ℓ^2 DECOUPLING THEOREM FOR SURFACES IN $\mathbb{R}^3$

LARRY GUTH, DOMINIQUE MALDAGUE, AND CHANGKEUN OH

Abstract. We identify a new way to divide the δ-neighborhood of surfaces $\mathcal{M} \subset \mathbb{R}^3$ into a finitely-overlapping collection of rectangular boxes S. We obtain a sharp (ℓ^2, L^p) decoupling estimate using this decomposition, for the sharp range of exponents $2 \leq p \leq 4$. Our decoupling inequality leads to new exponential sum estimates where the frequencies lie on surfaces which do not contain a line.

1. Introduction

Consider the manifold

$$\mathcal{M}_\phi := \{(\xi_1, \xi_2, \phi(\xi_1, \xi_2)) : \xi_1, \xi_2 \in [0,1]\}, \tag{1.1}$$

where $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Denote by $N_\delta(\mathcal{M}_\phi) \subset \mathbb{R}^3$ the δ-neighborhood of the manifold $\mathcal{M}_\phi$. Our main theorem is an ℓ^2 decoupling theorem for $\mathcal{M}_\phi$ into (ϕ, δ)-flat sets. This type of theorem was introduced by [BDK20].

Definition 1.1. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be smooth. We say that $S \subset [0,1]^2$ is (ϕ, δ)-flat if*

$$\sup_{u,v \in S} |\phi(u) - \phi(v) - \nabla\phi(u) \cdot (u - v)| \leq \delta. \tag{1.2}$$

For a measurable function $f : \mathbb{R}^3 \to \mathbb{C}$ and a measurable set $S \subset \mathbb{R}^2$, denote by f_S the Fourier restriction of f on the set $S \times \mathbb{R}$. We refer to Definition 2.1 for the rigorous definition.

Theorem 1.2. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Fix $\epsilon > 0$. Then there exists a sufficiently large number A depending on ϵ and ϕ satisfying the following.*

For any $\delta > 0$, there exists a collection $\mathcal{S}_\delta$ of finitely overlapping sets S such that

(1) the overlapping number is $O(\log \delta^{-1})$ in the sense that

$$\sum_{S \in \mathcal{S}_\delta} \chi_S \leq C_{\epsilon,\phi} \log(\delta^{-1}), \tag{1.3}$$

(2) S is $(\phi, A\delta)$-flat,

(3) for $2 \leq p \leq 4$ we have

$$\|f\|_{L^p} \leq C_{\epsilon,\phi} \delta^{-\epsilon} \Big(\sum_{S \in \mathcal{S}_\delta} \|f_S\|_{L^p}^2 \Big)^{1/2} \tag{1.4}$$

for all functions f whose Fourier supports are in $N_\delta(\mathcal{M}_\phi)$.

One important aspect of our theorem is that the collection $\mathcal{S}_\delta$ is not a strict partition of $[0,1]^2$ since the sets $S \in \mathcal{S}_\delta$ are $O(\log \delta^{-1})$-overlapping. The decoupling inequality (1.4) for the hyperbolic paraboloid is not true if a family $\mathcal{S}_\delta$ is a partition of $[0,1]^2$. In other words, the $O(\log \delta^{-1})$-overlapping property is necessary for the theorem to hold true (see Appendix A). The situation is different for curves $(\xi, \phi(\xi))$ in $\mathbb{R}^2$, where Yang proved a version of Theorem 1.2 using a strict partition [Yan21]. We realized that to prove an ℓ^2 decoupling inequality of the form (1.4), it is crucial to understand the number of possible choices of (ϕ, δ)-flat sets. For a curve $(\xi, \phi(\xi)) \in \mathbb{R}^2$ and a given point $p \in [0, 1]$, there is essentially one choice of (ϕ, δ)-flat set containing the point p with the maximal size. On the other hand, for the hyperbolic paraboloid in $\mathbb{R}^3$ and a given point $p \in [0, 1]^2$, there are essentially $O(\log \delta^{-1})$ many choices of (ϕ, δ)-flat sets containing the point p and with maximal size. This turns out to be the reason why a partition is not enough and we need (1.3). For the hyperbolic paraboloid in $\mathbb{R}^n$ with $n \geq 4$, the possible choices of (ϕ, δ)-flat sets containing a point is $\sim \delta^{-A_n}$ for some $A_n > 0$, which is why we do not have an analogous theorem without any δ^{-1}-power loss (see Appendix B).

Next we describe our main application of Theorem 1.2, which is to prove discrete restriction estimates for manifolds $\mathcal{M}_\phi$ not containing a line segment. For a function F and a set A, define

$$\|F\|^p_{L^p_{\#}(A)} := \frac{1}{|A|}\|F\|^p_{L^p(A)}. \tag{1.5}$$

Denote by Λ_δ a collection of δ-separated points in $[0, 1]^2$. Define $e(t) := e^{2\pi i t}$.

Corollary 1.3. *Let ϕ be a polynomial of degree d with coefficients bounded by 1. Suppose that the manifold $\mathcal{M}_\phi$ does not contain a line segment. Then for $2 \leq p \leq 4$, $\epsilon > 0$, and any sequence $\{a_\xi\}_{\xi\in\Lambda_\delta}$, we have*

$$\Big\| \sum_{\xi\in\Lambda_\delta} a_\xi e\big(x \cdot (\xi, \phi(\xi))\big)\Big\|_{L^p_{\#}(B_{\delta^{-d}})} \leq C_{\phi,\epsilon}\delta^{-\epsilon}\big(\sum_{\xi\in\Lambda_\delta} |a_\xi|^2\big)^{\frac{1}{2}}. \tag{1.6}$$

An (ℓ^2, L^p) discrete restriction estimate is an inequality of the form (1.6). Corollary 1.3 is sharp in the following senses. First, the range of p of (1.6) is optimal for the paraboloid, though we expect that a wider range of p is possible for generic polynomials ϕ. Second, the inequality (1.6) is false for any $p > 2$ for some collection Λ_δ if the manifold $\mathcal{M}_\phi$ contains a line segment.

The (ℓ^2, L^p) discrete restriction estimate (1.6) implies the sharp (ℓ^p, L^p) discrete restriction estimate. [LY21] obtained a sharp ℓ^p decoupling inequality for smooth manifolds, but their decoupling does not seem to imply (ℓ^p, L^p) discrete restriction estimates.

Our second application is about the Strichartz estimate for the nonelliptic Schrodinger equation on irrational tori. Let us introduce the conjecture. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha < 0$, define $\tilde{\Delta} := \frac{1}{2\pi}(\partial_{11} + \alpha\partial_{22})$. Let v be a solution of the partial differential equation

$$i\partial_t v - \tilde{\Delta} v = 0, \quad v(0, x) = f(x), \tag{1.7}$$

where the initial data f is defined on the torus $\mathbb{T}^2$. We let $e^{it\tilde{\Delta}} f(x) := v(x, t)$. The Strichartz estimate for the elliptic Schrodinger equation on irrational tori, which corresponds to the case $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha > 0$, was studied in [DGG17, DGGRM22].

Conjecture 1.4. [DGGRM22, Conjecture 1.1]

$$\|e^{it\tilde{\Delta}}f\|_{L^p([0,T]\times\mathbb{T}^2)} \leq C_\epsilon N^\epsilon \|f\|_{L^2(\mathbb{T}^2)} \begin{cases} T^{\frac{1}{p}} \text{ for } 2 \leq p \leq 4 \\ T^{\frac{1}{p}} + N^{1-\frac{4}{p}} \text{ for } 4 \leq p \leq 6 \\ T^{\frac{1}{p}} N^{\frac{1}{2}-\frac{3}{p}} + N^{1-\frac{4}{p}} \text{ for } 6 \leq p \leq 10 \\ T^{\frac{1}{p}} N^{1-\frac{8}{p}} + N^{1-\frac{4}{p}} \text{ for } 10 \leq p \end{cases} \tag{1.8}$$

for functions f whose Fourier support is in $[-N,N]^2$.

This conjecture is verified for $p > 8$ by [DGGRM22, Remark 1.3]. Corollary 1.5 gives partial progress on the conjecture for the range $2 \leq p \leq 4$.

Corollary 1.5. *Consider $2 \leq p \leq 4$. For $T \geq N$,*

$$\|e^{it\tilde{\Delta}}f\|_{L^p([0,T]\times\mathbb{T}^2)} \leq C_\epsilon N^\epsilon T^{\frac{1}{p}} \|f\|_{L^2(\mathbb{T}^2)} \tag{1.9}$$

for functions f whose Fourier support is in $[-N,N]^2$.

An analogous theorem of Corollary 1.5 for the elliptic Schrodinger equation is obtained as a corollary of an ℓ^2 decoupling theorem [BD15] for the paraboloid, which is observed by [DGG17].

We prove the corollaries in Section 4.

1.1. **Comparison to decoupling inequalities in the literature.** Let $\mathcal{M} \subset \mathbb{R}^3$ be a manifold in $\mathbb{R}^3$ with nonvanishing Gaussian curvature and let $\mathcal{P}_\delta(\mathcal{M})$ be a partition of $N_\delta(\mathcal{M})$ into approximate $\delta^{1/2} \times \delta^{1/2} \times \delta$ caps θ. In their foundational decoupling paper [BD15], Bourgain and Demeter proved that if $\mathcal{M}$ has everywhere positive Gaussian curvature and $2 \leq p \leq 4$, then

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon} \Big(\sum_{\theta \in \mathcal{P}_\delta(\mathcal{M})} \|f_\theta\|^2_{L^4(\mathbb{R}^3)} \Big)^{1/2}. \tag{1.10}$$

for all f with $\mathrm{supp}\widehat{f} \subset N_\delta(\mathcal{M})$. It is observed in [BD17a] that (1.10) is false for the hyperbolic paraboloid. The counterexample comes from the geometric fact that the hyperbolic paraboloid contains a line. Based on this observation, one might expect (1.10) to hold true for manifolds avoiding a line segment.

Conjecture 1.6. *Let ϕ be a polynomial of degree d. Assume that $\mathcal{M}_\phi$ does not contain a line segment and has everywhere negative Gaussian curvature. Define*

$$\mathcal{P}_\delta := \{[a, a+\delta^{\frac{1}{d}}] \times [b, b+\delta^{\frac{1}{d}}] : a, b \in \delta^{\frac{1}{d}}\mathbb{Z} \cap [0, 1-\delta^{\frac{1}{d}}]\}. \tag{1.11}$$

Then for $2 \leq p \leq 4$

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon} \Big(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|^2_{L^p(\mathbb{R}^3)} \Big)^{1/2} \tag{1.12}$$

for functions f whose Fourier supports are in $N_\delta(\mathcal{M})$.

Conjecture 1.6 implies Corollary 1.3 for the case that the determinant of the Hessian matrix of ϕ is nonnegative. Unfortunately, it is not clear how to prove Conjecture 1.6 using the current decoupling techniques. Instead of proving Conjecture 1.6, we introduce a modified version of the decoupling inequality, which is inspired by (ϕ, δ)-flat sets, and proved that this decoupling still implies Corollary 1.3. Both Conjecture 1.6 and Theorem 1.2 imply Corollary 1.3. But neither of them does not seem to imply the other.

Let us review decoupling for smooth manifolds to motivate the use of (ϕ,δ)-sets. Bourgain and Demeter [BD15, BD17a] proved that if $\mathcal{M}$ has everywhere nonzero Gaussian curvature and $2\le p\le 4$, then

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon}(\delta^{-1})^{\frac12-\frac1p}\Big(\sum_{\theta\in\mathcal{P}_\delta(\mathcal{M})}\|f_\theta\|^p_{L^p(\mathbb{R}^3)}\Big)^{1/p} \tag{1.13}$$

for all f with $\mathrm{supp}\widehat{f}\subset N_\delta(\mathcal{M})$. As mentioned before, if the Gaussian curvature is positive, then the above decoupling inequality may be refined to an (ℓ^2,L^p) decoupling (see [BD15]), which has the form

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon}\Big(\sum_{\theta\in\mathcal{P}_\delta(\mathcal{M})}\|f_\theta\|^2_{L^p(\mathbb{R}^3)}\Big)^{1/2}. \tag{1.14}$$

When $\mathcal{M}$ is the truncated paraboloid $\mathbb{P}^2=\{(\xi,|\xi|^2)\in\mathbb{R}^3:|\xi|\le 1\}$, $\mathcal{P}_\delta(\mathbb{P}^2)$ is the coarsest partition of $N_\delta(\mathbb{P}^2)$ so that each piece θ is essentially flat. Bourgain and Demeter [BD15] also proved sharp decoupling estimates for the cylinder $Cyl=\{(\xi_1,\xi_2,\xi_3):\xi_1^2+\xi_2^2=1,\ |\xi_3|\lesssim 1\}$ and the cone $\mathcal{C}=(\xi_1,\xi_2,\sqrt{\xi_1^2+\xi_2^2}):\frac14\le\xi_1^2+\xi_2^2\le 4\}$, where their δ-neighborhoods are partitioned into planks θ of dimensions about $1\times\delta^{1/2}\times\delta$. They observed that these are the coarsest (ϕ,δ)-flat sets. Based on this, Bourgain, Demeter, and Kemp [BDK20] introduced the notion of (ϕ,δ)-flat sets (Definition 1.1) for a general manifold $\mathcal{M}_\phi$, and asked if there is a partition of $N_\delta(\mathcal{M}_\phi)$, denoted by $\mathcal{P}_\delta(\mathcal{M}_\phi)$, so that each element is (ϕ,δ)-flat and for $2\le p\le 4$ the following ℓ^p decoupling is true

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon}|\mathcal{P}_\delta(\mathcal{M}_\phi)|^{\frac12-\frac1p}\Big(\sum_{\theta\in\mathcal{P}_\delta(\mathcal{M}_\phi)}\|f_\theta\|^p_{L^p(\mathbb{R}^3)}\Big)^{1/p} \tag{1.15}$$

for all functions f whose Fourier supports are in $N_\delta(\mathcal{M}_\phi)$. This question is answered affirmatively by [LY21] (see also the references therein).

One may ask if there is a partition of $N_\delta(\mathcal{M}_\phi)$ so that each element is (ϕ,δ)-flat and for $2\le p\le 4$ the following ℓ^2 decoupling is true

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon}\Big(\sum_{\theta\in\mathcal{P}_\delta(\mathcal{M}_\phi)}\|f_\theta\|^2_{L^p(\mathbb{R}^3)}\Big)^{1/2} \tag{1.16}$$

for all functions f whose Fourier supports are in $N_\delta(\mathcal{M}_\phi)$. Note that (1.16) implies (1.15) by Hölder's inequality. When S is the truncated hyperbolic paraboloid, Bourgain and Demeter proved that

$$\|f\|_{L^p(\mathbb{R}^3)} \lesssim_\epsilon \delta^{-\epsilon}\delta^{-\frac12(\frac12-\frac1p)}\Big(\sum_{\theta\in\mathcal{P}_\delta(\mathcal{M}_\phi)}\|f_\theta\|^2_{L^p(\mathbb{R}^3)}\Big)^{1/2}. \tag{1.17}$$

They also proved that the loss of δ on the right hand side is necessary up to the loss of $\delta^{-\epsilon}$, and the sharp example comes from the geometric fact that the hyperbolic paraboloid contains a line. At first, this might look like a counterexample to the question raised in (1.16). However, since there are multiple choices of (ϕ,δ)-flat sets, it does not give a counterexample. Also, given that ℓ^2 decoupling theorem for a curve in $\mathbb{R}^2$ is proved by [Yan21] (which is the two-dimensional version of (1.16)), one might still expect that (1.16) is true for some collection of (ϕ,δ)-flat sets. In Appendix A, we prove Theorem A.1, which says that for the hyperbolic paraboloid, there is no partition, whose elements are (ϕ,δ)-flat, such that (1.16) holds true. The proof of Theorem A.1 crucially uses the assumption that a collection of (ϕ,δ)-flat sets is a partition. However, in all of the known applications of decouplings we are aware of, it does not matter if a collection of (ϕ, δ)-flat sets is a partition or $O(\delta^{-\epsilon})$-finitely overlapping. Theorem 1.2 says that ℓ^2 decoupling theorem is true after allowing the collection $\mathcal{S}_\delta$ to be $O(\log \delta^{-1})$-finitely overlapping (see (1.3)). Our decoupling theorem is still useful in the sense that it implies Corollary 1.3 and 1.5, and Proposition 1.9. Also, by Hölder's inequality, our decoupling "essentially" implies the ℓ^p decoupling theorem (1.15) of Yang and Li.

