GENERALIZATIONS OF FREE MONOIDS

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This paper is dedicated to Gracinda Gomes, colleague and friend, on the occasion of her retirement.

ABSTRACT. We generalize free monoids by defining k-monoids. These are nothing other than the one-vertex higher-rank graphs used in C^*-algebra theory with the cardinality requirement waived. The 1-monoids are precisely the free monoids. We then take the next step and generalize k-monoids in such a way that self-similar group actions yield monoids of this type.

1. INTRODUCTION

The goal of this paper is to generalize free monoids to higher dimensions. We make no pretence of novelty since the monoids considered in this paper are nothing other than the one-vertex higher-rank graphs with the usual cardinality restriction waived. Higher-rank graphs were introduced in [10] (formalizing some ideas to be found in [25]) and the monoids within this class have been considered by a number of authors, such as [7, 8]. They are well-known within the operator algebra community, but, we maintain, they should also be interesting to those working within semigroup theory. Whereas free monoids are concretely monoids of strings, our monoids will have elements that we can regard as 'higher-dimensional strings'; for example, in two dimensions our elements can be regarded as rectangles. Our generalization of free monoids is called k-monoids; the 1-monoids will turn out to be precisely the free monoids. Classes of k-monoids were studied in [18] where they were used to construct groups via inverse semigroups. This work is summarized in Section 4. Our point of view is that any result for free monoids should be generalized to k-monoids.

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2. FREE MONOIDS

You can read all about free monoids in Lallement's book [11, Chapter 5] but we shall go over what we need here. Our goal is to motivate the definition of k-monoids which will be given in the next section. Let A be any set, called in this context an *alphabet* and whose elements will be called *letters.* We do not need to assume that A is finite and, although it could be empty, that is not a very interesting case. By a *string* over A we mean a finite sequence of elements of A. We shall dispense with brackets and so strings shall simply be written as words over the alphabet A. The empty string is denoted by ε. The set of all strings over A is denoted by A^*. The set A^* becomes a semigroup with operation $\cdot$ when we combine strings via concatenation: thus $x \cdot y = xy$. This really makes A^* into a semigroup and, in fact, a monoid with identity ε. As usual, we shall omit explicit reference to the semigroup operation $\cdot$. Monoids that are isomorphic to the monoids A^* are called *free monoids*. Free monoids are even simpler than free groups since there are no pesky inverses to deal with. The elements of A are called the *generators* of the free monoid. The simplest interesting free monoids are those with exactly one generator; such monoids are isomorphic to the monoid $(\mathbb{N}, +)$. So, arbitrary free monoids can be regarded as non-commutative arithmetic. The case where A is empty is special: the free monoid on no generators is just the one-element monoid.

Free monoids have some important algebraic properties but to state these, we need some definitions. We say that a monoid S is *equidivisible* if $xy = uv$ in S implies there exists an element $t \in S$ such that either $u = xt$ and $y = tv$ or $x = ut$ and $v = ty$. If x is any string in a free monoid, denote its *length* by $|x|$; this simply counts the total number of letters of A occurring in x including multiplicities. Free monoids are *cancellative*: this means that if $xy = xv$ then $y = v$ and if $xy = uy$ then $x = u$. Now, suppose that $xy = uv$ in a free monoid A^* and that $|x| < |u|$, in the first instance. Then $u = xt$ for some string t. We therefore have that $xy = xtv$. It follows from cancellation that $y = tv$. The alternative is that $|x| \geq |u|$. Then $x = ut$, for some string t thus $uty = uv$. It follows from cancellation that $ty = v$ This proves that free monoids are equidivisble.

Free monoids come equipped with a monoid homomorphism δ which associates with a string its length. Thus, there is a monoid homomorphism $\delta \colon A^* \to \mathbb{N}$ given by $\delta(x) = |x|$. It is possible to characterize free monoids by means of the properties of the monoid homomorphism δ. The following [11, Corollary V.1.6] was first proved by F. W. Levi in [23].

Theorem 2.1 (Levi)**.** *A monoid S is free if and only if S is equidivisible and there exists a homomorphism $\theta \colon S \to \mathbb{N}$ such that $\theta^{-1}(0)$ is the identity of S.*

We shall use the above theorem to obtain a new characterization of free monoids that will motivate this paper. Let $x \in A^*$. Then $|x| \in \mathbb{N}$. Suppose that $m, n \in \mathbb{N}$ such that $m + n = |x|$. Then there are unique elements $u, v \in A^*$ such that $x = uv$ where $|u| = m$ and $|v| = n$. To see that this is true, just remember that the elements of the free monoid are strings.

