dx$$

(using Hölder's and Poincaré inequality)

$$\leq \frac{\lambda|\Omega|^{1-\frac{p+1}{2}}}{p+1}\|w_{\lambda,\varepsilon}\|^{p+1} \leq \lambda\,c(\Omega,p)\|w_{\lambda,\varepsilon}\|_{H_0^1(\Omega)}^{p+1},$$

where $c(\Omega,p) > 0$ is a suitable constant only depending on Ω and on p. From this, since $w_{\lambda,\varepsilon} \not\equiv 0$ (and since $0 < p < 1$), we immediately derive the claimed (3.4).

STEP 2). In this second step we prove that, as $\varepsilon \to 0^+$, we have

a) $w_{\lambda,\varepsilon} \to w_\lambda$ weakly in $\mathcal{X}^{1,2}(\Omega)$;
b) $w_{\lambda,\varepsilon} \to w_\lambda$ strongly in $L^m(\Omega)$ for every $1 \leq m < 2^*$;

where $w_\lambda \in H_0^1(\Omega)$ is the unique solution of the purely local problem

$$(3.5) \qquad \begin{cases} -\Delta u = \lambda u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

To this end, we let $\{\varepsilon_j\}_j \subseteq (0,1]$ be any sequence converging to 0 as $j \to +\infty$, and we arbitrarily choose a subsequence $\{\varepsilon_{j_k}\}_k$ of $\{\varepsilon_j\}_j$. On account of (3.4), and using Remark 2.1 - 5), we know that there exists some function $\bar{w} \in \mathcal{X}^{1,2}(\Omega)$ such that, as $k \to +\infty$ and up to possibly choosing a further subsequence,

• $w_{\lambda,\varepsilon_{j_k}} \to \bar{w}$ weakly in $\mathcal{X}^{1,2}(\Omega)$;
• $w_{\lambda,\varepsilon_{j_k}} \to \bar{w}$ strongly in $L^m(\Omega)$ for every $1 \leq m < 2^*$;

from this, since $w_{\lambda,\varepsilon}$ is a global minimizer for $J_{\lambda,\varepsilon}$ (see (3.3)), we get

$$\begin{aligned} J_\lambda(\bar{w}) &= \frac{1}{2}\|\bar{w}\|_{H_0^1(\Omega)}^2 - \frac{\lambda}{p+1}\int_\Omega |\bar{w}|^{p+1}\,dx \\ &\leq \liminf_{k\to+\infty} J_{\lambda,\varepsilon_{j_k}}(w_{\lambda,\varepsilon_{j_k}}) \\ &\leq \liminf_{k\to+\infty} J_{\lambda,\varepsilon_{j_k}}(\varphi) = J_\lambda(\varphi) \quad \text{for every } \varphi \in H_0^1(\Omega), \end{aligned}$$

and thus $\bar{w}$ is a global minimizer for the functional J_λ naturally associated with the purely local, singular problem (3.5). As a consequence, since J_λ as a unique global minimizer which is the unique solution w_λ of (3.5) (that is, assertion i) also holds in the purely local case, see [19]), we derive that

$$\bar{w} = w_\lambda.$$

Due to arbitrariness of the sequence $\{\varepsilon_j\}_j \subseteq (0,1]$ of its subsequence $\{\varepsilon_{j_k}\}_k$, and since the limit *is always the same*, we conclude the validity of a) - b).

STEP 3) In this last step, we complete the proof of the assertion. First of all we observe that, since $w_{\lambda,\varepsilon}$ is a weak solution of problem (3.1), we readily derive that $w_{\lambda,\varepsilon}$ is a also *weak supersolution* (in the sense of Definition 2.2) of $\mathcal{L}_\varepsilon u = 0$ in Ω;

thus, by [13, Proposition 3.3] (and since $w_{\lambda,\varepsilon} \geq 0$ a.e. in $\mathbb{R}^n$) we get

$$(3.6) \qquad w_{\lambda,\varepsilon} \geq c \left(\fint_{B_r(x_0)} w_{\lambda,\varepsilon}^Q \, dx \right)^{1/Q} \qquad \text{a.e. on } B_r(x_0),$$

where Q, $c > 0$ do not depend on ε. On the other hand, since by Step 2) we know that $w_{\lambda,\varepsilon} \to w_\lambda$ as $\varepsilon \to 0$ strongly in $L^m(\Omega)$ for every $1 \leq m < 2^*$, we have

$$(3.7) \qquad \lim_{\varepsilon \to 0^+} \fint_{B_r(x_0)} w_{\lambda,\varepsilon}^Q \, dx = \fint_{B_r(x_0)} w_\lambda^Q \, dx.$$

Gathering (3.6)-(3.7), and recalling that w_λ is the unique solution of (3.5) (hence, $w_\lambda > 0$ a.e. in Ω), we can then fined some $\varepsilon_0 > 0$ such that

$$w_{\lambda,\varepsilon} \geq \frac{c}{2} \left(\fint_{B_r(x_0)} w_\lambda^Q \, dx \right)^{1/Q} > 0$$

a.e. in $B_r(x_0)$ and *for every* $0 < \varepsilon \leq \varepsilon_0$. This ends the proof. $\qquad\square$

We now turn to prove our first main result, namely Theorem 1.4. Before doing this, we establish the following auxiliary lemma.

Lemma 3.2. *Let $0 < \varepsilon \leq 1$, $\lambda > 0$ be fixed, and let $w_{\lambda,\varepsilon} \in \mathcal{X}^{1,2}(\Omega)$ be the unique solution of problem (3.1) (whose existence and uniqueness is guaranteed by Proposition 3.1). Then there exists a positive constant $\beta = \beta_{\lambda,\varepsilon} > 0$ such that*

$$(3.8) \qquad \rho_\varepsilon(\varphi)^2 - \lambda p \int_\Omega w_{\lambda,\varepsilon}^{p-1} \varphi \, dx \geq \beta_{\lambda,\varepsilon} \int_\Omega \varphi^2 \, dx, \quad \text{for all } \varphi \in \mathcal{X}^{1,2}(\Omega).$$

Proof. We first remind that, on account of Proposition 3.1-i), the function $w_{\lambda,\varepsilon}$ is the unique, global minimizer of the functional $J_{\lambda,\varepsilon}$ in (3.2), that is,

$$J_{\lambda,\varepsilon}(w_{\lambda,\varepsilon}) = \min_{u \in \mathcal{X}^{1,2}(\Omega)} J_{\lambda,\varepsilon}(u);$$

this, together with the fact that $w_{\lambda,\varepsilon} \in C^1(\overline{\Omega})$, see Proposition 3.1-ii), ensures that the *second variation* of the functional $J_{\lambda,\varepsilon}$ at $w_{\lambda,\varepsilon}$ is non-negative, namely

$$(3.9) \qquad \rho_\varepsilon(\varphi)^2 - p\lambda \int_\Omega w_{\lambda,\varepsilon}^{p-1} \varphi^2 \, dx \geq 0 \quad \text{for every } \varphi \in \mathcal{X}^{1,2}(\Omega).$$

As a consequence, we get

$$(3.10) \qquad \begin{aligned} &\lambda_1(\mathcal{L}_\varepsilon - \lambda \, p \, w_{\lambda,\varepsilon}^{p-1}) \\ &= \inf \left\{ \rho_\varepsilon(\varphi)^2 - \lambda \, p \int_\Omega w_{\lambda,\varepsilon}^{p-1} \varphi^2 \, dx : \varphi \in \mathcal{X}^{1,2}(\Omega), \, \|\varphi\|_{L^2(\Omega)} = 1 \right\} \geq 0. \end{aligned}$$

With (3.10) at hand, to prove (3.8) it then suffices to show that

$$(3.11) \qquad \lambda_1(\mathcal{L}_\varepsilon - \lambda \, p w_{\lambda,\varepsilon}^{p-1}) > 0;$$

in fact, if (3.11) holds, the desired (3.8) follows with the choice

$$\beta_{\lambda,\varepsilon} = \lambda_1(\mathcal{L}_\varepsilon - \lambda \, p w_{\lambda,\varepsilon}^{p-1}).$$

Hence, we turn to prove (3.11). To this end, we argue by contradiction assuming that (3.11) is not true; thus, by (3.10) we necessarily have that

$$\lambda_1(\mathcal{L}_\varepsilon - \lambda \, p w_{\lambda,\varepsilon}^{p-1}) = 0.$$

Taking into account Proposition 3.1‑ii), and reminding that $w_{\lambda,\varepsilon} > 0$ *pointwise in* Ω (as $w_{\lambda,\varepsilon} \in C^1(\overline{\Omega})$), we can find some $\delta = \delta_{\lambda,\varepsilon} > 0$ such that

$$w_{\lambda,\varepsilon}(x) \geq \delta_{\lambda,\varepsilon} \operatorname{dist}(x, \partial\Omega) \quad \text{for every } x \in \overline{\Omega};$$

from this, by arguing exactly as in [1, Remark 3.1], we infer the existence of a *non-negative eigenfunction* associate with $\lambda_1(\mathcal{L}_\varepsilon - a(x))$; this means, precisely, that there exists a function $\psi_{\lambda,\varepsilon} \in \mathcal{X}^{1,2}(\Omega)$, $\psi_{\lambda,\varepsilon} \gneqq 0$, such that

$$\mathcal{B}_\varepsilon(\psi_{\lambda,\varepsilon}, \varphi) - \lambda\, p \int_\Omega w_{\lambda,\varepsilon}^{p-1} \psi_{\lambda,\varepsilon}\, dx = 0 \quad \text{for every } \varphi \in \mathcal{X}^{1,2}(\Omega).$$

Now, choosing $\varphi = w_{\lambda,\varepsilon}$ in the above identity, we get

$$\mathcal{B}_\varepsilon(\psi_{\lambda,\varepsilon}, w_{\lambda,\varepsilon}) = \lambda\, p \int_\Omega w_{\lambda,\varepsilon}^p \psi_{\lambda,\varepsilon}\, dx; \tag{3.12}$$

on the other hand, if we choose $\varphi = \psi_{\lambda,\varepsilon}$ as a test function for the equation solved by $w_{\lambda,\varepsilon}$ appearing in problem (3.1) (see Definition 2.2), we have

$$\mathcal{B}_\varepsilon(\psi_{\lambda,\varepsilon}, w_{\lambda,\varepsilon}) = \lambda \int_\Omega w_{\lambda,\varepsilon}^p \psi_{\lambda,\varepsilon}\, dx. \tag{3.13}$$

Gathering (3.13)‑(3.12), we then conclude that

$$\lambda \int_\Omega w_{\lambda,\varepsilon}^p \psi_{\lambda,\varepsilon}\, dx = \lambda\, p \int_\Omega w_{\lambda,\varepsilon}^p \psi_{\lambda,\varepsilon}\, dx$$

which is a contradiction, since $p \in (0,1)$ (also recall that $w_{\lambda,\varepsilon} > 0$ pointwise in Ω, and that $\psi_{\lambda,\varepsilon} \gneqq 0$). This ends the proof. $\qquad\square$

With Lemma 3.2 at hand, we can provide the

Proof of Theorem 1.2. Let $\varepsilon \in (0,1]$ be fixed, and let $\beta_\varepsilon = \beta_{1,\varepsilon} > 0$ be as in Lemma 3.2, with $\lambda = 1$. Moreover, let $M_\varepsilon > 0$ be such that

$$p\, M_\varepsilon^{2^*-1} < \beta_\varepsilon. \tag{3.14}$$

We now prove that, *with this choice of* M_ε, assertion 1) is satisfied.

