ON PERSPECTIVE ABELIAN GROUPS

GRIGORE CĂLUGĂREANU, ANDREY CHEKHLOV

Abstract. As a special case of perspective R-modules, an Abelian goup is called *perspective* if isomorphic summands have a common complement. In this paper we describe many classes of such groups.

1. Introduction

This paper concerns about direct summands of Abelian groups. To simplify the writing, G will denote an (arbitrary) Abelian group and by some different letters we denote direct summands of G. In what follows, since *all summands we consider are direct*, we remove this adjective. Moreover, the word "complement" will be used only for direct complements. In trying to make a notation difference, by $\mathbb{Z}(m)$ we denote the Abelian group and by $\mathbb{Z}_m$ we denote the ring of integers modulo m. For an Abelian group G, by $\mathrm{End}(G)$ we simply denote $\mathrm{End}_{\mathbb{Z}}(G)$, that is, the endomorphism ring of G.

We start with the following general

Definition (see [5]). Let L be a bounded lattice. Two elements $x, y \in L$ are said to be *perspective* (in L) provided they have a common (direct) complement, i.e., an element $z \in L$ such that $x \vee z = y \vee z = 1$, $x \wedge z = y \wedge z = 0$. This definition comes back to John von Neumann.

Specializing for the submodule lattice of a module, two summands A, B of a module M will be denoted by $A \sim B$, if they have a common complement, i.e., there exists a submodule C such that $M = A \oplus C = B \oplus C$. It is clear that $A \sim B$ implies $A \cong B$. A module M is called *perspective* when $A \cong B$ implies $A \sim B$ for any two summands A, B of M.

A module ${}_RM$ over a ring R is said to satisfy *internal cancellation* (or we say M is internally cancellable; *IC*, for short) if, whenever $M = K \oplus N = K' \oplus N'$ (in the category of R-modules), $N \cong N' \Rightarrow K \cong K'$ [or $M/N \cong M/N'$].

It is clear that *perspective modules satisfy the internal cancellation property* in the sense that complements of isomorphic summands are isomorphic (see [6]).

The modules definition can be restricted to rings as follows

Definition A ring R is called *perspective* if isomorphic direct summands of ${}_RR$ have a common (direct) complement.

This property for rings turns out to be left–right symmetric, that is, ${}_RR$ is perspective if and only if R_R is perspective for any ring R and we call such ring a perspective ring.

In this paper, we characterize some large classes of perspective Abelian groups. In the sequel, the word "group" will always mean an "Abelian group".

2020 *Mathematics Subject Classification.* Primary: 20K10, 20K15, 20K27. Secondary: 20K21, 16D99.

Key words and phrases. Abelian group, perspective module, primary group, torsion-free group.

Our main results are the following.

1) For large classes of Abelian groups, we show that the perspectivity is equivalent to finite rank. These are: the divisible primary groups, the divisible torsion-free groups, the homogeneous completely decomposable groups with type $\mathbf{t}(\mathbb{Q}_p)$.

2) The only perspective rank 1 torsion-free groups are the rational groups. In particular, the only free perspective group is $\mathbb{Z}$.

3) Any finite p-group is perspective. Any bounded perspective p-group is finite.

4) A torsion group is perspective iff so are all its primary components.

5) (a) Let $G = T(G) \oplus F$ where $T(G)$ is the torsion part of G and F is torsion-free. Then G is perspective iff $T(G)$ and F are perspective

(b) Let $G = D(G) \oplus R$ where $D(G)$ is the divisible part of G and R is reduced. Then G is perspective iff $D(G)$ and R are perspective.

6) We describe the perspective torsion-free algebraically compact groups.

7) For any rational group H (i.e., rank 1 torsion-free group) we characterize when $H \oplus H$ is perspective.

Section 2 recalls a few general results on perspective modules from [4], which are used in the sequel. Section 3 contains our results on perspective Abelian groups, divided into subsection 3.1, about definitions, subsection 3.2, about reduction theorems, subsection 3.3, about perspective torsion groups and subsection 3.4, about perspective torsion-free groups. Some examples are given in the end.

2. Generalities on perspective modules

Definition 8.1 [8]. Let $\mathcal{P}$ be a module-theoretic property. We say that $\mathcal{P}$ is an *endomorphism ring property* (or **ER**-property for short) if, whenever A_R and A'_S are right modules (over possibly different rings R and S) such that $\mathrm{End}(A_R) \cong \mathrm{End}(A'_S)$ (as rings), A_R satisfies $\mathcal{P}$ implies that A'_S does (and then, of course, also conversely).

As an exhaustive reference on perspective R-modules we mention [4]. However, *not much* will be used when describing the perspective Abelian groups.

For further reference we list here from [4].

Theorem 2.1. *For a module M_S with $R = \mathrm{End}_S(M)$, the following conditions are equivalent:*

(1) M is perspective.

(4) If $erse = e$ for some $e^2 = e, r, s \in R$, then $erte = e$ for some $t \in R$ such that $ete \in U(eRe)$.

*In particular, M_S is perspective iff R_R is perspective, i.e., perspectivity is an **ER**-property.*

We mention here a useful consequence (not recorded in [4]) of the previous theorem.

Corollary 2.2. *Arbitrary products of perspective rings are perspective.*

Proof. Suppose R is direct product of R_i, $i \in I$ and $erse = e$ for some $e^2 = e, r, s \in R$. As $e = (e_i)$, $r = (r_i)$, $s = (s_i)$ so $erse = e$ iff $e_i r_i s_i e_i = e_i$ for each i. As each R_i is perspective, so there exists t_i such that $e_i r_i t_i e_i = e_i$ and $e_i t_i e_i \in U(e_i R_i e_i)$. If $t = (t_i)$, then $erte = e$ and $ete \in U(eRe)$. □

Proposition 2.3. *Any direct summand of a perspective module is perspective.*

Corollary 2.4. *If M and N are perspective R-modules with* $\mathrm{Hom}_R(M, N) = 0$, *then $M \oplus N$ is perspective.*

Remark 2.5. The group $\mathbb{Z} \oplus \mathbb{Z}$ is not perspective. For example, $(2, 5)\mathbb{Z}$ and $(1, 0)\mathbb{Z}$ are isomorphic direct summands of $\mathbb{Z}^2$ as a $\mathbb{Z}$-module, which do not have a common complement.

From [9] we recall a *method of constructing common complements* for some special direct sums.

Let $G = H \oplus K$. A subgroup D of G is called a *diagonal* in G (with respect to H and K) if $D + H = G = D + K$ and $D \cap H = 0 = D \cap K$.

Theorem 2.6. *Let $G = H \oplus K$. If $\delta : H \to K$ is an isomorphism then $D(\delta) = D(H, \delta) = \{x + \delta(x) | x \in H\} = (1 + \delta)(H)$ is a diagonal in G (with respect to H and K). Conversely, if D is a diagonal in G (with respect to H and K) there is a unique isomorphism $\delta : H \to K$ such that $D = D(\delta)$.*

Thus, there is a bijection between the diagonals (with respect to H and K) and isomorphisms of H and K.