More generally, we say that a monoid S has the *unique factorization property (UFP)* if it is equipped with a monoid homomorphism $\theta \colon S \to \mathbb{N}$ such that if $\theta(a) = m + n$ then there are unique elements $b, c \in S$ such that $a = bc$ where $\theta(b) = m$ and $\theta(c) = n$. We shall derive a couple of results about such monoids.

Lemma 2.2. *Let S be a monoid and let $\theta \colon S \to \mathbb{N}$ be a monoid homomorphism that satisfies the (UFP). Then S is equidivisible.*

Proof. Suppose that $xy = uv$. We shall compare $\theta(x)$ with $\theta(u)$. The set $\mathbb{N}$ is linearly ordered, so that either $\theta(x) < \theta(u)$ or $\theta(x) \geq \theta(u)$. Suppose first that $\theta(x) < \theta(u)$. Then, we can write $\theta(u) = \theta(x) + n$, for some natural number n. By the (UFP), we can write $u = xt$ where $t \in S$ is the unique element such that $\theta(t) = n$. It follows that $xy = xtv$. Take θ of both sides, and use basic algebra, to get that $\theta(y) = \theta(t) + \theta(v)$. By the (UFP), it follows that $y = tv$. the case where $\theta(x) \geq \theta(u)$ can be handled similarly. $\square$

Lemma 2.3. *Let S be a monoid and let $\theta \colon S \to \mathbb{N}$ be a monoid homomorphism that satisfies the (UFP). Then the identity of S is the only element that maps to 0.*

Proof. Because θ is a monoid homomorphism, we have that $\theta(1) = \mathbf{0}$. Now, let $e \in S$ be such that $\theta(e) = \mathbf{0}$. Then $e = 1e = e1$, since 1 is the identity. But $\theta(e) = \mathbf{0} + \mathbf{0}$. By the (UFP), we may write $e = e_1 e_2$ uniquely where $\theta(e_1) = \mathbf{0}$ and $\theta(e_2) = \mathbf{0}$. But $e = 1e = e1$, also. It follows that $e_1 = 1$ and $e_2 = 1$. We deduce that $e = 1$. $\square$

We can now provide a different characterization of free monoids which follows immediately by Theorem 2.1, Lemma 2.2 and Lemma 2.3.

Theorem 2.4. *The monoids that satisfy the unique factorization property are precisely the free monoids.*

The above theorem directly motivates the definition of the next section.

There are a couple of further definitions that will be useful to us later. Let A^* be a free monoid. We say that strings x and y are *(prefix) incomparable* if $xA^* \cap yA^* = \varnothing$; otherwise, we say they are *(prefix) comparable*. A subset $X \subseteq A^*$ is said to be a *prefix code* if its elements are prefix incomparable. A *maximal prefix code* can be characterized as a prefix code with the property that every element of A^* is prefix comparable with an element of the prefix code [1]. By [2], every right ideal of A^* is generated by a prefix code. The right ideal is essential (a term we shall define below) precisely when it is generated by a maximal prefix code.

3. k-monoids

The previous section set the scene for what we shall do in this section: essentially, we shall replace $\mathbb{N}$ by $\mathbb{N}^k$. We shall need a little notation first. The monoid $\mathbb{N}^k$ should be viewed as the positive cone of the lattice-ordered abelian group $\mathbb{Z}^k$. If $\mathbf{m} \in \mathbb{N}^k$ then

$$\mathbf{m} = (m_1, \ldots, m_i, \ldots, m_k)$$

and we define $\mathbf{m}_i = m_i$. The partial order in $\mathbb{Z}^k$ is defined componentwise: $\mathbf{m} \leq \mathbf{n}$ if and only if $\mathbf{m}_i \leq \mathbf{n}_i$ for $1 \leq i \leq k$. The join operation is $(\mathbf{m} \vee \mathbf{n})_i = \max(m_i, n_i)$ and the meet operation is $(\mathbf{m} \wedge \mathbf{n})_i = \min(m_i, n_i)$. Put $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$ both elements of $\mathbb{N}^k$. Define $\mathbf{e}_i$, where $1 \leq i \leq k$, to be that element of $\mathbb{N}^k$ which is zero everywhere except at i where it takes the value 1.

