CONNECTIONS BETWEEN METRIC DIFFERENTIABILITY AND RECTIFIABILITY

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Abstract. We combine Kirchheim's metric differentials with Cheeger charts in order to establish a non-embeddability principle for any collection $\mathcal{C}$ of Banach (or metric) spaces: if a metric measure space X bi-Lipschitz embeds in some element in $\mathcal{C}$, and if every Lipschitz map $X \to Y \in \mathcal{C}$ is differentiable, then X is rectifiable. This gives a simple proof of the rectifiability of Lipschitz differentiability spaces that are bi-Lipschitz embeddable in Euclidean space, due to Kell–Mondino. Our principle also implies a converse to Kirchheim's theorem: if all Lipschitz maps from a domain space to arbitrary targets are metrically differentiable, the domain is rectifiable. We moreover establish the compatibility of metric and w*-differentials of maps from metric spaces in the spirit of Ambrosio–Kirchheim.

1. Introduction

Going beyond geometric measure theory in Euclidean space, *metric differentiability*, introduced by Kirchheim [27] has become an indispensable tool in studying rectifiability of metric spaces. A metric measure space $X = (X, d, \mu)$ is called n-rectifiable if $\mu \ll \mathcal{H}^n$ and $\mu\left(X \setminus \bigcup_{i=1}^{\infty} \psi_i(E_i)\right) = 0$ for some countable family of Lipschitz maps $\psi_i \in \mathrm{LIP}(E_i, X)$ defined on $\mathscr{L}^n$-measurable sets $E_i \subset \mathbb{R}^n$. Kirchheim [27] showed that every Lipschitz map $f : E \to X$ from a measurable set $E \subset \mathbb{R}^n$ into a metric space is *metrically differentiable*: for $\mathscr{L}^n$-a.e. $x \in E$ there exists a seminorm $\mathrm{md}_x f$ on $\mathbb{R}^n$ so that

$$d(f(y), f(z)) = \mathrm{md}_x f(y - z) + o(d(y, x) + d(x, z)). \tag{1.1}$$

As an illustration of their use, metric differentials give rise to area and co-area formulae, and can be used to show that the maps ψ_i in the definition of rectifiability can be taken to be bi-Lipschitz, see [27, 1]. Recently, Bate [5] has obtained a characterization of rectifiability in terms of Gromov-Hausdorff approximations, a much weaker condition than (bi-)Lipschitz maps.

Parallel developments in analysis on metric spaces lead to the seminal work of Cheeger [10] introducing what have come to be known as Lipschitz differentiability spaces (LDS for short). These are spaces covered by countably many *Cheeger charts*. A Cheeger chart is a pair (U, φ)

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consisting of a Borel set $U \subset X$ with $\mu(U) > 0$ and $\varphi \in \mathrm{LIP}(X, \mathbb{R}^n)$ such that every $f \in \mathrm{LIP}(X)$ is differentiable μ-a.e. on U with respect to (U, φ): for μ-a.e. $x \in U$ there exists a unique linear map $\mathrm{d}_x f \in (\mathbb{R}^n)^*$ so that

$$f(y) - f(x) = \mathrm{d}_x f(\varphi(y) - \varphi(x)) + o(d(x, y)). \tag{1.2}$$

Cheeger showed that spaces endowed with a doubling measure supporting some Poincaré inequality (in short, PI-spaces) admit a countable covering by such charts, thus establishing a Rademacher-type almost everywhere differentiability result for Lipschitz functions from a metric space. This has lead to a rich theory of Sobolev spaces and first order calculus on PI-spaces [21, 23, 8] independently of any rectifiability assumptions. Indeed, one of the motivations in [10] is to obtain non-embedding results for purely unrectifiable PI-spaces such as Carnot groups and Laakso spaces, see e.g [10, Theorem 14.2], which have since received much attention [11, 12, 13, 16]. The general principle is that differentiability and embeddability imply rectifiability, providing an obstruction for unrectifiable PI-spaces to admit bi-Lipschitz embeddings. The connection between PI-spaces, Cheeger differentiability and rectifiability has also been thoroughly explored in the works of [29, 6, 17]. For further connections to PDE's and uniform rectifiability, we refer the reader to [3, 2].

In this note we combine Cheeger's idea of differentiability charts with Kirchheim's notion of metric differential. Throughout the paper, a *chart* refers to a pair (U, φ) consisting of a Borel set $U \subset X$ with $\mu(U) > 0$ and a Lipschitz map $\varphi : X \to \mathbb{R}^n$. The number n is called the dimension of the chart. Below, $\mathcal{S}(\mathbb{R}^n)$ denotes the set of all seminorms on $\mathbb{R}^n$, equipped with the metric $\delta(s, s') = \sup_{|v| \le 1} |s(v) - s'(v)|$.

Definition 1.1. *Given a map $f : A \subset X \to Y$ into a metric space Y, we say that f admits a metric differential with respect to the chart (U, φ) if there exists a Borel map $\mathrm{md}\, f : A \cap U \to \mathcal{S}(\mathbb{R}^n)$ satisfying, for μ-a.e. $x \in A \cap U$,*

$$\limsup_{A \ni y \to x} \frac{|d_Y(f(y), f(x)) - \mathrm{md}_x f(\varphi(y) - \varphi(x))|}{d(x, y)} = 0. \tag{1.3}$$

See also [14] for an alternative approach to metric differentiation of mappings between metric spaces which relies on metric differentiation along curves, and [20] for an extension of Kirchheim's result to maps defined on strongly rectifiable metric spaces. The definition above covers maps from a subset $A \subset X$. The case $A = U$ is the metric analogue of *weak* Cheeger charts while, if $\mu(A \cap U) = 0$, the condition is vacuous. We also remark that our definition (see also [1]) is slightly weaker than Kirchheim's original definition, where $d(f(y), f(z))) = \mathrm{md}_x f(z - y) + o(d(x, y) + d(x, z))$.