Every subgroup U of a direct sum $G = H \oplus K$ belongs to the direct product $\mathbf{L} = L(H) \times L(K)$ (i.e., has the form $H' \oplus K'$ for $H' \leq H$ and $K' \leq K$) or is a diagonal.

3. Perspective Abelian groups

First about the

3.1. **Definition.** As the Abelian groups analogue for $\mathbb{Z}$-modules, an (Abelian group) G is called *perspective* if isomorphic summands of G have a common complement.

In symbols: if $G = A \oplus H = B \oplus K$ with $A \cong B$, there exists (a summand) C such that $G = A \oplus C = B \oplus C$.

Nonexample. Let N be a group such that $N \ncong N \oplus N$ and let $G = N_1 \oplus N_2 \oplus N_3 \oplus ...$ countably many copies with $N_n = N$. Then G is *not* IC (and so nor perspective).

Indeed, $H = N_2 \oplus N_3 \oplus ...$ and $S = N_3 \oplus ...$ are isomorphic summands, but $G/H \cong N \ncong N \oplus N \cong G/S$.

Remarks. 1) Obviously, *if two summands of a group have a common complement, these are isomorphic.*

Indeed, if $G = H \oplus K = L \oplus K$ then $H \cong G/K \cong L$.

Therefore, perspectivity is a converse of this property.

2) *Indecomposable groups are trivially perspective* (e.g., any infinite cyclic group, any cocyclic group or rank 1 torsion-free group).

3) We mention (see Proposition **3.10**, [7]) that *for two idempotent endomorphisms ε, δ of G, $\varepsilon(G) \cong \delta(G)$ iff there exists endomorphisms α, β of G such that $\varepsilon = \alpha\beta$ and $\delta = \beta\alpha$.* Also equivalently, *the left* $\mathrm{End}(G)$*-modules* $\mathrm{End}(G)\varepsilon$ *and* $\mathrm{End}(G)\delta$ *are isomorphic to each other.* Since we deal only with direct summands, equivalently, we can deal only with the idempotent endomorphisms of the group. Thus, two "*isomorphic*" endomorphisms ε, δ [i.e., $\mathrm{im}(\varepsilon) \cong \mathrm{im}(\delta)$] of a group G are *perspective*, if there is an endomorphism γ such that $\mathrm{im}(\varepsilon) \oplus \mathrm{im}(\gamma) = G = \mathrm{im}(\delta) \oplus \mathrm{im}(\gamma)$.

More general, the group is *IC* if for any two endomorphisms ε, δ of G, $\mathrm{im}(\varepsilon) \cong \mathrm{im}(\delta)$ implies $G/\mathrm{im}(\varepsilon) \cong G/\mathrm{im}(\delta)$.

4) The group G is indecomposable iff $\mathrm{End}(G)$ has only the trivial idempotents (is *connected*). Such groups are trivially perspective.

Apparently there is another definition one could give for perspective Abelian groups.

Definition. An Abelian group G is called *e-perspective* if its endomorphism ring $\mathrm{End}(G)$ is (left or right) perspective.

However, it follows from Theorem 2.1 that for a module M_S with $R = \mathrm{End}_S(M)$, M_S is perspective iff R_R is perspective, i.e., perspectivity is an **ER**-property. Therefore, for $S = \mathbb{Z}$ it follows that

Proposition 3.1. *An Abelian group is perspective iff it is e-perspective.*

Hence, in the sequel we can use any of these two (equivalent) definitions.

3.2. **Reduction theorems.**

Proposition 3.2. *Summands of perspective groups are perspective.*

Proof. Suppose $G = H \oplus K$ and $H = S \oplus T = L \oplus N$ with $S \cong L$. Since these are direct summands also in G, by hypothesis, there is $M \leq^{\oplus} G$ such that $G = S \oplus M = L \oplus M$. Then, by modularity of the subgroup lattice: $H = G \cap H = (\underline{S} \oplus M) \cap \underline{H} \overset{\mathrm{mod}}{=} \underline{S} \oplus (M \cap \underline{H})$ (since $S \leq H$) and similarly $H = L \oplus (M \cap H)$, so $M \cap H$ is a common complement for S and L. □

Proposition 3.3. *Let $G = \bigoplus_{i \in I} H_i$ where each summand H_i is fully invariant in G. Then G is perspective iff all H_i, $i \in I$, are perspective.*

Proof. By Corollary 2.2, arbitrary products of perspective rings are perspective. It just remains to note that $\mathrm{End}(G) \cong \prod_{i \in I} \mathrm{End}(H_i)$ (see Theorem **106.1**, [3], the $I \times I$ matrices are diagonal). □

Corollary 3.4. *Let G be a torsion group. Then G is perspective iff so are all its primary components.*

Proof. A straightforward application of the previous proposition. □

As customarily, this reduces the study of perspective torsion groups to perspective p-groups, for any prime p.

We can use Corollary 2.4 whenever $\mathrm{Hom}(G, H) = 0$, that is, for: (i) G torsion, H torsion-free, (ii) G a p-group and H a q-group with different primes $p \neq q$, (iii) G divisible and H reduced. Thus

Corollary 3.5. *(a) Let $G = T(G) \oplus F$ where $T(G)$ is the torsion part of G and F is torsion-free. Then G is perspective iff $T(G)$ and F are perspective.*

(b) Let $G = D(G) \oplus R$ where $D(G)$ is the divisible part of G and R is reduced. Then G is perspective iff $D(G)$ and R are perspective.

Examples. $\mathbb{Z}_m \oplus \mathbb{Z}$, $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ or $\mathbb{Z}_{p^\infty} \oplus \mathbb{Q}$, all are perspective (splitting) mixed groups.

Therefore, the study of splitting mixed perspective (Abelian) groups reduces to perspective primary groups and to torsion-free groups. Moreover, it reduces to perspective divisible groups and to reduced groups.

According to these reductions, the study of perspective (Abelian) groups reduces to reduced perspective p-groups, reduced perspective torsion-free groups and mixed perspective not splitting groups. Some of such results are proved in the sequel. The reader can convince himself that even for highly predictable (for Abelian groups theorists) results, in the torsion and in the torsion-free cases, the proofs are not at all easy.

Therefore the nonsplitting mixed case was not addressed. Since all we know about the torsion part of such a group is that it is pure in the whole group, the question of which pure subgroups of perspective groups are also perspective (just partly addressed) becomes central.

3.3. **Perspective p-groups.** First, perspective divisible p-groups are described below. For the proof, recall that the socle of $\mathbb{Z}(p^\infty)$ is its smallest nonzero subgroup (having order p) and that each subsocle of divisible p-group D supports some summand of D.