Definition. A monoid S is said to be a *k-monoid* if there is a monoid homomorphism $\delta \colon S \to \mathbb{N}^k$ satisfying the *unique factorization property (UFP)*: if $\delta(x) = \mathbf{m} + \mathbf{n}$ then there exist unique elements x_1 and x_2 of S such that $x = x_1 x_2$ where $\delta(x_1) = \mathbf{m}$ and $\delta(x_2) = \mathbf{n}$. We call $\delta(a)$ the *size* of a.

Using the terminology we have introduced, we proved in the previous section that the 1-monoids are precisely the free monoids. Thus k-monoids really do generalize free monoids. However, the direct product of free monoids is not usually free. One particularly pleasant feature of k-monoids is that they are closed under finite direct products. To see this, let S be a k-monoid and let T be an l-monoid. We shall denote their respective monoid homomorphisms by $\delta_S \colon S \to \mathbb{N}^k$ and $\delta_T \colon T \to \mathbb{N}^l$. There is a natural isomorphism $\mu \colon \mathbb{N}^k \times \mathbb{N}^l \to \mathbb{N}^{k+l}$ where we take the ordered pair $((m_1, \ldots, m_k), (n_1, \ldots, n_l))$ to the single $(k+l)$-tuple $(m_1, \ldots, m_k, n_1, \ldots, n_l)$. Define $\delta \colon S \times T \to \mathbb{N}^{k+l}$ by $\delta(s,t) = \mu(\delta_S(s), \delta_T(t))$. In this way, it is easy to prove that $S \times T$ is a $(k+l)$-monoid. We have therefore established the following [10, Proposition 1.8].

Lemma 3.1. *If S is a k-monoid and T is an l-monoid then $S \times T$ is a $k+l$-monoid.*

Example 3.2. The direct product of k free monoids is therefore a k-monoid by Lemma 3.1. However, the product of free monoids is not free in general. For example, the free monoid on one generator is $\mathbb{N}$. But $\mathbb{N} \times \mathbb{N}$ is not isomorphic to the free monoid on one generator but it is abelian. Thus, it cannot be free.

We may therefore easily construct examples of k-monoids for any finite k. We say that a monoid S is *singly aligned* if $aS \cap bS \neq \varnothing$ implies that $aS \cap bS = cS$ for some $c \in S$. The proof of the following is easy.

Lemma 3.3. *The product of two singly aligned monoids is singly aligned.*

It follows that a direct product of k free monoids is always singly aligned. This tells us that these are special kinds of k-monoids.

The following portmanteau lemma (proved in [10]) summarizes some of the important algebraic properties of k-monoids. Recall that a monoid is said to be *conical* if its group of units is trivial.

Lemma 3.4. *Let S be a k-monoid with identity 1 and with monoid homomorphism $\delta \colon S \to \mathbb{N}^k$.*

(1) S is cancellative.
(2) $\delta(1) = \mathbf{0}$ and is the only element that is mapped by δ to $\mathbf{0}$.
(3) S is conical.

Proof. (1) Let $z = xy = xv$. We have that $\delta(y) = \delta(v)$ and so, by the (UFP), we have that $y = v$. Now, suppose that $xy = uy$. We apply the (UFP) again to deduce that $x = u$.

(2) We have that $11 = 1$ and so $\delta(11) = \delta(1) + \delta(1) = \delta(1)$. It follows that $\delta(1) = \mathbf{0}$. Now, suppose that $\delta(a) = \mathbf{0}$ where $a \in S$. Then $a = a_1 a_2$ where $\delta(a_1) = \mathbf{0} = \delta(a_2)$. But $a = a1 = 1a$. We have that $\delta(a) = \mathbf{0} + \mathbf{0}$. We deduce that $a = 11 = 1$.

(3) Suppose that $xy = 1$. Then $\delta(x) = \delta(y) = \mathbf{0}$. By part (2) above both x and y is equal to the identity. $\square$

The biggest difference between k-monoids, where $k \geq 2$, and free monoids lies in the fact that whereas the set $\mathbb{N}$ is linearly ordered the set $\mathbb{N}^k$ is not. Here is the appropriate analogue of Theorem 2.1.