To this end, we argue by contradiction assuming that, for some fixed $\lambda \in (0, \Lambda_\varepsilon)$, there exists a second solution $v_{\lambda,\varepsilon}$ of problem $(\mathrm{P}_\varepsilon)_\lambda$ such that

$$\|v_{\lambda,\varepsilon}\|_{L^\infty(\mathbb{R}^n)} \leq M_\varepsilon. \tag{3.15}$$

By Theorem 1.1 (which can be applied to $\mathcal{L}_\varepsilon$, see Lemma 2.6), we know that there exists a *minimal solution* $u_{\lambda,\varepsilon}$ of problem $(\mathrm{P}_\varepsilon)_\lambda$; hence, we have

$$v_{\lambda,\varepsilon} = u_{\lambda,\varepsilon} + g_\varepsilon,$$

for some function $g_\varepsilon \in \mathcal{X}^{1,2}(\Omega)$, $g_\varepsilon \geq 0$ in Ω.

We now define the auxiliary function $\zeta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ as

$$\zeta_\varepsilon(x) := \lambda^{1/(1-p)}\, w_{1,\varepsilon}(x),$$

where $w_{1,\varepsilon}$ is the unique solution of the purely sublinear problem (3.1) with $\lambda = 1$. A direct computation shows that ζ weakly solves

$$\mathcal{L}_\varepsilon \zeta_\varepsilon = \lambda\, \zeta_\varepsilon^p \quad \text{in } \Omega. \tag{3.16}$$

On the other hand, since $u_{\lambda,\varepsilon}$ is a weak solution of problem $(\mathrm{P}_\varepsilon)_\lambda$ (hence, in particular, $u_{\lambda,\varepsilon} > 0$ a.e. in Ω), this function is a *weak supersolution* of problem (3.1) (in the sense of Definition 2.2 with $f(t) = \lambda t^p$); as a consequence, since we have already

observed that ζ solves (3.1), we can use [2, Lemma 3.11] (which can certainly be applied to the operator $\mathcal{L}_\varepsilon$, the role of ε being immaterial), obtaining

$$(3.17) \qquad \zeta_\varepsilon = \lambda^{1/(1-p)} w_{1,\varepsilon} \le u_{\lambda,\varepsilon} \text{ in } \mathbb{R}^n.$$

With all this at hand we can now conclude. First,

$$
\begin{aligned}
\mathcal{L}_\varepsilon v_{\lambda,\varepsilon} &= \mathcal{L}_\varepsilon(u_{\lambda,\varepsilon} + g_\varepsilon) \\
&= \lambda(u_{\lambda,\varepsilon} + g_\varepsilon)^p + (u_{\lambda,\varepsilon} + g_\varepsilon)^{2^*-1} \quad (v_{\lambda,\varepsilon} \text{ is a solution of } (\mathrm{P}_\varepsilon)_\lambda) \\
&\le \lambda u_{\lambda,\varepsilon}^p + \lambda p\, u_{\lambda,\varepsilon}^{p-1} g_\varepsilon + (u_{\lambda,\varepsilon} + g_\varepsilon)^{2^*-1} \quad (\text{by concavity of } t \mapsto t^p);
\end{aligned}
$$

as a consequence, we have

$$
\begin{aligned}
\mathcal{L}_\varepsilon g_\varepsilon &\le \lambda u_{\lambda,\varepsilon}^p + \lambda p\, u_{\lambda,\varepsilon}^{p-1} g_\varepsilon + (u_{\lambda,\varepsilon} + g_\varepsilon)^{2^*-1} - \mathcal{L}_\varepsilon u_{\lambda,\varepsilon} \\
&= \lambda u_{\lambda,\varepsilon}^p + \lambda p\, u_{\lambda,\varepsilon}^{p-1} g_\varepsilon + (u_{\lambda,\varepsilon} + g_\varepsilon)^{2^*-1} - \lambda u_{\lambda,\varepsilon}^p - u_{\lambda,\varepsilon}^{2^*-1} \\
&= \lambda p\, u_{\lambda,\varepsilon}^{p-1} g_\varepsilon + (u_{\lambda,\varepsilon} + g_\varepsilon)^{2^*-1} - u_{\lambda,\varepsilon}^{2^*-1} \\
&\le p\, w_{1,\varepsilon}^{p-1} g_\varepsilon + (u_{\lambda,\varepsilon} + g_\varepsilon)^{2^*-1} - u_{\lambda,\varepsilon}^{2^*-1} \quad (\text{by (3.17), and since } 0 < p < 1) \\
&\le p\, w_{1,\varepsilon}^{p-1} g_\varepsilon + p\, M_\varepsilon^{2^*-1} g_\varepsilon \quad (\text{by (3.15)}) \\
&< p\, w_{1,\varepsilon}^{p-1} g_\varepsilon + \beta_\varepsilon\, g_\varepsilon \quad (\text{by (3.14)}).
\end{aligned}
$$

Summing up, the function g_ε satisfies (in the weak sense on Ω)

$$\mathcal{L}_\varepsilon g_\varepsilon - p\, w_{1,\varepsilon}^{p-1} g_\varepsilon < \beta_\varepsilon g_\varepsilon.$$

Testing it with g itself, and using (3.8) (with $\lambda = 1$), we get

$$\beta_\varepsilon \int_\Omega g_\varepsilon^2\, dx \le \rho_\varepsilon(g_\varepsilon)^2 - p \int_\Omega w_{1,\varepsilon}^{p-1} g_\varepsilon\, dx < \beta_\varepsilon \int_\Omega g_\varepsilon^2\, dx,$$

which is a contradiction. $\qquad\square$

4. Proof of Theorem 1.3

In this section we show that the famous result [18, Theorem 1] still holds in the mixed local-nonlocal setting. The proof is a quite simple adaptation of the original proof by Brezis and Nirenberg and the major role is played by the recent regularity results established in [3] and [38].

Proof of Theorem 1.3. By assumption, $u_0 \in \mathcal{X}^{1,2}(\Omega)$ is a local minimizer for Φ in the C^1-topology; this means, precisely, that there exists some $r > 0$ such that

$$(4.1) \qquad \Phi(u_0) \le \Phi(u_0 + v), \quad \text{for all } v \in C_0^1(\overline{\Omega}) \text{ s.t. } \|v\|_{C^1(\Omega)} \le r.$$

It follows that the Euler-Lagrange equation satisfied by u_0 is

$$
\begin{cases}
\mathcal{L}_\varepsilon v = f(x, v) & \text{in } \Omega, \\
v = 0 & \text{in } \mathbb{R}^n \setminus \Omega;
\end{cases}
$$

as a consequence, since (by assumption) f satisfies the growth condition (1.2), from Theorem 2.3 we derive that $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. In view of this fact, from now on we may assume without loss of generality that

$$u_0 = 0.$$

We now argue by contradiction, assuming that u_0 *is not a local minimizer for* Φ *in the* $\mathcal{X}^{1,2}$*-topology*; thus, for every $\delta > 0$ there exists $v_\delta \in \mathcal{X}^{1,2}(\Omega)$ such that

$$(4.2) \qquad \rho_\varepsilon(v_\delta) < \delta \quad \text{and} \quad \Phi(v_\delta) < \Phi(0).$$

We explicitly notice that use of $\rho_\varepsilon(\cdot)$ to define an open neighborhood of $u_0 = 0$ in (4.2) is motivated by the fact that, since $\varepsilon \in (0,1)$, by (2.6) we have

$$\|u\|_{H_0^1(\Omega)} \leq \rho_\varepsilon(u) \leq \rho(u) \leq \Theta\|u\|_{H_0^1(\Omega)}$$

for some $\Theta > 0$ only depending on n, s; as a consequence, the norm $\rho_\varepsilon(\cdot)$ is globally equivalent to $\rho(\cdot)$ (and to the H_0^1-norm), uniformly w.r.t. ε.

Now, since the closed set $K_\delta = \{u \in \mathcal{X}^{1,2}(\Omega) : \rho_\varepsilon(u) \leq \delta\} \subseteq$ is weakly sequentially compact, and since Φ is weakly lower semicontinuous and bounded from below (by the growth assumption (1.2) on f), we can find $w_\delta \in K_\delta$ such that

$$\Phi(w_\delta) = \min_{u \in K_\delta} \Phi(u).$$

In particular, by (4.2) we get

$$(4.3) \qquad \Phi(w_\delta) \leq \Phi(v_\delta) < \Phi(0) \quad \text{for every } \delta > 0.$$

We then claim that

$$(4.4) \qquad \text{i) } w_\delta \in C_0^1(\overline{\Omega}) \quad \text{and} \quad \text{ii) } \lim_{\delta \to 0} w_\delta = 0 \text{ in the } C^1(\overline{\Omega})\text{-topology.}$$

Taking this claim for granted for a moment, we can immediately complete the proof of the theorem: in fact, owing to (4.4), we see that there exists $\delta_0 > 0$ such that

$$\|w_\delta\|_{C^1(\overline{\Omega})} \leq r \quad \text{for every } 0 < \delta < \delta_0,$$

where $r > 0$ is as in (4.1); thus, by (4.1) (and since we are assuming $u_0 = 0$) we infer that $\Phi(0) \leq \Phi(w_\delta)$ for every $0 < \delta < \delta_0$, but this clearly contradicts (4.3).

We then turn to prove (4.4), and we proceed by steps.