Proposition 3.6. *A divisible p-group is perspective iff it has finite rank. As such it is isomorphic to a finite direct sum of $\mathbb{Z}(p^\infty)$.*

Proof. As already mentioned, an infinite rank direct sum of $\mathbb{Z}(p^\infty)$ is not perspective (see nonexample in the preceding section). Conversely, let D be a divisible p-group. The proof goes by induction on rank of D. If $r(D) = 1$ then D is indecomposable and so trivially perspective.

Let $r(D) = n+1$, $D = A \oplus B = C \oplus U$ and $A \cong C$. We go into several cases.

1) Let $A + C \neq D$. Then $D = (A + C) \oplus D'$ for some $D' \neq 0$ and so A and C are summands in a divisible group $A + C$ of rank $\leq n$. By induction $A + C = A \oplus K = C \oplus K$ for some K, whence $D = A \oplus (K \oplus D') = C \oplus (K \oplus D')$.

2) Let $A + C = D$.

a) If $A \cap C = 0$ then $D = A \oplus C$, so $B \cong C$. Denote an isomorphism by $f : B \to C$. Using Theorem 2.6, the subgroup $B' = \{b + f(b) \,|\, b \in B\}$ is a diagonal, so a summand of D such that $A \cap D' = C \cap B' = 0$ and $D = A \oplus B' = C \oplus B'$.

b) Let $A \cap C \neq 0$, so $(A \cap C)[p] \leq A[p], C[p]$ where $A[p] \cong C[p]$.

If $(A \cap C)[p] = A[p]$ then also $(A \cap C)[p] = C[p]$. Hence $A[p] = C[p]$, and by [2], $D = A \oplus B = C \oplus B$.

Next assume that $0 \neq (A \cap C)[p] < A[p]$, so $(A \cap C)[p] < C[p]$. There exist a summand A_1 of A with $A_1[p] = (A \cap C)[p]$, $A = A_1 \oplus A_2$. Similarly $C = C_1 \oplus C_2$, where $C_1[p] = (A \cap C)[p]$. Since $A \cong C$ and $A_1[p] = C_1[p]$ it follows that $A_1 \cong C_1$ and so $A_2 \cong C_2$.

3) Let $A_2 + C_2 \neq D$. Then as in case **1)**, $D = A_2 \oplus V = C_2 \oplus V$ for some V and so $A = A_2 \oplus (A \cap V)$ and $C = C_2 \oplus (C \cap V)$. Here V has rank $\leq n$ and $A \cap V$, $C \cap V$ are isomorphic summands, so by induction $V = (A \cap V) \oplus L = (C \cap V) \oplus L$ for some L. Finally $D = [A_2 \oplus (A \cap V)] \oplus L = [C_2 \oplus (C \cap V)] \oplus L$.

4) Let $A_2 + C_2 = D$. Since $A_2 \cap C_2 = 0$, as in case **2 a)**, $D = A_2 \oplus M = C_2 \oplus M$ for some M. Since $r(M) \leq n$, by induction, $A \cap M$ and $C \cap M$ are perspective in M, and as in case **3)**, A and C are perspective in D. □

If m is a cardinal and G is a group then $G^{(m)}$ denotes the direct sum of m copies of G.

The previous proposition has the following

Corollary 3.7. *For any cardinal m and any natural number n, the group $G = \mathbb{Z}(p^n)^{(m)}$ is perspective iff m is finite.*

Proof. As already mentioned, an infinite rank direct sum of $\mathbb{Z}(p^n)$ is not perspective (see nonexample in the preceding section). Conversely, if D is a divisible hull of G then $G = D[p^n]$. If $G = A \oplus B = C \oplus K$, $A \cong C$, then by Proposition 3.6, $D = D_A \oplus U = D_C \oplus U$ for some $U \leq D$, where D_A, D_C are the divisible hulls of A, C, respectively. So $G = D[p^n] = D_A[p^n] \oplus U[p^n] = D_C[p^n] \oplus U[p^n]$, where $D_A[p^n] = A$, $D_C[p^n] = C$. □

Next, about finite or bounded p-groups, we have

Proposition 3.8. *The finite p-groups are perspective.*

Proof. The proof goes by induction on the order $|G|$ of the group G. Let $G = G_1 \oplus G_2$, where G_1 is a direct sum of finitely many groups $\mathbb{Z}(p^n)$, where p^n is the maximal order of elements in G, $G = A \oplus B = C \oplus K$ and $A \cong C$.

By Corollary 3.7 G_1 is perspective. Note that if $A \cap G_1 \neq 0$ then also $C \cap G_1 \neq 0$. Otherwise, if $C \cap G_1 = 0$, since G_1 is an absolute direct summand (see [2], Exercise 8 of §9) of G, we can suppose that $C \leq G_2$. However, in this case C would not have any element of order p^n but in A such elements exist in view of $A \cap G_1 \neq 0$. This would contradict the isomorphism $A \cong C$.

Since in a direct sum of cyclic groups each subsocle supports some summand of this group (see [3], Exercise 3 of §66) it follows that $A = A_1 \oplus A_2$, $C = C_1 \oplus C_2$, where $A_1[p] = (A \cap G_1)[p]$, $C_1[p] = (C \cap G_1)[p]$. These direct decompositions of cyclic groups are isomorphic, so from $A \cong C$ it follows that $A_1 \cong C_1$ and so $A_2 \cong C_2$. Hence by Corollary 3.7, $G_1 = A_1 \oplus U = C_1 \oplus U$ for some U. Then $A = A_1 \oplus A_3$, $C = C_1 \oplus C_3$, where $A_3 = [A \cap (U \oplus G_2)]$, $C_3 = [C \cap (U \oplus G_2)]$. So A_3, C_3 are isomorphic direct summands in $U \oplus G_2$. Since $|U \oplus G_2| < |G|$, by induction $U \oplus G_2 = A_3 \oplus V = C_3 \oplus V$. Hence $G = G_1 \oplus G_2 = (A_1 \oplus U) \oplus G_2 = (A_1 \oplus A_3) \oplus V = (C_1 \oplus C_3) \oplus V$, where $A_1 \oplus A_3 = A$ and $C_1 \oplus C_3 = C$, as desired. □

Since summands of perspective groups are perspective it follows from the nonexample mentioned before that the Ulm-Kaplanski invariants $f_n(G)$ of perspective reduced p-groups G are finite for all integer $n \geq 0$. Thus a basic subgroup of G is countable and so $|G| \leq 2^{\aleph_0}$ [2]. Therefore

Corollary 3.9. *Any perspective bounded p-group is finite.*

For (Abelian) groups we can introduce the following

Definition. A group is called *finitely perspective* if it is perspective with respect to finite (direct) summands. Then we can prove a surprising (specific for Abelian groups) result.