Lemma 3.5. *Let S be a k-monoid. Let $xy = uv$ and suppose that $\delta(x) \geq \delta(u)$. Then there is $t \in S$ such that $x = ut$ and $v = ty$. In particular, if $\delta(x) = \delta(u)$ then $x = y$.*

Proof. Put $z = xy = uv$. There exists $\mathbf{r} \in \mathbb{N}^k$ such that $\delta(x) = \delta(u) + \mathbf{r}$. By the (UFP), we have that $x = u't$ where $\delta(u) = \delta(u')$ and $\delta(t) = \mathbf{r}$. We therefore have that $u'ty = uv$. We now apply the (UFP) again, to deduce that $u' = u$ and $v = ty$. If $\delta(x) = \delta(u)$ then we get $x = u$ from $x = ut$. $\square$

We return briefly to 1-monoids. Let S be a free monoid on the alphabet A. The associated monoid homomorphism $\delta \colon S \to \mathbb{N}$ is always surjective except in the case where A is empty. This can be generalized.

Lemma 3.6. *Suppose that $\delta \colon S \to \mathbb{N}^k$ is a k-monoid. If δ is not surjective then there exists a monoid homomorphism $\delta' \colon S \to \mathbb{N}^{k-1}$ such that S is a $(k-1)$-monoid.*

Proof. Suppose that there exists $a \in S$ such that $\delta(a)$ does not have a zero appear in any component. Thus $\delta(a) = (m_1, \ldots, m_k)$ where $m_i > 0$ for all $1 \leq i \leq k$. Then, for any positive natural number n, we have that $\delta(a^n) = (nm_1, \ldots, nm_k)$. Let $\mathbf{n} = (n_1, \ldots, n_k)$ be any element of $\mathbb{N}^k$. Choose n big enough so that $n_i \leq nm_i$ for all $1 \leq i \leq k$. It then follows by the (UFP), that every element of $\mathbb{N}^k$ is in the image of δ. Thus δ is surjective. It follows that there exists an element of $\mathbf{m} \in \mathbb{N}^k$ which is not in the image of δ and which has a component which is zero. By permutating the components if necessary, we can assume, without loss of generality, that it is the kth component of $\mathbf{m}$ which is 0. Observe that no element in the image of δ can have a non-zero entry in the kth position. Suppose to the contrary that $\delta(a)$ has a non-zero entry in the kth position. Then, for some n, we have that $\delta(a^n) \geq \mathbf{m}$. By the (UFP), there is an element $b \in S$ such that $\delta(b) = \mathbf{m}$. This is a contradiction. Define $\delta' \colon S \to \mathbb{N}^{k-1}$ to be δ followed by the map from

$\mathbb{N}^k \to \mathbb{N}^{k-1}$ given by $(n_1, \ldots, n_{k-1}, n_k) \mapsto (n_1, \ldots, n_{k-1})$. Then δ' is a monoid homorphism and satisfies the (UFP). $\square$

An *atom* in a monoid is an element a such that if $a = bc$ then at least one of b or c is invertible.

Lemma 3.7. *Let S be a k-monoid.*

(1) The atoms in S are the elements a where $\delta(a) = \mathbf{e}_i$.
(2) Every non-identity element is a product of atoms

Proof. (1) Suppose that a is such that $\delta(a) = \mathbf{e}_i$. We prove that a is an atom. Suppose that $a = bc$. Then $\delta(a) = \delta(b) + \delta(c)$. But $\delta(a) = \mathbf{e}_i$. Thus either $\delta(b) = \mathbf{0}$ or $\delta(c) = \mathbf{0}$. It follows that either b is invertible or c is invertible (in fact, the identity).

(2) Let a be a non-identity element. Then we may write $\delta(a) = m_1\mathbf{e}_1 + \ldots + m_k\mathbf{e}_k$ where $m_1, \ldots, m_k$ are natural numbers. Using the fact that $m_1\mathbf{e}_1 = \mathbf{e}_1 + \ldots + \mathbf{e}_1$ and so on and the (UFP), we may now use (1) and deduce that every non-identity element is a product of atoms. $\square$