Let us finish this subsection by mentioning higher dimensional manifolds. It is conceivable that ℓ^p decoupling conjecture is still true in high dimensions. Let us introduce some notations to state the conjecture. For a function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, we say $S \subset [0,1]^{n-1}$ is (ϕ, δ)-set if (1.2) is true. Consider the manifold

$$\mathcal{M}_\phi := \{(\xi, \phi(\xi)) : \xi \in [0,1]^{n-1}\}. \tag{1.18}$$

For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ and a measurable set $S \subset \mathbb{R}^{n-1}$, denote by f_S the Fourier restriction of f on the set $S \times \mathbb{R}$.

Conjecture 1.7. *Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth function. Fix $\epsilon > 0$. Then there exists a sufficiently large number A depending on ϵ satisfying the following.*

For any $\delta > 0$, there exists a collection $\mathcal{S}_\delta$ of finitely overlapping sets $S \subset [0,1]^{n-1}$ such that

(1) the overlapping number is $O(\log \delta^{-1})$ in the sense that

$$\sum_{S \in \mathcal{S}_\delta} \chi_S \leq C_\epsilon \log(\delta^{-1}). \tag{1.19}$$

(2) S is $(\phi, A\delta)$-flat.

(3) for $2 \leq p \leq 2(n+1)/(n-1)$ we have

$$\|f\|_{L^p} \leq C_\epsilon \delta^{-\epsilon} (\#\mathcal{S}_\delta)^{\frac{1}{2}-\frac{1}{p}} \Big(\sum_{S \in \mathcal{S}_\delta} \|f_S\|_{L^p}^p \Big)^{\frac{1}{p}} \tag{1.20}$$

for all functions f whose Fourier supports are in $N_\delta(\mathcal{M}_\phi)$.

Conjecture 1.7 is known for the paraboloid and hyperbolic paraboloid, but would be new for general manifolds. The ℓ^p decoupling inequality for manifolds in $\mathbb{R}^2$ and $\mathbb{R}^3$ are proved by [Yan21] and [LY21]. It remains open for higher dimensions. One main ingredient of ℓ^p decoupling for manifolds in $\mathbb{R}^3$ is ℓ^2 decoupling for a curve in $\mathbb{R}^2$. Since we now have ℓ^2 decoupling for manifolds in $\mathbb{R}^3$, it might be possible to prove ℓ^p decoupling for manifolds in $\mathbb{R}^4$.

1.2. **Tomas-Stein for the hyperbolic paraboloid.** We state the following version of the Stein-Tomas restriction theorem, which is recorded in Theorem 1.16 and Proposition 1.27 of [Dem20]. Let $\mathbb{H}$ be the truncated hyperbolic paraboloid.

Theorem 1.8 (Stein-Tomas [Tom75, Ste86])**.** *For $4 \leq p \leq \infty$, we have*

$$\|\widehat{F}\|_{L^p(B_{\delta^{-1}})} \lesssim \delta^{\frac{1}{2}} \|F\|_{L^2(N_\delta(\mathbb{H}))}$$

for each $0 < \delta < 1$, each ball $B_{\delta^{-1}} \subset \mathbb{R}^3$, and each $F \in L^2(N_\delta(\mathbb{H}))$ supported in $N_\delta(\mathbb{H})$.

Proposition 1.9. *Theorem 2.5 implies Theorem 1.8 up to an $\delta^{-\epsilon}$ loss.*

Proof of Proposition 1.9. Theorem 1.8 with $p=\infty$ is trivial, so it suffices to prove the $p=4$ case and invoke interpolation for the remaining exponents $p \in (4,\infty)$. Apply Theorem 2.5 with $p=4$ to obtain the estimate

$$\|\widehat{F}\|_{L^4(B_{\delta^{-1}})} \lesssim_\epsilon \delta^{-\epsilon}\Big(\sum_{m=-2^{-1}\log\delta^{-1}}^{2^{-1}\log\delta^{-1}} \sum_{S\in\mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}} \|(\widehat{F})_S\|^2_{L^4(\mathbb{R}^3)}\Big)^{\frac12}. \tag{1.21}$$

Note that $(\widehat{F})_S = \widehat{F\chi_S}$. Using Hölder's inequality, we have for each $S\in\mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}$ that

$$\begin{aligned}\|\widehat{F\chi_S}\|_{L^4(\mathbb{R}^3)} &\leq \|\widehat{F\chi_S}\|^{\frac12}_{L^\infty(\mathbb{R}^3)}\|\widehat{F\chi_S}\|^{\frac12}_{L^2(\mathbb{R}^3)} \\ &\leq \|F\|^{\frac12}_{L^1(N_\delta(S))}\|F\|^{\frac12}_{L^2(N_\delta(S))} \\ &\leq |N_\delta(S)|^{\frac14}\|F\|_{L^2(N_\delta(S))}.\end{aligned}$$

Using the above inequality to bound the right hand side of (1.21), we have

$$\begin{aligned}\|\widehat{F}\|_{L^4(B_{\delta^{-1}})} &\lesssim_\epsilon \delta^{-\epsilon}\Big(\sum_{m=-2^{-1}\log\delta^{-1}}^{2^{-1}\log\delta^{-1}} \sum_{S\in\mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}} |N_\delta(S)|^{\frac12}\|F\|^2_{L^2(N_\delta(S))}\Big)^{\frac12} \\ &\lesssim_\epsilon \delta^{-\epsilon} \max_{\substack{|m|\lesssim\log\delta^{-1} \\ S\in\mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}}} |N_\delta(S)|^{\frac14}\|F\|_{L^2(N_\delta(\mathbb{H}))}.\end{aligned}$$

It remains to note that $|N_\delta(S)| \leq \delta^2$ for each $S\in\mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}$ and each $|m|\lesssim\log\delta^{-1}$. □

1.3. **Proof strategy/structure of the paper.** In Section 2, we prove ℓ^2 decoupling inequalities for perturbed hyperbolic paraboloids using a collection of $O(\log\delta^{-1})$-overlapping (ϕ,δ)-flat sets. Our result is new even for the hyperbolic paraboloid. The main new idea for the hyperbolic paraboloid is an iterative way of using broad-narrow analysis, which is combined with the induction on scale argument by [BD15, BDG16]. To prove theorem for perturbed hyperbolic paraboloids (Theorem 2.2), we combine the aforementioned argument with a restriction theory for perturbed hyperbolic paraboloids developed by [BMV20, BMV22, BMV23, GO23]. In Section 3, we use the ℓ^2 decoupling for the perturbed hyperbolic paraboloids as a black box, and prove the ℓ^2 decoupling for general polynomials. This argument is essentially the same as that for [LY21]. In Section 4, we prove Corollary 1.3 and 1.5 from the decoupling theorem. In Appendix A, we prove that there is no ℓ^2 decoupling for the hyperbolic paraboloid in $\mathbb{R}^3$ using a partition. This explains why it is necessary to introduce $\log(\delta^{-1})$ many partitions in Theorem 1.2. In Appendix B, we prove that there does not exist a collection of (ϕ,δ)-flat rectangles such that ℓ^2 decoupling for the collection is true in high dimensions without any additional power. In appendix C, we prove the sharpness of Theorem 1.2.

1.4. **Notations.** For a function F and a set A, define

$$\|F\|^p_{L^p_{\#}(A)} := \frac{1}{|A|}\|F\|^p_{L^p(A)}. \tag{1.22}$$

For real numbers a_i, define

$$\Big|\prod\!\!\!\prod_{i=1}^{3} a_i\Big| := \prod_{i=1}^{3} |a_i|^{\frac{1}{3}}. \tag{1.23}$$

For a ball B_K of radius K centered at $c(B_K)$, define

$$w_{B_K}(x) := \big(1 + \big|\frac{x - c(B_K)}{K}\big|^2\big)^{-100}. \tag{1.24}$$

For a measurable set A and a function f, define

$$\fint_A |f(x)|\, dx := \frac{1}{|A|} \int_A |f(x)|\, dx. \tag{1.25}$$

For a ball B, we define

$$\|f\|^p_{L^p_{\#}(w_B)} := |B|^{-1} \|f\|^p_{L^p(w_B)}. \tag{1.26}$$

For a rectangular box T, we denote by CT the dilation of T by a factor of C with respect to its centroid.

For two non-negative numbers A_1 and A_2, we write $A_1 \lesssim A_2$ to mean that there exists a constant C such that $A_1 \leq CA_2$. We write $A_1 \sim A_2$ if $A_1 \lesssim A_2$ and $A_2 \lesssim A_1$. We also write $A_1 \lesssim_\epsilon A_2$ if there exists C_ϵ depending on a parameter ϵ such that $A_1 \leq C_\epsilon A_2$.

Acknowledgements. The authors would like to thank Yuqiu Fu for discussion at the early stage of the project. LG is supported by a Simons Investigator award. DM is supported by the National Science Foundation under Award No. 2103249.

2. Perturbed hyperbolic paraboloids

Consider the manifold $\mathcal{M}_\phi$ associated with

$$\phi(\xi_1, \xi_2) := \xi_1\xi_2 + a_{2,0}\xi_1^2 + a_{0,2}\xi_2^2 + \sum_{3 \leq j+k \leq d} a_{j,k}\xi_1^j\xi_2^k, \tag{2.1}$$

where the coefficients $a_{j,k}$ satisfy

$$|a_{j,k}| \leq 10^{-10d} \tag{2.2}$$

for $d \geq 3$.

When $d = 2$, the manifold $\mathcal{M}_\phi$ is associated with $\phi(\xi_1, \xi_2) := \xi_1\xi_2$. Note that this is a special of case of (2.1) as we allow $a_{j,k}$ to be zero.

Definition 2.1. *Let S be a rectangle in $\mathbb{R}^2$. Take a smooth function $\Xi_S : \mathbb{R}^2 \to \mathbb{R}$ such that*

1. *the support of $\widehat{\Xi_S}$ is contained in $2S$.*
2. *$0 \leq \widehat{\Xi_S}(\xi_1, \xi_2) \leq 1$ for all $(\xi_1, \xi_2) \in \mathbb{R}^2$.*
3. *$\widehat{\Xi_S}$ is greater than 1/10 on the set S.*
4. *$\|\Xi_S\|_{L^1} \leq 1000$.*

Given a function $f : \mathbb{R}^3 \to \mathbb{C}$, define $f_S : \mathbb{R}^3 \to \mathbb{C}$ by

$$f_S(x_1, x_2, x_3) := \int_{\mathbb{R}^2} f(x_1 - y_1, x_2 - y_2, x_3)\Xi_S(y_1, y_2)\, dy_1 dy_2. \tag{2.3}$$

Note that this is a convolution of f and Ξ_S for the first two variables.

Theorem 2.2 (Uniform ℓ^2 decoupling for perturbed hyperbolic paraboloids)**.** *Fix $d \geq 2$ and $\epsilon > 0$. Then there exists a sufficiently large number A depending on d and ϵ satisfying the following.*

Let $\mathcal{M}_\phi$ be a manifold of the form (2.1) satisfying (2.2). Then for any $\delta > 0$, there exists a family $\mathcal{S}_\delta$ of rectangles $S \subset \mathbb{R}^2$ such that

(1) the overlapping number is $O(\log \delta^{-1})$ in the sense that

$$\sum_{S \in \mathcal{S}_\delta} \chi_S \leq C_{d,\epsilon} \log(\delta^{-1}), \tag{2.4}$$

(2) every S is $(\phi, A\delta)$-flat,

(3) for all f whose Fourier support is in $N_\delta(\mathcal{M}_\phi)$,

$$\|f\|_{L^4} \leq C_{d,\epsilon} \delta^{-\epsilon} \Big(\sum_{S \in \mathcal{S}_\delta} \|f_S\|_{L^4}^2 \Big)^{1/2}. \tag{2.5}$$

The constant $C_{d,\epsilon}$ is independent of the choice of ϕ.

The first step of the proof is to construct the family $\mathcal{S}_\delta$. A restriction theory for perturbed hyperbolic paraboloids is developed by [BMV20, BMV22, BMV23, GO23]. We use this theory, in particular, a language in [BMV23] to construct the family.

2.1. Construction of the family $\mathcal{S}_\delta$.

To construct a family $\mathcal{S}_\delta$, we will first define null vectors of the Hessian matrix of ϕ. Let us recall the notions in Section 3.1.1 of [BMV23]. Consider two vectors at $\xi \in [0,1]^2$ defined by

$$w_\xi := (-A(\xi), 1), \quad v_\xi := (1, -B(\xi)). \tag{2.6}$$

Here A and B are given by

$$A(\xi) = \frac{\phi_{22}(\xi)}{\phi_{12}(\xi) + \sqrt{|H_\phi(\xi)|}}, \quad B(\xi) = \frac{\phi_{11}(\xi)}{\phi_{12}(\xi) + \sqrt{|H_\phi(\xi)|}}, \tag{2.7}$$

and $H_\phi(\xi)$ is the determinant of the Hessian matrix of ϕ. The functions $\phi_{ij}(\xi)$ is defined by $\partial_i \partial_j \phi(\xi)$. Note that by (2.2) we have

$$|A(\xi)| \leq 10^{-5d}, \quad |B(\xi)| \leq 10^{-5d}. \tag{2.8}$$

By (3.7) of [BMV23], the vectors w_ξ and v_ξ have the following property.

$$(w_\xi) H_\phi(\xi) (w_\xi)^T = 0, \quad (v_\xi) H_\phi(\xi) (v_\xi)^T = 0. \tag{2.9}$$

Moreover, by (2.2), w_ξ and v_ξ are linearly independent.

We are now ready to define a family $\mathcal{S}_\delta$. Assume that δ^{-1} is a dyadic number. Let us first define $\mathcal{R}_\delta$. The family $\mathcal{S}_\delta$ will be a subset of it.

$$\mathcal{R}_\delta := \bigcup_{1 \leq \alpha \leq \delta^{-1/2} : \alpha \in 2^{\mathbb{Z}}} \mathcal{R}_{\delta\alpha, \alpha^{-1}}. \tag{2.10}$$

We need to define $\mathcal{R}_{\delta\alpha,\alpha^{-1}}$. Every element in the set will be a rectangle with dimension $\delta\alpha \times \alpha^{-1}$. Introduce a parameter β which will be related to the angle of the long direction of the rectangle and the ξ_1-axis. Define

$$\mathcal{R}_{\delta\alpha,\alpha^{-1}} := \bigcup_{\beta \in \mathbb{Z} : 0 \leq \beta \leq \delta^{-1}\alpha^{-2}\pi} \mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}. \tag{2.11}$$

Here $\mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$ is a collection of the translated copies of the rectangle of dimension $\delta\alpha \times \alpha^{-1}$ with the angle $\delta\alpha^2\beta$ with respect to the ξ_1-axis so that the elements of $\mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$ are disjoint and their union contains $[0,1]^2$. So we have defined

$$\mathcal{R}_\delta = \bigcup_{1\leq\alpha\leq\delta^{-1/2}:\alpha\in 2^{\mathbb{Z}}} \bigcup_{\beta\in\mathbb{Z}:0\leq\beta\leq\delta^{-1}\alpha^{-2}\pi} \mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}. \tag{2.12}$$

The family $\mathcal{S}_\delta$ in Theorem 2.2 will be a subcollection of $\mathcal{R}_\delta$.

Definition 2.3. *Let A be a sufficiently large constant, which will be the constant A in Theorem 2.2. We say two rectangles R_1, R_2 are comparable if*

$$R_1 \subset 2AR_2, \quad \text{and} \quad R_2 \subset 2AR_1. \tag{2.13}$$

Remark 2.4. *Suppose that R_1, R_2 are comparable. Let L be an affine transformation so that $L(R_1), L(R_2)$ are rectangles. Then $L(R_1), L(R_2)$ are also comparable.*

Let us now explain how to choose rectangles. Let A be a large number. Let $R \in \mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$. If the two following conditions are satisfied, we add R to $\mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$.

(1) R is $(\phi, A\delta)$-flat.
(2) For any point $z \in R$, we consider a rectangle centered at z, of dimension $\delta\alpha \times \alpha^{-1}$, with a long direction parallel to the vector w_z. This rectangle is comparable to R.