Proposition 3.10. *Each p-group G is finitely perspective.*

Proof. Let $A \cong C$ be finite summands of G. Then $p^m A = 0$ for some integer $m \geq 1$, and so also $p^m C = 0$ and $p^m(A + C) = 0$. Hence $A + C \leq H$ for a $p^m G$-high subgroup H. By a theorem of Khabbaz (see [2], Theorem 27.7), H is a summand of $G = H \oplus F$. We have $H = H_1 \oplus \cdots \oplus H_m$, where H_i is a direct sum of groups $\mathbb{Z}(p^i)$ whenever $H_i \neq 0$. Let $\pi_i G \to H_i$ be the projections for $i = 1, \ldots, m$. Since $A + C$ is finite, it follows that each $\pi_i(A + C)$ is finite, and $A + C \leq \pi_1(A + C) \oplus \cdots \oplus \pi_m(A + C)$. Each $\pi_i(A + C)$ is contained in some finite summand H_i' of H_i and so $H' = H_1' \oplus \cdots \oplus H_m'$ is a finite summand in $H = H' \oplus H''$. By Proposition 3.8, $H' = A \oplus U = C \oplus U$ for some U. Then $G = A \oplus (U \oplus H'' \oplus F) = C \oplus (U \oplus H'' \oplus F)$. $\square$

We just mention that if $G = A \oplus C$, where $A \cong C$ then $G = A \oplus U = C \oplus U$, for any diagonal U with respect to A and C.

Let $\mathfrak{A}$ be a class of (Abelian) groups and $G \in \mathfrak{A}$. A relativization of our main property can be defined.

Definition. We call A *perspective in class* $\mathfrak{A}$ if for $G = A \oplus B = C \oplus K$, where $A \cong C$ and $G \in \mathfrak{A}$ it follows that $G = A \oplus U = C \oplus U$ for some U.

Then we can prove a result on torsion-complete (for several equivalent definitions for reduced groups, see [3], Theorem 68.4) p-groups

Proposition 3.11. *The torsion-complete p-groups A with finite Ulm-Kaplanski invariants are perspective in the class of separable p-groups.*

Proof. Let $G = A \oplus B = C \oplus K$, where $A \cong C$ and let G be a separable p-group. Then $G = (A_1 \oplus \cdots \oplus A_n) \oplus (A_n^* \oplus B) = (C_1 \oplus \cdots \oplus C_n) \oplus (C_n^* \oplus K)$, where $A = (A_1 \oplus \cdots \oplus A_n) \oplus A_n^*$, $C = (C_1 \oplus \cdots \oplus C_n) \oplus C_n^*$, and $A_1 \oplus \cdots \oplus A_n$, $C_1 \oplus \cdots \oplus C_n$ respectively, are summands of the basic subgroups of A, C (A_k, C_k are direct sums of cyclic groups of order p^k). By Proposition 3.10, $G = (A_1 \oplus \cdots \oplus A_n) \oplus U^{(n)}$, $G = (C_1 \oplus \cdots \oplus C_n) \oplus U^{(n)}$, where we can choose the $U^{(n)}$'s, such that $U^{(n+1)}$ is a summand in $U^{(n)}$ and $U^{(n)}/U^{(n+1)}$ is a direct sum of cyclic groups of order p^{n+1}. So G has a basic subgroup of type $(\bigoplus_{n\geq 1} A_n) \oplus (\bigoplus_{n\geq 1} V_n^{(n)}) = (\bigoplus_{n\geq 1} C_n) \oplus (\bigoplus_{n\geq 1} V_n^{(n)})$, where $V_n^{(n)}$ is a summand in $U_n^{(n)}$, each $U_n^{(n)}$ is a summand in $U^{(n)}$ and is a direct sum of cyclic groups of order p^n. So by [3],Theorem 71.3, $\overline{G} = A \oplus U = C \oplus U$, where $\overline{X}$ is the torsion completion of X and $A = \overline{(\bigoplus_{n\geq 1} A_n)}$, $C = \overline{(\bigoplus_{n\geq 1} C_n)}$, $U = \overline{(\bigoplus_{n\geq 1} V_n^{(n)})}$. Hence $G = A \oplus (G \cap U) = C \oplus (G \cap U)$, and the proof is complete. $\square$

3.4. **Perspective torsion-free groups.** As it is well known, the divisible torsion-free groups are the direct sums of $\mathbb{Q}$.

Proposition 3.12. *A torsion-free divisible group is perspective iff it has finite rank. As such, it is isomorphic to a finite direct sum of $\mathbb{Q}$.*

Proof. As already mentioned, an infinite rank direct sum of $\mathbb{Q}$ is not perspective. Conversely, for a torsion-free divisible group D, the proof goes by induction on the rank of D. Let $r(D) = n + 1$, A and B are isomorphic summands of D.

If $A+B < D$ then $D = (A+B) \oplus C$ and $r(A+B) \leq n$, so $A+B = A \oplus H = B \oplus H$, since by induction the divisible group $A+B$ is perspective. Then $D = A \oplus (H \oplus C) = B \oplus (H \oplus C)$.

Assume now that $A + B = D$. Only two cases are possible.

1) If $K := A \cap B \neq 0$ then K is divisible and $D = K \oplus L$ for some $L \leq D$. We have $A = K \oplus (A \cap L)$, $B = K \oplus (B \cap L)$. Here $A \cap L \cong B \cap L$, so by induction $L = (A \cap L) \oplus C$ and $L = (B \cap L) \oplus C$ whence $D = [K \oplus (A \cap L)] \oplus C$ and $D = [K \oplus (B \cap L)] \oplus C$, where $[K \oplus (A \cap L)] = A$, $[K \oplus (B \cap L)] = B$.

2) If $A \cap B = 0$ then $D = A \oplus B$ and since $A \cong B$ it follows that $A = \mathbb{Q}a_1 \oplus \cdots \oplus \mathbb{Q}a_m$ and $B = \mathbb{Q}b_1 \oplus \cdots \oplus \mathbb{Q}b_m$ for some $0 \neq a_1, \ldots, a_m, b_1, \ldots, b_m \in D$ (with A

and B considered as vector spaces on field $\mathbb{Q}$). Then $D = A \oplus H = B \oplus H$, where $H = \mathbb{Q}(a_1 + b_1) \oplus \cdots \oplus \mathbb{Q}(a_m + b_m)$. $\square$

It was already mentioned that $\mathbb{Z}$ is perspective since it is indecomposable and that $\mathbb{Z} \oplus \mathbb{Z}$ is not perspective (see Remark 2.5). It follows

Proposition 3.13. *The only perspective free group is $\mathbb{Z}$.*

Since the rational groups (the subgroups of $\mathbb{Q}$) are also indecomposable it follows

Proposition 3.14. *The only perspective rank 1 torsion-free groups are the rational groups.*

Clearly

Proposition 3.15. *If a torsion-free group has a summand isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, it is not perspective.*

Next we characterize some perspective homogeneous completely decomposable groups.