In free monoids A^*, the atoms are nothing other than the letters. But, whereas free monoids have one alphabet, k-monoids have k, which we now define. Fix k and let $1 \leq l \leq k$. Define $\pi_l \colon \mathbb{N}^k \to \mathbb{N}$ by $(m_1, \ldots, m_l, \ldots, m_k) \mapsto m_l$. This is a monoid homomorphism. Let $\delta \colon S \to \mathbb{N}^k$ be a k-monoid. Define S_l to consist of all elements $a \in S$ such that $\delta(a)$ has 0's eveywhere except possibly at the lth component. The set S_l is non-empty since it must contain the identity. Thus S_l is clearly a submonoid of S. Observe that $\delta_l = \pi_l\delta \colon S_l \to \mathbb{N}$ shows that S_l is a 1-monoid, and so free. If $s \in S$ then $\delta(s) = m_1\mathbf{e}_1 + \ldots + m_k\mathbf{e}_k$. We may write $S = S_1 \ldots S_l$ uniquely. It follows that k-monoids are constructed from k free monoids. In the case $k = 2$, we can construct 2-monoids from certain Zappa-Szép products [4] of free monoids. For each $1 \leq l \leq k$, define $X_l = \delta^{-1}(\mathbf{e}_l)$. We call $(X_1, \ldots, X_k)$ the k *alphabets* associated with the k-monoid S. Each set X_l (which could be empty) is a set of free generators of the free monoid S_l.

The proof of the following lemma is immediate by the (UFP); it shows that in a k-monoid there are certain relationships between the atoms.

Lemma 3.8. *Let S be a k-monoid with the above notation. If a and b are atoms with $a \in X_i$ and $b \in X_j$, where $i \neq j$, then there exist unique atoms $a' \in X_i$ and $b' \in X_j$ such that $ab = b'a'$.*

It follows immediately from the above lemma, that $X_iX_j \subseteq X_jX_i$ for all $i \neq j$. An extreme case of the above lemma leads to finite direct products of free monoids.

Lemma 3.9. *Let S be a k-monoid with k alphabets $(X_1 \ldots, X_k)$. Suppose that $a \in X_i$ and $b \in X_j$, where $i \neq j$, implies that $ab = ba$. Then S is isomorphic to a finite direct product of free monoids.*

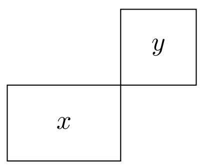
Proof. It is enough to prove that $S \cong S_1 \times \ldots \times S_k$. For each $s \in S$ let $s = u_1 \ldots u_k$ be the unique representation of s as a product of elements on S_i where $1 \leq i \leq k$. We define $\theta(s) = (u_1, \ldots, u_k)$. Clearly, this is a bijection. Suppose that $t = v_1 \ldots v_k$. Then $st = u_1 \ldots u_k v_1 \ldots v_k$. By our assumption, v_1 commutes with all the elements $u_2, \ldots, u_k$. Thus we may write $st = (u_1v_1)u_2 \ldots u_kv_2 \ldots v_k$. Repeating, we obtain that $st = (u_1v_1)(u_2v_2) \ldots (u_kv_k)$. This calculation shows that θ is a homomorphism. $\square$

The following describes the 'degenerate' k-monoids.

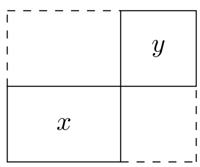
Proposition 3.10. *Let S be a k-monoid whose associated monoid homomorphism is surjective and with associated k-alphabets $(X_1, \ldots, X_k)$. Then $S \cong \mathbb{N}^k$ if and only if $|X_1| = |X_2| = \ldots = |X_k| = 1$.*

Proof. Only one direction needs proving. Suppose that $|X_1| = |X_2| = \ldots = |X_k| = 1$. Then $X_i^* \cong \mathbb{N}$. The homomorphism $d \colon S \to \mathbb{N}^k$ is already surjective and it is injective by the (UFP). $\square$

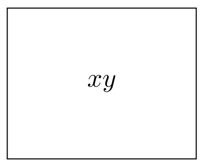
We shall now obtain a more geometric way of thinking about the elements of k-monoids. We start with free monoids. Let A be a (non-empty) alphabet. We shall now regard elements of A as directed (to the right) line segments of unit length labelled by an element of A. A (non-empty) string a over A can now be regarded as the concatenation of $|a|$ such directed line segments each of length 1. Thus the total length of the line segment that results is $|a|$. We therefore regard the elements of A^* as being labelled lines. Suppose now that $k = 2$. Let S be a 2-monoid. There are two alphabets X_1 and X_2 and each element of $a \in S$ can be written uniquely as $a = uv$ where $u \in X_1^*$ and $v \in X_2^*$. It is therefore tempting to view a as consisting of two strings over different alphabets. But we shall argue that it makes more sense to regard a as a *rectangle* with area $m \times n$ where $\delta(a) = (m, n)$. Observe, first, that any representation of a as a product of atoms must contain exactly m atoms from X_1 and exactly n atoms from X_2. None of these ways of representing a is privileged. Represent elements of X_1 by horizontal line segments directed to the right and represent elements of X_2 by vertical line segments directed upwards. Then each way of representing an element a can be regarded as a sequence of directed line segments some horizontal and some vertical. These can be labelled by elements of $X_1 \cup X_2$. We therefore obtain an $m \times n$ grid in which the horizontal line segments are labelled by elements of X_1 and the vertical line segments are labelled by elements of X_2. We can use this geometrical way of regarding the elements of k-monoids to demonstrate how the multiplication works. We are given x and y to calculate xy. We place the top right-hand corner of x against the bottom left-hand corner of y:



We do not yet have a rectangle, so we use the (UFP) to fill in the gaps



and now we get the result xy represented as a rectangle

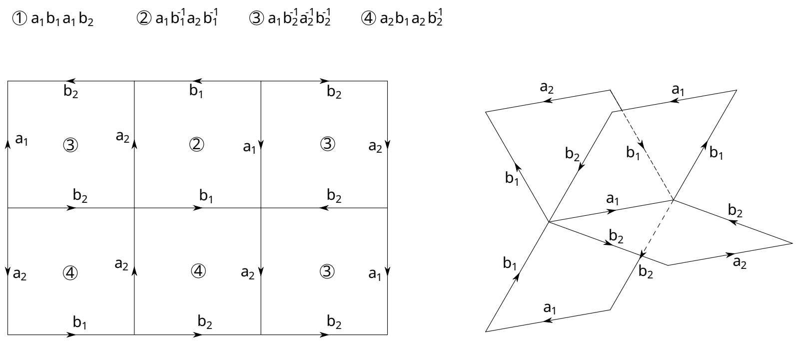


What we have said for $k = 2$ applies to any value of k. We can therefore regard the elements of a k-monoid as being k-boxes. It is in this way, that we can regard k-monoids as being *higher dimensional free monoids*. It is therefore natural to view the elements of k-monoids as being k-strings.

A good source of examples of k-monoids can be obtained by considering simply transitive groups actions on products of trees, as constructed in [29] and [26].

Proposition 3.11. *Let $\mathcal{B}$ be a product of k trees equipped with a simply transitive action of a group G. Then the rectangular subsets of apartments in $\mathcal{B}$, decorated by the action of G, are k-strings.*

Example 3.12. The group with relations 1,2,3,4 below acts on a product of two trees. The relations can be viewed as two-dimensional letters, where each relation gives four letters.



Example 3.13. Let

$$X_1 = \{a, a_1, a_2, a_3\}, \quad X_2 = \{b, b_1, b_2, b_3, b_4\}, \text{ and } X_3 = \{c, c_1, c_2, c_3\}.$$

We suppose that the following relations are satisfied: $ab = b_1 a_1$, $bc = c_2 b_2$, $a_1 c = c_1 a_2$, $ac_2 = c_3 a_3$, $a_3 b_2 = b_3 a_2$ and $b_1 c_1 = c_3 b_4$, with all other relations being of the form $xy = yx$ where $x \in X_i$ and $y \in X_j$ and $i \neq j$. Let S be the monoid generated by the set $X_1 \cup X_2 \cup X_3$ subject to the above relations. We shall prove that S is not a 3-monoid. We calculate abc in two ways. First, $abc = (ab)c = b_1 a_1 c = b_1 c_1 a_2$. Second, $abc = a(bc) = ac_2 b_2 = c_3 a_3 b_2$. By associativity, we must have that $b_1 c_1 a_2 = c_3 a_3 b_2$. Now $(b_1 c_1) a_2 = c_3 b_4 a_2 = c_3 a_2 b_4$. In S, we therefore have that $c_3 a_3 b_2 = c_3 a_2 b_4$. If S were a k-monoid, it would be cancellative. This would imply that $a_3 b_2 = a_2 b_4$. But this contradicts the fact that in a 3-monoid such a factorization would have to be unique.

There is one other property that will be crucial when we come to describe presentations of k-monoids.