Charts with respect to which every $f \in \mathrm{LIP}(U)$ admits a metric differential are weak Cheeger charts, see Proposition 3.1. Moreover, metric differentials are compatible with w^*-differentials, whenever both exist, see Section 5. Notice that we do not impose uniqueness of the metric differential in Definition 1.1. Indeed, uniqueness among seminorms is a much stronger requirement than uniqueness among linear maps, implying in particular the density of directions realised by φ. In Theorem 1.2 we will instead impose the a priori weaker condition

$$\mathrm{Lip}(v \cdot \varphi|_U) > 0 \ \mu\text{-a.e. on } U \text{ for any } v \in S^{n-1}, \tag{1.4}$$

which is equivalent to uniqueness of linear differentials, see [7, Lemma 2.1]. We remark that if (U, φ) is a weak Cheeger chart, then the density of directions holds (see [4, Lemma 9.1]) and thus, a posteriori, metric differentials are unique, see Section 3.

Our main result is a rectifiability criterion which relates metric differentiability of maps into a given target class to bi-Lipschitz embeddability in the same class. Our proof gives a conceptually simple argument covering several non-embeddability results known in the literature, see Corollary 1.3.

Theorem 1.2. *Let (U, φ) be an n-dimensional chart in X satisfying* (1.4)*, and $\mathcal{C}$ a collection of metric spaces so that some $Y \in \mathcal{C}$ contains a non-trivial geodesic. If every Lipschitz map $U \to Y \in \mathcal{C}$ admits a metric differential with respect to (U, φ), and U bi-Lipschitz embeds into some $Y \in \mathcal{C}$, then $(U, d, \mu|_U)$ is n-rectifiable.*

More precisely, $\mu|_U \simeq \mathcal{H}^n|_U$ and there are disjoint Borel sets U_i with $\mu(U \setminus \bigcup_i U_i) = 0$ so that $\varphi|_{U_i}$ is bi-Lipschitz for each i.

We make a few remarks:

1) A non-trivial geodesic in a metric space Z is an isometric embedding $\gamma : [a, b] \to Z$ for some $a < b$. The existence of one in some space $Z \in \mathcal{C}$ guarantees (under the hypotheses of Theorem 1.2) that all real valued Lipschitz functions admit a metric differential (see Lemma 4.1), a fact which self-improves to the existence of *linear* differentials (Proposition 3.1). We give an elementary argument in Section 3, but this can also be shown along the lines of [4].

2) We could alternatively require that every Lipschitz map $X \to Y \in \mathcal{C}$ admits a metric differential and that there exists $f \in \mathrm{LIP}(X, Y)$ such that $f|_U$ is bi-Lipschitz for some $Y \in \mathcal{C}$. In particular, if we assume that (X, Y) has the Lipschitz extension property for each $Y \in \mathcal{C}$, the claim of the theorem is true assuming metric differentiability of every Lipschitz map $X \to Y \in \mathcal{C}$.

3) A noteworthy consequence of the fact that (U, φ) is a weak Cheeger chart (Proposition 3.1) is that $\varphi_\#(\mu|_U) \ll \mathscr{L}^n$. This is by now well-known (see [18] for a proof) and follows from the deep result of De Philippis–Rindler [19]. Together with the bi-Lipschitz decomposition, this implies the mutual absolute continuity of $\mu|_U$ and $\mathcal{H}^n|_U$, and completes the proof of rectifiability of $(U, d, \mu|_U)$.

Now, we list some straightforward consequences of Theorem 1.2. Notice that part (a) of the following corollary provides a simpler proof of [26, Theorem 3.7].

Corollary 1.3.

(a) *If X is an LDS admitting a bi-Lipschitz embedding into a Euclidean space, then (X, d, μ) is rectifiable.*
(b) *More generally, let V be a Banach space. If every Lipschitz map $X \to V$ is metrically differentiable, and X bi-Lipschitz embeds into V, then X is rectifiable.*
(c) *If every Lipschitz map $f : X \to c_0$ is metrically differentiable, then X is rectifiable.*
(d) *If every Lipschitz map from X to an arbitrary target is metrically differentiable, then X is rectifiable.*

In particular, since non-Abelian Carnot groups and the Laakso space are purely unrectifiable RNP-Lipschitz differentiability spaces (see [1, 28, 12, 16]), they do not admit a bi-Lipschitz embedding into an RNP-Banach space. Note that (a), (b) and (d) are immediate consequences of Theorem 1.2 with $\mathcal{C} = \{\mathbb{R}^n\}, \{V\}, \{X \times \mathbb{R}\}$, respectively, and (c) follows readily from Theorem 1.2 and the fact that every separable metric space bi-Lipschitz embeds into c_0 [22, Theorem 3.12]. A similar conclusion holds with c_0 replaced by ℓ^∞ or $C[0, 1]$, since both contain isometric copies of every separable metric space. Since all three of these are non-RNP Banach spaces, not all Lipschitz maps into them can be (linearly) differentiable.

2. Preliminaries

Throughout this paper the triplet (X, d, μ) denotes a complete separable metric space endowed with a measure μ which is Borel regular and finite on bounded sets. In particular, μ is Radon (see [23, Corollary 3.3.47]).

For a mapping $f : (X, d_X) \to (Y, d_Y)$ between metric spaces we define the pointwise Lipschitz constant as

$$\operatorname{Lip} f(x) := \limsup_{y \to x} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

We denote by $\mathrm{LIP}(X, Y)$ the space of Lipschitz mappings and, for the particular case of $Y = \mathbb{R}$, we use the notation $\mathrm{LIP}(X)$.

In order to give a formal definition of rectifiability, we briefly recall the notion of *Hausdorff measure.* For $s > 0$, first fix $\delta > 0$ and consider, for any set $E \subset X$,

$$\mathcal{H}^s_\delta(E) := \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(E_i)^s \right\},$$

where the infimum is taken over all countable covers of E by sets $E_i \subset X$ with $\operatorname{diam}(E_i) < \delta$ for all $i \in \mathbb{N}$. Then the *s-dimensional Hausdorff measure* is defined as

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

Given two measures μ and ν, we say that μ is *absolutely continuous* with respect to ν (denoted $\mu \ll \nu$) if whenever $A \subset X$ such that $\nu(A) = 0$ then $\mu(A) = 0$.