Note that, similarly to Proposition 3.3, we can prove the following

Proposition 3.16. *Let $G = \prod_{i \in I} H_i$, where each subgroup H_i is fully invariant in G. Then G is perspective iff all H_i, $i \in I$, are perspective.*

Let p be a prime number and $\mathbb{Q}_p$ the group (ring) of all rational numbers with denominator coprime with p.

Proposition 3.17. *Let G be a homogeneous completely decomposable group with type $\mathbf{t}(G) = \mathbf{t}(\mathbb{Q}_p)$. The group G is perspective iff G has finite rank.*

Proof. As already mentioned, only one way needs a proof. Suppose the rank of G is finite and $G = A \oplus B = C \oplus K$, where $A \cong C$. The proof goes by induction on $n = r(G)$ the rank of G. We distinguish several cases.

1) $A_1 = A \cap C \neq 0$. Then A_1, as pure subgroup, is a summand in G (see [3], Lemma **86.8**), say $A = A_1 \oplus A_2$, $C = A_1 \oplus C_2$, where $C_2 = (A_2 \oplus B) \cap C$. So $A_2 \oplus B = C_2 \oplus V$ for some V. By the induction hypothesis $A_2 \oplus U = C_2 \oplus U$ for some U. Hence $G = (A_1 \oplus A_2) \oplus U = (A_1 \oplus C_2) \oplus U$, where $A_1 \oplus A_2 = A$ and $A_1 \oplus C_2 = C$.

2) $F = B \cap K \neq 0$. Then $B = B' \oplus F$ and $K = K' \oplus F$ for some B', K' and $(A \oplus F) \oplus B' = (C \oplus F) \oplus K'$, where $B' \cap K' = 0$. If now $(A \oplus F) \oplus U = (C \oplus F) \oplus U$ then $A \oplus (F \oplus U) = C \oplus (F \oplus U)$. So this case reduces to $B \cap K = 0$.

3) $A_K = A \cap K \neq 0$. Then $A = A_K \oplus A_2$, $K = A_K \oplus K_2$ for some A_2, K_2 such that $A_2 \leq C \oplus K_2$. Let $\pi : C \oplus K_2 \to C$ be the projection and let $C_2 = \langle \pi(A_2) \rangle$ be the pure hull of $\pi(A_2)$ in C. Since $A_2 \cap K_2 = 0$ then $C_2 \cong A_2$ and it follows that $A_2 \leq C_2 \oplus K_2$, and so $C_2 \oplus K_2 = A_2 \oplus V$ for some V. Since $r(C_2 \oplus K_2) < n$ by induction hypothesis $C_2 \oplus U = A_2 \oplus U$ for some U. So $(C_1 \oplus C_2) \oplus (A_K \oplus U) = (A_K \oplus A_2) \oplus (C_1 \oplus U)$, where $A_K \cong C_1$. Hence if H is a diagonal in $A_K \oplus C_1$ (with respect to A_K and C_1) then $C \oplus (H \oplus U) = A \oplus (H \oplus U)$. The case $C \cap B \neq 0$ is similar.

4) $A \cap C = A \cap K = B \cap K = C \cap B = 0$. It follows that $r(A) = r(B) = r(K)$. Let $R = \mathbb{Q}_p$. Then G is a free R-module of rank $n = 2m$. We have $A = \bigoplus_{i=1}^m Ra_i$, $B = \bigoplus_{i=1}^m Rb_i$, $C = \bigoplus_{i=1}^m Rc_i$, and $K = \bigoplus_{i=1}^m Rk_i$. Since the pure submodules are summands in G, we can choose $c_i = a_i' + b_i'$ for some $a_i' = r_i a_i$, $b_i' = s_i b_i$, where $r_i, s_i \in R$.

Assume $s_i \in pR$, $i = 1, \dots, l$ for some $l \leq m$ and $s_i \in R \setminus pR$ for $i = l+1, \dots, m$. Note that if $s_i \in pR$ then $r_i \in R \setminus pR$, and similarly if $r_i \in pR$ then $s_i \in R \setminus pR$. Let $H = \bigoplus_{i=1}^{l} R(a_i + b_i)$. Then $r_i^{-1} c_i = a_i + s_i' b_i$, where $s_i' = r_i^{-1} s_i$ for $i = 1, \dots, l$.

Observe that $H \cap C = 0$. Indeed, if $h \in H$ and $h = t_1(a_1+b_1)+\cdots+t_l(a_l+b_l) \in C$ for some $t_1, \dots, t_l \in R$, then $[t_1(a_1 + b_1) + \cdots + t_l(a_l + b_l)] - [t_1(a_1 + s_i' b_1) + \cdots + t_l(a_l + s_i' b_l)] = t_1(1 - s_1')b_1 + \cdots + t_l(1 - s_l)b_l \in C \cap B = 0$. So $t_1 = \cdots = t_l = 0$, i.e. $h = 0$.

Assume also that $r_i \in pR$ for $i = l+1, \dots, l+r$, when $l+r \leq m$ and $r_i \in R \setminus pR$ for $i = l + r + 1, \dots, m$ when $l + r < m$. Then as before $L \cap (C \oplus H) = 0$, where $L = \bigoplus_{i=l+1}^{l+r} R(a_i + b_i)$. If $l + r < m$ then let $M = \bigoplus_{i=l+r+1}^{m} R(a_i + c_i)$. We show that $A \oplus (H \oplus L \oplus M) = C \oplus (H \oplus L \oplus M) = G$. Indeed, this follows from the fact that these sums of summands are direct, as for the construction we used linear independent subsystems and these sums equal G since these contain the basis of G.

From the construction of H and L it follows that $b_i \in A \oplus H \oplus L$ for $i = 1, \dots, l+r$. Since $c_i = r_i a_i + s_i b_i$, where s_i are invertible in R for $i = l + r + 1, \dots, m$ it follows that $b_i \in A \oplus M$ for the specified i. Hence $G = A \oplus (H \oplus L \oplus M)$. It remains to prove that $a_i, b_i \in C \oplus (H \oplus L \oplus M)$ for all $i = 1, \dots, m$. Indeed, if for example, $i = 1, \dots, l$, then $c_i - (r_i a_i + r_i b_i) = (r_i a_i + s_i b_i) - (r_i a_i + r_i b_i) = (s_i - r_i) b_i \in C \oplus H$. Since $s_i \in pR$ and $r_i \in R \setminus pR$ we get $s_i - r_i \in R \setminus pR$, so this element is invertible in R, whence b_i and so a_i belong to $C \oplus H \leq C \oplus (H \oplus L \oplus M)$ for $i = 1, \dots, l$. Similarly we can prove that all $a_i, b_i \in C \oplus (H \oplus L \oplus M)$ for $i = 1, \dots, m$, i.e. $G = C \oplus (H \oplus L \oplus M)$. Since the sum of the ranks of H, L, M equals m the latter sum is direct. □

Remark. Since the pure subgroups of group G in the previous proposition are summands of G, this is a trivial example of groups with all pure subgroups being perspective.