Lemma 3.14. *Let S be a k-monoid with k alphabets $(X_1, \ldots, X_k)$. Let X_i, X_j and X_k be distinct alphabets. Let $f \in X_i$, $g \in X_j$ and $h \in X_k$. Suppose that $fg = g^1 f^1$, $f^1 h = h^1 f^2$, $g^1 h^1 = h^2 g^2$, $gh = h_1 g_1$, $fh_1 = h_2 f_1$ and $f_1 g_1 = g_2 f_2$. Then $f^2 = f_2$, $g^2 = g_2$, and $h^2 = h_2$.*

We shall now work quite generally although motivated by the Lemmas 3.8 and 3.14. Let X be any non-empty set partitioned into k-blocks $X_1, \ldots, X_k$, each of which is assumed non-empty. We shall refer to the elements of block X_i as having the 'colour i'. For each ordered pair $(a,b) \in X_i \times X_j$, where $i \neq j$ (thus a and b have different colours), we suppose that we are given a unique ordered pair $(b',a') \in X_j \times X_i$. Put R equal to the binary relation on X^* which contains precisely the ordered pairs $(ab, b'a')$ and $(b'a', ab)$. We call such a binary relation R a *complete set of squares over* X. We need an extra condition on R. Suppose that $(ab, b^1a^1), (a^1c, c^1a^2), (b^1c^1, c^2b^2) \in R$ and also $(bc, c_1b_1), (ac_1, c_2a_1), (a_1b_1, b_2a_2) \in R$ together imply that $a^2 = a_2$, $b^2 = b_2$ and $c^2 = c_2$. We call this the *asociativity condition.*

The following result was proved in the more general setting of category theory in [9]. It gives a monoid presentation of k-monoids with surjective monoid homomorphisms.

Theorem 3.15. *Let X be any non-empty set partitioned into k-blocks $X_1, \ldots, X_k$. Let R be any complete set of squares over X that satisfies the associativity condition. Let ρ be the congruence generated by R. Then $S = X^*/\rho$ is a k-monoid, and every k-monoid (whose associated monoid homomorphism is surjective) is isomorphic to one of this type. Specifically, the quotient map from X^* to S is injective on X, and so we may identify X with its image in S. There is a unique monoid homomorphism $\delta \colon S \to \mathbb{N}^k$ such that $\delta(a) = \mathbf{e}_i$ whenever $a \in X^i$.*

4. How to construct Thompson-Higman type groups

The main goal of this section is to explain how to construct groups from certain kinds of k-monoids. This work goes back to a paper by Birget [2], was developed using free monoids in [14, 15] and then generalized to classes of k-monoids in [18, 20]. The paper [17] provides a retrospective on the older work. In the papers [14, 15], it was shown that the Thompson group $G_{n,1}$ arises from the free monoid on n-generators. In unpublished work, John Fountain described how the Thompson groups might be generalized but a suitable generalization of free monoids was lacking. A class of k-monoids provides just what we are looking for.

The intersection of principal right ideals in a free monoid, if non-empty, is always a principal right ideal. More generally, we say that a k-monoid S is *finitely aligned* if the intersection $aS \cap bS$ is either empty or finitely generated as a right ideal. The notion of a monoid being finitely aligned was first defined by Gould [6], and then further studied in [5]. Independently, it became very important in the theory of higher-rank graphs and their C^*-algebras [24, 27, 28]. It will be useful to have some definitions centred around principal right ideals. If $xS \cap yS \neq \varnothing$, we say that x and y are *comparable* whereas if $xS \cap yS = \varnothing$, we say that x and y are *incomparable.* Incomparable subsets of k-monoids are analogues of prefix codes in free monoids, so we call incomparable sets *generalized prefix codes.* Let $a \in S$. We say that it is *dependent on* a set X, if $au = xv$ for some $x \in X$ and $u, v \in S$. A generalized prefix code X is said to be *maximal* if every element of S is dependent on an element of X. In free monoids, maximal prefix codes are precisely those prefix codes maximal in the above sense.

We investigate first the intersection of principal right ideals in complete generality. The following is a folklore result known to many working on higher-rank graphs. It is therefore important but not original.

Lemma 4.1. *Let S be a k-monoid. Assume that a and b are comparable and let $c \in aS \cap bS$. Then there exists an element $d \in S$ such that $c = dt$, for some $t \in S$, where $d \in aS \cap bS$ and $\delta(d) = \delta(a) \vee \delta(b)$.*

Proof. By assumption, $c = ax = by$ for some $x, y \in S$. It follows that $\delta(c) \geq \delta(a), \delta(b)$. Thus $\delta(c) \geq \delta(a) \vee \delta(b)$. We may therefore write $\delta(c) = (\delta(a) \vee \delta(b)) + \mathbf{m}$ for some $\mathbf{m} \in \mathbb{N}^k$. Thus by the (UFP), we may write $c = dt$ where $\delta(d) = d(a) \vee d(b)$

and $\delta(t) = \mathbf{m}$. Now $c = dt = ax$. But $\delta(d) = \delta(a) \vee \delta(b) = \delta(a) + \mathbf{n}$ for some $\mathbf{n} \in \mathbb{N}^k$. By the (UFP), we may therefore write $d = a'u$ where $\delta(a') = \delta(a)$ and $\delta(u) = \mathbf{n}$. Thus $a'ut = ax$. It now follows by the (UFP), that $a = a'$. We have therefore proved that $d \in aS$. We may similarly prove that $d \in bS$. $\square$

The proof of the following is now immediate by the above.