Definition 2.1. (n-rectifiable space) *We say that (X, d, μ) is* n-rectifiable *if $\mu \ll \mathcal{H}^n$ and there exists a countable collection of Lipschitz maps $\psi_i : E_i \to X$ from $\mathscr{L}^n$-measurable sets $E_i \subset \mathbb{R}^n$ with $\mu\left(X \setminus \bigcup_{i=1}^{\infty} \psi_i(E_i)\right) = 0$.*

An *n-dimensional chart* on (X, d, μ) is a pair (U, φ) such that $U \subset X$ is Borel and $\varphi : X \to \mathbb{R}^n$ is Lipschitz.

Definition 2.2. (weak Cheeger chart) *We say that an n-dimensional chart (U, φ) on (X, d, μ) is a* weak Cheeger chart *if for every mapping $f \in \mathrm{LIP}(X)$ and $\mu-$almost every $x \in U$ there exists a unique linear map $\mathrm{d}_x f \in (\mathbb{R}^n)^*$ such that*

$$\limsup_{U \ni y \to x} \frac{|f(y) - f(x) - \mathrm{d}_x f(\varphi(y) - \varphi(x))|}{d(x, y)} = 0. \tag{2.1}$$

As mentioned in the introduction, if (2.1) holds without the restriction $y \in U$ then (U, φ) is called a *Cheeger chart.* If X can be decomposed, up to a μ-null set, into a countable union of (weak) Cheeger charts then X is called a (weak) *Lipschitz differentiability space*, or (weak) LDS in short. It is interesting to notice that, if porous sets in X have zero measure, a weak Cheeger chart is automatically a Cheeger chart (see [25, Remark 2.11] and [7, Proposition 2.8]). Also, a countable union of LDS's might not be an LDS (see [6] for an example), whereas countable unions of weak LDS are trivially a weak LDS.

In order to study mappings into a metric space where linearity is absent, a generalized definition of differentiability is necessary, and leads to the concept of *metric differential* introduced by Kirchheim in [27] (see Definition 1.1). In this scenario, the metric differential $\mathrm{md}_x f$ of a mapping $f : X \to Y$ cannot be a linear map, as defined in a (weak) Cheeger chart. Instead, it is substituted by a *seminorm* in $\mathbb{R}^n$, that is, a subadditive non-negative function. More precisely, the distances $d_Y(f(y), f(x))$ behave like a seminorm (equation (1.3)) instead of $f(y) - f(x)$ behaving linearly (equation (1.2)), in both cases up to a first order error.

In Section 5 we will also consider the notion of w^**-differential*, a weaker version of differentiability for maps taking values in duals to separable Banach spaces. Namely, if V is a separable Banach space, we can consider the w^**-topology* in V^*, that is,

$$w^* - \lim_{j\to\infty} w_j = w \iff \lim_{j\to\infty} \langle w_j, v\rangle = \langle w, v\rangle \quad \forall v \in V,$$

whenever $w_j, w \in V^*$. Here $\langle\cdot,\cdot\rangle$ denotes the standard duality $\langle v, w\rangle = w(v)$ for $v \in V$ and $w \in V^*$.

Definition 2.3. (w^*-differentiability) *Let (U, φ) be a chart and V a separable Banach space. Given a map $f : U \to V^*$, we say that f is* w^*-differentiable *with respect to the chart (U, φ) if for μ-a.e. $x \in U$ there exists a unique linear map $D_x f : \mathbb{R}^n \to V^*$ such that*

$$\limsup_{U\ni y\to x} \frac{|\langle v, f(y) - f(x) - D_x f(\varphi(y) - \varphi(x))\rangle|}{d(y, x)} = 0 \quad \textit{for all } v \in V. \tag{2.2}$$

3. Metric and linear differentials

Let (U, φ) be an n-dimensional chart satisfying (1.4). Note that for every $x \in U$ and any sequence $(x_j) \subset U \setminus \{x\}$ with $x_j \to x$, the Lipschitz property of φ gives that $\left(\frac{\varphi(x_j) - \varphi(x)}{d(x_j, x)}\right)$ has a convergent subsequence. We denote by $L(\varphi, x) \subset \mathbb{R}^n$ the set of limit points of sequences $\left(\frac{\varphi(x_j) - \varphi(x)}{d(x_j, x)}\right)$ with (x_j) as above.

From (1.4) it follows that $L(\varphi, x)$ spans $\mathbb{R}^n$ for μ-a.e. $x \in U$ (since $\mathrm{Lip}(v \cdot \varphi|_U)(x) = 0$ for any $v \perp L(\varphi, x)$). It moreover follows from (1.4) that a function can have at most one (linear) differential with respect to (U, φ), see [4, Lemma 3.3] and [7, Lemma 2.1]. If on the other hand $f \in \mathrm{LIP}(X)$ admits a metric differential with respect to (U, φ) in the sense of Definition 1.1 it follows that, for μ-a.e. $x \in U$, the metric differential $\mathrm{md}_x f$ is uniquely determined on $L(\varphi, x)$. In fact, if $w = \lim_{j\to\infty} \frac{\varphi(x_j) - \varphi(x)}{d(x_j, x)} \in L(\varphi, x)$, then

$$\mathrm{md}_x f(w) = \lim_{j\to\infty} \frac{\mathrm{md}_x f(\varphi(x_j) - \varphi(x))}{d(x_j, x)} = \lim_{j\to\infty} \frac{|f(x_j) - f(x)|}{d(x_j, x)}.$$

However if $\mathbb{R}L(\varphi, x)$ is not dense in $\mathbb{R}^n$, there may exist many seminorms s on $\mathbb{R}^n$ with $s|_{L(\varphi,x)} = \mathrm{md}_x f|_{L(\varphi,x)}$. This is in contrast with linear maps, which are uniquely determined by their values on a spanning set. The density of $\mathbb{R}L(\varphi, x)$ holds for weak Cheeger charts by [4, Lemma 9.1] whose proof uses Alberti representations.

Below we show that metric differentiability of *every* Lipschitz function self-improves to the existence of linear differentials.

Proposition 3.1. *Suppose (U,φ) is an n-dimensional chart satisfying* (1.4). *If every $f \in \mathrm{LIP}(U)$ admits a metric differential with respect to (U,φ), then (U,φ) is a weak Cheeger chart.*

Combining this with [4, Lemma 9.1] we have the following immediate corollary.