The next result draws attention on rank 2 summands of homogeneous torsion-free groups of finite rank.

Lemma 3.18. *If G is a torsion-free homogeneous group of finite rank then G is perspective iff G has a perspective rank 2 summand.*

Proof. The condition is necessary since summands of perspective groups are perspective.

Conversely, since equal rank summands of G are isomorphic, it suffices to focus on any summand. Let $G = A \oplus B = C \oplus K$, where $A \cong C$. As in Proposition 3.17, we can assume that $A \cap C = A \cap K = 0$. Let $0 \neq a \in A$ and $a = c + x$, where $c \in C$, $x \in K$ $(c, x \neq 0)$. If $A_1 = \langle a \rangle_*$, $C_1 = \langle c \rangle_*$, $K_1 = \langle x \rangle_*$ then A_1, C_1 and K_1 are (rank 1) summands, so $A = A_1 \oplus A_2$, $C = A_1 \oplus C_2$, $K = K_1 \oplus K_2$ and $G = (C_1 \oplus K_1) \oplus (C_2 \oplus K_2)$. By hypothesis, $C_1 \oplus K_1 = C_1 \oplus W_1 = A_1 \oplus W_1$ for some W_1, so $G = C_1 \oplus (W_1 \oplus C_2 \oplus K_2) = A_1 \oplus (W_1 \oplus C_2 \oplus K_2)$. Since $A = A_1 \oplus A \cap (W_1 \oplus C_2 \oplus K_2)$, consider$A_2 = A \cap (W_1 \oplus C_2 \oplus K_2)$. Hence $W_1 \oplus C_2 \oplus K_2 = A_2 \oplus L$ for some L and we complete the proof by induction: $W_1 \oplus C_2 \oplus K_2 = C_2 \oplus W = A_2 \oplus W$ so $G = C_1 \oplus (W_1 \oplus C_2 \oplus K_2) = (C_1 \oplus C_2) \oplus W = A_1 \oplus (W_1 \oplus C_2 \oplus K_2) = (A_1 \oplus A_2) \oplus W$. □

Using Proposition 3.3 and Proposition 3.16 it follows that

Corollary 3.19. *Let* $G = \bigoplus_{p\in P} G_p$ $(G = \prod_{p\in P} G_p)$*, where* P *is some subset of prime numbers and* G_p *are homogeneous completely decomposable groups of finite rank with type* $t(G_p) = t(\mathbb{Q}_p)$*. Then the group* G *is perspective.*

Next, we describe the perspective torsion-free algebraically compact groups. Denote by $\hat{\mathbb{Z}}_p$ be the ring (group) of p-adic integers and by $\mathbb{P}$ the set of all prime numbers.

Proposition 3.20. *A non-zero torsion-free algebraically compact group* G *is perspective iff* $G = \prod_{p\in\pi} G_p$*, where* $\varnothing \neq \pi \subseteq \mathbb{P}$ *and* G_p *is a finite direct product of copies of the group* $\hat{\mathbb{Z}}_p$*.*

Proof. To show that the condition is necessary, first recall that any torsion-free algebraically compact group G has the form $G = \prod_{p\in\pi} G_p$, where G_p is a p-adic algebraically compact group (see [2], Proposition **40.1**), and in particular, is $\hat{\mathbb{Z}}_p$-module. So the rank of each G_p is finite, i.e. G_p is a free $\hat{\mathbb{Z}}_p$-module of finite rank.

To show that the condition is sufficient, also recall that intersections of summands in torsion-free p-adic algebraically compact groups are also summands of this group. Next note that in the ring $\hat{\mathbb{Z}}_p$, all elements of $\hat{\mathbb{Z}}_p \setminus p\hat{\mathbb{Z}}_p$ are invertible, and that in any $\hat{\mathbb{Z}}_p$-module of finite rank, pure submodules are summands. As in Proposition 3.17, one can prove that p-adic algebraically compact modules of finite rank are perspective. Then using Proposition 3.16, the proof is complete. □

Example 3.21. *Let* $G = G_1 \oplus \cdots \oplus G_n$*, where* G_i*,* $1 \leq i \leq n$ *are perspective groups and* $\mathrm{Hom}(G_i, G_j) = 0$ *for* $i = 2, \ldots, n$ *and* $1 \leq j \leq i-1$*. Then* G *is perspective.*

Proof. The proof goes by induction on n. If $G_2 \oplus \cdots \oplus G_n$ is perspective then since it is fully invariant in G, it is perspective by Corollary 2.4. □

Next, we give some examples and nonexamples.

Recall that a torsion-free group G is called *cohesive* if G/H is divisible for all pure subgroups $H \neq 0$ of G. For some facts about such groups see [3], **88**, exercise 17.

Example 3.22. *A perspective direct sum of pure subgroups of a perspective group.*

By Proposition 3.20, the group $G = \prod_{p\in\mathbb{P}} \hat{\mathbb{Z}}_p$ is perspective. Let A_i, $i = 1, \ldots, n$, be pure subgroups of G such that $p^\omega A_i = 0$ for all $p \in \mathbb{P}$ and $r(A_i) = m_i$, where $1 \leq m_1 < \cdots < m_n \leq 2^{\aleph_0}$. Then A_i are cohesive groups (see [7], **32**). If $A = A_1 \oplus \cdots \oplus A_n$, then A is perspective.

Indeed, each A_i is perspective as indecomposable group and $\bigoplus_{j=i}^n A_j$ is fully invariant in A for all $i = 2, \ldots, n-1$. This follows from the fact that $\mathrm{Hom}(A_j, A_i) = 0$ for every $i < j$, owing that since $r(A_i) < r(A_j)$, each homomorphism $f : A_j \to A_i$ has a non-zero kernel and so $f(A_j)$ is divisible, whence $f = 0$.

In general we can ask

Question. Which *pure subgroups* of a perspective group are perspective ?

Example 3.23. *A subgroup of a perspective group which is not perspective.*

Let G be the torsion-free group of rank $n \geq 3$ as in [3], **88**, exercise 8. Then G is perspective as an indecomposable group but all the subgroups in G of rank $n-1$ are free. So the proper subgroups of rank ≥ 2 of G are not perspective.

Example 3.24. *A factor group of perspective groups which is not perspective.*

Let G be a torsion-free cohesive group with $r(G) \geq \aleph_0$ and let H be a pure subgroup of G such that $r(G/H) \geq \aleph_0$ (e.g., a pure subgroup H of rank 1). Then G/H is not perspective being divisible torsion-free group of infinite rank, but G and H are perspective being indecomposable groups.