If the two following conditions are satisfied, we also add R to $\mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$.

(1) R is $(\phi, A\delta)$-flat.
(2) For any point $z \in R$, we consider a rectangle centered at z, of dimension $\delta\alpha \times \alpha^{-1}$, with a long direction parallel to the vector v_z. This rectangle is comparable to R.

Recall that v_z, w_z are defined in (2.6). After this process, we finally have

$$\mathcal{S}_\delta := \bigcup_\alpha \bigcup_\beta \mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}. \tag{2.14}$$

We define a partition of unity associated with $\mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$. Namely, we take smooth functions $\{\Xi_R\}_{R\in\mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}}$ so that

(1) the support of $\widehat{\Xi_R}$ is contained in $2R$.
(2) $0 \leq \widehat{\Xi_R}(\xi_1,\xi_2) \leq 1$ for all $(\xi_1,\xi_2) \in \mathbb{R}^2$.
(3) $\widehat{\Xi_R}$ is greater than $1/10$ on the set R.
(4) for any $\xi \in [0,1]^2$ we have

$$\sum_{R\in\mathcal{R}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}} \widehat{\Xi_R}(\xi) = 1. \tag{2.15}$$

(5) $\|\Xi_R\|_{L^1} \leq 1000$.

For a function f, define f_R by

$$f_S(x_1,x_2,x_3) := \int_{\mathbb{R}^2} f(x_1-y_1, x_2-y_2, x_3)\Xi_R(y_1,y_2)\, dy_1 dy_2. \tag{2.16}$$

2.2. **Example: Hyperbolic paraboloid.** Our theorem is even new for the hyperbolic paraboloid. Let us state this special case.

Theorem 2.5. *Consider $\phi(\xi_1,\xi_2)=\xi_1\xi_2$. For $2\le p\le 4$,*

$$\|f\|_{L^4}\le C_\epsilon \delta^{-\epsilon}\Big(\sum_{m=-2^{-1}\log\delta^{-1}}^{2^{-1}\log\delta^{-1}}\sum_{S\in\mathcal{R}_{2^m\delta^{1/2},2^{-m}\delta^{1/2},0}}\|f_S\|_{L^4}^2\Big)^{\frac12} \tag{2.17}$$

for all functions f whose Fourier support is in $N_\delta(\mathcal{M}_\phi)$.

While the ℓ^p decoupling for the hyperbolic paraboloid does not imply the Tomas-Stein theorem for the manifold, our theorem implies the Tomas-Stein theorem for the manifold. As an application of Theorem 2.5 to exponential sum estimates, we will prove Corollary 1.5 in Section 4.

Theorem 2.5 itself does not imply ℓ^p decoupling by [BD17a]. However, in the proof of the theorem, we proved a slightly stronger inequality, which is a mixture of ℓ^2 and ℓ^p norms for all the intermediate scales. We do not state this as a theorem as it is very involved. But this inequality likely implies the ℓ^p decoupling by [BD17a].

2.3. **Properties of partitions.** We have defined partitions $\mathcal{S}_\delta$ (see (2.14)). Let us study some properties of it.

Proposition 2.6. *Let A be a sufficiently large constant. Then*

$$\mathcal{R}_{\delta^{1/2},\delta^{1/2},0}\subset\mathcal{S}_\delta. \tag{2.18}$$

This says that our partitions contain the squares with the canonical scale. As a remark, for the hyperbolic paraboloid, note that the following decoupling is false

$$\|f\|_{L^4}\le C_\epsilon\delta^{-\epsilon}\Big(\sum_{S\in\mathcal{R}_{\delta^{1/2},\delta^{1/2},0}}\|f_S\|_{L^4}^2\Big)^{1/2} \tag{2.19}$$

for some functions f whose Fourier transforms are supported on the δ-neighborhood of the hyperbolic paraboloid.

Proof. Take $R\in\mathcal{R}_{\delta^{1/2},\delta^{1/2},0}$. Since R is a square, by the definition of $\mathcal{S}_\delta$, it suffices to prove that R is $(\phi,A\delta)$-flat. By definition, we need to prove that

$$\sup_{u,v\in R}|\phi(u)-\phi(v)-\nabla\phi(u)\cdot(u-v)|\le A\delta. \tag{2.20}$$

Since R is a square of side length $\delta^{1/2}$ and ϕ has the Hessian matrix whose determinant is bounded by two (see (2.1) and (2.2)), (2.20) follows from an application of Taylor's theorem. □

Proposition 2.7 (Finitely overlapping property)**.** *For any $\xi\in[0,1]^2$*

$$\#\{S\in\mathcal{S}_\delta:\xi\in S\}\lesssim A\log(\delta^{-1}). \tag{2.21}$$

This proposition gives a proof of the first property of the family $\mathcal{S}_\delta$ in Theorem 2.2 (recall that A is a constant depending only on d and ϵ). Let us give a proof.

Proof. Fix $\xi\in[0,1]^2$. Since the number of dyadic numbers α with $1\le\alpha\le\delta^{-1/2}$ is $O(\log(\delta^{-1}))$, it suffices to show that given α,

$$\#\{\beta\in\mathbb{Z}:\text{there exist } S\in\mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}\text{ s.t. }\xi\in S\}\lesssim A. \tag{2.22}$$

Denote by $S_{1,\xi}$ (and $S_{2,\xi}$) the rectangle centered at z, of dimension $\delta\alpha \times \alpha^{-1}$, with a long direction parallel to the vector w_ξ (and v_ξ).

Suppose that $\xi \in S$ for some $S \in \mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$. Then S is comparable to either $S_{1,\xi}$ or $S_{2,\xi}$. Without loss of generality, we may assum that S is comparable to $S_{1,\xi}$. Then the angle between the long direction of S and w_ξ is $\lesssim A\delta\alpha^2$. Since $\{\delta\alpha^2\beta\}_{\beta\in\mathbb{Z}}$ is $\delta\alpha^2$-separated, there are only $\sim A$ many β satisfying the property. This gives the proof. $\square$

The next lemma is a technical lemma. This says that the angle between the longest direction of a rectangle S and ξ_1-axis (or ξ_2-axis) is small. If S is a square, it is not clear how to define "the longest direction of S". So we prove such a statement under the condition (2.23) to guarantee that S looks like a rectangle quantitatively.

Lemma 2.8. *Let A be a sufficiently large number. Let $S \in \mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$. Suppose that*

$$\frac{\text{Length of the long direction of S}}{\text{Length of the short direction of S}} \geq A^2. \tag{2.23}$$

Then we have

$$\delta\alpha^2\beta \leq \frac{1}{10^{5d}}. \tag{2.24}$$

Proof. Suppose that $S \in \mathcal{S}_{\delta\alpha,\alpha^{-1},\delta\alpha^2\beta}$. Let us follow the proof of Proposition 2.7. Fix $\xi \in S$. Without loss of generality, we may assume that S is comparable to $S_{1,\xi}$. Then by Euclidean geometry, the angle between the long direction of S and w_ξ is $\lesssim A\delta\alpha^2$. The condition (2.23) says that $\delta\alpha^2 \leq A^{-2}$. So the angle is bounded by $\lesssim A^{-1}$. On the other hand, by (2.6) and the normalization condition (2.2), the angle between w_ξ and ξ_1-axis is bounded by 10^{-9d}. Since A is sufficiently large, the angle between the long direction of S and ξ_1-axis is bounded by 10^{-5d}. Therefore, $\delta\alpha^2\beta$, which indicates the angle, is bounded by 10^{-5d}. $\square$

Definition 2.9. *Define $D(\delta)$ to be the smallest constant such that*

$$\|f\|_{L^4} \leq D(\delta)\Big(\sum_{R\in\mathcal{S}_\delta} \|f_R\|_{L^4}^2\Big)^{\frac{1}{2}} \tag{2.25}$$

for all manifolds $\mathcal{M}_\phi$ of the form (2.1) *satisfying* (2.2) *and functions f whose Fourer supports are in $N_\delta(\mathcal{M}_\phi)$.*

One of the key properties of the partitions is the parabolic rescaling lemma. This type of rescalings is very crucial in the work of [BD15] and [LY21]. We observe that the same rescaling lemma still holds true for our collections. The proof uses a basic property of perturbed hyperbolic paraboloids (for example, see Lemma 3.2 of [GO23]).

Lemma 2.10 (Parabolic rescaling)**.** *Let $\delta \leq \sigma \leq 1$. Let $R' \in \mathcal{S}_\sigma$. Then we have*

$$\|f_{R'}\|_{L^4} \leq CD(\sigma^{-1}\delta)\Big(\sum_{R\in\mathcal{S}_\delta} \|f_R\|_{L^4}^2\Big)^{\frac{1}{2}} \tag{2.26}$$

for all manifolds $\mathcal{M}_\phi$ of the form (2.1) *satisfying* (2.2) *and functions f whose Fourer supports are in $N_\delta(\mathcal{M}_\phi)$.*

Proof. Suppose that R' has dimension $\sigma\alpha \times \alpha^{-1}$ for some $1 \leq \alpha \leq \sigma^{-1}$. Then the Fourier support of $f_{R'}$ is contained in $2R'$. By doing the triangle inequality, we may assume that the support of $f_{R'}$ is contained in a box with dimension $\sigma\alpha \times \alpha^{-1}$. By translation and rotation, we may assume that the support of $f_{R'}$ is $[0, \alpha^{-1}] \times [0, \sigma\alpha]$. By abusing the notation, let us still call this box R'. After the change of variables, our phase function changes. Let us denote the new phase function by

$$\phi_0(\xi_1, \xi_2) := \xi_1\xi_2 + b_{2,0}\xi_1^2 + b_{0,2}\xi_2^2 + \sum_{3 \leq j+k \leq d} b_{j,k}\xi_1^j\xi_2^k. \tag{2.27}$$

For convenience, we introduce $b_{1,1} = 0$. Note that

$$|b_{j,k}| \lesssim 1, \quad 2 \leq j + k \leq d. \tag{2.28}$$

Since R' is $(\phi_0, A\sigma)$-flat, by definition,

$$\sup_{u,v \in R'} |\phi_0(u) - \phi_0(v) - \nabla\phi_0(u) \cdot (u - v)| \leq A\sigma. \tag{2.29}$$

Take $u = (0, 0)$ and $v = (t, 0)$. Since $\nabla\phi_0(u) = 0$, this inequality becomes

$$\sup_{0 \leq t \leq \alpha^{-1}} |\sum_{j=2}^{d} b_{j,0}t^j| \leq A\sigma. \tag{2.30}$$

By [LY21, Proposition 7.1], this inequality gives

$$|b_{j,0}| \lesssim \sigma\alpha^j, \;\; 2 \leq j \leq d. \tag{2.31}$$

Similarly, we take $u = (0, 0)$ and $v = (0, t)$ where $0 \leq t \leq \sigma\alpha$. Then by the same reasoning, we obtain

$$|b_{0,j}| \lesssim \sigma(\sigma\alpha)^{-j}, \;\; 2 \leq j \leq d. \tag{2.32}$$

We next do rescaling. Define $L(\xi, \eta) := (\alpha^{-1}\xi, \sigma\alpha\eta)$ and

$$\widetilde{\phi}(\xi, \eta) := \sigma^{-1}\phi_0(L(\xi, \eta)). \tag{2.33}$$

The function $\tilde{\phi}(\xi, \eta)$ can be rewritten as

$$\xi_1\xi_2 + (\sigma^{-1}\alpha^{-2}b_{2,0})\xi_1^2 + (\sigma\alpha^2 b_{0,2})\xi_2^2 + \sum_{3 \leq j+k \leq d} (\sigma^{k-1}\alpha^{k-j}b_{j,k})\xi_1^j\xi_2^k. \tag{2.34}$$

To apply the induction hypothesis, we will prove

$$|b_{j,k}| \lesssim \sigma^{-k+1}\alpha^{-k+j}, \quad 2 \leq j + k \leq d. \tag{2.35}$$

The cases for $j = 0$ or $k = 0$ follow from (2.31) and (2.32). Hence, we may assume that $j \geq 1$ and $k \geq 1$.

Let us consider the subcase that $k \leq j$. By the inequality $1 \leq \alpha \leq \sigma^{-1}$, we have $1 \leq \alpha^{-k+j}$. So what we need to prove follows from

$$|b_{j,k}| \lesssim \sigma^{-k+1}. \tag{2.36}$$

Since $k \geq 1$ and $\sigma \leq 1$, this follows from (2.28). Let us next consider the subcase that $k \geq j$. By the inequality $1 \leq \alpha \leq \sigma^{-1}$, we have $\sigma^{k-j} \leq \alpha^{-k+j}$. So what we need to prove follows from

$$|b_{j,k}| \lesssim \sigma^{-j+1}. \tag{2.37}$$

Since $j \geq 1$ and $\sigma \leq 1$, this follows from (2.28).

We have now proved (2.35). So the function (2.34) can be rewritten as

$$\xi_1\xi_2 + c_{2,0}\xi_1^2 + c_{0,2}\xi_2^2 + \sum_{3\le j+k\le d} c_{j,k}\xi_1^j\xi_2^k, \tag{2.38}$$

where

$$|c_{j,k}| \lesssim 1, \quad 2 \le j+k \le d. \tag{2.39}$$

By applying an linear transformation, we may assume that $c_{2,0} = c_{0,2} = 0$. Moreover, by doing some triangle inequality, we may assume that $|c_{j,k}| \le 10^{-10d}$. By abusing the notation, let us still denote by $\tilde{\phi}$ the new phase function. By (2.33), and change of variables on physical variables, the δ-neighborhood of $\mathcal{M}_\phi$ becomes the $\sigma^{-1}\delta$-neighborhood of $\mathcal{M}_{\tilde{\phi}}$. For functions g whose Fourier supports are in $N_{\sigma^{-1}\delta}(\mathcal{M}_{\tilde{\phi}})$, we have

$$\|g\|_{L^4} \le D(\sigma^{-1}\delta)\Big(\sum_{R''\in\mathcal{S}_{\sigma^{-1}\delta}} \|g_{R''}\|_{L^4}^2\Big)^{\frac12}. \tag{2.40}$$

We next rescale back to the original variables. Given $R'' \in \mathcal{S}_{\sigma^{-1}\delta}$, by definition, we have

$$\sup_{u,v\in R''} |\widetilde{\phi}(u) - \widetilde{\phi}(v) - \nabla\widetilde{\phi}(u)\cdot(u-v)| \le A\sigma^{-1}\delta. \tag{2.41}$$

By (2.33), we can see that $L^{-1}(R'')$ is $(\phi, A\delta)$-flat. To conclude that $L^{-1}(R'') \in \mathcal{S}_\delta$, it remains to show the following property: For any point $z \in L^{-1}(R'')$, we consider a rectangle centered at z, of dimension $\sigma\alpha \times \alpha^{-1}$, with a long direction parallel to the vector v_z (let us denote by R_z). This rectangle is comparable to $L^{-1}(R'')$.

Let us prove the property. Fix $z \in L^{-1}(R'')$, and let v_z be a vector associated with the phase ϕ. By Remark 2.4, it suffices to show that $L(R_z)$ is comparable to R''. After calculating the Hessian matrix, we see that $L(v_z)$ is comparable to the vector $v_{L(z)}$ associated with the phase function $\widetilde{\phi}$. Since R'' belongs to $\mathcal{S}_{\sigma^{-1}\delta}$, this gives the desired property. □

2.4. **The broad-narrow analysis.** In this subsection, we introduce a multilinear decoupling, and show that a linear decoupling constant is comparable to a multilinear decoupling constant up to epsilon loss (Theorem 2.13). This framework is introduced by [BD15].

Definition 2.11. *Let $n(\xi)$ be the unit normal vector of $\mathcal{M}_\phi$ at the point $(\xi, \phi(\xi))$. We say three points $\xi_{(1)}, \xi_{(2)}, \xi_{(3)} \in [0,1]^2$ are N^{-1}-transverse if*

$$|n(\xi_{(1)}) \wedge n(\xi_{(2)}) \wedge n(\xi_{(3)})| \ge N^{-1}. \tag{2.42}$$

Three squares $\tau_1, \tau_2, \tau_3 \subset [0,1]^2$ are called N^{-1}-transverse if every triple $\xi_{(i)} \in \tau_i$ is N^{-1}-transverse.