Corollary 3.2. *Under the hypotheses of Proposition 3.1, $\mathbb{R}L(\varphi,x)$ is dense in $\mathbb{R}^n$ for μ-a.e. $x \in U$.*

Proof of Proposition 3.1. Let $f \in \mathrm{LIP}(X)$. By assumption there exists a μ-null set $N \subset U$ such that $L(\varphi,x)$ spans $\mathbb{R}^n$ and there are seminorms $\mathrm{md}_x(f+w\cdot\varphi)$ satisfying

$$\limsup_{U\ni y\to x} \frac{||f(y)-f(x)+w\cdot(\varphi(y)-\varphi(x))| - \mathrm{md}_x(f+w\cdot\varphi)(\varphi(y)-\varphi(x))|}{d(y,x)} = 0$$

for every $x \in U\setminus N$ and $w \in \mathbb{Q}^n$. We fix $x \in U\setminus N$.

If $v = \lim_{j\to\infty}\frac{\varphi(x_j)-\varphi(x)}{d(x_j,x)} \in L(\varphi,x)$, and $a(f,x,v) = \lim_{j\to\infty}\frac{f(x_j)-f(x)}{d(x_j,x)}$ exists, then

$$|a(f,x,v)+w\cdot v| = \mathrm{md}_x(f+w\cdot\varphi)(v)$$

for all $w \in \mathbb{Q}^n$. If we consider now a different sequence $x'_j \to x$ such that $v = \lim_{j\to\infty}\frac{\varphi(x'_j)-\varphi(x)}{d(x'_j,x)}$ and $a'(f,x,v) = \lim_{j\to\infty}\frac{f(x'_j)-f(x)}{d(x'_j,x)}$ exists, then

$$|a'(f,x,v)+w\cdot v| = |a(f,x,v)+w\cdot v|$$

for all $w \in \mathbb{Q}^n$ and thus we conclude that $a'(f,x,v) = a(f,x,v)$. It follows that the map $L_f : L(\varphi,x) \to \mathbb{R}$ given by

$$L_f(v) := \lim_{j\to\infty}\frac{f(x_j)-f(x)}{d(x_j,x)}, \text{ whenever } v = \lim_{j\to\infty}\frac{\varphi(x_j)-\varphi(x)}{d(x_j,x)}$$

is well-defined and satisfies

$$\mathrm{md}_x(f+w\cdot\varphi)(v) = |L_f(v)+w\cdot v| \text{ for all } w \in \mathbb{Q}^n. \tag{3.1}$$

We prove that

(a) $L_f(tv) = tL_f(v)$ if $v, tv \in L(\varphi,x)$, and
(b) $L_f(v+v') = L_f(v)+L_f(v')$ if $v, v', v+v' \in L(\varphi)$.

If $v, v' \in L(\varphi,x)$ satisfy $v' = tv$, $t \in \mathbb{R}$, then

$$|L_f(v')+w\cdot v'| = \mathrm{md}_x(f+w\cdot\varphi)(v') = |t|\,\mathrm{md}_x(f+w\cdot\varphi)(v) = |t||L_f(v)+w\cdot v|.$$

Thus $|L_f(v')+tw\cdot v| = |tL_f(v)+tw\cdot v|$ for all $w \in \mathbb{Q}^n$, implying (a). Next suppose $v, v', v+v' \in L(\varphi,x)$. From (3.1) and the fact that $\mathrm{md}_x(f+w\cdot\varphi)$ is a seminorm we obtain that

$$|L_f(v+v')+w\cdot(v+v')| \le |L_f(v)+w\cdot v| + |L_f(v')+w\cdot v'|, \quad w \in \mathbb{Q}^n.$$

If $v' = -v$, then $L_f(v+v') = 0 = L_f(v)+L_f(v')$ by (a). Otherwise, there exists $w \in \mathbb{Q}^n$ such that $w\cdot z > 0$ for $z = v, v', v+v'$. By multiplying w by a sufficiently large positive number we find w^+ such that $L_f(z)+w^+\cdot z > 0$, $z = v, v', v+v'$. From the inequality above we obtain $L_f(v+v') \le L_f(v)+L_f(v')$. Similarly, by multiplying w by a suitably large (in absolute value) negative number we find w^- such that $L_f(z)+w^-\cdot z < 0$, $z = v, v', v+v'$, yielding $-L_f(v+v') \le -L_f(v)-L_f(v')$. These two inequalities together prove (b).

Since $L(\varphi,x)$ spans $\mathbb{R}^n$ and L_f satisfies (a) and (b), there exist a linear map $L : \mathbb{R}^n \to \mathbb{R}$ such that $L|_{L(\varphi,x)} = L_f$. It follows that L is the Cheeger differential of f at x with respect to (U,φ). Indeed, otherwise there would exist $\varepsilon_0 > 0$ and a sequence $U \ni x_j \to x$ with $v := \lim_{j\to\infty} \frac{\varphi(x_j)-\varphi(x)}{d(x_j,x)}$ such that

$$\varepsilon_0 \leq \frac{|f(x_j) - f(x) - L(\varphi(x_j) - \varphi(x))|}{d(x_j, x)} \xrightarrow{j\to\infty} |L_f(v) - L(v)|,$$

contradicting the fact that $L_f(v) = L(v)$. ■

4. Metric differential and rectifiability

Throughout this section, we fix an n-dimensional chart (U,φ) satisfying (1.4).