STEP I). In this first step we prove that there exist $\delta_0 > 0$ and a constant $c_0 > 0$, *independent of* δ, such that the following estimate holds:

$$(4.5) \qquad \|w_\delta\|_{L^\infty(\Omega)} \leq c_0 \quad \text{for every } 0 < \delta < \delta_0.$$

To this end we first notice that, since w_δ is a global minimizer for Φ on K_δ, by the Lagrange Multiplier Theorem, there exists $\mu_\delta \in \mathbb{R}$ such that

$$(4.6) \qquad \mathcal{B}_\varepsilon(w_\delta, \varphi) - \int_\Omega f(x, w_\delta)\varphi \, dx = \mu_\delta \mathcal{B}_\varepsilon(w_\delta, \varphi) \quad \text{for all } \varphi \in \mathcal{X}^{1,2}(\Omega).$$

If $\rho_\varepsilon(v_\delta) < \delta$ (that is, if $w_\delta \in \text{int}(K_\delta)$), then $\mu_\delta = 0$; if, instead, $\rho_\varepsilon(v_\delta) = \delta$ (that is, if $w_\delta \in \partial K_\delta$), by choosing $\varphi = v_\delta$ in (4.6) we derive that

$$\mu_\delta \, \delta^2 = \mathcal{B}_\varepsilon(w_\delta, w_\delta) = \mathcal{B}_\varepsilon(w_\delta, w_\delta) - \int_\Omega f(x, w_\delta)w_\delta \, dx$$

$$= \Phi(w_\delta) + \int_\Omega F(x, w_\delta) \, dx - \int_\Omega f(x, w_\delta)w_\delta \, dx$$

$$\text{(using (4.3), and since } \Phi(0) = 0)$$

$$< \int_\Omega \left(F(x, w_\delta) \, dx - \int_\Omega f(x, w_\delta)w_\delta \right) dx \leq 0,$$

where we have used the growth condition (1.2). Summing up, we derive that $\mu_\delta \leq 0$ and that $w_\delta \in \mathcal{X}^{1,2}(\Omega)$ is a of the Dirichlet problem

$$(4.7) \qquad \begin{cases} \mathcal{L}_\varepsilon u = \dfrac{f(x,u)}{1-\mu_\delta} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

(in the sense of Definition 2.2). Now, since f satisfies the growth condition (1.2) and since $\mu_\delta \leq 0$, it is readily seen that the function

$$(4.8) \qquad g_\delta(x,t) = \frac{f(x,t)}{1-\mu_\delta} \qquad (x \in \Omega,\, t \in \mathbb{R})$$

satisfies *the same growth condition* (1.2), with the same constant $C_f > 0$; we are then entitled to apply [38, Theorem 1.1], thus giving

$$(4.9) \qquad \|w_\delta\|_{L^\infty(\Omega)} \leq \kappa_0 \Big(1 + \int_\Omega |w_\delta|^{2^*\beta}\, dx\Big)^{\frac{1}{2^*(\beta-1)}},$$

where $\beta = (2^*+1)/2$ and $\kappa_0 > 0$ is a constant independent of δ. With (4.9) at hand, to complete the proof of (4.5) we need to show that

$$(4.10) \qquad \int_\Omega |w_\delta|^{2^*\beta}\, dx \leq c$$

for some constant $c > 0$ independent of δ, provided that δ is sufficiently small. To prove (4.10), we closely follow the approach in [38, Lemma 3.2]: defining

$$\varphi(t) := \begin{cases} -\beta T^{\beta-1}(t+T) + T^\beta, & t \leq -T \\ |t|^\beta, & -T < t < T \\ \beta T^{\beta-1}(t-T) + T^\beta, & t \geq T. \end{cases}$$

(with $T > 0$), and arguing exactly as in [38, Lemma 3.2], we get

$$(4.11) \quad \Big(\int_\Omega |\varphi(w_\delta)|^{2^*}\, dx\Big)^{2/2^*} \leq c\beta \Big(\int_\Omega |w_\delta|^{2^*}\, dx + \int_\Omega (\varphi(w_\delta))^2 |w_\delta|^{2^*-2}\, dx\Big),$$

which is exactly to [38, Equation (3.5)] (here, $c > 0$ is a constant only depending on n, Ω and on the constant C_f). On the other hand, we have

$$\int_\Omega (\varphi(w_\delta))^2 |w_\delta|^{2^*-2}\, dx$$
$$= \int_{\{|w_\delta|\leq 1\}} (\varphi(w_\delta))^2 |w_\delta|^{2^*-2}\, dx + \int_{\{|w_\delta|>1\}} (\varphi(w_\delta))^2 |w_\delta|^{2^*-2}\, dx$$
$$\leq \int_{\{|w_\delta|\leq 1\}} \frac{\varphi(w_\delta)^2}{|w_\delta|}\, dx + \int_\Omega (\varphi(w_\delta))^2 |w_\delta|^{2^*-2}\, dx = (\bigstar);$$

then, using Hölder's and Sobolev's inequality, we obtain

$$(\bigstar) \leq \int_{\{|w_\delta|\leq 1\}} \frac{|\varphi(w_\delta)|^2}{|w_\delta|}\, dx + \Big(\int_\Omega |\varphi(w_\delta)|^{2^*}\, dx\Big)^{2/2^*} \Big(\int_\Omega |w_\delta|^{2^*}\, dx\Big)^{\frac{2^*-2}{2^*}}$$
$$\leq \int_{\{|w_\delta|\leq 1\}} \frac{|\varphi(w_\delta)|^2}{|w_\delta|}\, dx + S_n^{\frac{2-2^*}{2}} \Big(\int_\Omega |\varphi(w_\delta)|^{2^*}\, dx\Big)^{2/2^*} \|w_\delta\|_{H_0^1(\Omega)}^{2^*-2}$$
$$(\text{since } \rho_\varepsilon(\cdot) \geq \|\cdot\|_{H_0^1}, \text{ and since } w_\delta \in K_\delta)$$

$$\leq \int_{\{|w_\delta|\leq 1\}} \frac{|\varphi(w_\delta)|^2}{|w_\delta|}\, dx + \delta^{2^*-2} S_n^{\frac{2-2^*}{2}} \Big(\int_\Omega |\varphi(w_\delta)|^{2^*}\, dx \Big)^{2/2^*}.$$

where $S_n > 0$ is the best constant in the (local) Sobolev inequality. Using this last estimate in the above (4.11), we then derive

$$(4.12) \qquad \begin{aligned} \Big(\int_\Omega |\varphi(w_\delta)|^{2^*}\, dx \Big)^{2/2^*} &\leq c\beta \Big(\int_\Omega |w_\delta|^{2^*}\, dx + \int_{\{|w_\delta|\leq 1\}} \frac{|\varphi(w_\delta)|^2}{|w_\delta|}\, dx \Big) \\ &\quad + c\beta \delta^{2^*-2} S_n^{\frac{2-2^*}{2}} \Big(\int_\Omega |\varphi(w_\delta)|^{2^*}\, dx \Big)^{2/2^*} \Big). \end{aligned}$$

With (4.12) at hand, we can proceed toward the end of the proof of (4.10): indeed, if we choose $\delta_0 \in (0,1)$ so small that

$$c\beta \delta^{2^*-2} S_n^{\frac{2-2^*}{2}} < \frac{1}{2}$$

and if we let $0 < \delta < \delta_0$, we can reabsorb the last integral in the right-hand side of the cited (4.12), thus giving

$$\Big(\int_\Omega |\varphi(w_\delta)|^{2^*}\, dx \Big)^{2/2^*} \leq 2c\beta \Big(\int_\Omega |w_\delta|^{2^*}\, dx + \int_{\{|w_\delta|\leq 1\}} \frac{|\varphi(w_\delta)|^2}{|w_\delta|}\, dx \Big);$$

from this, by letting $T \to +\infty$ and by arguing as in [38, Lemma 3.2], we get

$$\Big(\int_\Omega |w_\delta|^{2^*\beta}\, dx \Big)^{2/2^*} \leq 4c\beta \int_\Omega |w_\delta|^{2^*}\, dx \quad \text{for every } 0 < \delta < \delta_0.$$

Finally, by using once again the Sobolev inequality, we conclude that

$$\Big(\int_\Omega |w_\delta|^{2^*\beta}\, dx \Big)^{2/2^*} \leq 4c\beta\, S_n^{-2^*/2} \rho_\varepsilon(w_\delta) \leq 4c\beta\, S_n^{-2^*/2},$$

and this estimate holds for every $0 < \delta < \delta_0$ (since $\delta_0 < 1$). Recalling that $c > 0$ only depends on n, Ω and on the constant C_f, this completes the proof of (4.10).

STEP II). In this second step we complete the proof of (4.4)-i). To this end it suffices to observe that, since the function g_δ in (4.8) satisfies the growth condition (1.2) *with the same constant C_f* (independent of δ), we are entitled to apply [38, Theorem 1.3]; this, together with (4.5), ensures that

1) $w_\delta \in C^{1,\alpha}(\overline{\Omega})$ for every $0 < \alpha < \min\{1, 2-2s\}$;
2) there exists $c > 0$, independent of δ, such that

$$(4.13) \qquad \|w_\delta\|_{C^{1,\alpha}(\overline{\Omega})} \leq c(1 + \|w_\delta\|_{L^\infty(\Omega)}^{2^*}) \leq c \quad \text{for every } 0 < \delta < \delta_0.$$

In particular, since $w_\delta \in \mathcal{X}^{1,2}(\Omega)$, we get $w_\delta \equiv 0$ pointwise on $\partial\Omega$.

STEP III). In this last step we give the proof of (4.4)-ii). To this end we observe that, on account of (4.13), by the Arzelà-Ascoli Theorem we can find a sequence $\{\delta_j\} \subseteq (0, \delta_0)$ converging to 0 as $j \to +\infty$ and $w_0 \in C_0^1(\overline{\Omega})$ such that

$$\|w_{\delta_j} - w_0\|_{C^1(\overline{\Omega})} \to 0 \text{ as } j \to +\infty;$$

thus, since $w_{\delta_j} \to 0$ in $\mathcal{X}^{1,2}(\Omega)$ as $j \to +\infty$ (recall that $w_\delta \in K_\delta$ and that the two norms $\rho_\varepsilon(\cdot)$, $\rho(\cdot)$ are globally equivalent in $\mathcal{X}^{1,2}(\Omega)$), we conclude that

$$w_0 \equiv 0.$$

This ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Proof of Theorem 1.4

Following the approach in [1], in this section we show how Theorem 1.3 can be used in order to establish Theorem 1.4, that is, the existence of (at least) *two distinct solutions* of $(\mathrm{P}_\varepsilon)_\lambda$ provided that $\varepsilon \in (0,1)$ is small enough.

We now turn to prove that, if $\varepsilon \in (0,1]$ is fixed and if $\lambda \in (0, \mu_\sharp)$ (with $\mu_\sharp > 0$ as in Lemma 2.6), there exists a solution $u_{\lambda,\varepsilon}$ of problem $(\mathrm{P}_\varepsilon)_\lambda$ which is a *local minimizer* for the functional associated with the problem, that is,

$$(5.1) \qquad I_{\lambda,\varepsilon}(u) = \frac{1}{2}\rho_\varepsilon(u)^2 - \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx,$$

where $\rho_\varepsilon(u)^2 := \|u\|_{H_0^1(\Omega)} + \varepsilon\,[u]_{s,\mathbb{R}^n}^2$.