Definition 2.12. *Define $D_{\mathrm{mul}}(\delta, N^{-1})$ to be the smallest constant satisfying the following: for any N^{-1}-transverse squares $\tau_1, \tau_2, \tau_3 \in \mathcal{R}_{a,a,0}$ for $\delta^{1/2} \le a \le 1$,*

$$\Big\|\prod_{i=1}^{3} f_{\tau_i}\Big\|_{L^4} \le D_{\mathrm{mul}}(\delta, N^{-1})\Big(\sum_{R\in\mathcal{S}_\delta} \|f_R\|_{L^4}^2\Big)^{\frac12} \tag{2.43}$$

for all manifolds $\mathcal{M}_\phi$ of the form (2.1) satisfying (2.2) and functions f whose Fourer supports are in $N_\delta(\mathcal{M}_\phi)$.

Recall that the definition of f_{τ_i} is given in the paragraph below (2.14). The main theorem of this subsection is as follows.

Theorem 2.13. *Given $\epsilon > 0$, there exists a sufficiently large number N depending on ϵ so that for all $\delta > 0$ we have*

$$D(\delta) \leq C_\epsilon \delta^{-\epsilon}\Big(\sup_{\delta \leq \delta' \leq 1} D_{\mathrm{mul}}(\delta', N^{-1}) + 1\Big). \tag{2.44}$$

Proof. We do a broad-narrow analysis by [BG11]. Let K be a sufficiently large number, which will be determined later. Take a sufficiently large constant K_1 such that K is sufficiently large compared to K_1. Fix a ball B_K. We write

$$f = \sum_{\tau \in \mathcal{R}_{K^{-1/2},K^{-1/2},0}} f_\tau. \tag{2.45}$$

Recall the definition in (2.11). Each τ is a square of side length $K^{-1/2}$. Consider a collection of squares

$$\mathcal{C} := \{\tau \in \mathcal{R}_{K^{-1/2},K^{-1/2},0} : \|f_\tau\|_{L^4(B_K)} \geq K^{-100}\|f\|_{L^4(B_K)}\}. \tag{2.46}$$

We will consider several cases. By a pigeonholing argument, we can see that $\mathcal{C}$ is nonempty. Take $\tau_1 \in \mathcal{C}$. Consider the first case that all the squares $\tau \in \mathcal{C}$ are in the $(K_1)^{-1}$-neighborhood of τ_1. Denote by τ_1' the neighborhood. Then by the triangle inequality we have

$$\|f\|_{L^4(B_K)} \leq \Big\| \sum_{\tau \in \mathcal{R}_{K^{-1/2},K^{-1/2},0} : \tau \cap \tau_1' \neq \emptyset} f_\tau \Big\|_{L^4(B_K)} + \sum_{\tau \notin \mathcal{C}} \|f_\tau\|_{L^4(B_K)}. \tag{2.47}$$

By the definition of $\mathcal{C}$, this gives

$$\|f\|_{L^4(B_K)} \leq 2\Big\| \sum_{\tau \in \mathcal{R}_{K^{-1/2},K^{-1/2},0} : \tau \cap \tau_1' \neq \emptyset} f_\tau \Big\|_{L^4(B_K)}. \tag{2.48}$$

The set τ_1' depends on a choice of B_K. Hence, by summing over all possible squares, we have

$$\|f\|_{L^4(B_K)} \lesssim \big(\sum_{\tau' \in \mathcal{R}_{K_1^{-1},K_1^{-1},0}} \|f_{\tau'}\|_{L^4(B_K)}^4 \big)^{\frac{1}{4}}. \tag{2.49}$$

We next consider the case that there exists $\tau_2 \in \mathcal{C}$ outisde of the $(K_1)^{-1}$-neighborhood of τ_1. There are two subcases. Suppose that there are $\tau_3 \in \mathcal{C}$ such that τ_1, τ_2, τ_3 are $10K^{-1}$-transverse. Then we have

$$\|f\|_{L^4(B_K)} \lesssim K^{100} \prod \big\|f_{\tau_i}\big\|_{L^4(B_K)}. \tag{2.50}$$

By an application of randomization argument and uncertainty principle, we have

$$\|f\|_{L^4(B_K)} \lesssim K^{200} \| \prod \widetilde{f}_{\tau_i} \|_{L^4(w_{B_K})}, \tag{2.51}$$

where $\widetilde{f}_{\tau_i}$ is a modulation of f_{τ_i}. The definition of w_{B_K} is given in (1.24). We refer to (2.5) – (2.8) of [GMO23] for the details of the argument.

Lastly, suppose that there does not exist such τ_3. Denote by $\xi_{(i)}$ the center of τ_i for $i = 1, 2$. By Definition 2.11, all the elements $\tau \in \mathcal{C}$ are contained in the set

$$Z_0 := \{\xi \in [0,1]^2 : |n(\xi_{(1)}) \wedge n(\xi_{(2)}) \wedge n(\xi)| < 100K^{-1}\}. \tag{2.52}$$

Here $n(\xi)$ is the normal vector of the manifold (2.1) at the point $(\xi, \phi(\xi))$. By the definition of $\mathcal{C}$, we can obtain

$$\|f\|_{L^4(B_K)} \lesssim \Big\| \sum_{\tau \subset Z_0} f_\tau \Big\|_{L^4(B_K)}. \tag{2.53}$$

We apply a uniform ℓ^2 decoupling theorem for the set Z_0 of [LY21, Theorem 3.1], and obtain

$$\|f\|_{L^4(B_K)} \lesssim_\epsilon K^\epsilon \big(\sum_S \|f_S\|^2_{L^4(w_{B_K})} \big)^{\frac{1}{2}} \tag{2.54}$$

for any $\epsilon > 0$. Here $S \subset [0,1]^2$ is a rectangular box, and the collection $\{S\}_S$ has bounded overlap. Moreover,

$$\bigcup_S S \subset \{\xi \in [0,1]^2 : |n(\xi_{(1)}) \wedge n(\xi_{(2)}) \wedge n(\xi)| \lesssim K^{-1}\}. \tag{2.55}$$

The function f_S is the Fourier restriction of f to the rectangle $S \times \mathbb{R}$.[1] Define

$$Z := \{\xi \in [0,1]^2 : |n(\xi_{(1)}) \wedge n(\xi_{(2)}) \wedge n(\xi)| = 0\}. \tag{2.56}$$

Then by the separation between τ_1 and τ_2, and by the normalization conditions (2.1) and (2.2), one can see that

$$(2.55) \subset N_{CK_1/K}(Z). \tag{2.57}$$

Let us state this as a lemma.

Lemma 2.14.

$$\{\xi \in [0,1]^2 : |n(\xi_{(1)}) \wedge n(\xi_{(2)}) \wedge n(\xi)| \lesssim K^{-1}\} \subset N_{CK_1/K}(Z). \tag{2.58}$$

We postpone the proof of this lemma and continue the proof of Theorem 2.13. By the lemma, each S in (2.55) is contained in a rectangle with dimension $1 \times CK_1K^{-1}$. Let us fix S. Suppose that

$$\frac{\text{Length of the long direction of S}}{\text{Length of the short direction of S}} \leq K_1. \tag{2.59}$$

Then we decompose S into squares of side length equal to the length of the short direction of S. Then the number of squares is bounded by $O(K_1)$. By Proposition 2.6, we have

$$\|f_S\|_{L^4} \lesssim (K_1)^{\frac{1}{4}} \big(\sum_{S' \in \mathcal{S}_X : S' \cap 2S \neq \emptyset} \|f_{S'}\|^2_{L^4} \big)^{\frac{1}{2}}, \tag{2.60}$$

where $K^{-2} \lesssim X \lesssim (K_1K^{-1})^2$. Since K is sufficiently large compared to K_1, the term $(K_1)^{1/4}$ on the right hand side will not make any trouble.

Let us consider the case that

$$\frac{\text{Length of the long direction of S}}{\text{Length of the short direction of S}} \geq K_1. \tag{2.61}$$

[1]There is a minor technical issue. In the work of [LY21], they used a characteristic function for the cutoff function. The proof can be modified so that the cutoff is smooth. In our application, we use a smooth cutoff function.

Assume that K_1 is sufficiently large so that $K_1 > A^2$. By translation and rotation, we may assume that S is contained in $S' := [0,1] \times [0, CK_1K^{-1}]$. By Lemma 2.8, after some change of variables, we may write the new phase function by

$$\xi_1\xi_2 + a\xi_1^2 + b\xi_2^2 + \sum_{3 \le j+k \le d} b_{j,k}\xi_1^j\xi_2^k, \tag{2.62}$$

where

$$|a| + |b| + \sum_{3 \le j+k \le d} |b_{j,k}| \le 10^{-d}. \tag{2.63}$$

We will apply the following lemma.

Lemma 2.15. *Let K_2 be a number such that $1 \le K_2 \le \delta^{-1/2}$. Let $S' = [0,1] \times [0, K_2^{-1}]$. Suppose that the phase function is*

$$\Phi(\xi_1,\xi_2) := \xi_1\xi_2 + a\xi_1^2 + b\xi_2^2 + \sum_{3 \le j+k \le d} b_{j,k}\xi_1^j\xi_2^k \tag{2.64}$$

where

$$|a| + |b| + \sum_{3 \le j+k \le d} |b_{j,k}| \le 10^{-d}. \tag{2.65}$$

Then for any $\epsilon > 0$ we have

$$\|f_{S'}\|_{L^4(B_{K_2})} \le C_{\epsilon,d,A} K_2^{\epsilon} \sup_{K_2 \le L \le (K_2)^2} \Big(\sum_{T \in \mathcal{S}_{L-1} : T \subset 2S'} \|f_T\|^2_{L^4(w_{B_{K_2}})} \Big)^{\frac12} \tag{2.66}$$

for all functions f whose Fourier support is in $N_\delta(\mathcal{M}_\Phi)$. Here $f_{S'}$ is the Fourier restriction of f to the rectangle $S' \times \mathbb{R}$.

Let us assume this lemma and finish the proof. By (2.54) and Lemma 2.15, we have

$$\|f\|_{L^4(B_K)} \le C_\epsilon (K_1^{-1}K)^\epsilon K^\epsilon \sup_{(K_1^{-1}K) \le L \le (K)^2} \Big(\sum_{T \in \mathcal{S}_{L-1}} \|f_T\|^2_{L^4(w_{B_K})} \Big)^{\frac12}. \tag{2.67}$$

To summarize, by considering all possible cases, we have

$$\begin{aligned} \|f\|_{L^4} \lesssim_\epsilon & \Big(\sum_{\tau' \in \mathcal{R}_{K_1^{-1},K_1^{-1},0}} \|f_{\tau'}\|^4_{L^4} \Big)^{\frac14} \\ & + K^{200} \Big\| \prod_{i=1}^{3} \widetilde{f}_{\tau_i} \Big\|_{L^4} + K^\epsilon \sup_{K_1^{-1}K \le L \le (K)^2} \big(\sum_{T \in \mathcal{S}_L} \|f_T\|^2_{L^4} \big)^{\frac12} \end{aligned} \tag{2.68}$$

for some transverse τ_i. By Definition 2.12 and Lemma 2.10, we have

$$D(\delta) \lesssim_\epsilon D(K_1^2\delta) + K^{200} D_{\mathrm{mul}}(\delta, 10K^{-1}) + K^\epsilon \sup_{(K_1^{-1}K) \le L \le (K)^2} D(L\delta). \tag{2.69}$$

We apply this inequality to the first and third terms on the right hand side of (2.69) repeatedly, and obtain the desired result. □

Proof of Lemma 2.14. For simplicity, we introduce $F(\xi) := n(\xi_{(1)}) \wedge n(\xi_{(2)}) \wedge n(\xi)$. By the definition of the normal vector $n(\xi)$, this can be rewritten as

$$F(\xi) = \begin{vmatrix} \partial_1\phi(\xi_{(1)}) & \partial_2\phi(\xi_{(1)}) & -1 \\ \partial_1\phi(\xi_{(2)}) & \partial_2\phi(\xi_{(2)}) & -1 \\ \partial_1\phi(\xi) & \partial_2\phi(\xi) & -1 \end{vmatrix}. \tag{2.70}$$

After routine computations, this is equal to

$$-\partial_1\phi(\xi)\big(\partial_2\phi(\xi_{(1)})-\partial_2\phi(\xi_{(2)})\big)+\partial_2\phi(\xi)\big(\partial_1\phi(\xi_{(1)})-\partial_1\phi(\xi_{(2)})\big) \\ -\partial_1\phi(\xi_{(1)})\partial_2\phi(\xi_{(2)})+\partial_1\phi(\xi_{(2)})\partial_2\phi(\xi_{(1)}). \tag{2.71}$$

Recall that $\xi_{(1)}$ and $\xi_{(2)}$ are fixed. For simplicity, we write the function as

$$F(\xi)=A\partial_1\phi(\xi)+B\partial_2\phi(\xi)+C. \tag{2.72}$$

We first claim that

$$\max\Big(|\partial_2\phi(\xi_{(1)})-\partial_2\phi(\xi_{(2)})|,|\partial_1\phi(\xi_{(1)})-\partial_1\phi(\xi_{(2)})|\Big)\gtrsim\frac{1}{K_1}. \tag{2.73}$$

The claim is equivalent to

$$\big|\nabla\phi(\xi_{(1)})-\nabla\phi(\xi_{(2)})\big|\gtrsim\frac{1}{K_1}. \tag{2.74}$$

By Taylor's theorem, note that

$$|\nabla^2\phi(\xi_{(1)})\cdot(\xi_{(1)}-\xi_{(2)})|\sim\big|\nabla\phi(\xi_{(1)})-\nabla\phi(\xi_{(2)})\big|. \tag{2.75}$$

Since the eigenvalues of the Hessian matrix of ϕ is comparable to one, and by the separation $|\xi_{(1)}-\xi_{(2)}|\gtrsim\frac{1}{K_1}$, we have

$$|\nabla^2\phi(\xi_{(1)})\cdot(\xi_{(1)}-\xi_{(2)})|\gtrsim\frac{1}{K_1}. \tag{2.76}$$

This completes the proof of the claim.

Let us continue the proof of Lemma 2.14. Recall (2.72). Let us consider the case that $|A|\leq|B|$. The case $|A|\geq|B|$ can be dealt with similarly. To prove Lemma 2.14, let us fix $\xi\in[0,1]^2$ and suppose that

$$F(\xi)=X,\ \ |X|\lesssim K^{-1}. \tag{2.77}$$

It suffices to show that there exists $\alpha\in\mathbb{R}$ such that

- $|\alpha|\lesssim K_1K^{-1}$.
- $F\big(\xi+(\alpha,0)\big)=0$.

By Taylor's theorem, since F is a polynomial of degree $d-1$,

$$F\big(\xi+(\alpha,0)\big)=F(\xi)+\partial_1F(\xi)\alpha+\sum_{2\leq j\leq d-1}\frac{\partial_1^jF(\xi)}{j!}\alpha^j. \tag{2.78}$$

By the expression (2.72),

$$\partial_1^jF(\xi)=A\partial_1^{j+1}\phi(\xi)+B\partial_1^j\partial_2\phi(\xi). \tag{2.79}$$

By the normalization (2.1) and (2.2), we have

$$\begin{aligned} F\big(\xi+(\alpha,0)\big)&=F(\xi)+(\partial_1F(\xi))\alpha+O\big(B\alpha^2\big)\\ &=X+(\partial_1F(\xi))\alpha+O\big(B\alpha^2\big). \end{aligned} \tag{2.80}$$

Note that $|\partial_1F(\xi)|\sim|B|$. By (2.73), we have $|B|\gtrsim K_1^{-1}$ and we can find $|\alpha|\lesssim K_1K^{-1}$ such that

$$X+(\partial_1F(\xi))\alpha+O\big(B\alpha^2\big)=0. \tag{2.81}$$

This finishes the proof. $\square$

Proof of Lemma 2.15. We use an induction on K_2. By taking $C_{\epsilon,d,A}$ sufficiently large, we may assume that Lemma 2.15 is true for sufficiently large K_2 (compared to A).