Lemma 4.1. *Suppose $\mathcal{C}$ is a collection of metric spaces containing a space with a non-trivial geodesic, and that every Lipschitz map $U \to Y \in \mathcal{C}$ admits a metric differential with respect to (U,φ). Then every $f \in \mathrm{LIP}(U)$ admits a metric differential with respect to (U,φ).*

Proof. Let $\gamma : [a,b] \to Y \in \mathcal{C}$ be a non-trivial geodesic, and let $h : \mathbb{R} \to (a,b)$ be a Lipschitz diffeomorphism. By assumption, for any $f \in \mathrm{LIP}(U)$ the map $\tilde{f} = \gamma \circ h \circ f$ admits a metric differential. Since $d_Y(\tilde{f}(y), \tilde{f}(x)) = |h(f(y)) - h(f(x))|$ for each $x, y \in U$, it follows that $h \circ f \in \mathrm{LIP}(U)$ admits a metric differential. However if $\mathrm{md}(h\circ f)$ denotes the metric differential of $h\circ f$, we have for μ-a.e. $x \in U$

$$\begin{aligned}
|f(y) - f(x)| &= |h^{-1}(h \circ f(y)) - h^{-1}(h \circ f(x))| \\
&= |(h^{-1})'(h \circ f(x))(h \circ f(y) - h \circ f(x))| + o(|h \circ f(y) - h \circ f(x)|) \\
&= |(h^{-1})'(h \circ f(x))| \,\mathrm{md}_x(h \circ f)(\varphi(y) - \varphi(x)) + o(d(y,x))
\end{aligned}$$

which implies that $\mathrm{md}\, f := |(h^{-1})' \circ h \circ f| \,\mathrm{md}(h \circ f)$ is a metric differential of f. This proves the claim. ■

We now give the proof of the main result.

Proof of Theorem 1.2. By hypothesis, there exists a bi-Lipschitz embedding $f : U \to Y$ of U into some $(Y, d_Y) \in \mathcal{C}$, which admits a metric differential $\mathrm{md}\, f$ with respect to (U,φ). Denote the bi-Lipschitz constant of f by L. In particular by (1.3) we have

$$\lim_{U\ni y\to x} \frac{|d_Y(f(x), f(y)) - \mathrm{md}_x f(\varphi(y) - \varphi(x))|}{d(x,y)} = 0 \tag{4.1}$$

for $\mu-$almost every $x \in U$. Let $N \subset U$ be a null set such that (4.1) holds for all $x \in U\backslash N$ and rewrite the limit in (4.1) as follows:

$$\lim_{j\to\infty} F_j(x) = 0 \quad \text{where} \quad F_j(x) := \sup_{y\in B(x,\frac{1}{j})\cap U} \frac{|d_Y(f(x), f(y)) - \mathrm{md}_x f(\varphi(x) - \varphi(y))|}{d(x,y)}.$$

By Egorov's theorem for every $\varepsilon > 0$ there exists a set $K \subset U \setminus N$ with $\mu(U \setminus K) < \varepsilon$ so that $\mathrm{md}_x f$ exists for every $x \in K$, and $F_j \to 0$ uniformly on K. Since μ is Radon we may further assume that K is compact. Let $j_0 \in \mathbb{N}$ be such that

$$\sup_{y\in B(x,\frac{1}{j_0})\cap K} \frac{|d_Y(f(x), f(y)) - \mathrm{md}_x f(\varphi(x) - \varphi(y))|}{d(x,y)} \leq F_{j_0}(x) \leq \frac{1}{2L}, \quad x \in K.$$

In particular, for any $x, y \in K$ with $d(x,y) < 1/j_0$ we have

$$\frac{1}{L}d(x,y) - \mathrm{md}_x f(\varphi(x)-\varphi(y)) \le d_Y(f(y),f(x)) - \mathrm{md}_x f(\varphi(x)-\varphi(y)) \le \frac{1}{2L}d(x,y)$$

from which we obtain

$$d(x,y) \le 2L\,\mathrm{md}_x\, f(\varphi(y)-\varphi(x)) \le 2LC|\varphi(x)-\varphi(y)|,$$

where $C := \sup_{|v|\le 1}\mathrm{md}_x\, f(v)$. Consider a covering of K by balls $\{B(x,\frac{1}{2j_0})\}_{x\in K}$. By compactness, there exist $x_1, x_2, \cdots, x_N$ such that $K \subset \bigcup_{i=1}^N B(x_i, \frac{1}{2j_0})$. Choose $x_0 \in \{x_1, x_2, \cdots, x_N\}$ such that $\mu(K \cap B(x_0, \frac{1}{2j_0})) > 0$, and define

$$A_k := \{x \in K : \frac{1}{k} \le \sup_{|v|\le 1}\mathrm{md}_x\, f(v) \le k\}.$$

Recall that f is the restriction of a bi-Lipschitz mapping on K, so that $\sup_{|v|\le 1}\mathrm{md}_x\, f(v) \neq 0$ for all $x \in K$, implying $\bigcup_{k=1}^\infty A_k = K$. Thus there exists $k \ge 1$ such that $\mu(A_k \cap K \cap B(x, \frac{1}{2j_0})) > 0$. Now, let $x, y \in A_k \cap K \cap B(x_0, \frac{1}{2j_0})$. Since $y \in A_k \cap K \cap B(x, \frac{1}{j_0})$ we have that

$$d(x,y) \le 2Lk|\varphi(x)-\varphi(y)|,$$

that is, φ is injective on $A_k \cap K \cap B(x_0, \frac{1}{2j_0})$ and φ^{-1} is $2Lk$-Lipschitz on $\varphi(A_k \cap K \cap B(x_0, \frac{1}{2j_0}))$. In particular, φ is bi-Lipschitz on $A_k \cap K \cap B(x_0, \frac{1}{2j_0})$. By [24, Proposition 3.1.1], there exists a countable decomposition

$$U = Z \cup \bigcup_i V_i,$$

where $\mu(Z) = 0$ and $\{V_i\}$ is a collection of mutually disjoint measurable sets such that $\varphi|_{V_i}$ is bi-Lipschitz for each i.

To finish the proof, we note that (U,φ) is a weak Cheeger chart of dimension n by Lemma 4.1 and Proposition 3.1. In particular $\varphi_\#(\mu|_U) \ll \mathscr{L}^n$ [18, Theorem 1.1]. Writing $\varphi_\#(\mu|_{V_i}) = \rho_i\mathscr{L}^n$ for each V_i as above, we obtain that

$$\mu|_{V_i} = (\rho_i \circ \varphi^{-1})\varphi_\#^{-1}(\mathscr{L}^n|_{\varphi(V_i)}) \simeq \mathcal{H}^n|_{V_i}.$$

Consequently $\mu|_U = \sum_i \mu|_{V_i} \simeq \sum_i \mathcal{H}^n|_{V_i} = \mathcal{H}^n|_U$. ■

Next we prove a "converse" [1] of Corollary 1.3 (d), namely that Lipschitz maps from a rectifiable space to arbitrary targets admit metric differentials. This result can be considered folklore, but we record the statement and its proof below for the reader's convenience.