Example 3.25. *A factor group of a not perspective group may be perspective.*

Let $X = \{a_n : n \in \mathbb{N}^*\}$ and let $G = \langle X \rangle$ be free of countable rank. Consider the function $f : X \to \mathbb{Q}$, $f(a_n) = \frac{1}{n!}$ for every $n \in \mathbb{N}^*$. The group G being free, f extends to a group homomorphism $\overline{f} : G \to \mathbb{Q}$, obviously surjective (as $\mathbb{Q} = \left\langle \frac{1}{n!} : n \in \mathbb{N}^* \right\rangle$). So G is not perspective but $\mathbb{Q} \cong G/\ker(\overline{f})$ is perspective.

As in the module case, conditions for a perspective group G which assure that $G \oplus G$ is also perspective, are of interest (and difficult to find).

In closing, we address this problem for rank 1 torsion-free groups, which are perspective as indecomposable groups.

First, a simple lemma which is well known

Lemma 3.26. *Each subgroup A of torsion-free group G of rank 1, isomorphic to the whole group, is of form nG, for some integer $n \geq 1$.*

Proof. If $f : G \to A$ is an isomorphism then f acts as multiplication with some rational number n/m, where $mG = G$, and so $f(G) = nG = A$. □

Next, a characterization.

Proposition 3.27. *Let G be a torsion-free group of rank 1. Then $G \oplus G$ is perspective iff for all coprime integers $m, n \geq 1$ and all integers $k, t \geq 0$, such that*

(i) at least one of k, t is non-zero, and

(ii) if one of k, t is zero then the other is equal to 1, and

(iii) if both k, t are non-zero these are coprime,

in each of the following cases:

1) $mG \neq G$, $k = 1$, $t = 0$,

2) $nG \neq G$, $k = 0$, $t = 1$,

3) $mG \neq G$, $tG \neq G$, where $t \neq 0$,

4) $mG, nG \neq G$,

there exist coprime integers s, l with $(ml - sn)G = G$, $(kl - st)G = G$.

Proof. To show that the conditions are necessary, we present $G \oplus G$ as $F = Ra \oplus bR$, where R is an additive subgroup with 1 of $\mathbb{Q}$, isomorphic to G. Since m, n are coprime, the subgroup $R(ma+nb)$ is pure, so its is a summand of F. Let $\pi : F \to Ra$ and $\theta : F \to Rb$ be the projections and let A and C be isomorphic rank 1 summands of F. Hence $A = R(ma + nb)$ and $C = R(ka + tb)$ for some $m, n, k, t \in \mathbb{Z}$, where at least one of $\{m, n\}$ is 1 and the corresponding integer of $\{k, t\}$ is non-zero. It suffices to consider the case $m, n \neq 0$. Since A is a summand of F, $dG = G$ for $d = \gcd(m, n)$, so we can consider $d = 1$. Similarly, if $t = 0$ and $C = Rka$ then $kG = G$, so we can assume $k = 1$.

1) Let $mG \neq G$. Assume that $F = R(ma + nb) \oplus U = Ra \oplus U$ for some U, i.e. in this case $k = 1$, $t = 0$. Clearly $U \neq Ra, Rb$ and so $\pi(U), \theta(U) \neq 0$, whence $\pi(U) = Rsa$, $\theta(U) = Rlb$ for some coprime integers $s, l \geq 1$ (by Lemma 3.26) and so $U = R(sa + lb)$. We have $-s(ma + nb) + m(sa + lb) = (ml - sn)b$ and $-sa + (sa + lb) = lb$. Since the presentation of elements is unique in direct sums and $Rb \leq F$, it follows that $(ml - sn)G = lG = G$.

2) Let $nG \neq G$. Assume that $F = R(ma+nb) \oplus U = Rb \oplus U$ for some U, i.e. in this case $t = 1$, $k = 0$. Clearly $U \neq Ra, Rb$ and as above we can show that $(ml - sn)G = sG = G$ for some $m, l \in \mathbb{Z}$ with coprime s, l.

3) Let $mG \neq G$, $tG \neq G$ for $t \neq 0$. Clearly $U \neq Ra, Rb$, so $U = R(sa+lb)$, with coprime s, l. We have $-s(ma+nb) + m(sa+lb) = (ml - sn)b$ and $-s(ka+tb) + k(sa+lb) = (kl - st)b$. Hence $(ml - sn)G = G$, $(kl - st)G = G$.

4) Let $mG, nG \neq G$. Then $U \neq Ra, Rb$, so, as in case 3), $(ml - sn)G = G$, $(kl - st)G = G$ for some coprime $s, l \in \mathbb{Z}$.

To show that the conditions are sufficient, let $F = A \oplus B = C \oplus K$, $A \cong C$ and $r(A) = 1$. We consider several cases.

I. a) $A = Ra$, $C = Rb$. We can choose $U = R(a+b)$.

b) $A = R(m'a + n'b)$, where $m'G = n'G = G$. If $C = Rb$ we can take $U = Ra$, and if $C = Ra$ we can take $U = Rb$.

c) $A = R(m'a + n'b)$, $C = R(k'a + t'b)$, where $m'G = n'G = G$, $k'G = t'G = G$. Then we can choose $U = Ra$ or $U = Rb$.

II. $A = R(ma+nb)$, where $mG \neq G$ or $nG \neq G$.

If $A = R(ma+nb)$, $C = R(ka+tb)$, $mG \neq G$, $nG = G$, $tG = G$ and $kG = G$ or $kG \neq G$, $k \neq 0$, then in both cases we can take $U = Ra$. Also $U = Ra$ if $C = Rb$.

1) $A = R(ma+nb)$, where $mG \neq G$ and $C = Ra$, i.e. $k = 1$, $t = 0$. Since $mG \neq G$ then $n \neq 0$ (otherwise A is not summand of F). Then by hypothesis there exist coprime $s, l \in \mathbb{Z}$ with $(ml - sn)G = G$, $lG = G$. If $U = R(sa+lb)$ then $-s(ma+nb)+m(sa+lb) = (ml-sn)b$, $-sa+(sa+lb) = lb$ and $-l(ma+nb)+n(sa+lb) = (ns - lm)a$, where $(ml - sn)G = G$ and $lG = G$, so $Ra, Rb \leq A \oplus U, C \oplus U$ whence $F = A \oplus U = C \oplus U$.