Theorem 5.1. *Let $\varepsilon \in (0,1]$ be fixed, and let $\lambda \in (0, \mu_\sharp)$ (with $\mu_\sharp > 0$ as in Lemma 2.6). Then, there exists a solution $u_{\lambda,\varepsilon} \in \mathcal{X}^{1,2}(\Omega)$ of problem $(\mathrm{P}_\varepsilon)_\lambda$ which is a local minimizer for $I_{\lambda,\varepsilon}$, that is, there exists $\varrho_0 > 0$ such that*

$$I_{\lambda,\varepsilon}(u) \geq I_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}) \quad \text{for all } u \in \mathcal{X}^{1,2}(\Omega) \text{ with } \rho(u - u_{\lambda,\varepsilon}) < \varrho_0.$$

Moreover, the following assertions hold:

1) *there exists a constant $C > 0$, independent of ε but possibly depending on the fixed $\lambda \in (0, \mu_\sharp)$, such that*

$$(5.2) \qquad\qquad\qquad \|u_{\lambda,\varepsilon}\|_{L^\infty(\Omega)} \leq C;$$

2) *if $w_{\lambda,\varepsilon} \in \mathcal{X}^{1,2}(\Omega)$ is the unique solution of (3.1), we have*

$$(5.3) \qquad\qquad\qquad u_{\lambda,\varepsilon} \geq w_{\lambda,\varepsilon} \quad \text{a.e. in } \Omega.$$

Proof. The proof is analogous to that of [1, Lemma 4.1], but we present it here for the sake of completeness. To begin with, we fix $\lambda_1, \lambda_2 \in (0, \mu_\sharp)$ such that

$$(5.4) \qquad\qquad\qquad \lambda_1 < \lambda < \lambda_2;$$

accordingly, we let $u_i = \bar{u}_{\lambda_i,\varepsilon} \in \mathcal{X}^{1,2}(\Omega)$ be the minimal solution of problem $(\mathrm{P}_\varepsilon)_{\lambda_i}$ (with $i = 1, 2$), whose existence is guaranteed by Lemma 2.6. We then observe that, on account of Theorem 2.3, we have that $u_1, u_2 \in C^1(\overline{\Omega})$; moreover, since the map $t \mapsto \bar{u}_{t,\varepsilon}$ is non-decreasing on $(0, \Lambda_\varepsilon)$, by (5.4) we also have

$$u_1 \leq u_2 \text{ pointwise in } \Omega.$$

In view of these facts, and since u_i solves $(\mathrm{P}_\varepsilon)_{\lambda_i}$ (for $i = 1, 2$), we infer that

  a) $w = u_2 - u_1 \in \mathcal{X}^{1,2}(\Omega) \cap C^1(\overline{\Omega})$;
  b) for every test function $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ on Ω, we have

$$\mathcal{B}_\varepsilon(w, \varphi) = \int_\Omega (\lambda_2 u_2^p + u_2^{2^*-1})\varphi\,dx - \int_\Omega (\lambda_1 u_1^p + u_1^{2^*-1})\varphi\,dx$$

$$\text{(by (5.4), and since } u_2 \geq u_1 \geq 0)$$

$$\geq \lambda_1 \int_\Omega (u_2^p - u_1^p)\varphi\,dx + \int_\Omega (u_2^{2^*-1} - u_1^{2^*-1})\varphi\,dx \geq 0$$

  and this proves that w is a (non-negative) *weak supersolution* (in the sense of Definition 2.2) of the following problem

$$\begin{cases} \mathcal{L}_\varepsilon u = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

Since $w \not\equiv 0$ in Ω (as $\lambda_1 < \lambda_2$), by Theorem 2.4 we conclude that

(5.5) $\qquad w = u_2 - u_1 > 0$ pointwise in Ω and $\partial_\nu(u_2 - u_1) < 0$ on $\partial\Omega$.

With (5.5) at hand, we now define the function

$$f(x,t) = \begin{cases} \lambda u_1(x)^p + u_1(x)^{2^*-1} & \text{if } t \leq u_1(x) \\ \lambda t^p + t^{2^*-1} & \text{if } u_1(x) < t < u_2(x) \\ \lambda u_2(x)^p + u_2(x)^{2^*-1} & \text{if } t \geq u_2(x) \end{cases}$$

(for $x \in \Omega$ and $t \in \mathbb{R}$), and we consider the functional

$$J(u) = \frac{1}{2}\rho_\varepsilon(u)^2 - \int_\Omega F(x,u)\,dx, \qquad \text{with } F(x,t) = \int_0^t f(x,s)\,ds.$$

Since $u_1, u_2 \in C^1(\overline{\Omega})$, the function f is *globally bounded* on $\Omega \times \mathbb{R}$; as a consequence of this fact, it is easy to check that the functional J achieves its (global) minimum at some $u \in \mathcal{X}^{1,2}(\Omega)$, which is a weak solution of

(5.6) $$\begin{cases} \mathcal{L}_\varepsilon u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Moreover, from Theorem 2.3 we get $u \in C^1(\overline{\Omega})$. We then claim that

(5.7) $\qquad$ i) $u_1 < u < u_2$ pointwise in Ω;

$\qquad\qquad$ ii) $\partial_\nu(u - u_1) < 0$ and $\partial_\nu(u - u_2) > 0$ pointwise on $\partial\Omega$.

In fact, let $w_1 = u - u_1$. On the one hand, we have $w_1 \in \mathcal{X}^{1,2}(\Omega) \cap C^1(\overline{\Omega})$ (as the same is true of both u and u_1); on the other hand, since u_1 solves $(\text{P})_{\lambda_1,\varepsilon}$ and since u solves (5.6), by definition of f (and by recalling (5.5)) we also have

$$\mathcal{B}_\varepsilon(w_1, \varphi) = \int_\Omega \big(f(x,u) - \lambda u_1^p - u_1^{2^*-1}\big)\varphi\,dx \geq 0$$

for every test function $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ on Ω. Hence, we see that w_1 is a weak supersolution (in the sense of Definition 2.2) of the problem

$$\begin{cases} \mathcal{L}_\varepsilon u = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

Since $w_1 \not\equiv 0$ in Ω (as $\lambda_1 < \lambda$), again by Theorem 2.4 we then get

$$w_1 = u - u_1 > 0 \text{ pointwise in } \Omega \quad \text{and} \quad \partial_\nu(u - u_1) < 0 \text{ on } \partial\Omega.$$

In a totally analogous way one can prove that $w_2 = u_2 - u > 0$ pointwise in Ω and that $\partial_\nu w_2 < 0$ on $\partial\Omega$, and this completes the proof of (5.7).

Now we have established (5.7), we are finally ready to complete the demonstration of the theorem. Indeed, owing to (5.7), there exists $\varrho_0 > 0$ so small that

$$\|v - u\|_{C^1(\overline{\Omega})} < \varrho_0 \implies u_1 \leq v \leq u_2 \text{ pointwise in } \Omega;$$

from this, taking into account that $J(v) = I_{\lambda,\varepsilon}(v)$ for every $v \in \mathcal{X}^{1,2}(\Omega)$ satisfying $u_1 \leq v \leq u_2$ in Ω (see the explicit definition of f), we derive that

$$I_{\lambda,\varepsilon}(v) = J(v) \geq J(u) = I_{\lambda,\varepsilon}(u) \quad \text{for all } v \in C^1(\overline{\Omega}) \text{ with } \|v - u\|_{C^1(\overline{\Omega})} < \varrho_0,$$

and this proves that u is a local minimizer of $I_{\lambda,\varepsilon}$ in the C^1-topology. This, together with Theorem 1.3, ensures that $u \in \mathcal{X}^{1,2}(\Omega) \cap C^1(\overline{\Omega})$ is a local minimizer of the functional $I_{\lambda,\varepsilon}$ also in the $\mathcal{X}^{1,2}$-topology, as desired.

We now turn to prove the two assertions 1)-2).

- *Proof of assertion* 1). It is a direct consequence of (5.7)-i) and of estimate (2.17) in Lemma 2.6, taking into account that $u_1 = \bar{u}_{\lambda_1,\varepsilon}$.

- *Proof of assertion* 2). First of all we observe that, since $u_{\lambda,\varepsilon}$ is a weak solution of problem $(P_\varepsilon)_\lambda$ (hence, in particular, $u_{\lambda,\varepsilon} > 0$ a.e. in Ω), this function is a *weak supersolution* of problem (3.1) (in the sense of Definition 2.2 with $f(t) = \lambda t^p$); thus, since $w_{\lambda,\varepsilon}$ solves (3.1), we can use [2, Lemma 3.11] (which can certainly be applied to the operator $\mathcal{L}_\varepsilon$, the role of ε being immaterial), obtaining

$$w_{\lambda,\varepsilon} \leq u_{\lambda,\varepsilon} \quad \text{a.e. in } \Omega.$$

We explicitly mention that [2, Lemma 3.11] can be applied, since $f(t)/t = \lambda t^{p-1}$ is decreasing, and since $w_{\lambda,\varepsilon}\, u_{\lambda,\varepsilon} \in L^{2^*}(\Omega)$, see Remark 2.1-4).

This ends the proof. $\square$

Now we have established Theorem 5.1, we are ready to embark on the demonstration of Theorem 1.4. To keep the notation as simple as possible, throughout what follows we avoid to keep explicit track of the dependence on ε of the functional $I_{\lambda,\varepsilon}$ in (5.1) and of the solution $u_{\lambda,\varepsilon}$ obtained in Theorem 5.1. Thus, as in the previous sections we simply write I_λ and u_λ, the dependence on ε being understood.

We begin with the following key lemma.

Lemma 5.2. *Let $\lambda \in (0, \mu_\sharp)$ be fixed (with $\mu_\sharp > 0$ as in (2.18)). Then, there exist $\varepsilon_0 \in (0,1)$, $R_0 > 0$ and a positive function $\Psi \in \mathcal{X}^{1,2}(\Omega)$ such that*

$$(5.8) \quad \begin{cases} I_\lambda(u_\lambda + R\Psi) < I_\lambda(u_\lambda) & \text{for all } \varepsilon \in (0,\varepsilon_0) \text{ and } R \geq R_0, \\ I_\lambda(u_\lambda + tR_0\Psi) < I_\lambda(u_\lambda) + \frac{1}{n}S_n^{n/2} & \text{for all } \varepsilon \in (0,\varepsilon_0) \text{ and } t \in [0,1]. \end{cases}$$

Proof. First of all, we choose a *Lebesgue point* of u_λ in Ω, say y, and we let $r > 0$ be such that $B_r(y) \Subset \Omega$; we then choose a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that

(∗) $0 \leq \varphi \leq 1$ in Ω;
(∗) $\varphi \equiv 1$ on $B_r(y)$;

and we consider the one-parameter family of functions

$$(5.9) \qquad U_\varepsilon = V_\varepsilon\, \varphi, \qquad \text{where } V_\varepsilon(x) = \frac{\varepsilon^{\frac{\alpha(n-2)}{2}}}{(\varepsilon^{2\alpha} + |x-y|^2)^{\frac{n-2}{2}}},$$

where $\alpha > 0$ will be appropriately chosen later on. Notice that this family $\{V_\varepsilon\}_\varepsilon$ is the well-known family of the *Aubin-Talenti functions*, which are the unique (up to translation) extremals in the (local) Sobolev inequality; this means that

$$(5.10) \qquad\qquad \frac{\|\,|\nabla V_\varepsilon|\,\|_{L^2(\mathbb{R}^n)}^2}{\|V_\varepsilon\|_{L^{2^*}(\mathbb{R}^n)}^2} = S_n,$$

where $S_n > 0$ is the *best constant* in the Sobolev inequality.