Proposition 4.2. *Suppose (U,φ) be a n-dimensional chart in a metric measure space (X,d,μ) such that $\varphi|_U$ is bi-Lipschitz and $\mu|_U \ll \mathcal{H}^n$ Then, for any metric space Y, every Lipschitz map $f \in \mathrm{LIP}(U,Y)$ admits a metric differential with respect to (U,φ).*

Proof. Let $g = f \circ \varphi^{-1} : \varphi(U) \to Y$. Notice that g is a composition of Lipschitz mappings, so it is also Lipschitz. By Kirchheim's Rademacher Theorem [27, Theorem 2], for $\mathcal{H}^n$-almost every $z \in \varphi(U)$ there exists a unique seminorm $\mathrm{md}_z g$ on $\mathbb{R}^n$ such that

$$\lim_{\substack{y\to z\\ y\in\varphi(U)}} \frac{|d_Y(g(z),g(y)) - \mathrm{md}_z g(y-z)|}{|y-z|} = 0. \tag{4.2}$$

[1] Assuming porous sets in X have measure zero, if X is rectifiable, it can be decomposed into a countable union of charts with respect to which every $f : X \to Y$ admits a metric differential.

On the other hand, $g(\varphi(x)) = f(x)$ for each $x \in U$. Fix $x_0 \in U$ such that for $z_0 := \varphi(x_0)$ there exists a unique seminorm $\mathrm{md}_{z_0} g$ on $\mathbb{R}^n$ such that (4.2) holds. As φ is continuous, if $x \in U$ and $x \to x_0$, then $\varphi(x) \to z_0$. Therefore

$$\begin{aligned} &\lim_{\substack{x \to x_0 \\ x \in U}} \frac{|d_Y(f(x_0), f(x)) - \mathrm{md}_z g(\varphi(x) - \varphi(x_0))|}{d(x, x_0)} \\ \leq &\lim_{\substack{z \to z_0 \\ z = \varphi(x)}} \frac{|d_Y(g(z_0), g(z)) - \mathrm{md}_{z_0} g(z - z_0)|}{\frac{1}{C}|z - z_0|} = 0. \end{aligned}$$

where C is the Lipschitz constant of φ. Then $\mathrm{md}_{x_0} f := \mathrm{md}_{\varphi(x_0)} g$ is the metric differential of f at $x_0 \in U$. We finish the proof by noticing that, because $\varphi_\#(\mu_{|U}) \ll \mathcal{H}^n_{|\varphi(U)}$ and

$$\mathcal{H}^n(\{z_0 \in \varphi(U) : \mathrm{md}_{z_0} g \text{ does not exist}\}) = 0,$$

we conclude that

$$\mu(\{x_0 \in U : \mathrm{md}_{x_0} f \text{ does not exist}\}) = 0.$$

■

5. Metric and w^*-differentials

Differentiability of real valued Lipschitz functions gives rise to w^*-differentials of Lipschitz maps into the dual of a separable Banach spaces. This insight was made explicit[2] in [1], where it was shown that the w^*-differential is compatible with the metric differential of Kirchheim, see [1, Theorem 3.5]. In Propositions 5.1 and 5.2 below we establish the compatibility of metric and w^*-differentials in the setting of (weak) Cheeger charts.

Proposition 5.1. *Let (U, φ) be a weak Cheeger chart and V a separable Banach space. Given $f \in \mathrm{LIP}(U, V^*)$, f admits a w^*-differential $D_x f$ with respect to (U, φ) for μ-a.e. $x \in U$.*

Proof. Let $D \subset V$ be a countable dense vector space over $\mathbb{Q}$, and $N \subset U$ a μ-null set such that the unique differential $L_x(v) := \mathrm{d}_x\langle v, f\rangle \in (\mathbb{R}^n)^*$ of $\langle v, f\rangle$ with respect to (U, φ) exists for every $v \in D$ whenever $x \in U \setminus N$. We fix $x \in U \setminus N$. Since

$$\begin{aligned} \langle v + w, f(y) - f(x)\rangle &= \langle v, f(y) - f(x)\rangle + \langle w, f(y) - f(x)\rangle \\ &= (L_x(v) + L_x(w))(\varphi(y) - \varphi(x)) + o(d(x, y)), \end{aligned}$$

it follows by the uniqueness of the differential that $L_x(v + w) = L_x(v) + L_x(w)$ for $v, w \in D$. Similarly $L_x(av) = aL_x(v)$. These identities together with the estimate

$$\mathrm{Lip}(L_x(v) \circ \varphi|_U)(x) = \mathrm{Lip}(\langle v, f\rangle)(x) \leq \|v\| \operatorname{Lip} f(x)$$

show that $v \mapsto L_x(v)$ is a bounded linear map $D \to ((\mathbb{R}^n)^*, |\cdot|_x^*)$ and thus extends to a bounded linear map $L_x : V \to (\mathbb{R}^n)^*$. Here $|\lambda|_x^* = \mathrm{Lip}(\lambda \circ \varphi|_U)(x)$ is a norm on $(\mathbb{R}^n)^*$, in light of the fact that a weak Cheeger chart satisfies (1.4).