We write our phase function (2.64) as

$$A(\xi_1)+B(\xi_1,\xi_2):=\big(a\xi_1^2+\sum_{3\leq j\leq d}b_{j,0}\xi_1^j\big)+(\xi_1\xi_2+b\xi_2^2+\cdots). \tag{2.82}$$

On the set $(\xi_1,\xi_2)\in S'$,

$$A(\xi_1)+B(\xi_1,\xi_2)=A(\xi_1)+O(K_2^{-1}). \tag{2.83}$$

By [LY21, Proposition 7.1],

$$\sup_{0\leq\xi_1\leq 1}|A(\xi_1)|\sim_d \max\big(|a|,\max_j|b_{j,0}|\big). \tag{2.84}$$

Let us consider the case that the left hand side of (2.84) is smaller than AK_2^{-1} for some fixed large number A. We would like to show that $S'\in\mathcal{S}_{(K_2)^{-1}}$. First, note that S' is (ϕ,K_2^{-1})-flat. Also, for any $z\in S'$, the angle between ξ_1-axis and v_z is smaller than $O(AK_2^{-1})$. Hence, the rectangle, centered at z of dimension $1\times K_2^{-1}$ with a long direction parallel to the vector v_z, is comparable to S'. So we see that $S'\in\mathcal{S}_{(K_2)^{-1}}$. This already gives the desired result (2.66).

We next consider the case that the left hand side of (2.84) is larger than $AK_2^{-1}/2$. Write our function as

$$f(x)=\int_{\mathbb{R}^3}\hat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi. \tag{2.85}$$

We do a change of variables $x_3\mapsto M^{-1}x_3$ and $\xi_3\mapsto M\xi_3$ so that the Fourier support of f is contained in the δM-neighborhood of

$$\{M(A(\xi_1)+B(\xi_1,\xi_2)):\xi_1\in[0,1],\,\xi_2\in[0,K_2^{-1}]\} \tag{2.86}$$

for some $M^{-1}\gtrsim AK_2^{-1}$ so that $M\sup_{0\leq\xi_1\leq1}|A(\xi_1)|\sim 1$. Note that

$$M(A(\xi_1)+B(\xi_1,\xi_2))=MA(\xi_1)+O(A^{-1}). \tag{2.87}$$

After the change of variables, the ball B_{K_2} on the physical ball becomes a ball of radius K_2/M, which is still larger than A. Then we use a uniform ℓ^2 decoupling for a curve of [Yan21, Theorem 1.4] by ignoring ξ_2-variable, and obtain

$$\|f_{S'}\|_{L^4}\leq C_{\epsilon/2}^{\mathrm{curve}}(A')^{\epsilon/2}\big(\sum_{T'}\|f_{T'}\|_{L^4}^2\big)^{\frac12}. \tag{2.88}$$

Here the sidelength of the longest direction of T' is $(A')^{-1}$, which is smaller than or equal to $A^{-1/d}$. The set T' has dimension $(A')^{-1}\times(K_2)^{-1}$. We next do a isotropic rescaling (with translation), and T' becomes a rectangle with dimension $1\times A'/K_2$. Since $\frac{A'}{K_2}>\frac{1}{K_2}$, by applying the induction hypothesis on K_2, and rescaling back, the left hand side of (2.88) is bounded by

$$C_{\epsilon/2}^{\mathrm{curve}}C_\epsilon(A')^{\epsilon/2}(\frac{K_2}{A'})^{\epsilon}\big(\sum_{T}\|f_T\|_{L^4}^2\big)^{\frac12}. \tag{2.89}$$

Recall that $A'\geq A^{1/d}$. Since A is sufficiently large, we have $(A')^{-\epsilon/2}\leq(C_{\epsilon/2}^{\mathrm{curve}})^{-1}$, and we can close the induction. $\square$

2.5. **Bourgain-Demeter type iteration.** We have shown that the linear decoupling constant is bounded by the multilinear decoupling constant (Theorem 2.13). The next step is to bound the multilinear decoupling constant. Here is an ingredient.

Lemma 2.16 (Ball inflation lemma)**.** *Let ϕ be a function* (2.1) *satisfying* (2.2). *Let $p = 4$, $q = 8/3$, and $N > 0$. Let τ_1, τ_2, τ_3 be N^{-1}-transverse squares. For any $\rho > 0$, $\epsilon > 0$ and $x_0 \in \mathbb{R}^3$,*

(2.90)
$$\Big(\fint_{B(x_0,\rho^{-2})} \Big(\prod_{i=1}^{3} \big(\sum_{J \in \mathcal{R}_{\rho,\rho,0}: J \subset \tau_i} \|f_J\|^2_{L^q_\#(w_{B_{\rho^{-1}}(x)})} \big)^{\frac12} \Big)^p dx \Big)^{\frac1p} \\ \leq C_{\epsilon,N} \rho^{-\epsilon} \prod_{i=1}^{3} \Big(\sum_{J \in \mathcal{R}_{\rho,\rho,0}: J \subset \tau_i} \|f_J\|^2_{L^q_\#(w_{B_{\rho^{-2}}(x_0)})} \Big)^{\frac12}$$

for all functions f whose Fourier supports are in $N_{\rho^2}(\mathcal{M}_\phi)$.

The proof of the ball inflation lemma is standard. We refer to [BDG16, Theorem 6.6] for the details (see also [BD17b, Theorem 9.2]).

We are ready to give a proof of Theorem 2.2. Let Γ be the smallest constant such that for every $\epsilon > 0$, we have

$$D(\delta) \leq C_\epsilon \delta^{-\Gamma-\epsilon}, \text{ for every dyadic } \delta < 1, \tag{2.91}$$

where C_ϵ is a constant depending on ϵ. Our goal is to prove $\Gamma = 0$.

We introduce some notations. Take $q := 8/3$, and note that

$$\frac{1}{q} = \frac{1/2}{2} + \frac{1/2}{4}. \tag{2.92}$$

Define

$$\begin{aligned} \tilde{A}_2(b) &:= \Big\| \prod_{i=1}^{3} \Big(\sum_{J \in \mathcal{R}_{\delta^{b/2},\delta^{b/2},0}: J \subset \tau_i} \|f_J\|^2_{L^2_\#(w_{B(x,\delta^{-b})})} \Big)^{\frac12} \Big\|_{L^4_{x\in\mathbb{R}^3}}, \\ \tilde{A}_q(b) &:= \Big\| \prod_{i=1}^{3} \Big(\sum_{J \in \mathcal{R}_{\delta^{b/2},\delta^{b/2},0}: J \subset \tau_i} \|f_J\|^2_{L^q_\#(w_{B(x,\delta^{-b/2})})} \Big)^{\frac12} \Big\|_{L^4_{x\in\mathbb{R}^3}}, \\ \tilde{A}_4(b) &:= \Big\| \prod_{i=1}^{3} \Big(\sum_{J \in \mathcal{R}_{\delta^{b/2},\delta^{b/2},0}: J \subset \tau_i} \|f_J\|^2_{L^4_\#(w_{B(x,\delta^{-b})})} \Big)^{\frac12} \Big\|_{L^4_{x\in\mathbb{R}^3}}, \end{aligned} \tag{2.93}$$

where $0 < b < 1$. For $0 < b < 1$ and $* = 2, q, 4$, we let $a_*(b)$ the infimum over all exponents a satisfying that

$$\tilde{A}_*(b) \lesssim_{a,N} \delta^{-a} \Big(\sum_{R \in \mathcal{S}_\delta} \|f_R\|^2_{L^4(\mathbb{R}^3)} \Big)^{\frac12} \tag{2.94}$$

for every $\delta > 0$, every N^{-1}-trasverse squares τ_i, and every choice of a function f. Lastly, define

$$a_* := \liminf_{b \to 0} \frac{\Gamma - a_*(b)}{b} \tag{2.95}$$

for $* = 2, q, 4$.

Lemma 2.17. *We have the following inequalities.*

1. (setup) $a_* < \infty$ *for* $* = 2, q, 4$
2. (rescaling) $a_4 \geq \Gamma$
3. (L^2-orthogonality) $a_2 \geq 2a_q$
4. (ball inflation) $a_q \geq a_4/2 + a_2/2$

Notice that by combining all the inequalities in the lemma, we obtain

$$\Gamma \leq a_4 \leq 2a_q - a_2 \leq 0. \tag{2.96}$$

Hence, in order to prove Theorem 2.2, it suffice to prove Lemma 2.17.

Proof of Lemma 2.17. Let us show the item (1). It suffices to show

$$\Gamma \leq Cb + a_*(b) \tag{2.97}$$

for some constant C. By the essentially constant property and Bernstein's inequality, we have

$$\Big\| \prod_{i=1}^{3} f_{\tau_i} \Big\|_{L^4} \lesssim \delta^{-Cb} \tilde{A}_*(b) \tag{2.98}$$

for any $* = 2, q, 4$ and $0 < b < 1$. By the definition of $a_*(b)$ (see (2.94)), we have

$$\Big\| \prod_{i=1}^{3} f_{\tau_i} \Big\|_{L^4} \lesssim \delta^{-Cb-a_*(b)} \Big(\sum_{R \in \mathcal{S}_\delta} \|f_R\|_{L^4(\mathbb{R}^3)}^2 \Big)^{\frac{1}{2}}. \tag{2.99}$$

Hence, we have

$$D_{\mathrm{mul}}(\delta, N^{-1}) \lesssim \delta^{-Cb-a_*(b)}. \tag{2.100}$$

Combining this with Theorem 2.13 gives

$$D(\delta) \lesssim_\epsilon \delta^{-\epsilon} \delta^{-Cb-a_*(b)}. \tag{2.101}$$

Since $\epsilon > 0$ is arbitrary, by the definition of Γ, we obtain (2.97).

Let us next show the item (2). By the definition of a_4, it follows from

$$a_4(b) \leq \Gamma \cdot (1-b). \tag{2.102}$$

By Hölder's inequality and Minkowski's inequality, we have

$$\tilde{A}_4(b) \lesssim \prod_{i=1}^{3} \Big(\sum_{J \in \mathcal{R}_{\delta^{b/2}, \delta^{b/2}, 0} : J \subset \tau_i} \|f_J\|_{L^4(\mathbb{R}^3)}^2 \Big)^{\frac{1}{2}}. \tag{2.103}$$

By the parabolic rescaling lemma (Lemma 2.10), this is further bounded by

$$\begin{aligned} &\lesssim D(\delta^{1-b}) \prod_{i=1}^{3} \Big(\sum_{J \in \mathcal{S}_\delta : J \subset \tau_i} \|f_J\|_{L^4(\mathbb{R}^3)}^2 \Big)^{\frac{1}{2}} \\ &\lesssim D(\delta^{1-b}) \Big(\sum_{J \in \mathcal{S}_\delta} \|f_J\|_{L^4(\mathbb{R}^3)}^2 \Big)^{\frac{1}{2}}. \end{aligned} \tag{2.104}$$

By (2.91) and (2.94), we have

$$\delta^{-a_4(b)} \lesssim \delta^{-\Gamma \cdot (1-b)}. \tag{2.105}$$

This gives (2.102).

Let us move onto the item (3). By the L^2-orthogonality and Hölder's inequality, we obtain

$$\begin{aligned}\tilde{A}_2(b) &\lesssim \Big\| \prod_{i=1}^{3} \Big(\sum_{J \in \mathcal{R}_{\delta^b,\delta^b,0}: J \subset \tau_i} \|f_J\|^2_{L^2_\#(w_{B(x,\delta^{-b})})} \Big)^{\frac{1}{2}} \Big\|_{L^4_{x\in\mathbb{R}^3}} \\ &\lesssim \Big\| \prod_{i=1}^{3} \Big(\sum_{J \in \mathcal{R}_{\delta^b,\delta^b,0}: J \subset \tau_i} \|f_J\|^2_{L^q_\#(w_{B(x,\delta^{-b})})} \Big)^{\frac{1}{2}} \Big\|_{L^4_{x\in\mathbb{R}^3}}. \end{aligned} \tag{2.106}$$

The last expression is $\tilde{A}_q(2b)$, so we have

$$\delta^{-a_2(b)} \lesssim \delta^{-a_q(2b)}, \tag{2.107}$$

or equivalently,

$$a_2(b) \leq a_q(2b). \tag{2.108}$$

After some computations, this gives

$$a_2 \geq 2a_q. \tag{2.109}$$

This completes the proof of the item (3).

Lastly, let us show the item (4). By the ball inflation lemma (Lemma 2.16),

$$\tilde{A}_q(b) \lesssim_\epsilon \delta^{-\epsilon} \Big\| \prod_{i=1}^{3} \Big(\sum_{J \in \mathcal{R}_{\delta^{b/2},\delta^{b/2},0}: J \subset \tau_i} \|f_J\|^2_{L^q_\#(w_{B(x,\delta^{-b})})} \Big)^{\frac{1}{2}} \Big\|_{L^4_{x\in\mathbb{R}^3}}. \tag{2.110}$$

By Hölder's inequality and (2.92),

$$\tilde{A}_q(b) \lesssim_\epsilon \delta^{-\epsilon} \tilde{A}_2(b)^{\frac{1}{2}} \tilde{A}_4(b)^{\frac{1}{2}}. \tag{2.111}$$

By the definition of $a_*(b)$, this gives

$$\delta^{-a_q(b)} \lesssim \delta^{-\epsilon} \delta^{-\frac{1}{2}(a_2(b)+a_4(b))}. \tag{2.112}$$

After some computations, this gives

$$a_q \geq a_4/2 + a_2/2. \tag{2.113}$$

This completes the proof of the item (4). □

3. General manifolds

We have proved the ℓ^2 decoupling for perturbed hyperbolic paraboloids. We will use this decoupling as a black box, and prove Theorem 3.1. This section does not contain any novelty, and we simply follow the argument of [LY21] in Section 5.

The proof of Theorem 1.2 can be reduced to that for ℓ^2 decoupling for polynomials (Theorem 3.1). We refer to Section 2.3 of [LY21] for the details.

Theorem 3.1. *Fix $d \geq 2$ and $\epsilon > 0$. Then there exists a sufficiently large number A depending on d and ϵ satisfying the following.*

Let ϕ be a polynomial of two variables of degree d with coefficients bounded by one. For any $\delta > 0$, there exists a collection $\mathcal{S}_\delta$ of finitely overlapping sets S such that

(1) the overlapping number is $O(\log \delta^{-1})$ in the sense that

$$\sum_{S \in \mathcal{S}_\delta} \chi_S \leq C_{d,\epsilon} \log(\delta^{-1}). \tag{3.1}$$

(2) S is $(\phi, A\delta)$-flat.
(3) we have

$$\|f\|_{L^4} \le C_{d,\epsilon}\delta^{-\epsilon}\Big(\sum_{S\in\mathcal{S}_\delta}\|f_S\|_{L^4}^2\Big)^{1/2} \tag{3.2}$$

for all f whose Fourier support is in $N_\delta(\mathcal{M}_\phi)$.
The constant $C_{d,\epsilon}$ is independent of the choice of ϕ.

To prove Theorem 3.1, we use Theorem 2.2 as a black box. Theorem 2.2 is a decoupling theorem for a phase function (2.1) satisfying (2.2), but by a simple change of variables, this theorem can be generalized to that for a polynomial function ϕ with bounded coefficients such that

$$|H_\phi(\xi)| \gtrsim 1 \tag{3.3}$$

for all $\xi \in [0,1]^2$. Here H_ϕ is the determinant of the Hessian matrix of ϕ.

3.1. **Sketch of the proof of Theorem 3.1.** Let M be a sufficiently large number, which will be determined later (see the end of Section 3). The proof is composed of three steps.

Step 1. Dichotomy: a curved part and a flat part. Decompose $[0,1]^2$ according to the size of $|H_\phi(\xi)|$.

$$\begin{aligned} S_{\text{curved}} &:= \{(\xi_1,\xi_2)\in[0,1]^2 : |H_\phi(\xi_1,\xi_2)| > M^{-1}\},\\ S_{\text{flat}} &:= \{(\xi_1,\xi_2)\in[0,1]^2 : |H_\phi(\xi_1,\xi_2)| < M^{-1}\}.\end{aligned} \tag{3.4}$$

Introduce a parameter $M \ll M_1$. Decompose $[0,1]^2$ into squares of side length M_1. Then on each square, one can see that $|H_\phi| > M^{-1}/2$ because $M_1^{-1} \ll M^{-1}$ and ∇H_ϕ is bounded. By the triangle inequality,

$$\|f\|_{L^4} \le \Big\|\sum_{\substack{|\tau|=M_1^{-1}:\\ \tau\cap S_{\text{curved}}\neq\emptyset}} f_\tau\Big\|_{L^4} + \Big\|\sum_{\substack{|\tau|=M_1^{-1}:\\ \tau\subset S_{\text{flat}}}} f_\tau\Big\|_{L^4}. \tag{3.5}$$

After applying the triangle inequality, we apply Theorem 2.2 (see also the discussion below Theorem 3.1) to the first term on the right hand side, and this gives

$$\begin{aligned}\Big\|\sum_{\substack{|\tau|=M_1^{-1}:\\ \tau\cap S_{\text{curved}}\neq\emptyset}} f_\tau\Big\|_{L^4} &\lesssim M_1\Big(\sum_{\substack{|\tau|=M_1^{-1}:\\ \tau\cap S_{\text{curved}}\neq\emptyset}}\|f_\tau\|_{L^4}^2\Big)^{\frac12}\\ &\lesssim_\epsilon M_1\delta^{-\epsilon}\Big(\sum_{S\in\mathcal{S}_\delta}\|f_S\|_{L^4}^2\Big)^{1/2}.\end{aligned} \tag{3.6}$$

Since M_1 is a fixed number independent of δ, this already gives the desired bound. It remains to bound the second term on the right hand side of (3.5).