We now recall that, owing to [16, Lemma 1.1], we have (as $\varepsilon \to 0^+$)

(5.11)
$$\text{i) } \|U_\varepsilon\|^2_{H^1_0(\Omega)} = \varepsilon^{\alpha(n-2)}\Big(\frac{K_1}{\varepsilon^{\alpha(n-2)}} + O(1)\Big) = K_1 + o(\varepsilon^{\frac{\alpha(n-2)}{2}});$$

$$\text{ii) } \|U_\varepsilon\|^{2^*}_{L^{2^*}(\Omega)} = \varepsilon^{\alpha n}\Big(\frac{K_2}{\varepsilon^{\alpha n}} + O(1)\Big) = K_2 + o(\varepsilon^{\alpha n/2});$$

where the constants $K_1, K_2 > 0$ are given, respectively, by

$$K_1 = \||\nabla V_1|\|^2_{L^2(\mathbb{R}^n)}, \qquad K_2 = \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^n}\, dx$$

and they satisfy $K_1/K_2^{1-2/n} = S_n$, see the above (5.10). On the other hand, by combining (5.11)-ii) with [10, Lemma 4.11], we also have

(5.12)
$$\varepsilon\, [U_\varepsilon]^2_s \le \varepsilon \|U_\varepsilon\|^2_{L^{2^*}(\Omega)} \cdot O(\varepsilon^{\alpha(2-2s)}) = \|U_\varepsilon\|^2_{L^{2^*}(\Omega)} \cdot O(\varepsilon^{1+\alpha(2-2s)})$$
$$= \big(K_2 + o(\varepsilon^{\alpha n/2})\big)^{2/2^*} \cdot O(\varepsilon^{1+\alpha(2-2s)}) = o(\varepsilon^{1+\alpha(1-s)}).$$

Defining $w = u_\lambda + tRU_\varepsilon$ (for $R \ge 1$ and $t \in [0,1]$), we then obtain the following estimate (as $\varepsilon \to 0^+$):

$$I_{\lambda,\varepsilon}(w) = I_{\lambda,\varepsilon}(u_\lambda) + \frac{t^2R^2}{2}\rho_\varepsilon(U_\varepsilon)^2 + tR\mathcal{B}_\rho(u_\lambda, U_\varepsilon)$$
$$- \frac{1}{2^*}\big(\|u_\lambda + tRU_\varepsilon\|^{2^*}_{L^{2^*}(\Omega)} - \|u_\lambda\|^{2^*}_{L^{2^*}(\Omega)}\big)$$
$$- \frac{\lambda}{p+1}\Big(\int_\Omega |u_\lambda + tRU_\varepsilon|^{p+1}\, dx - \int_\Omega |u_\lambda|^{p+1}\, dx\Big),$$

and recalling that u_λ solves $(\mathrm{P}_\varepsilon)_\lambda$, we get

$$I_{\lambda,\varepsilon}(w) = I_{\lambda,\varepsilon}(u_\lambda) + \frac{t^2R^2}{2}\rho_\varepsilon(U_\varepsilon)^2 + tR\int_\Omega (\lambda u_\lambda^p + u_\lambda^{2^*-1})U_\varepsilon\, dx$$
$$- \frac{1}{2^*}\big(\|u_\lambda + tRU_\varepsilon\|^{2^*}_{L^{2^*}(\Omega)} - \|u_\lambda\|^{2^*}_{L^{2^*}(\Omega)}\big)$$
$$- \frac{\lambda}{p+1}\Big(\int_\Omega |u_\lambda + tRU_\varepsilon|^{p+1}\, dx - \int_\Omega |u_\lambda|^{p+1}\, dx\Big)$$
$$= I_{\lambda,\varepsilon}(u_\lambda) + \frac{t^2R^2}{2}\rho_\varepsilon(U_\varepsilon)^2 - \frac{t^{2^*}R^{2^*}}{2^*}\|U_\varepsilon\|^{2^*}_{L^{2^*}(\Omega)}$$
$$- t^{2^*-1}R^{2^*-1}\int_\Omega U_\varepsilon^{2^*-1}u_\lambda\, dx - \mathcal{R}_\varepsilon - \mathcal{D}_\varepsilon$$

where we have introduced the notation

$$(*)\ \mathcal{R}_\varepsilon = \frac{1}{2^*}\int_\Omega \Big\{|u_\lambda + tRU_\varepsilon|^{2^*} - u_\lambda^{2^*} - (tRU_\varepsilon)^{2^*}$$
$$- 2^* u_\lambda(tRU_\varepsilon)\big(u_\lambda^{2^*-2} + (tRU_\varepsilon)^{2^*-2}\big)\Big\}dx;$$

$$(*)\ \mathcal{D}_\varepsilon = \frac{\lambda}{p+1}\int_\Omega \Big\{|u_\lambda + tRU_\varepsilon|^{p+1} - |u_\lambda|^{p+1} - tR(p+1)u_\lambda^p U_\varepsilon\Big\}dx.$$

We start estimating $\mathcal{R}_\varepsilon$. To this aim, we follow [17, Proof of Theorem 1] (from equation (17) on) where the main difference is due to the fact that u_λ is actually

dependent on ε as well. However, using the uniform upper bound of the L^∞-norm of u_λ in Theorem 5.1, one can get

$$(5.13) \qquad |-\mathcal{R}_\varepsilon| \leq R^\beta o(\varepsilon^{\frac{\alpha(n-2)}{2}}) \quad \text{as } \varepsilon \to 0^+,$$

for some $\beta \in (0, 2^*)$. Regarding $\mathcal{D}_\varepsilon$, since the map

$$[0, +\infty) \ni r \mapsto \gamma(r) = r^{p+1}$$

is *convex on* $[0, +\infty)$, for every $\varepsilon \in (0, 1)$ we have

$$
\begin{aligned}
(5.14) \qquad -\mathcal{D}_\varepsilon &= \frac{\lambda}{p+1} \int_\Omega \left\{ tR(p+1)u_\lambda^p U_\varepsilon + |u_\lambda|^{p+1} - |u_\lambda + tRU_\varepsilon|^{p+1} \right. \\
&= -\frac{\lambda}{p+1} \int_\Omega \left\{ \gamma(u_\lambda + tRU_\varepsilon) - \gamma(u_\lambda) - \gamma'(u_\lambda)(tRU_\varepsilon) \right\} dx \leq 0.
\end{aligned}
$$

Finally, combining once again (5.3) and Proposition 3.1 - iii), and arguing as in the proof of [10, Lemma 4.11], we get

$$
\begin{aligned}
(5.15) \qquad \int_\Omega U_\varepsilon^{2^*-1} u_\lambda \, dx &\geq \int_{B_r(y)} V_\varepsilon^{2^*-1} w_{\lambda,\varepsilon} \, dx \geq c_2 \int_{B_r(y)} V_\varepsilon^{2^*-1} \, dx \\
&\geq \varepsilon^{\alpha(n-2)/2} D_0 + o(\varepsilon^{\alpha(n-2)/2}) \quad \text{as } \varepsilon \to 0^+,
\end{aligned}
$$

where $D_0 > 0$ is a constant only depending on n.

Gathering (5.11)-(5.12), we obtain

$$
\begin{aligned}
(5.16) \qquad I_{\lambda,\varepsilon} &\leq I_{\lambda,\varepsilon}(u_\lambda) + \frac{t^2 R^2}{2} K_1 - \frac{t^{2^*} R^{2^*}}{2^*} K_2 \\
&\quad - t^{2^*-1} R^{2^*-1} D_0 \varepsilon^{\alpha(n-2)/2} - R^\beta o(\varepsilon^{\alpha(n-2)/2}) + C(tR + t^2 R^2) o(\varepsilon^{\frac{\alpha(n-2)}{2}}) \\
&\quad + (t^2 R^2 + t^{2^*} R^{2^*})(o(\varepsilon^{1+\alpha(1-s)}) + o(\varepsilon^{\alpha(n-2)/2})).
\end{aligned}
$$

Thus, if we choose $\alpha \in (0, 1]$ so small that $1 + \alpha(1-s) > \alpha(n-2)/2$ (notice that this is always possible, since $n \geq 3$ and $s \in (0, 1)$), we get

$$
\begin{aligned}
(5.17) \qquad I_{\lambda,\varepsilon}(u_\lambda + tRU_\varepsilon) &\leq I_{\lambda,\varepsilon}(u_\lambda) + \frac{t^2 R^2}{2} K_1 - \frac{t^{2^*} R^{2^*}}{2^*} K_2 \\
&\quad - t^{2^*-1} R^{2^*-1} D_0 \varepsilon^{\alpha(n-2)/2} \\
&\quad + C(t^2 R^2 + t^{2^*} R^{2^*}) o(\varepsilon^{\alpha(n-2)/2}).
\end{aligned}
$$

Now, thanks to estimate (5.17) we are finally ready to complete the proof of the lemma: in fact, starting from this estimate and repeating word by word the argument in [39, Lemma 3.1], we find $\varepsilon_0 > 0$ and $R_0 > 0$ such that

$$
\begin{cases}
I_\lambda(u_\lambda + RU_\varepsilon) < I_\lambda(u_\lambda) & \text{for all } \varepsilon \in (0, \varepsilon_0) \text{ and } R \geq R_0, \\
I_\lambda(u_\lambda + tR_0 U_\varepsilon) < I_\lambda(u_\lambda) + \frac{1}{n} S_n^{n/2} & \text{for all } \varepsilon \in (0, \varepsilon_0) \text{ and } t \in [0, 1].
\end{cases}
$$

Thus, the lemma is proved by choosing $\Psi = U_\varepsilon$ (with $\varepsilon < \varepsilon_0$ and y, a as above). $\qquad \square$

Thanks to Lemma 5.2, we can now proceed toward the proof of Theorem 1.4: in fact, we turn to show that problem $(\mathrm{P}_\varepsilon)_\lambda$ possesses a second solution $v_\lambda \neq u_\lambda$, provided that $0 < \lambda < \lambda_*$ and $0 < \varepsilon < \varepsilon_0$, *where $\varepsilon_0 > 0$ is as in Lemma 5.2.*

Let then $0 < \varepsilon < \varepsilon_0$ be arbitrarily but fixed (with $\varepsilon_0 > 0$ as in Lemma 5.2), and let $0 < \lambda < \lambda_*$. Since we know that u_λ is a local minimizer of the functional I_λ, there exists some $0 < \varrho_0 = \varrho_0(\varepsilon) \leq \rho_\varepsilon(u_\lambda)$ such that

$$(5.18) \qquad I_\lambda(u) \geq I_\lambda(u_\lambda) \quad \text{for every } u \in \mathcal{X}^{1,2}(\Omega) \text{ with } \rho_\varepsilon(u - u_\lambda) < \varrho_0.$$

As a consequence of (5.18), if we consider the cone

$$T = \{u \in \mathcal{X}^{1,2}(\Omega) : \, u \geq u_\lambda > 0 \text{ a.e. in } \Omega\},$$

only one of the following two cases hold:

    A) $\inf\{I_\lambda(u) : \, u \in T \text{ and } \rho_\varepsilon(u - u_\lambda) = \varrho\} = I_\lambda(u_\lambda)$ for every $0 < \varrho < \varrho_0$;

    B) there exists $\varrho_1 \in (0, \varrho_0)$ such that

$$(5.19) \qquad \inf\{I_\lambda(u) : \, u \in T \text{ and } \rho_\varepsilon(u - u_\lambda) = \varrho_1\} > I_\lambda(u_\lambda).$$

We explicitly notice that use of $\rho_\varepsilon(\cdot)$ to define an open neighborhood of u_λ in (5.18) is motivated by the fact that the norm $\rho_\varepsilon(\cdot)$ is globally equivalent to $\rho(\cdot)$ (and to the H_0^1-norm), uniformly w.r.t. ε, see (2.6).

We then turn to consider the two cases A)-B) separately. In doing this, we will repeatedly use the following elementary result of Real Analysis.