2) $A = R(ma+nb)$, where $nG \neq G$, so $m \neq 0$, and $C = Rb$, i.e. $k = 0$, $t = 1$. Let $s, l \in \mathbb{Z}$ be coprime, $(ml - sn)G = G$, $tG = G$ and $U = R(sa+lb)$. As in the previous case 1), $F = A \oplus U = C \oplus U$. The remaining cases 3) and 4) are similar and since this way all the possible cases are covered, the proof is complete. ☐

Corollary 3.28. *Let G be a torsion-free group of rank 1 such that $G \oplus G$ is perspective. Then G is p-divisible at least for one prime number p.*

Proof. As in Proposition 3.27, assume $(ml - sn)G = G$ and $(kl - st)G = G$. Moreover, assume that $ml - sn = \pm 1$ and $kl - st = \pm 1$. Then if $t = 0$ we have $l = \pm 1$ and so $\pm m \pm 1 = sn$. Since we can choose coprime m and n such that $(\pm m \pm 1)/n \notin \mathbb{Z}$, it follows G is not divisible only by ± 1. ☐

The converse fails as shows the following

Example 3.29. *If G is a torsion-free group of rank 1, with $2G, 5G \neq G$ and G is divisible only by 11, then $G \oplus G$ is not perspective.*

By contradiction, suppose $G \oplus G$ is perspective. Then according to Proposition 3.27, $5l - 2s = \pm 11^a$, $kl - st = \pm 11^b$, for some integers $a, b \geq 0$. Taking $l = 0$ we can suppose $t = 1$ and so $s = \pm 11^b$ and $5l \pm 2 \cdot 11^b = \pm 11^a$. Since $(l, 11^b) = 1$ we get $b = 0$ or $a = 0$. If $b = 0$ then the equation $5l \pm 2 = \pm 11^a$ has no solutions since the last digit of the RHS 1 but is 2 or 7 in the LHS. If $a = 0$ then the equation $5l \pm 1 = \pm 2 \cdot 11^b$ has no solutions since the last digit of the RHS is 2 but is 1 or 6 in the LHS.

A result of the same sort is the following

Proposition 3.30. *If G is a torsion-free homogeneous of rank 1 group such that G is divisible for all prime numbers except two coprime numbers p and q then $G \oplus G$ is perspective.*

Proof. Let $F = G \oplus G$ and $F = A \oplus B = C \oplus K$, where $A \cong C$, $r(A) = 1$. As in the previous proposition 3.27, we take $A = R(ma + nb)$, $C = R(ka + tb)$, and in view of the sufficiency part, we can suppose $m, n, k, t \neq 0$. We are searching U such that $U = R(sa + lb)$, where $(s, l) = 1$. Consider all the possible cases with respect to divisibility of G by m, n, k, t (in the next table, the sign "+" means divisibility by the corresponding number, sign "–" not divisibility).

	1	2	3	4	5	6	7	8	9	10	11	12
m	+	+	+	+	+	−	−	−	−	+	−	−
n	+	+	+	+	−	+	−	−	−	−	+	+
k	+	+	−	−	+	+	−	−	+	+	+	−
t	+	−	+	−	+	+	−	+	+	−	−	+

As the cases $\{2, 3, 5, 6\}$, $\{4, 9\}$ and $\{10, 12\}$ are respectively similar, it suffices to check the cases 2, 4 and 10.

1)–2) $mG = nG = kG = G$ and $tG = G$ or $tG \neq G$. If $U = Rb$ then $F = R(ma + ng) \oplus U = R(ka + tb) \oplus U$.

4) $mG = nG = G$, and $kG, tG \neq G$. Since $\gcd(k, t) = 1$, let $p \mid k$, $q \mid t$ and $q \nmid k$, $p \nmid t$. If now $U = R(qa + pb)$, i.e. $s = q$, $l = p$, then $p, q \nmid (mp - qn)$ and $p, q \nmid (kp - qt)$, so $(mp - qn)G = G$, $(kp - qt)G = G$. Consequently, by Proposition 3.27, $F = A \oplus U = C \oplus U$. If $p \mid t$, $q \mid k$, where $q \nmid t$, $p \nmid k$ and $U = R(pa + qb)$ then $p, q \nmid mq - pn$, $p, q \nmid kq - pt$.

7) $mG, nG, kG, tG \neq G$.

a) $p \mid m, t$ and $q \mid n, k$. If $U = R(a + b)$ then $p, q \nmid m - n$ and $p, q \nmid k - t$.

b) $p \mid m, k$ and $q \mid n, t$. Let $U = R(qa + pb)$, then $p, q \nmid mp - qn$ and $p, q \nmid kp - qt$.

8) $p \mid m, k$ and $q \mid n$, $tG = G$.

a) $q \nmid k$. If $U = R(qa + b)$ then $p, q \nmid m - qn$ and $p, q \nmid k - qt$.

b) $q \mid k$. If $U = R(a + b)$, then $p, q \nmid m - n$ and $p, q \nmid k - t$.

10) $mG, kG = G$ and $nG, tG \neq G$. If $U = Rb$ then $F = A \oplus U = C \oplus U$.

11) a) $q \mid m, t$ and $p \nmid m, t$, but $nG = kG = G$. Let $U = R(pa + b)$, i.e. $s = p$, $l = 1$. Then $p, q \nmid m - pn$ and $p, q \nmid k - pt$.

b) $p, q \mid m, t$. If $U = R(a + b)$ then $p, q \nmid m - n$, and $p, q \nmid k - t$.

c) $p \nmid m$, $q \mid m$ and $q \nmid t$, $p \mid t$. If $U = R(pa + qb)$, i.e. $s = p$, $l = q$ then $p, q \nmid mq - pn$ and $p, q \nmid kq - pt$. ☐

Acknowledgement 3.31. The second author was supported by the Ministry of Science and Higher Education of Russia (agreement No. 075-02-2023-943

References

[1] A. R. Chekhlov *Abelian groups with normal endomorphism rings*. Algebra and Logic **48** (2009), 298-308.

[2] L. Fuchs *Infinite abelian groups*, vol. 1, Pure and Applied Mathematics, vol. 36, Academic Press, New York, London 1970.

[3] L. Fuchs *Infinite abelian groups*, vol. 2, Pure and Applied Mathematics, vol. 36-II, Academic Press, New York, London 1973.

[4] S. Garg, H. K. Grover, D. Khurana *Perspective rings*. J of Algebra **415** (2014), 1-12.

[5] K.R. Goodearl *Von Neumann Regular Rings*. Krieger Publishing Comp., Second Edition (1991).

[6] D. Khurana, T. Y. Lam *Rings with internal cancellation*. J. Algebra **284** (2005), 203-235.

[7] P. A. Krylov, A. V. Mikhalev, A. A. Tuganbaev *Endomorphism rings of abelian groups*. Algebras and Applications, vol. 2. Kluwer Academic Publishers, Dordrecht, 2003.
[8] T. Y. Lam *A crash course on stable range, cancellation, substitution and exchange*. J. of Algebra and Appl. **3** (3) (2004), 301-343.
[9] R. Schmidt *Subgroup lattices of groups*. de Gruyter Expositions in Mathematics, 14. Walter de Gruyter, 1994.

Department of Mathematics, Babeş-Bolyai University, Cluj-Napoca, 400084, Romania
Email address: calu@math.ubbcluj.ro

Department of Mathematics, Faculty of Mechanics and Mathematics, Tomsk State University, Tomsk, Russia
Email address: cheklov@math.tsu.ru