Step 2. Analysis of the flat part. We next bound the term

$$\Big\|\sum_{\substack{|\tau|=M_1^{-1}:\\ \tau\subset S_{\text{flat}}}} f_\tau\Big\|_{L^4}. \tag{3.7}$$

By the generalized 2D uniform ℓ^2 decoupling for polynomials (Theorem 3.1 of [LY21]), we can cover S_{flat} by rectangles T satisfying

$$T \subset \{(\xi_1, \xi_2) \in [0,1]^2 : |H_\phi(\xi_1, \xi_2)| \lesssim M^{-1}\}, \tag{3.8}$$

and

$$\Big\| \sum_{\substack{|\tau|=M_1^{-1}: \\ \tau \subset S_{\text{flat}}}} f_\tau \Big\|_{L^4} \lesssim_\epsilon M^\epsilon \big(\sum_T \|f_T\|_{L^4}^2 \big)^{\frac{1}{2}}. \tag{3.9}$$

Let us fix T. Take an affine transformation L such that it maps $[0,1]^2$ to T. Take $\phi_1 := \phi \circ L$. Then one can see that

$$|H_{\phi_1}(\xi)| \lesssim M^{-1} \tag{3.10}$$

for all $\xi \in [0,1]^2$. By [LY21, Proposition 7.1], all the coefficients of $H_{\phi_1}(\xi)$ are bounded above by $\sim M^{-1}$. Hence, we can apply [LY21, Theorem 3.2], and obtain a rotation ρ such that

$$\phi_2(\xi_1, \xi_2) := \phi_1 \circ \rho(\xi_1, \xi_2) = a(\xi_1) + M^{-\alpha} b(\xi_1, \xi_2). \tag{3.11}$$

Here a, b have coefficients bounded above by ~ 1, and α is a positive number. By the uncertainty principle, the term $M^{-\alpha}B(\xi_1, \xi_2)$ is negligible on the ball of radius M^α on the physical side. Hence, we apply the 2D uniform ℓ^2 decoupling of [LY21, Theorem 4.4] to the one variable polynomial $a(\xi_1)$, and partition $[0,1]$ into intervals I so that each rectangle is $(\phi_2, AM^{-\alpha})$-flat. Combining all the inequalities we have obtained so far, we have

$$\begin{aligned} \Big\| \sum_{\substack{|\tau|=M_1^{-1}: \\ \tau \subset S_{\text{flat}}}} f_\tau \Big\|_{L^4} &\lesssim_\epsilon M^\epsilon \big(\sum_T \|f_T\|_{L^4}^2 \big)^{\frac{1}{2}} \\ &\lesssim_\epsilon M^\epsilon \big(\sum_{S' \in \mathcal{S}_{M^{-\alpha}}} \|f_{S'}\|_{L^4}^2 \big)^{\frac{1}{2}}. \end{aligned} \tag{3.12}$$

Step 3. Iteration. By (3.5), (3.6), and (3.12), we obtain

$$\|f\|_{L^4} \leq C_\epsilon \delta^{-\epsilon} \Big(\sum_{S \in \mathcal{S}_\delta} \|f_S\|_{L^4}^2 \Big)^{1/2} + C_\epsilon (M^\alpha)^\epsilon \big(\sum_{S' \in \mathcal{S}_{M^{-\alpha}}} \|f_{S'}\|_{L^4}^2 \big)^{\frac{1}{2}}. \tag{3.13}$$

The first term is already of the desired form. To bound the second term, we fix S' and do rescaling. Let L be an affine transformation mapping $[0,1]^2$ to S'. Since S' is $(\phi, AM^{-\alpha})$-flat, we have

$$\sup_{u,v \in S'} |\phi(u) - \phi(v) - \nabla\phi(u) \cdot (u - v)| \leq AM^{-\alpha}. \tag{3.14}$$

Fix any point $u_0 \in S'$ and define

$$\widetilde{\phi}(\xi) := A^{-1} M^\alpha \big(\phi(Lu_0) - \phi(L\xi) - \nabla\phi(Lu_0) \cdot (Lu_0 - L\xi) \big). \tag{3.15}$$

Then by (3.14), we have $\sup_{\xi \in [0,1]^2} |\widetilde{\phi}(\xi)| \leq 1$. By [LY21, Proposition 7.1], all the coefficients of $\widetilde{\phi}$ is bounded by one. So this phase function satisfies the hypothesis of Theorem 3.1. We apply (3.13) to the function, and obtain

$$\begin{aligned} \|f_{S'}\|_{L^4} &\leq C_\epsilon \delta^{-\epsilon}\Big(\sum_{S\in\mathcal{S}_\delta : S\cap S'\neq\emptyset} \|f_S\|_{L^4}^2\Big)^{1/2} \\ &\quad + C_\epsilon (M^\alpha)^\epsilon \big(\sum_{S'\in\mathcal{S}_{M^{-2\alpha}} : S\cap S'\neq\emptyset} \|f_{S'}\|_{L^4}^2\big)^{\frac{1}{2}}. \end{aligned} \tag{3.16}$$

By (3.13) and (3.16), we have

$$\begin{aligned} \|f\|_{L^4} &\leq (C_\epsilon + C_\epsilon^2 (M^\alpha)^\epsilon)\delta^{-\epsilon}\Big(\sum_{S\in\mathcal{S}_\delta} \|f_S\|_{L^4}^2\Big)^{1/2} \\ &\quad + C_\epsilon^2 (M^{2\alpha})^\epsilon \big(\sum_{S'\in\mathcal{S}_{M^{-2\alpha}}} \|f_{S'}\|_{L^4}^2\big)^{\frac{1}{2}}. \end{aligned} \tag{3.17}$$

We repeat this process $O(\log_M \delta^{-1})$-times, and we obtain

$$\|f\|_{L^4} \lesssim_\epsilon (\delta^{-2\epsilon} + \delta^{-\log_M C_\epsilon})\delta^{-\epsilon}\Big(\sum_{S\in\mathcal{S}_\delta} \|f_S\|_{L^4}^2\Big)^{1/2}. \tag{3.18}$$

We take M sufficiently large so that $\log_M C_\epsilon \leq \epsilon$. This completes the proof.

4. Proof of Corollary 1.3 and 1.5

In this section, we prove Corollary 1.3 and 1.5. We give a remark that Corollary 1.3 holds true under a general condition.

Remark 4.1. *Fix $d \geq 2$. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Given a straight line l intersecting $[0,1]^2$, we parametrize the line by $\gamma(t)$ with the unit speed. Assume that*

$$\phi(\gamma(t)) = a_0 + a_1 t + a_2 t^2 + \cdots + a_d t^d + E(t), \tag{4.1}$$

where

$$|E(t)| \leq c_\phi t^{d+1} \tag{4.2}$$

and

$$|a_2| + \cdots + |a_d| \geq C_\phi > 0. \tag{4.3}$$

Here, c_ϕ and C_ϕ are independent of the choice of the line l. Then (1.6) *is true.*

Corollary 1.5 is stated using a language in a partial differential equation. By Fourier series, we can write the operator $e^{it\tilde{\Delta}} f$ as follows.

$$e^{it\tilde{\Delta}} f(x) = \sum_{\xi\in[-N,N]^2\cap\mathbb{Z}^2} \hat{f}(\xi) e\big((x_1,x_2,x_3)\cdot(\xi_1,\xi_2,\xi_1^2-\alpha\xi_2^2)\big). \tag{4.4}$$

Also by Parseval's identity, we have

$$\|f\|_{L^2(\mathbb{T}^2)}^2 \sim \sum_{\xi\in[-N,N]^2\cap\mathbb{Z}^2} |\hat{f}(\xi)|^2. \tag{4.5}$$

Hence, Corollary 1.5 can be rephrased as follows.

Corollary 4.2. *Let α be irrational. Suppose that*

$$\phi(\xi_1,\xi_2)=\xi_1^2-\xi_2^2, \quad \Lambda_\delta=\big(\delta\mathbb{Z}\times\alpha\delta\mathbb{Z}\big)\cap[0,1]^2. \tag{4.6}$$

Then for $2\le p\le 4$ and $\epsilon>0$, we have

$$\Big\|\sum_{\xi\in\Lambda_\delta}a_\xi e\big(x\cdot(\xi,\phi(\xi))\big)\Big\|_{L^p_{\#}(B_{\delta^{-3}})}\le C_\epsilon\delta^{-\epsilon}\big(\sum_{\xi\in\Lambda_\delta}|a_\xi|^2\big)^{\frac12}. \tag{4.7}$$

4.1. **Proof of Corollary 1.3.** Suppose that the manifold $\mathcal{M}_\phi$ does not contain a line. Fix $\epsilon>0$. By Theorem 3.1 with δ replaced by δ^d, we have

$$\|f\|_{L^4}\le C_{d,\epsilon}\delta^{-\epsilon}\Big(\sum_{S\in\mathcal{S}_{\delta^d}}\|f_S\|_{L^4}^2\Big)^{1/2} \tag{4.8}$$

for all f whose Fourier support is in $N_{\delta^d}(\mathcal{M}_\phi)$. Since ϕ does not contain a line, by a compactness argument, for any l, we can parametrize it as a function of t, and write

$$\phi(t,at+b)=a_0+a_1t+a_2t^2+\cdots+a_dt^d \tag{4.9}$$

where

$$\max(|a_2|,|a_3|,\cdots,|a_d|)\sim 1. \tag{4.10}$$

Suppose that S is a rectangle and is $(\phi,A\delta^d)$-flat. We claim that the length of a long direction of S is smaller or equal to $C_\phi\delta$ for some constant C_ϕ depending on the choice of ϕ. We may assume that the angle between the long direction of S and ξ_1-axis is smaller than or equal to $\pi/4$. Let l be a line segment passing through the center of S and parallel to the long direction of S but contained in S. For convenience, we introduce a parametrization of the line l; $\gamma(t)=(t,at+b)$ where $t\in[b_0,b_1]$. To prove the claim, it suffices to show that

$$|b_1-b_0|\le C_{\phi,A}\delta. \tag{4.11}$$

By the definition of $(\phi,A\delta^d)$-flat, we have

$$\sup_{t\in[b_0,b_1]}|\phi(\gamma(b_0))-\phi(\gamma(t))-\nabla\phi(\gamma(b_0))\cdot(\gamma(b_0)-\gamma(t))|\le A\delta^d. \tag{4.12}$$

By (4.9) and (4.10), after some computations, this can be rewritten as

$$\sup_{t\in[b_0,b_1]}|\widetilde{a_1}(t-b_0)+\widetilde{a_2}(t-b_0)^2+\cdots+\widetilde{a_d}(t-b_0)^d|\le A\delta^d, \tag{4.13}$$

where

$$\max(|\widetilde{a_2}|,|\widetilde{a_3}|,\cdots,|\widetilde{a_d}|)\sim 1. \tag{4.14}$$

By [LY21, Proposition 7.1], (4.13) gives that

$$\max(|\widetilde{a_2}(b_0-b_1)^2|,|\widetilde{a_3}(b_0-b_1)^3|,\cdots,|\widetilde{a_d}(b_0-b_1)^d|)\lesssim A\delta^d. \tag{4.15}$$

The condition (4.14) gives that $|b_1-b_0|\lesssim\delta$, and this finishes the proof of the claim.

We have proved the claim. Corollary 1.3 will follow by counting the number of frequencies in Λ_δ contained in a δ^d neighborhood of each $S\in\mathcal{S}_\delta$. The claim says that each S contains at most $\lesssim 1$ many frequencies in Λ_δ, and this gives the desired result. Let us give more details. We take f to be a Fourier transform of

$$\sum_{\xi\in\Lambda_\delta}\psi_\xi \tag{4.16}$$

where ψ_ξ is a smooth bump function supported on the ball of radius δ^d centered at the point $(\xi, \phi(\xi))$. By the claim, we have

$$\Big(\sum_{S\in\mathcal{S}_{\delta^d}} \|f_S\|_{L^4}^2\Big)^{1/2} \lesssim \Big(\sum_{\xi\in\Lambda_\delta} \|\widehat{\psi_\xi}\|_{L^4}^2\Big)^{\frac12}. \tag{4.17}$$

On the other hand, by the definition of f, we have

$$\|f\|_{L^4} \sim \Big\|\sum_{\xi} \widehat{\psi_\xi}\Big\|_{L^4}. \tag{4.18}$$

Note that

$$\widehat{\psi}_\xi(x) = \delta^{3d} e(\xi, \phi(\xi))\psi_{B(0,\delta^{-d})}, \tag{4.19}$$

where ψ_B is a smooth function essentially supported on the ball B. Corollary 1.3 follows from this, (4.8), (4.9), and (4.10). This completes the proof.

4.2. **Proof of Corollary 1.5.** As we discussed, it suffices to prove Corollary 4.2. For simplicity, we only consider $\alpha = \sqrt{2}$. The general case can be proved identically. For $\phi(\xi_1, \xi_2) = \xi_1^2 - \xi_2^2$ and $\Lambda_\delta = \delta\mathbb{Z} \times \sqrt{2}\delta\mathbb{Z}$, our goal is to show that

$$\Big\|\sum_{\xi\in\Lambda_\delta} a_\xi e\big(x\cdot(\xi,\phi(\xi))\big)\Big\|_{L^p_{\#}(B_{\delta^{-3}})} \le C_\epsilon \delta^{-\epsilon}\big(\sum_{\xi\in\Lambda_\delta}|a_\xi|^2\big)^{\frac12}. \tag{4.20}$$

This will follow from Theorem 2.5 by counting the number of frequencies in Λ_δ contained in a δ^3 neighborhood of each $S \in \mathcal{S}_\delta$. Suppose that $(m_i\delta, n_i\sqrt{2}\delta) \in \Lambda_\delta \cap N_{\delta^3}(S)$ for $i = 1, 2$. Then using the definition of (ϕ, δ^3)-flat, we have

$$|m_1^2\delta^2 - 2n_1^2\delta^2 - (m_2^2\delta^2 - 2n_2^2\delta^2) - 2(m_1\delta, -\sqrt{2}n_1\delta)\cdot((m_1-m_2)\delta, \sqrt{2}(n_1-n_2)\delta)| \lesssim \delta^3.$$

This simplifies to

$$|(m_1-m_2)^2 - 2(n_1-n_2)^2| \lesssim \delta,$$

so

$$|[m_1 - m_2 - \sqrt{2}(n_1-n_2)][m_1-m_2+\sqrt{2}(n_1-n_2)]| \lesssim \delta.$$

By Lemma 4 of Section 2, Chapter 2 of [Cas72], we have $\frac{1}{b^{1+\epsilon'}} \lesssim_{\epsilon'} |a+\sqrt{2}b|$ for any $a, b \in \mathbb{Z}$. Therefore, if $(m_1, n_1) \neq (m_2, n_2)$, the above displayed inequality implies that $|n_1 - n_2| \gtrsim_{\epsilon'} \delta^{-1+\epsilon'}$. Since the elements of $\Lambda_\delta \cap N_{\delta^3}(S)$ are $\gtrsim \delta$-separated, there are fewer than $\lesssim_{\epsilon'} \delta^{-\epsilon'}$ frequencies in $\Lambda_\delta \cap N_{\delta^3}(S)$, as desired.