Lemma 5.3. *Let $A \subseteq \mathbb{R}^n$ be an arbitrary measurable set, and let $1 \leq m < \infty$. Moreover, let $f \in L^m(A)$ and let $\{f_j\}_j \subseteq L^m(A)$ be such that*

$$f_j \to f \text{ in } L^m(A) \text{ as } j \to +\infty.$$

Then, for every $0 < \vartheta \leq m$ we have

$$(5.20) \qquad \lim_{j \to +\infty} \int_A |f_j|^\vartheta \varphi \, dx = \int_A |f|^\vartheta \varphi \, dx \quad \text{for every } \varphi \in L^{m'_\vartheta}(A),$$

where $m'_\vartheta = m/(m - \vartheta)$ (with the convention $m'_\vartheta = \infty$ if $\vartheta = m$).

Proof. We preliminary observe that, taking into account the very definition of m'_ϑ, a direct application of Hölder's inequality shows that

$$(5.21) \qquad |u|^\vartheta |v| \in L^1(A) \quad \text{for every } u \in L^m(A) \text{ and every } v \in L^{m'_\vartheta}(A);$$

hence, all the integrals in (5.20) are well-defined and finite.

With (5.21) at hand, we now turn to establish (5.20). To this end, we arbitrarily choose a sub-sequence $\{f_{j_k}\}_k$ of $\{f_j\}_j$ and we prove that, by possibly choosing a further sub-sequence, identity (5.20) holds for $\{f_{j_k}\}_k$.

Let then $\varphi \in L^{m'_\vartheta}(A)$ be fixed. Since, by assumption, $f_j \to f$ strongly in $L^m(A)$ as $j \to +\infty$, we can find a function $h \in L^m(A)$ such that (up to a sub-sequence)

    i) $f_{j_k} \to f$ a.e. in A as $k \to +\infty$;

    ii) $0 \leq |f_{j_k}| \leq h$ a.e. in A, for every $k \in \mathbb{N}$;

thus, since $\vartheta > 0$, by assertion ii) we have the estimate

$$(5.22) \qquad 0 \leq |f_{j_k}|^\vartheta |\varphi| \leq h^\vartheta |\varphi| \text{ a.e. in } A \text{ and for every } k \in \mathbb{N}.$$

Now, by (5.21) we know that $g = h^\vartheta |\varphi| \in L^1(A)$; this, together with assertion i) and estimate (5.22), allows us to apply the Lebesgue Theorem, giving

$$\lim_{k \to +\infty} \int_A |f_{j_k}|^\vartheta \varphi \, dx = \int_A |f|^\vartheta \varphi \, dx.$$

This ends the proof. $\qquad \square$

We are now ready to prove the following propositions.

Proposition 5.4. *Assume that* CASE A) *holds. Then, for every* $\varrho \in (0, \varrho_0)$ *there exists a solution* v_λ *of problem* $(\mathrm{P}_\varepsilon)_\lambda$ *such that*

$$\rho_\varepsilon(u_\lambda - v_\lambda) = \varrho.$$

In particular, $v_\lambda \not\equiv u_\lambda$.

Proof. Let $0 < \varrho < \varrho_0$ be arbitrarily fixed. Since we are assuming that CASE A) holds, we can find a sequence $\{u_k\}_k \subseteq T$ satisfying the following properties:

   a) $\rho_\varepsilon(u_k - u_\lambda) = \varrho$ for every $k \geq 1$;
   b) $I_\lambda(u_k) \to I_\lambda(u_\lambda) =: \mathbf{c}_\lambda$ as $k \to +\infty$.

We then choose $\delta > 0$ so small that $\varrho - \delta > 0$ and $\varrho + \delta < \varrho_0$ and, accordingly, we consider the subset of T defined as follows:

$$X = \{u \in T : \varrho - \delta \leq \rho_\varepsilon(u - u_\lambda) \leq \varrho + \delta\} \subseteq T$$

(note that $u_k \in X$ for every $k \geq 1$, see a)). Since it is *closed*, this set X is a *complete metric space* when endowed with the distance induced by ρ; moreover, since I_λ is a *real-valued and continuous functional* on X, and since

$$\inf_X I_\lambda = I_\lambda(u_\lambda)$$

we are entitled to apply the Ekeland Variational Principle (see [5]) to the functional I_λ on X, providing us with a sequence $\{v_k\}_k \subseteq X$ such that

(5.23)
   i) $I_\lambda(v_k) \leq I_\lambda(u_k) \leq I_\lambda(u_\lambda) + 1/k^2$,
   ii) $\rho_\varepsilon(v_k - u_k) \leq 1/k$,
   iii) $I_\lambda(v_k) \leq I_\lambda(u) + 1/k \cdot \rho(v_k - u)$ for every $u \in X$.

We now observe that, since $\{v_k\}_k \subseteq X$ and since the set X is *bounded* in $\mathcal{X}^{1,2}(\Omega)$, there exists $v_\lambda \in \mathcal{X}^{1,2}(\Omega)$ such that (as $k \to +\infty$ and up to a sub-sequence)

(5.24)
   i) $v_k \to v_\lambda$ weakly in $\mathcal{X}^{1,2}(\Omega)$;
   ii) $v_k \to v_\lambda$ strongly in $L^m(\Omega)$ for every $1 \leq m < 2^*$;
   iii) $v_k \to v_\lambda$ pointwise a.e. in Ω.

where we have also used the *compact embedding* $\mathcal{X}^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$. To complete the proof, we then turn to prove the following two facts:

   1) v_λ is a weak solution of $(\mathrm{P}_\varepsilon)_\lambda$;
   2) $\rho_\varepsilon(v_\lambda - u_\lambda) = \varrho > 0$.

Proof of 1). To begin with, we fix $w \in T$ and we choose $\nu_0 = \nu_0(w, \lambda) > 0$ so small that $u_\nu = v_k + \nu(w - v_k) \in X$ for every $0 < \nu < \nu_0$. We explicitly stress that the existence of such an ν_0 easily follows from (5.23)-ii) and property a) above.

On account of (5.23)-iii) (with $u = u_\nu$), we have

$$\frac{I_\lambda(v_k + \nu(w - v_k)) - I_\lambda(v_k)}{\nu} \geq -\frac{1}{k} \rho_\varepsilon(w - v_k);$$

From this, by letting $\nu \to 0^+$ we obtain

(5.25)
$$-\frac{1}{k} \rho_\varepsilon(w - v_k) \leq \mathcal{B}_\varepsilon(v_k, w - v_k) - \int_\Omega v_k^{2^*-1}(w - v_k)\,dx$$
$$- \lambda \int_\Omega v_k^p(w - v_k)\,dx,$$

Now, given any $\varphi \in C_0^\infty(\Omega)$ and any $\nu > 0$, we define

(∗) $\psi_{k,\nu} = v_k + \nu\varphi - u_\lambda$ and $\phi_{k,\nu} = (\psi_{k,\nu})_-$;
(∗) $\psi_\nu = v_\lambda + \nu\varphi - u_\lambda$ and $\phi_\nu = (\psi_\nu)_-$.

Since, obviously, $w = v_k + \nu\varphi + \phi_{k,\nu} \in T$, by exploiting (5.25) we get

$$(5.26) \qquad -\frac{1}{k}\rho_\varepsilon(\nu\varphi + \phi_{k,\nu}) \leq \mathcal{B}_\varepsilon(v_k, \nu\varphi + \phi_{k,\nu}) - \int_\Omega v_k^{2^*-1}(\nu\varphi + \phi_{k,\nu})\,dx$$
$$- \lambda \int_\Omega v_k^p(\nu\varphi + \phi_{k,\nu})\,dx.$$

Then, we aim to pass to the limit as $k \to +\infty$ and $\nu \to 0^+$ in the above (5.26). To this end we first observe that, on account of (5.24)-iii), we have

$$(5.27) \qquad \phi_{k,\nu} \to \phi_\nu \text{ pointwise a.e. in } \Omega \text{ as } k \to +\infty;$$

moreover, by the very definition of $\phi_{k,\nu}$ we also have the following estimate

$$|\phi_{k,\nu}| = (u_\lambda - \nu\varphi - v_k) \cdot \mathbf{1}_{\{u_\lambda - \nu\varphi - v_k \geq 0\}} \leq u_\lambda + \nu|\varphi|;$$

as a consequence, we get

$$(5.28) \qquad \begin{array}{l} \text{i) } v_k^{2^*-1}|\phi_{k,\nu}| = v_k^{2^*-1}(u_\lambda - \nu\varphi - v_k) \cdot \mathbf{1}_{\{u_\lambda - \nu\varphi - v_k \geq 0\}} \leq (u_\lambda + \nu|\varphi|)^{2^*}; \\[2mm] \text{ii) } v_k^p|\phi_{k,\nu}| = v_k^p(u_\lambda - \nu\varphi - v_k) \cdot \mathbf{1}_{\{u_\lambda - \nu\varphi - v_k \geq 0\}} \leq (u_\lambda + \nu|\varphi|)^{p+1}. \end{array}$$

Using a standard dominated-convergence argument based on (5.27)-(5.28), jointly with Lemma 5.3 (see (5.24)-ii) and remind that $\varphi \in C_0^\infty(\Omega)$), we then obtain

$$(5.29) \qquad \lim_{k \to +\infty} \left(\int_\Omega v_k^{2^*-1}(\nu\varphi + \phi_{k,\nu})\,dx + \lambda \int_\Omega v_k^p(\nu\varphi + \phi_{k,\nu})\,dx \right)$$
$$= \int_\Omega v_\lambda^{2^*-1}(\nu\varphi + \phi_\nu)\,dx + \lambda \int_\Omega v_\lambda^p(\nu\varphi + \phi_\nu)\,dx.$$

As regards the *operator term* $\mathcal{B}_\varepsilon(v_k, \nu\varphi + \phi_{k,\nu})$ by proceeding *exactly* as in [13, Proposition 5.2] (where the same operator $\mathcal{L}_\varepsilon$ is considered), we get

$$(5.30) \qquad \mathcal{B}_\varepsilon(v_k, \nu\varphi + \phi_{k,\nu}) \leq \mathcal{B}_\varepsilon(v_\lambda, \nu\varphi + \phi_\nu) + o(1) \qquad \text{as } k \to +\infty,$$

where we have used the fact that $v_k \to v_\lambda$ weakly in $\mathcal{X}^{1,2}(\Omega)$.