We denote by $D_x f : (\mathbb{R}^n, |\cdot|_x) \to V^*$ the adjoint operator ($|\cdot|_x$ the dual norm of $|\cdot|_x^*$) and note that it satisfies $\langle D_x f(z), v\rangle = \langle L_x(v), z\rangle$ for all $z \in \mathbb{R}^n$ and $v \in V$. To see that $D_x f$ is the w^*-differential of f, observe that if $v \in V$, we have

$$\mathrm{Lip}(\langle v, f - D_x f \circ \varphi|_U\rangle)(x) \leq \mathrm{Lip}(\langle v_i, f - D_x f \circ \varphi|_U\rangle)(x) + \|v_i - v\|_V(\operatorname{Lip} f(x) + \mathrm{Lip}(D_x f \circ \varphi|_U)(x))$$

[2]The authors state in [1] that w^*-differentiability of Lipschitz maps from $\mathbb{R}^n$ is a folklore result.

$$\leq \operatorname{Lip}(\langle v_i, f\rangle - L_x(v_i)\circ\varphi|_U)(x) + \|v_i - v\|_V(\operatorname{Lip} f(x) + \operatorname{Lip}(D_x f\circ\varphi|_U)(x))$$
$$= 0 + \|v_i - v\|_V(\operatorname{Lip} f(x) + \operatorname{Lip}(D_x f\circ\varphi|_U)(x))$$

for any $v_i \in D$. Taking $v_i \to v$ we obtain (2.2). ■

Proposition 5.2. *Suppose (U,φ) is a weak Cheeger chart and $f \in \operatorname{LIP}(U, V^*)$, where V is a separable Banach space. If f admits a metric differential* $\operatorname{md} f$ *with respect to (U,φ), then for μ-a.e. $x \in U$ we have $\operatorname{md}_x f(z) = \|D_x f(z)\|_{V^*}$ for all $z \in \mathbb{R}^n$.*

The proof is a modification of the argument in [1, Theorem 3.5], and uses curve fragments and Alberti representations. A *curve fragment* in X is a bi-Lipschitz map $\gamma : \operatorname{dom}(\gamma) \to X$ where $\operatorname{dom}(\gamma) \subset \mathbb{R}$ is compact, and the set $\operatorname{Fr}(X)$ of curve fragments in X is equipped with the topology arising from the Hausdorff metric on their graphs, see [4, Definition 2.1]. An *Alberti representation* $\mathcal{A} = \{\nu_\gamma, \mathbb{P}\}$ of a (Radon) measure ν on X consists of a finite positive measure $\mathbb{P}$ on $\operatorname{Fr}(X)$ and a family $\{\nu_\gamma\}$ of probability measures on X such that

(a) $\nu_\gamma \ll \mathcal{H}^1|_{\operatorname{Im}(\gamma)}$ $\mathbb{P}$-a.e. γ;
(b) $\gamma \mapsto \nu_\gamma(B)$ is $\mathbb{P}$-measurable and $\nu(B) = \displaystyle\int \nu_\gamma(B)\,\mathrm{d}\,\mathbb{P}(\gamma)$ for every Borel $B \subset X$.

Given $\varphi \in \operatorname{LIP}(X, \mathbb{R}^n)$, $z \in S^{n-1}$, $\varepsilon > 0$, and a cone $C(z,\varepsilon) := \{p \in \mathbb{R}^n : z\cdot p \geq (1-\varepsilon)|p|\}$, we say that the Alberti representation $\mathcal{A}$ is in the φ-direction of $C(z,\varepsilon)$ if $(\varphi\circ\gamma)'(t) \in C(z,\varepsilon)$ a.e. $t \in \operatorname{dom}(\gamma)$ for $\mathbb{P}$-a.e. $\gamma \in \operatorname{Fr}(X)$. Note that if $p \in C(z,\varepsilon)$, then $|z - p/|p|| < 2\varepsilon$. See [4, Section 2, Definition 5.7, and Definition 7.3] for the definition of independence, δ-speed and ξ-separation of Alberti representations used in the proof below. Alberti representations have the following very useful property. If $\Gamma_0 \subset \operatorname{Fr}(X)$ is $\mathbb{P}$-null, $E_\gamma \subset \operatorname{dom}(\gamma)$ is $\mathscr{L}^1$-null for each $\gamma \notin \Gamma_0$, and $\{(\gamma, t) : \gamma \notin \Gamma_0,\ t \in E_\gamma\} \subset \operatorname{Fr}(X)\times\mathbb{R}$ is $\mathbb{P}\times\mathscr{L}^1$-measurable, then for ν-a.e. $x \in X$ there exists $\gamma \notin \Gamma_0$ and $t \in \operatorname{dom}(\gamma)\setminus E_\gamma$ with $\gamma_t = x$, cf. [4, Proposition 2.9].

The following facts, which will be used in the proof pf Proposition 5.2, can be established as in the proof of Proposition 5.1. Let $U \subset X$ be a Borel set and $f \in \operatorname{LIP}(U, V^*)$. If $\mathcal{A}$ is an Alberti representation of $\mu|_U$, then the limit

$$(f\circ\gamma)'_t = w^* - \lim_{\operatorname{dom}(\gamma)\ni t'\to t} \frac{f(\gamma_{t'}) - f(\gamma_t)}{t' - t} \in V^*$$

exists for a.e. $t \in \operatorname{dom}(\gamma)$ for $\mathbb{P}$-a.e. γ. If (U,φ) is a weak Cheeger chart, then $D_{\gamma_t} f((\varphi\circ\gamma)'_t) = (f\circ\gamma)'_t$ for a.e. $t \in \operatorname{dom}(\gamma)$ for $\mathbb{P}$-a.e. γ. Finally, for $\mathbb{P}$-a.e. γ, we have that

$$\|D_{\gamma_t} f((\varphi\circ\gamma)'_t)\|_{V^*} = \lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} \chi_{\operatorname{dom}(\gamma)}(s)\|D_{\gamma_s} f((\varphi\circ\gamma)'_s)\|_{V^*}\,\mathrm{d}\,s$$

a.e. $t \in \operatorname{dom}(\gamma)$ since $t \mapsto \chi_{\operatorname{dom}(\gamma)}(t)\|D_{\gamma_t} f((\varphi\circ\gamma)'_t)\|_{V^*}$ is integrable for $\mathbb{P}$-a.e. γ, see [15, Theorem 1.4] and [9, Theorem 3.5].