Appendix A. Partition is not enough

In this appendix, we prove that there is no ℓ^2 decoupling for the hyperbolic paraboloid using a partition. Note that in Theorem 2.5 we introduce $O(\log\delta^{-1})$ many partitions to obtain ℓ^2 decoupling for the hyperbolic paraboloid. The following theorem shows that it is necessary to introduce many partitions.

Theorem A.1. *Let $\mathcal{M}_\phi$ be the hyperbolic paraboloid given by $\phi(\xi_1,\xi_2) = \xi_1\xi_2$. Fix $0 < \epsilon < \frac{1}{100}$ and $A \ge 1$. Then the following is false: for any $\delta > 0$, there exists a family $\mathcal{S}_\delta$ of rectangles $S \subset \mathbb{R}^2$ such that*

(1) the interiors of rectangles are disjoint

$$\mathrm{int}(S_1) \cap \mathrm{int}(S_2) = \emptyset, \quad S_1, S_2 \in \mathcal{S}_\delta \tag{1.1}$$

(2) every S is $(\phi, A\delta)$-flat

(3) for all f whose Fourier support is in $N_\delta(\mathcal{M}_\phi)$,

$$\|f\|_{L^4} \le C_{\epsilon,A}\delta^{-\epsilon}\Big(\sum_{S\in\mathcal{S}_\delta}\|f_S\|_{L^4}^2\Big)^{1/2}. \tag{1.2}$$

Proof. Fix $\delta > 0$. For simplicity, assume that δ^{-1} is a dyadic number. Suppose that such a family $\mathcal{S}_\delta$ exists for a contradiction. Let us first consider the partition

$$\begin{aligned} [0,1]^2 &= \bigcup_{a\in\mathbb{Z}\cap[0,\delta^{-1}-1]}\Big([0,1]\times[a\delta, a\delta+\delta]\Big) =: \bigcup_a C_a, \\ [0,1]^2 &= \bigcup_{b\in\mathbb{Z}\cap[0,\delta^{-1}-1]}\Big([b\delta, b\delta+\delta]\times[0,1]\Big) =: \bigcup_b D_b. \end{aligned} \tag{1.3}$$

Let us fix C_a. Consider a collection of elements of $\mathcal{S}_\delta$ which have a large intersection with C_a

$$\mathcal{S}_{\delta,C_a} := \{S\in\mathcal{S}_\delta : |S\cap C_a| \ge \delta^1\delta^{\frac{1}{2}-\epsilon}\}. \tag{1.4}$$

Define $\mathcal{S}_{\delta,C_b}$ similarly. By abusing notations, let us denote by $\mathcal{S}_{\delta,a}$ and $\mathcal{S}_{\delta,b}$ the sets $\mathcal{S}_{\delta,C_a}$ and $\mathcal{S}_{\delta,D_b}$. Recall that elements of the set $\mathcal{S}_{\delta,a}$ are disjoint by the first condition (1.1).

We claim that

$$\Big|\bigcup_{S\in\mathcal{S}_{\delta,a}} S\cap C_a\Big| \ge \frac{99\delta}{100}, \quad \Big|\bigcup_{S\in\mathcal{S}_{\delta,b}} S\cap D_b\Big| \ge \frac{99\delta}{100}. \tag{1.5}$$

Let us assume this claim for a moment and finish the proof of the theorem. By taking the union over a to (1.5), we have

$$\Big|\bigcup_{a\in\mathbb{Z}\cap[0,\delta^{-1}-1]}\bigcup_{S\in\mathcal{S}_{\delta,a}} S\cap C_a\Big| \ge \frac{99}{100}. \tag{1.6}$$

By pigeonholing, there exists $D_b\in\mathcal{S}_{\delta,b}$ such that

$$\Big|D_b\cap\Big(\bigcup_{a\in\mathbb{Z}\cap[0,\delta^{-1}-1]}\bigcup_{S\in\mathcal{S}_{\delta,a}} S\cap C_a\Big)\Big| \ge \frac{99\delta}{100}. \tag{1.7}$$

This implies

$$\Big|D_b\cap\Big(\bigcup_{a\in\mathbb{Z}\cap[0,\delta^{-1}-1]}\bigcup_{S\in\mathcal{S}_{\delta,a}} S\Big)\Big| \ge \frac{99\delta}{100}. \tag{1.8}$$

We claim that if $S\in\mathcal{S}_{\delta,a}$ for some a then

$$|D_b\cap S| \lesssim \delta^1\delta^{\frac{1}{2}+\epsilon}. \tag{1.9}$$

Note that this means that S does not belong to $\mathcal{S}_{\delta,D_b}$. Since the area of D_b is δ, the first inequality of (1.5) gives

$$\Big|\bigcup_{S\in\mathcal{S}_{\delta,D_b}} S\cap D_b\Big| \le \frac{\delta}{100}. \tag{1.10}$$

This contradicts with the second inequality of (1.5). Let us give a proof of (1.9). Suppose that $S\in\mathcal{S}_{\delta,a}$. It suffices to prove that

$$S\subset(\alpha,\beta)+\Big([0,1]\times[0,C\delta^{\frac{1}{2}+\epsilon}]\Big) \tag{1.11}$$

for some $\alpha, \beta \in [0,1]$. To prove this, we use the assumption that S is $(\phi, A\delta)$-flat. By the definition of $\mathcal{S}_{\delta,a}$, the rectangle S contains a line segment parallel to ξ_1-axis with length $\delta^{\frac{1}{2}-\epsilon}$. Since our manifold is translation invariant, for simplicity, assume that the line is $[0, \delta^{\frac{1}{2}-\epsilon}] \times \{0\}$. We will show that if $p = (p_1, p_2)$ is an element of S then $|p_2| \lesssim \delta^{\frac{1}{2}+\epsilon}$. This gives the proof of (1.9). To show the bound of p_2, we set $u = (0,0)$. Then by the definition of $(\phi, A\delta)$-flat (definition 1.1),

$$|p_1 p_2| \leq A\delta. \tag{1.12}$$

Similarly, we use $u = (\delta^{\frac{1}{2}-\epsilon}, 0)$ and this gives

$$|p_1 p_2 - \delta^{\frac{1}{2}-\epsilon} p_2| \leq A\delta. \tag{1.13}$$

Combining these two gives $|p_2| \lesssim \delta^{\frac{1}{2}+\epsilon}$.

It remains to prove (1.5). By symmetry, let's show only the first inequality. We will use the assumption (1.2). Fix C_a. Let us first show that there exists $S \in \mathcal{S}_\delta$ such that

$$|S \cap C_a| \geq (\log \delta^{-1})^{-10} \delta^{1+4\epsilon}. \tag{1.14}$$

Suppose that such S does not exist for a contradiction. We take f such that

(1) $\hat{f}$ is supported on the δ-neighborhood of the set $\mathcal{M}_\phi \cap (C_a \times \mathbb{R})$.
(2) $\hat{f}$ is equal to one on the $\delta/2$-neighborhood of the set $\mathcal{M}_\phi \cap (C_a \times \mathbb{R})$.

Here $\mathcal{M}_\phi$ is the hyperbolic paraboloid. Then f is essentially supported on a box with dimension $\delta^{-1} \times 1 \times \delta^{-1}$ and has amplitude $\sim \delta^2$. So we have

$$\|f\|_{L^4} \sim \delta^2 \delta^{-2/4} \sim \delta^{3/2}. \tag{1.15}$$

On the other hand, by pigeonholing, there exists j such that

$$\Big(\sum_{S \in \mathcal{S}_\delta} \|f_S\|_{L^4}^2 \Big)^{1/2} \lesssim (\log \delta^{-1})^{\frac{1}{2}} \Big(\sum_{S \in \mathcal{S}_\delta^j} \|f_S\|_{L^4}^2 \Big)^{1/2} \tag{1.16}$$

where the elements S of $\mathcal{S}_\delta^j$ satisfies

$$|S \cap C_a| \sim \delta 2^{-j}. \tag{1.17}$$

Here, we used the fact that for large j the right hand side of (1.16) is negligible (which is proved in (1.18)). Since we are assuming that there does not exist S satisfying (1.14), (1.16) is true for $j > 4\epsilon \log_2 \delta^{-1} + 10 \log_2 \log \delta^{-1}$. Note that f_S is essentially constant on a box with volume $(\delta\delta 2^{-j})^{-1}$ and has amplitude $\delta\delta 2^{-j}$. Since $\mathcal{S}_\delta$ is a partition (see (1.1)), the cardinality of $\mathcal{S}_\delta^j$ is bounded by 2^j. So we have

$$\begin{aligned} \Big(\sum_{S \in \mathcal{S}_\delta^j} \|f_S\|_{L^4}^2 \Big)^{1/2} &\lesssim (\#\mathcal{S}_\delta^j)^{\frac{1}{2}} \max_S \|f_S\|_{L^4} \\ &\lesssim 2^{j/2} (\delta^2 2^{-j}) (\delta^2 2^{-j})^{-\frac{1}{4}} \sim \delta^{\frac{3}{2}} 2^{-\frac{j}{4}}. \end{aligned} \tag{1.18}$$

(1.2), (1.15), and (1.18) gives

$$\delta^{\frac{3}{2}} \lesssim (\log \delta^{-1})^{\frac{1}{2}} \delta^\epsilon \delta^{\frac{3}{2}} 2^{-\frac{j}{4}}. \tag{1.19}$$

We get a contradiction from the assumption that $j > 4\epsilon \log_2 \delta^{-1} + 10 \log_2 \log \delta^{-1}$. The same argument works for a general situation. Let $C \subset C_a \subset [0,1]^2$ be a convex set (polygon). Then we can show that there exists $S \in \mathcal{S}_\delta$ such that

$$|S \cap C| \gtrsim (\log \delta^{-1})^{-10} \delta^{4\epsilon} |C|. \tag{1.20}$$

Let us explain the proof. First, since C is a convex set, we can find a rectangle $\widetilde{C} \subset C$ such that $|\widetilde{C}| \gtrsim |C|$ (for example, by Kovner–Besicovitch theorem). Since $\widetilde{C}$ is a rectangle in C_a, we can repeat the proof of (1.14) and find $S \in \mathcal{S}_\delta$ such that

$$|S \cap C| \geq |S \cap \widetilde{C}| \geq (\log \delta^{-1})^{-10} \delta^{4\epsilon} |\widetilde{C}| \gtrsim (\log \delta^{-1})^{-10} \delta^{4\epsilon} |C|. \tag{1.21}$$

This gives (1.20).

We have shown that there exists $S_1 \in \mathcal{S}_\delta$ such that (1.14) holds true. In particular, $S_1 \in \mathcal{S}_{\delta,a}$ (see (1.4) for the definition of $\mathcal{S}_{\delta,a}$ and note that we used the assumption that ϵ is small). If we have $|S_1 \cap C_a| \geq 99\delta/100$, then this gives (1.5). So suppose that

$$\delta^{1+4\epsilon} \leq |S_1 \cap C_a| \leq \frac{99}{100}\delta. \tag{1.22}$$

Since S_1 is a rectangle, the set $C_a \setminus (S_1)^c$ is either a convex set or a union of two convex sets. Let us write it as

$$C_a \setminus (S_1)^c = C_{11} \cup C_{12}. \tag{1.23}$$

Here C_{11} and C_{12} are disjoint convex sets and

$$|C_{11}| + |C_{12}| \leq |C_a|(1 - \delta^{4\epsilon}). \tag{1.24}$$

This finishes the first round of the iteration. Let us explain how to proceed. If C_{11} satisfies $|C_{11}| \leq \delta\delta^{\frac{1}{4}}$ then we leave this set. If $|C_{11}| > \delta\delta^{\frac{1}{4}}$ then we apply (1.20) to the convex set C_{11}, and obtain the set $S_2 \in \mathcal{S}_\delta$ such that

$$|S_2 \cap C_{11}| \gtrsim (\log \delta^{-1})^{-10} \delta^{4\epsilon} |C_{11}|. \tag{1.25}$$

By the stopping time condition $|C_{11}| > \delta\delta^{\frac{1}{4}}$ and the assumption that ϵ is small, we have $S_2 \in \mathcal{S}_{\delta,C_a}$ (see (1.4) for the definition). We write $C_{11} \setminus (S_2)^c = C_{111} \cup C_{112}$. Then we have

$$|C_{111}| + |C_{112}| \leq |C_{11}|(1 - \delta^{4\epsilon}), \tag{1.26}$$

where C_{11j} are convex sets. Repeat this process to C_{12}. If $|C_{12}| < \delta\delta^{\frac{1}{4}}$, then we have

$$|C_{111}| + |C_{112}| + |C_{12}| \leq |C_a|(1 - \delta^{4\epsilon})^2 + |C_{12}|\delta^{4\epsilon}. \tag{1.27}$$

If $|C_{12}| > \delta\delta^{\frac{1}{4}}$, then we have

$$|C_{111}| + |C_{112}| + |C_{113}| + |C_{114}| \leq |C_a|(1 - \delta^{4\epsilon})^2. \tag{1.28}$$

This finishes the second round of the iteration. We repeat this process M-times with $M \gtrsim \delta^{-6\epsilon}$. Denote by $\{C_X\}_X$ and $\{S_Y\}_Y$ a collection of convex sets and a collection of rectangles $S \in \mathcal{S}_\delta$ that we obtained via this process. Note that

$$\sum_X |C_X| + \sum_Y |S_Y \cap C_a| = |C_a| = \delta. \tag{1.29}$$

We have

$$\begin{aligned}\sum_X |C_X| &= \sum_{X:|C_X|\geq \delta\delta^{\frac14}} |C_X| + \sum_{X:|C_X|<\delta\delta^{\frac14}} |C_X| \\ &\leq |C_a|(1-\delta^{4\epsilon})^M + \Big|\bigcup_{X:|C_X|<\delta\delta^{\frac14}} C_X\Big|\delta^{4\epsilon} \leq \frac{1}{100}|C_a|.\end{aligned} \tag{1.30}$$

The last two inequalities follow from the fact that all C_X are disjoint and $C_X \subset C_a$. This completes the proof of the claim (1.5). □

Appendix B. Higher dimensions

In this appendix, we prove that an analogous result to Theorem 1.2 is false in higher dimensions. For convenience, let us consider only a manifold in $\mathbb{R}^4$. Higher dimensional manifolds can be proved in a similar way. Definitions 1.1 and 2.1 can be naturally generalized to higher dimensions. We will not state them. Recall that the critical exponent of p of decoupling for the hyperbolic paraboloid in $\mathbb{R}^4$ is 10/3.

Theorem B.1. *Consider $\phi(\xi_1,\xi_2,\xi_3)=\xi_1^2+\xi_2^2-\xi_3^2$. Fix $0<\epsilon<\frac{1}{1000}$. Let A be a constant. Then the following statement is false:*

For any $\delta>0$, there exists a collection $\mathcal{S}_\delta$ of rectangular boxes $S\subset[0,1]^3$ such that

(1) the overlapping number is $O(\log\delta^{-1})$ in the sense that

$$\sum_{S\in\mathcal{S}_\delta}\chi_S \leq C_{d,\epsilon}\log(\delta^{-1}). \tag{2.1}$$

(2) S is $(\phi, A\delta)$-flat.

(3) we have

$$\|f\|_{L^{10/3}} \leq C_{d,\epsilon}\delta^{-\epsilon}\Big(\sum_{S\in\mathcal{S}_\delta}\|f_S\|_{L^{10/3}}^2\Big)^{1/2} \tag{2.2}$$

for all f whose Fourier support is in $N_\delta(\mathcal{M}_\phi)$.

Proof. Let us prove by contradiction. Fix $0<\epsilon<10^{-10}$. Define

$$\mathcal{C} := \{(\xi_1,\xi_2,\xi_3)\in[0,1]^3 : \xi_1^2+\xi_2^2-\xi_3^2=0\}. \tag{2.3}$$

We cover it by $\sim\delta^{-1/2}$ many canonical blocks P with dimension $1\times\delta^{1/2}\times\delta$. Let us denote by $\mathcal{P}$ the collection of blocks. Note that each block is $(\phi,A\delta)$-flat for some large constant A. For given $P\in\mathcal{P}$, we take translated copies and cover $[0,1]^3$ so that they are disjoint. Denote by $\mathcal{P}_P$ the collection of translated copies of P. Note that the cardinality of $\mathcal{P}_P$ is comparable to $\delta^{-3/2}$.