Gathering (5.29) and (5.30), and taking into account that $\rho_\varepsilon(\phi_{k,\nu})$ is *uniformly bounded* with respect to k (as the same is true of v_k), we can finally pass to the limit as $k \to +\infty$ in (5.26), obtaining

$$(5.31) \qquad \mathcal{B}_\varepsilon(v_\lambda, \nu\varphi + \phi_\nu) \geq \int_\Omega v_\lambda^{2^*-1}(\nu\varphi + \phi_\nu)\,dx + \lambda \int_\Omega v_\lambda^p(\nu\varphi + \phi_\nu)\,dx.$$

With (5.31) at hand, we can now exploit once again the computations carried out in [13, Proposition 5.2], getting

$$
\mathcal{B}_\varepsilon(v_\lambda, \varphi) - \lambda \int_\Omega v_\lambda^p \varphi\, dx - \int_\Omega v_\lambda^{2^*-1} \varphi\, dx
$$

$$
\geq -\frac{1}{\nu}\Big(\mathcal{B}_\varepsilon(v_\lambda, \phi_\nu) - \lambda \int_\Omega v_\lambda^p \phi_\nu\, dx - \int_\Omega v_\lambda^{2^*-1}\phi_\nu\, dx\Big)
$$

(since u_λ is a solution of $(\mathrm{P}_\varepsilon)_\lambda$)

$$
= \frac{1}{\nu}\Big(-\mathcal{B}_\varepsilon(v_\lambda - u_\lambda, \phi_\nu) + \lambda \int_\Omega (v_\lambda^p - u_\lambda^p)\phi_\nu\, dx
$$

(5.32)
$$
+ \int_\Omega (v_\lambda^{2^*-1} - u_\lambda^{2^*-1})\phi_\nu\, dx\Big)
$$

(since $v_\lambda = \lim_{k\to+\infty} v_k \geq u_\lambda$)

$$
\geq \frac{1}{\nu}\Big(-\int_{\{v_\lambda + \nu\varphi \leq u_\lambda\}} \nabla(v_\lambda - u_\lambda)\cdot\nabla(v_\lambda - u_\lambda + \nu\varphi)\, dx
$$

$$
- \varepsilon \iint_{\mathbb{R}^{2n}} \frac{((v_\lambda - u_\lambda)(x) - (v_\lambda - u_\lambda)(y))(\phi_\nu(x) - \phi_\nu(y))}{|x-y|^{n+2s}}\, dx\, dy\Big)
$$

$$
\geq o(1) \text{ as } \nu \to 0^+;
$$

as a consequence, by letting $\nu \to 0^+$ in (5.32), we obtain

$$
\mathcal{B}_\varepsilon(v_\lambda, \varphi) - \lambda \int_\Omega v_\lambda^p \varphi\, dx - \int_\Omega v_\lambda^{2^*-1}\varphi\, dx \geq 0.
$$

This, together with the *arbitrariness* of the fixed $\varphi \in C_0^\infty(\Omega)$, finally proves that the function v_λ is a weak solution of problem $(\mathrm{P}_\varepsilon)_\lambda$ as claimed.

In particular, from Theorem 2.3 we derive that

$$
(5.33) \qquad\qquad v_\lambda \in L^\infty(\Omega).
$$

Proof of 2). To prove assertion 2) it suffices to show that

$$
(5.34) \qquad\qquad v_k \to v_\lambda \text{ strongly in } \mathcal{X}^{1,2}(\Omega) \text{ as } k \to +\infty.
$$

In fact, owing to property a) of $\{u_k\}_k$ we have

$$
\varrho - \rho_\varepsilon(u_k - v_k) \leq \rho_\varepsilon(v_k - u_\lambda) \leq \rho_\varepsilon(v_k - u_k) + \varrho;
$$

this, together with (5.34) and (5.23)-ii), ensures that $\rho_\varepsilon(u_\lambda - v_\lambda) = \varrho$. Hence, we turn to to prove (5.34), namely the *strong convergence* of $\{v_k\}_k$ to v_λ.

First of all, by (5.24) and the Brezis‑Lieb Lemma [15], we have

(5.35)
$$
\begin{aligned}
&\text{i) } \|v_k - v_\lambda\|_{L^{p+1}(\Omega)} \to 0 \text{ as } k \to +\infty,\\
&\text{ii) } \|v_k\|_{L^{2^*}(\Omega)}^{2^*} = \|v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + \|v_k - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + o(1);\\
&\text{iii) } \rho_\varepsilon(v_k)^2 = \rho_\varepsilon(v_\lambda)^2 + \rho_\varepsilon(v_k - v_\lambda)^2 + o(1).
\end{aligned}
$$

In particular, from (5.35)‑i) we get

$$
(5.36) \qquad\qquad \int_\Omega v_k^{p+1}\, dx = \int_\Omega v_\lambda^{p+1}\, dx + o(1) \quad \text{as } k \to +\infty.
$$

Owing to (5.35), and choosing $w = v_\lambda \in T$ in (5.25), we then get

$$\rho_\varepsilon(v_k - v_\lambda)^2 = -\mathcal{B}_\varepsilon(v_k, v_\lambda - v_k) + \mathcal{B}_\varepsilon(v_\lambda, v_\lambda - v_k)$$

$$\leq \frac{1}{k}\rho_\varepsilon(v_k - v_\lambda) + \lambda \int_\Omega v_k^p (v_k - v_\lambda)\, dx$$

$$+ \int_\Omega v_k^{2^*-1}(v_k - v_\lambda)\, dx + \mathcal{B}_\varepsilon(v_\lambda, v_\lambda - v_k)$$

(since $\{v_k\}_k$ is bounded and $v_k \to v_\lambda$ weakly in $\mathcal{X}^{1,2}(\Omega)$)

$$= \lambda \int_\Omega v_k^{p+1}\, dx + \int_\Omega v_k^{2^*-1}(v_k - v_\lambda)\, dx - \lambda \int_\Omega v_k^p v_\lambda\, dx + o(1)$$

$$= \lambda \int_\Omega v_\lambda^{p+1}\, dx + \|v_k - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + \|v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} - \int_\Omega v_k^{2^*-1}v_\lambda\, dx$$

$$- \lambda \int_\Omega v_k^p v_\lambda\, dx + o(1) \quad \text{as } k \to +\infty.$$

On the other hand, owing to (5.24)-(5.33) (and reminding that $0 < p < 1$), we can exploit once again Lemma 5.3, thus deriving

$$\int_\Omega v_k^p v_\lambda\, dx \to \|v_\lambda\|_{L^{p+1}(\Omega)}^{p+1} \qquad \text{and} \qquad \int_\Omega v_k^{2^*-1}v_\lambda\, dx \to \|v_\lambda\|_{L^{2^*}(\Omega)}^{2^*}$$

as $k \to +\infty$; as a consequence, we obtain

$$(5.37) \qquad \rho_\varepsilon(v_k - v_\lambda)^2 \leq \|v_k - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + o(1) \quad \text{as } k \to +\infty.$$

To proceed further, we now choose $w = 2v_k \in T$ in (5.25): this yields

$$\rho_\varepsilon(v_k)^2 - \|v_k\|_{L^{2^*}(\Omega)}^{2^*} - \lambda \int_\Omega v_k^{p+1}\, dx \geq -\frac{1}{k}\rho_\varepsilon(v_k)^2 = o(1);$$

thus, recalling that v_λ is a weak solution of problem $(\mathrm{P}_\varepsilon)_\lambda$, we get

$$\rho_\varepsilon(v_k - v_\lambda)^2 = \rho_\varepsilon(v_k)^2 - \rho_\varepsilon(v_\lambda)^2 + o(1)$$

$$(5.38) \qquad \geq \left(\|v_k\|_{L^{2^*}(\Omega)}^{2^*} + \lambda \int_\Omega v_k^{p+1}\, dx\right) - \mathcal{B}_\varepsilon(v_\lambda, v_\lambda)$$

$$= \|v_k\|_{L^{2^*}(\Omega)}^{2^*} + \lambda \int_\Omega v_k^{p+1}\, dx - \|v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} - \lambda \int_\Omega v_\lambda^{p+1}\, dx$$

$$= \|v_k - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + o(1) \quad \text{as } k \to +\infty,$$

where we have also used (5.36). Gathering (5.37)-(5.38), we then obtain

$$(5.39) \qquad \rho_\varepsilon(v_k - v_\lambda)^2 = \|v_k - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + o(1) \quad \text{as } k \to +\infty.$$

With (5.39) at hand, we can finally end the proof of (5.34). In fact, assuming (to fix the ideas) that $I_\lambda(u_\lambda) \leq I_\lambda(v_\lambda)$, from (5.23) and (5.35)-i) we get

$$I_\lambda(v_k - v_\lambda) = \frac{1}{2}\rho_\varepsilon(v_k - v_\lambda)^2 + \frac{\lambda}{p+1}\int_\Omega |v_k - v_\lambda|^{p+1}\, dx + \frac{1}{2^*}\|v_k - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*}$$

$$= I(v_k) - I(v_\lambda) + o(1) \leq I(u_\lambda) - I(v_\lambda) + \frac{1}{k^2} + o(1)$$

$$\leq o(1) \qquad \text{as } k \to +\infty;$$

this, together with (5.35)-i), gives

$$
\frac{1}{2}\rho_\varepsilon(v_k - v_\lambda)^2 - \frac{1}{2^*}\|v_k - v_\lambda\|^{2^*}_{L^{2^*}(\Omega)}
$$

(5.40)

$$
= I_\lambda(v_k - v_\lambda) + \frac{\lambda}{p+1}\int_\Omega |v_k - v_\lambda|^{p+1}\, dx \le o(1).
$$

Thus, by combining (5.39)-(5.40), we easily obtain

$$
\lim_{k\to+\infty} \|v_k - v_\lambda\|^{2^*}_{L^{2^*}(\Omega)} = \lim_{k\to+\infty} \rho_\varepsilon(v_k - v_\lambda)^2 = 0,
$$

and this proves (5.34). □

Proposition 5.5. *Assume that* CASE B) *holds. Then, there exists a second solution* v_λ *of problem* $(P_\varepsilon)_\lambda$ *such that* $v_\lambda \not\equiv u_\lambda$.

Proof. To begin with, we consider the set

$$
\Gamma = \big\{\eta \in C([0,1]; T) :\ \eta(0) = u_\lambda,\ I_\lambda(\eta(1)) < I_\lambda(u_\lambda) \text{ and } \rho_\varepsilon(\eta(1) - u_\lambda) > \varrho_1\big\},
$$

(where $\varrho_1 > 0$ is as in (5.19)), and we claim that $\Gamma \ne \varnothing$.

In fact, since the fixed ε satisfies $0 < \varepsilon < \varepsilon_0$ (with $\varepsilon_0 > 0$ as in Lemma 5.2), by the cited Lemma 5.2 we know that there exists $R_0 > 0$ such that

(5.41)
$$
\begin{cases} I_\lambda(u_\lambda + RU_\varepsilon) < I_\lambda(u_\lambda) & \text{for all } R \ge R_0, \\ I_\lambda(u_\lambda + tR_0U_\varepsilon) < I_\lambda(u_\lambda) + \frac{1}{n}S_n^{n/2} & \text{for all } t \in [0,1]. \end{cases}
$$

In particular, from (5.41) we easily see that

$$
\eta_0(t) = u_\lambda + tR_0U_\varepsilon \in \Gamma
$$

(by enlarging R_0 if needed), and thus $\Gamma \ne \varnothing$, as claimed.