Proof of Proposition 5.2. By passing to a subset we may assume that (U,φ) is a λ-structured chart and $\mu|_U$ has n ξ-separated Alberti representations with speed strictly greater than δ, for some numbers $\lambda, \xi, \delta > 0$. Let $z \in S^{n-1}$ and $\varepsilon > 0$. By [4, Theorem 9.5] $\mu|_U$ has an Alberti representation in the φ-direction $C(z,\varepsilon)$ with speed greater than $\tau = \tau(n,\lambda,\xi,\delta) > 0$. Thus, for μ-a.e. $x \in U$, there exists a curve fragment $\gamma : \operatorname{dom}(\gamma) \to X$ in the φ-direction of $C(z,\varepsilon)$ and with φ-speed at least τ, and $t \in \operatorname{dom}(\gamma)$ with $\gamma_t = x$, $(\varphi\circ\gamma)'_t \in C(z,\varepsilon)$, and $|(\varphi\circ\gamma)'_t| \geq \tau \operatorname{Lip}\varphi(x)|\gamma'_t|$

such that $\mathrm{md}_x f$, $D_x f, (\varphi\circ\gamma)'_t, (f\circ\gamma)'_t$ exist and satisfy $(f\circ\gamma)'_t = D_x f((\varphi\circ\gamma)'_t)$. Moreover, we may assume that

(1) $(f\circ\gamma)'_s = D_{\gamma_s} f((\varphi\circ\gamma)'_s)$ a.e. $s\in\mathrm{dom}(\gamma)$;

(2) we have $\|D_{\gamma_t} f((\varphi\circ\gamma)'_t)\|_{V^*} = \lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h}\chi_{\mathrm{dom}(\gamma)}(s)\|D_{\gamma_s} f((\varphi\circ\gamma)'_s)\|_{V^*}\,\mathrm{d}\,s$, and $\lim_{h\to 0^+}\frac{|[t,t+h]\cap\mathrm{dom}(\gamma)|}{h} = 1$.

Indeed, (1) and (2) can be assumed to hold by the discussion before the proof.

From the lower semicontinuity of the norm with respect to w^*-convergence we obtain

$$\|D_x f((\varphi\circ\gamma)'_t)\|_{V^*} = \|(f\circ\gamma)'_t\|_{V^*} \le \mathrm{md}_x\, f((\varphi\circ\gamma)'_t).$$

Denoting $z_\gamma := z - \frac{(\varphi\circ\gamma)'_t}{|(\varphi\circ\gamma)'_t|}$ we have $\|D_x f(z)\|_{V^*} \le \mathrm{md}_x\, f(z) + \|D_x f(z_\gamma)\|_{V^*} + \mathrm{md}_x\, f(z_\gamma)$. Since, for μ-a.e. $x\in U$, we have that for every $\varepsilon>0$ there exist γ and $t\in\mathrm{dom}(\gamma)$ as above with $|z_\gamma|<2\varepsilon$, we obtain the inequality $\|D_x f(z)\|_{V^*}\le \mathrm{md}_x\, f(z)$ for μ-a.e. $x\in U$.

We prove the opposite inequality. Let $f_\gamma : [a,b]\to V^*$ be the extension of $f\circ\gamma : \mathrm{dom}(\gamma)\to V^*$ to the smallest interval $[a,b]$ containing $\mathrm{dom}(\gamma)$ obtained by extending linearly into the gaps. Writing $[a,b]\setminus\mathrm{dom}(\gamma) = \bigcup_i (a_i,b_i)$, we have that $f'_\gamma(s) = \frac{f(\gamma_{b_i})-f(\gamma_{a_i})}{b_i-a_i}$, $s\in(a_i,b_i)$, so that $\|f'_\gamma\|_{V^*}\le \mathrm{LIP}(f\circ\gamma)$ on $[a,b]\setminus\mathrm{dom}(\gamma)$ and $f'_\gamma = (f\circ\gamma)' = D_\gamma f((\varphi\circ\gamma)')$ a.e. on $\mathrm{dom}(\gamma)$. For $v\in V$ with $\|v\|_V\le 1$ and $h>0$ we have

$$\Big\langle \frac{f(\gamma_{t+h})-f(\gamma_t)}{h}, v\Big\rangle = \frac{1}{h}\int_t^{t+h}\chi_{\mathrm{dom}(\gamma)}(s)\langle D_{\gamma_s} f((\varphi\circ\gamma)'_s), v\rangle\,\mathrm{d}\,s + \frac{1}{h}\int_t^{t+h}\chi_{\mathbb{R}\setminus\mathrm{dom}(\gamma)}(s)\langle f'_\gamma(s), v\rangle\,\mathrm{d}\,s.$$

Taking supremum over $v$with $\|v\|_V\le 1$ yields the estimate

$$\frac{\|f(\gamma_{t+h})-f(\gamma_t)\|_{V^*}}{h} \le \frac{1}{h}\int_t^{t+h}\chi_{\mathrm{dom}(\gamma)}(s)\|D_{\gamma_s} f((\varphi\circ\gamma)'_s)\|_{V^*}\,\mathrm{d}\,s + \mathrm{LIP}(f\circ\gamma)\frac{|[t,t+h]\setminus\mathrm{dom}(\gamma)|}{h}.$$

Letting $h\to 0^+$ and using (2) we obtain

$$\mathrm{md}_x\, f((\varphi\circ\gamma)'_t) = \lim_{h\to 0^+}\frac{\|f(\gamma_{t+h})-f(\gamma_t)\|_{V^*}}{h} \le \|D_x f((\varphi\circ\gamma)'_t)\|_{V^*}.$$

Thus $\mathrm{md}_x\, f(z) \le \|D_x f(z)\|_{V^*} + \mathrm{md}_x\, f(z_\gamma) + \|D_x f(z_\gamma)\|_{V^*}$, where $z_\gamma = z - \frac{(\varphi\circ\gamma)'_t}{|(\varphi\circ\gamma)'_t|}$ satisfies $|z_\gamma|<2\varepsilon$. Arguing as above we get $\mathrm{md}_x\, f(z)\le\|D_x f(z)\|_{V^*}$ μ-a.e. $x\in U$.

By choosing a countable dense set $D\subset\mathbb{R}^n$ it follows from the argument above that μ-a.e. $x\in U$ we have $\mathrm{md}_x\, f(z) = \|D_x f(z)\|_{V^*}$ for all $z\in D$. For such x, the equality holds for all $z\in\mathbb{R}^n$ by continuity and 1-homogeneity. This completes the proof. ■

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