Fix $P'\in\mathcal{P}_P$ for some $P\in\mathcal{P}$. We next decompose P' into parallel tubes with radius δ^X, where X is sufficiently large number. Let us denote by $\mathcal{P}_{P,P'}$ the collections of tubes P''. We claim that for given $P''\in\mathcal{P}_{P,P'}$ there is an element S of $\mathcal{S}_\delta$ such that the diameter of the set $S\cap P''$ is larger than or equal to $(\log\delta^{-1})\delta^{5\epsilon}$. To prove the claim, let us use the hypothesis (2.2). Take a function f so that

$$\hat{f}(\xi) := \sum_{j=1}^{(\log\delta^{-1})\delta^{-5\epsilon}} f_j(\xi) := \sum_{j=1}^{(\log\delta^{-1})\delta^{-5\epsilon}} 1_{N_{\delta^X}(B_j)}(\xi) \tag{2.4}$$

where $\{B_j\}_{j=1}^{(\log\delta^{-1})\delta^{-5\epsilon}}$ is an arithmetic progression with difference $(\log\delta^{-1})\delta^{5\epsilon}$, and $\bigcup_j B_j \subset P''$. By the direct computations, one can see that

$$\|f\|_{L^{10/3}} \gtrsim (\log\delta^{-1})^{\frac{1}{5}}\delta^{-5\epsilon(\frac{1}{2}-\frac{1}{10/3})}(\sum_j \|f_j\|_{L^{10/3}}^2)^{1/2}. \tag{2.5}$$

Hence, (2.2) says that there must be an element S of $\mathcal{S}_\delta$ containing at least two B_j. This means that the diameter of $S\cap P''$ is larger than or equal to $(\log\delta^{-1})\delta^{5\epsilon}$. This completes the proof of the claim.

For given $P' \in \mathcal{P}_P$, denote by $\mathcal{S}_{P'}$ the elements of $S\in\mathcal{S}_\delta$ satisfying the property: for some $P''\in\mathcal{P}_{P,P'}$ the diameter of $S\cap P''$ is larger than or equal to $(\log\delta^{-1})\delta^{5\epsilon}$. The claim says that $\mathcal{S}_{P'}$ is non-empty.

We next claim that if S is $(\phi, A\delta)$-flat and $S\in\mathcal{S}_\delta$, then S is contained in a box with dimension $\delta^{\frac{1}{2}-\frac{5\epsilon}{2}}\times\delta^{1-5\epsilon}\times\delta^{5\epsilon}$. Let us give a proof. First of all, by an affine transformation, we may assume that S contains a line $\{0\}\times\{0\}\times[0,\delta^{5\epsilon}]$ and our new phase function is $\widetilde{\phi}(\xi)=\xi_1^2+\xi_2\xi_3$. Let $v=(v_1,v_2,v_3)\in S$. We will show that

$$|v_1|\lesssim\delta^{\frac{1}{2}-\frac{5\epsilon}{2}}, \quad |v_2|\lesssim\delta^{1-5\epsilon}. \tag{2.6}$$

By the definition of $(\phi, A\delta)$-flat set with $u=(0,0,0)$ we obtain

$$|v_1^2+v_2v_3|\le A\delta. \tag{2.7}$$

We next use $u=(0,0,\delta^{5\epsilon})$ and obtain

$$|v_1^2+v_2v_3-\delta^{5\epsilon}v_2|\le A\delta. \tag{2.8}$$

These two inequalities give $|v_2|\lesssim\delta^{1-5\epsilon}$. This bound and (2.7) give $|v_1|\lesssim\delta^{\frac{1}{2}-\frac{5\epsilon}{2}}$. This proves the claim.

By the claim, if $P_1'\in\mathcal{P}_{P_1}$ and $P_2'\in\mathcal{P}_{P_2}$, and the angle of the longest directions of P_1 and P_2 is greater than $\delta^{\frac{1}{2}-100\epsilon}$, then any two sets $S_1\in\mathcal{S}_{P_1'}$ and $S_2\in\mathcal{S}_{P_2'}$ are distinct. Note also that

$$\Big|\bigcup_{S'\in\mathcal{S}_{P'}} S'\cap P'\Big|\gtrsim\delta^{5\epsilon}|P'|. \tag{2.9}$$

Let us finish the proof of the theorem. To get a contradiction, we need to prove that there exists $\xi\in[0,1]^3$ such that the number of $S\in\mathcal{S}_\delta$ containing ξ is greater than the right hand side of (2.1). To prove this, it suffices to show that there exists $\xi\in[0,1]^3$ satisfying

$$\sum_{P\in\mathcal{P}}\sum_{P'\in\mathcal{P}_P}\sum_{S'\in\mathcal{S}_{P'}} 1_{S'\cap P'}(\xi)\gtrsim\delta^{-\alpha-100\epsilon} \tag{2.10}$$

for some number $\alpha>0$. This follows from a pigeonholing argument with

$$\begin{aligned}\int_{[0,1]^3}\sum_{P\in\mathcal{P}}\sum_{P'\in\mathcal{P}_P}\sum_{S'\in\mathcal{S}_{P'}} 1_{S'\cap P'}(\xi)\,d\xi &= \sum_{P\in\mathcal{P}}\sum_{P'\in\mathcal{P}_P}\int_{[0,1]^3}\sum_{S'\in\mathcal{S}_{P'}} 1_{S'\cap P'}(\xi)\,d\xi \\ &\gtrsim \delta^{-\frac{3}{2}}\delta^{-\frac{1}{2}}\delta^{5\epsilon}\delta^{\frac{3}{2}}\sim\delta^{-\frac{1}{2}+5\epsilon}.\end{aligned} \tag{2.11}$$

This completes the proof. $\square$

Appendix C. Sharpness of Theorem 1.2

Let us discuss the sharpness of Theorem 1.2. More precisely, we would like to discuss if the range of p for which (1.4) holds true is sharp. An example of a manifold that the aforementioned range of p is not sharp is as follows.

$$\phi(\xi_1,\xi_2)=\xi_1^2. \tag{3.1}$$

The ℓ^2 decoupling for (3.1) is proved for $2 \le p \le 6$ by [BD15], and this range of p is sharp. Motivated by this example, let us introduce the following definition.

Definition C.1. *We say $\phi(\xi_1,\xi_2)$ depends only on a variable if there exists an affine transformation L such that*

$$\phi(L(\xi_1,\xi_2))=\psi(\xi_1)+a\xi_2 \tag{3.2}$$

for some function ψ and some $a\in\mathbb{R}$. Second, we say $\phi(\xi_1,\xi_2)$ does not depend on any variable if

$$\phi(\xi_1,\xi_2)=a+b\xi_1+c\xi_2 \tag{3.3}$$

for some $a,b,c\in\mathbb{R}$. Lastly, we say $\phi(\xi_1,\xi_2)$ depends on two variables if it is not a form of (3.3) *and there does not exist L such that* (3.2) *holds true.*

Here is a complete characterization of the ℓ^2 decoupling theorem.

Proposition C.2. *Consider a polynomial $\phi := \phi(\xi_1,\xi_2)$. Suppose that $\mathcal{S}_\delta$ is a family constructed in Subsection 2.1 and Section 3.*

1. *Let ϕ depend on two variables. Then* (1.4) *is true for $2\le p\le 4$.*
2. *Let ϕ depend only on a variable. Then* (1.4) *is true for $2\le p\le 6$.*
3. *Let ϕ not depend on any variable. Then* (1.4) *is true for $2\le p<\infty$.*

The ranges of p stated in items (1), (2), and (3) are sharp.

Proof. Let us first prove Item (1). Theorem 1.2 gives (1.4) for $2\le p\le 4$, so it suffices to prove that the range is sharp. Let $H_\phi(\xi)$ be the Hessian matrix of ϕ. By [dBvdE04, Theorem 3.1], $H_\phi(\xi)$ is not identically zero. Consider

$$Z_\phi := \{\xi\in\mathbb{R}^2 : \det(H_\phi(\xi))=0\}. \tag{3.4}$$

Since $\det H_\phi(\xi)$ is not identically zero, it is a zero set of a polynomial. So there exists a square τ not intersecting Z_ϕ. Let us fix such τ. Assume that δ is smaller than the sidelength of τ. Let $\hat f$ be a smooth bump function of the δ-neighborhood of

$$\{(\xi_1,\xi_2,\phi(\xi_1,\xi_2)) : \xi\in\tau\}. \tag{3.5}$$

One can see that if $\xi\in\tau$, then

$$|\det H_\phi(\xi)|\gtrsim 1. \tag{3.6}$$

So the multiplication of eigenvalues of $H_\phi(\xi)$ does not change the sign over $\xi\in\tau$.

Suppose that the eigenvalues of $H_\phi(\xi)$ have the same sign for all $\xi\in\tau$. Then each $S\in\mathcal{S}_\delta$ intersecting τ is a square with length $\delta^{1/2}\times\delta^{1/2}$. So we have

$$f=\sum_S f_S, \quad |f_S(x)|=\delta^2 1_{T_S} \tag{3.7}$$

where T_S is a tube with dimension $\delta^{-1/2}\times\delta^{-1/2}\times\delta^{-1}$. Then

$$\delta^2\delta^{-1}\lesssim \|f\|_{L^p(B_1)}\lesssim \|f\|_{L^p(\mathbb{R}^3)}. \tag{3.8}$$

On the other hand,

$$\Big(\sum_{S}\|f_S\|_{L^p}^2\Big)^{\frac{1}{2}} \lesssim \delta^{-\frac{1}{2}}\delta^2|T_S|^{\frac{1}{p}} \sim \delta^{-\frac{1}{2}}\delta^2\delta^{-\frac{2}{p}}. \tag{3.9}$$

This shows that the decoupling inequality is false for $p>4$.

Suppose that the eigenvalues have different signs. Then as in the previous case, we have

$$\delta \lesssim \|f\|_{L^p(\mathbb{R}^3)}. \tag{3.10}$$

We need to calculate

$$\Big(\sum_{S\in\mathcal{S}_\delta}\|f_S\|_{L^p}^2\Big)^{1/2}. \tag{3.11}$$

By the construction of $\mathcal{S}_\delta$, we have $|f_S| = \delta^2 1_{T_S}$ where T_S has dimension $\delta^A \times \delta^{1-A}\times\delta$ for some $0\leq A\leq 1$. Here the number A depends on the choice of S. Also, the cardianlity of $\mathcal{S}_\delta$ is comparable to δ^{-1} up to $\delta^{-\epsilon}$ losses. So we have

$$\Big(\sum_{S\in\mathcal{S}_\delta}\|f_S\|_{L^p}^2\Big)^{1/2} \lesssim_\epsilon \delta^{-\epsilon}\delta^{-\frac{1}{2}}\delta^2|T_S|^{\frac{1}{p}} \sim \delta^{-\epsilon}\delta^{-\frac{1}{2}}\delta^2\delta^{-\frac{2}{p}}. \tag{3.12}$$

This shows that the decoupling inequality is false for $p>4$, and completes the proof of Item (1).

Let us move on to Item (2). Since ϕ depends only on a variable, after some change of variables and abusing notations, we may assume that $\phi(\xi_1,\xi_2)=\psi(\xi_1)$. Fix x_2 and define

$$g(x_1,x_3) := f(x_1,x_2,x_3). \tag{3.13}$$

Then (1.4) simply follows from [Yan21, Theorem 1.4]. Let us show the sharpness. Since ψ is a one-variable polynomial, the zeros of ψ are finite. So we can find an open interval $I\subset[0,1]$ such that for every $\xi_1\in I$, we have $|\psi''(\xi_1)|\gtrsim 1$. We write I as a union of intervals J with length $\delta^{1/2}$. Take $\hat{f}$ to be a smooth bump function of the δ-neighborhood of

$$\{(\xi_1,\xi_2,\phi(\xi_1,\xi_2)) : \xi_1\in I, \xi_2\in[0,1]\}. \tag{3.14}$$

Then similar calculations in the proof of Item (1) give the sharpness for p. We leave out the details.

Lastly, let us prove Item (3). If ϕ does not depend on any variable, then according to our construction, $f_S=f$. So (1.4) is true for $2\leq p<\infty$. This completes the proof of Proposition C.2. □

References

[BD15] Jean Bourgain and Ciprian Demeter. The proof of the l^2 decoupling conjecture. *Ann. of Math. (2)*, 182(1):351–389, 2015.

[BD17a] Jean Bourgain and Ciprian Demeter. Decouplings for curves and hypersurfaces with nonzero Gaussian curvature. *J. Anal. Math.*, 133:279–311, 2017.

[BD17b] Jean Bourgain and Ciprian Demeter. A study guide for the l^2 decoupling theorem. *Chin. Ann. Math. Ser. B*, 38(1):173–200, 2017.

[BDG16] Jean Bourgain, Ciprian Demeter, and Larry Guth. Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three. *Ann. of Math. (2)*, 184(2):633–682, 2016.

[BDK20] Jean Bourgain, Ciprian Demeter, and Dominique Kemp. Decouplings for real analytic surfaces of revolution. In *Geometric aspects of functional analysis. Vol. I*, volume 2256 of *Lecture Notes in Math.*, pages 113–125. Springer, Cham, [2020] ©2020.

[BG11] Jean Bourgain and Larry Guth. Bounds on oscillatory integral operators based on multilinear estimates. *Geom. Funct. Anal.*, 21(6):1239–1295, 2011.

[BMV20] Stefan Buschenhenke, Detlef Müller, and Ana Vargas. A Fourier restriction theorem for a perturbed hyperbolic paraboloid. *Proc. Lond. Math. Soc. (3)*, 120(1):124–154, 2020.

[BMV22] Stefan Buschenhenke, Detlef Müller, and Ana Vargas. A Fourier restriction theorem for a perturbed hyperbolic paraboloid: polynomial partitioning. *Math. Z.*, 301(2):1913–1938, 2022.

[BMV23] Stefan Buschenhenke, Detlef Müller, and Ana Vargas. Fourier restriction for smooth hyperbolic 2-surfaces. *Math. Ann.*, 387(1-2):17–56, 2023.

[Cas72] J. W. S. Cassels. *An introduction to Diophantine approximation*, volume No. 45 of *Cambridge Tracts in Mathematics and Mathematical Physics*. Hafner Publishing Co., New York, 1972. Facsimile reprint of the 1957 edition.

[dBvdE04] Michiel de Bondt and Arno van den Essen. Singular Hessians. *J. Algebra*, 282(1):195–204, 2004.

[Dem20] Ciprian Demeter. *Fourier restriction, decoupling, and applications*, volume 184 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2020.

[DGG17] Yu Deng, Pierre Germain, and Larry Guth. Strichartz estimates for the Schrödinger equation on irrational tori. *J. Funct. Anal.*, 273(9):2846–2869, 2017.

[DGGRM22] Yu Deng, Pierre Germain, Larry Guth, and Simon L. Rydin Myerson. Strichartz estimates for the Schrödinger equation on non-rectangular two-dimensional tori. *Amer. J. Math.*, 144(3):701–745, 2022.

[GMO23] Larry Guth, Dominique Maldague, and Changkeun Oh. Small cap decoupling for paraboloid in $\mathbb{R}^n$, 2023.

[GO23] Shaoming Guo and Changkeun Oh. A restriction estimate for surfaces with negative gaussian curvatures, 2023.

[LY21] Jianhui Li and Tongou Yang. Decoupling for smooth surfaces in $\mathbb{R}^3$, 2021.

[Ste86] E. M. Stein. Oscillatory integrals in Fourier analysis. In *Beijing lectures in harmonic analysis (Beijing, 1984)*, volume 112 of *Ann. of Math. Stud.*, pages 307–355. Princeton Univ. Press, Princeton, NJ, 1986.

[Tom75] Peter A. Tomas. A restriction theorem for the Fourier transform. *Bull. Amer. Math. Soc.*, 81:477–478, 1975.

[Yan21] Tongou Yang. Uniform l^2-decoupling in $\mathbb{R}^2$ for polynomials. *J. Geom. Anal.*, 31(11):10846–10867, 2021.

Larry Guth, Deparment of Mathematics, Massachusetts Institute of Technology, USA

Email address: `lguth@math.mit.edu`

Dominique Maldague, Deparment of Mathematics, Massachusetts Institute of Technology, USA

Email address: `dmal@mit.edu`

Changkeun Oh, Deparment of Mathematics, Massachusetts Institute of Technology, USA, and Department of Mathematical Sciences and RIM, Seoul National University, Republic of Korea

Email address: `changkeun.math@gmail.com`