Now we have proved that $\Gamma \ne \varnothing$, we can proceed towards the end of the proof. To this end we first observe that, since it is non-empty, this set Γ is a *complete metric space*, when endowed with the distance

$$
d_\Gamma(\eta_1, \eta_2) := \max_{0\le t\le 1} \rho_\varepsilon\big(\eta_1(t) - \eta_2(t)\big);
$$

moreover, since I_λ is *real-valued and continuous* on $\mathcal{X}^{1,2}(\Omega)$, it is easy to recognize that the functional $\Phi : \Gamma \to \mathbb{R}$ defined as

$$
\Phi(\eta) := \max_{0\le t\le 1} I_\lambda(\eta(t)),
$$

is (well-defined and) continuous on Γ. In view of these facts, we are then entitled to apply the Ekeland Variational Principle to this functional Φ on Γ: setting

$$
\gamma_0 := \inf_\Gamma \Phi(\eta),
$$

there exists a sequence $\{\eta_k\}_k \subseteq \Gamma$ such that

(5.42)
$$
\begin{aligned} &\text{i) } \Phi(\eta_k) \le \gamma_0 + 1/k, \\ &\text{ii) } \Phi(\eta_k) \le \Phi(\eta) + 1/k\, d_\Gamma(\eta_k, \eta) \quad \text{for every } \eta \in \Gamma. \end{aligned}
$$

Now, starting from (5.42) and proceeding exactly as in the proof of [6, Lemma 3.5], we can find another sequence

$$
v_k = \eta_k(t_k) \in T
$$

(for some $t_k \in [0,1]$) such that

   a) $I_\lambda(v_k) \to \gamma_0$ as $k \to +\infty$;

   b) there exists some $C > 0$ such that, for every $w \in T$, one has

$$
(5.43) \qquad
\begin{aligned}
&\mathcal{B}_\varepsilon(v_k, w - v_k) - \lambda \int_\Omega v_k^p(w - v_k)\,dx \\
&\qquad - \int_\Omega v_k^{2^*-1}(w - v_k)\,dx \geq -\frac{C}{k}(1 + \rho_\varepsilon(w)).
\end{aligned}
$$

In particular, choosing $w = 2v_k$ in (5.43), we get

$$
(5.44) \qquad \rho_\varepsilon(v_k)^2 - \lambda \int_\Omega v^{p+1}\,dx - \int_\Omega v_k^{2^*}\,dx \geq -\frac{C}{k}(1 + 2\rho_\varepsilon(v_k)).
$$

By combining (5.44) with assertion a), and exploiting Hölder's and Sobolev's inequalities, we then obtain the following estimate

$$
(5.45) \qquad
\begin{aligned}
\gamma_0 + o(1) &= \frac{1}{2}\rho_\varepsilon(v_k)^2 - \frac{\lambda}{p+1}\int_\Omega v_k^{p+1}\,dx - \frac{1}{2^*}\int_\Omega v_k^{2^*}\,dx \\
&\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\rho_\varepsilon(v_k)^2 - \lambda\left(\frac{1}{p+1} - \frac{1}{2^*}\right)\int_\Omega v_k^{p+1}\,dx \\
&\quad - \frac{C}{2^* k}(1 + 2\rho_\varepsilon(v_k)) \\
&\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\rho_\varepsilon(v_k)^2 - C\left(\rho_\varepsilon(v_k)^{p+1} - 2\rho_\varepsilon(v_k) - 1\right),
\end{aligned}
$$

where $C > 0$ is a constant depending on n and on $|\Omega|$. Since, obviously,

$$
c_0 = \frac{1}{2} - \frac{1}{2^*} > 0,
$$

it is readily seen from (5.45) that the sequence $\{v_k\}_k$ is *bounded in* $\mathcal{X}^{1,2}(\Omega)$ (otherwise, by possibly choosing a sub-sequence we would have $\rho_\varepsilon(v_k) \to +\infty$, and hence the right-hand side of (5.45) would diverges as $k \to +\infty$, which is not possible).

In view of this fact, we can thus proceed as in the proof of Lemma 5.4 to show that $\{v_k\}_k$ weakly converges (up to a sub-sequence) to a *weak solution* $v_\lambda \in \mathcal{X}^{1,2}(\Omega)$ of problem $(P_\varepsilon)_\lambda$, further satisfying the identity

$$
(5.46) \qquad \rho_\varepsilon(v_k - v_\lambda)^2 - \|v_k - v_\lambda\|_{L^{2^*}(\Omega)}^{2^*} = o(1) \quad \text{as } k \to +\infty.
$$

In view of these facts, to complete the proof we are left to show that $v_\lambda \not\equiv u_\lambda$. To this end we first observe that, given any $\eta \in \Gamma$, we have

$$
\rho_\varepsilon(\eta(0) - u_\lambda) = 0 \quad \text{and} \quad \rho_\varepsilon(\eta(1) - u_\lambda) > \varrho_1,
$$

and hence there exists a point $t_\eta \in [0,1]$ such that $\rho_\varepsilon(\eta(t_\eta) - u_\lambda) = \varrho_1$; as a consequence, since *we are assuming that* CASE B) holds, we obtain

$$
\begin{aligned}
\gamma_0 = \inf_{\eta \in \Gamma} \Phi(\eta) &\geq \inf\left\{I_\lambda(\eta(t_\eta)) : \eta \in \Gamma\right\} \\
&\geq \inf\{I_\lambda(u) : u \in T \text{ and } \rho_\varepsilon(u - u_\lambda) = \varrho_1\} > I_\lambda(u_\lambda).
\end{aligned}
$$

On the other hand, since we already know that $\eta_0(t) = u_\lambda + tR_0 U_\varepsilon \in \Gamma$, from (5.41) (and the very definition of γ_0) we derive the following estimate

$$
\gamma_0 \leq \Phi(\eta_0) = \max_{0 \leq t \leq 1} I_\lambda(\eta_0(t)) < I_\lambda(u_\lambda) + \frac{1}{n}S_n^{n/2}.
$$

Summing up, we have

$$(5.47) \qquad I_\lambda(u_\lambda) < \gamma_0 < I_\lambda(u_\lambda) + \frac{1}{n} S_n^{n/2}.$$

Now, since the sequence $\{v_k\}_k$ weakly converges in $\mathcal{X}^{1,2}(\Omega)$ to v_λ as $k \to +\infty$, *the same assertions* in (5.35) hold also in this context; this, together with (5.47) and the above property a) of the sequence $\{v_k\}_k$, gives

$$
\begin{aligned}
(5.48) \qquad & \frac{1}{2}\rho_\varepsilon(v_k - v_\lambda)^2 - \frac{1}{2^*}\|v_k - v_\lambda\|_{L^{2^*}(\Omega)}^2 \\
& = \frac{1}{2}\big(\rho_\varepsilon(v_k)^2 - \rho_\varepsilon(v_\lambda)^2\big) - \frac{1}{2^*}\big(\|v_k\|_{L^{2^*}(\Omega)}^{2^*} - \|v\|_{L^{2^*}(\Omega)}^{2^*}\big) + o(1) \\
& = I_\lambda(v_k) - I_\lambda(u_\lambda) + o(1) = \gamma_0 - I_\lambda(u_\lambda) + o(1) \\
& < \frac{1}{n} S_n^{n/2} - \delta_0,
\end{aligned}
$$

for some $\delta_0 > 0$ such that $1/n\, S_n^{n/2} - \delta_0 > 0$ (provided that k is large enough).

Gathering (5.46)-(5.48), and arguing as in [39, Proposition 3.1], it is then easy to recognize that $v_k \to v_\lambda$ *strongly in* $\mathcal{X}^{1,2}(\Omega)$; as a consequence, by the continuity of the functional I_λ and by (5.43)-(5.47), we get

$$I_\lambda(u_\lambda) < \gamma_0 = \lim_{k \to +\infty} I_\lambda(v_k) = I_\lambda(v_\lambda),$$

and this finally proves that $v_\lambda \not\equiv u_\lambda$, as desired. $\qquad\square$

6. Proof of Theorem 1.6

As anticipated in the Introduction, in this last section we exploit Theorem 1.2 to prove Theorem 1.6. The main reason why we postponed this proof at the very end of the paper is *philosophical*: since Theorem 1.6 concerns the behavior of the L^∞-norm of any solution of problem $(\mathrm{P}_\varepsilon)_\lambda$ *different from its minimal solution* $\bar{u}_{\lambda,\varepsilon}$, we need to know that there exists (at least) *one bounded solution of* $(\mathrm{P}_\varepsilon)_\lambda$ *distinct from such* $\bar{u}_{\lambda,\varepsilon}$; this is precisely the content of our Theorem 1.4 (at least for $\varepsilon \ll 1$), taking into account Theorem 2.3.

Proof (of Theorem 1.6). We closely follow the approach in [1, Theorem 2.4].

Arguing by contradiction, we assume that there exist a sequence $\{\lambda_j\}_j \subseteq (0, \Lambda_\varepsilon)$ and a family $\{v_j = v_{\lambda_j,\varepsilon}\}_j$ of weak solutions of problem $(\mathrm{P}_\varepsilon)_{\lambda_j}$ such that

    a) $\lambda_j \to 0$ as $j \to +\infty$;

    b) $v_j \not\equiv \bar{u}_{\lambda_j,\varepsilon}$ (where $\bar{u}_{\lambda_j,\varepsilon}$ is the unique minimal solution of $(\mathrm{P}_\varepsilon)_{\lambda_j}$), and

$$(6.1) \qquad \|v_j\|_{L^\infty(\Omega)} \le \mathbf{c}.$$

      for some constant $\mathbf{c} > 0$ (possibly depending on ε).

In particular, we have

$$(6.2) \qquad
\begin{aligned}
& \bullet) \; v_j > 0 \text{ pointwise in } \Omega \text{ and } u = 0 \text{ pointwise in } \partial\Omega; \\
& \bullet) \; \partial_\nu v_j < 0 \text{ pointwise on } \partial\Omega
\end{aligned}
$$

We then observe that, since v_j is a weak solution of problem $(\mathrm{P}_\varepsilon)_{\lambda_j}$ different from its minimal solution $\bar{u}_{\lambda_j,\varepsilon}$ (that is, problem by $(\mathrm{P}_\varepsilon)_{\lambda_j}$ possesses at least two distinct solutions), by Theorem 1.2 we necessarily have that

$$(6.3) \qquad \|v_j\|_{L^\infty(\Omega)} \ge M_\varepsilon,$$

for some constant $M_\varepsilon > 0$ (independent of j, but possibly depending on ε).

On the other hand, by combining (6.1) with the global regularity result proved in [38, Theorem 1.3] (see also the proof of Theorem 2.3), we derive that

$$(6.4) \qquad \|v_j\|_{C^{1,\alpha}(\overline{\Omega})} \leq c\left(1 + \|v_j\|_{L^\infty(\Omega)}^{2^*}\right) \leq c_1,$$

and such an estimate holds for every fixed $\alpha \in (0, \min\{1, 2-2s\})$, with a suitable constant $c > 0$ independent of j. Now, owing to (6.4) (and bearing in mind (6.2)), we easily derive that the sequence $\{v_j\}_j$ *uniformly converges* (up to a sub-sequence) as $j \to +\infty$ to some *non-negative function*

$$v_0 \in \mathcal{X}^{1,2}(\Omega) \cap C^1(\overline{\Omega}),$$

which is a weak solution of the *purely critical problem*

$$(\star) \qquad \begin{cases} \mathcal{L}_\varepsilon u = |u|^{2^*-1} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

(recall that $\lambda_j \to 0$ as $j \to +\infty$); moreover, by (6.4) we get

$$\|v_0\|_{L^\infty(\Omega)} \geq M_\varepsilon \implies v_0 \not\equiv 0.$$

Summing up, the function v_0 is a *non-trivial and non-negative weak solution of* $(\star)$ (in the sense of Definition 2.2), but this is in contradiction with [10, Theorem 1.3] (since, by assumption, Ω is star-shaped). This ends the proof. $\qquad\square$

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