Corollary 4.2. *Let S be a k-monoid. Suppose that $aS \cap bS \neq \varnothing$. Then $aS \cap bS = \bigcup_{d \in aS \cap bS,\, \delta(d) = \delta(a) \vee \delta(b)} dS$.*

Our goal is to construct a group from a suitable k-monoid. We shall do this by going via inverse monoids. But there is a problem, which we now explain. If T is an inverse semigroup with zero then T/σ, where σ is the minimum group congruence [13], is trivial. In order to contruct interesting groups, we need to focus on 'large' elements of T. This is why we need the following definition. Let T be any inverse monoid with a zero. A non-zero idempotent e of T to be *essential* if for any non-zero idempotent f we have that $ef \neq 0$. We say an element $a \in T$ is *essential* if both idempotents $a^{-1}a$ and aa^{-1} are essential. The *essential part* of T, denoted by T^e, is the set of all essential elements of T. It is an inverse monoid; see [15]. The key point is that the zero is not an essential element and so does not belong to the essential part of our inverse monoid.

Lemma 4.3. *Let S be a finitely aligned k-monoid. Let $R = XS$ be a finitely generated right ideal of S. Then the identity function on R is an essential idempotent if and only if every element of S is dependent on an element of X.*

Proof. Suppose first that the identity function on R is an essential idempotent. Let $a \in S$ be any element. Then aS is a right ideal. It follows that the identity function on aS is an idempotent. By assumption, this has a non-zero product with the identity function on R. It follows that $aS \cap R \neq \varnothing$. Thus there is some element $u \in S$ such that $au \in R$. Whence, $au = xv$ for some $x \in X$ and $v \in S$. The converse is proved from the observation that a is dependent on an element of X if and only if $aS \cap R \neq \varnothing$. $\square$

Let S be a k-monoid which is finitely aligned. We shall now show how to construct a group from S. Let R_1 and R_2 be right ideals of S. A function $\theta \colon R_1 \to R_2$ is called a *morphism* if $\theta(as) = \theta(a)s$ for all $a \in R_1$ and $s \in S$. A bijective morphism is called an *isomorphism.* Define $\mathsf{R}(S)$ to be the set of all isomorphisms between the finitely generated right ideals of S. If S is finitely aligned then the intersection of any two finitely generated right ideals is either empty or again a finitely generated right ideal. With this condition, $\mathsf{R}(S)$ is an inverse monoid. We define the group associated with S as follows:

$$\boxed{\mathscr{G}(S) = \mathsf{R}(S)^e/\sigma.}$$

The only problem with this group is that we can, in general, say nothing about it. So, we shall now define an apparently different group using generalized prefix codes.

Lemma 4.4. *Let S be a finitely aligned k-monoid. Then if $aS \cap bS \neq \varnothing$ the right ideal $aS \cap bS$ is generated by a finite generalized prefix code.*

Proof. By assumption, $aS \cap bS = XS$ where X is a finite set. By Lemma 4.1, for each $x \in X$ there exists an element d_x such that $x = d_x t$ for some $t \in S$, where $d_x \in aS \cap bS$ and $\delta(d_x) = \delta(a) \vee \delta(b)$. Put $D = \{d_x \colon x \in X\}$. This is a finite set since X is a finite set. We claim that $aS \cap bS = DS$. By design, we have that $DS \subseteq aS \cap bS$. If $s \in aS \cap bS$ then $s = xu$ for some $x \in X$ and $u \in S$. It follows that $s = d_x tu$ and so $s \in DS$. We have therefore proved that $aS \cap bS = DS$. But any two elements of D have the same size. Suppose that $d, d' \in D$ are comparable. Then $dp = d'q$ for some $p, q \in S$. By Lemma 3.5, it follows that $d = d'$. $\square$

Let S be a finitely aligned k-monoid. If $a, b \in S$ define $a \vee b = \varnothing$ if a and b are incomparable, otherwise, define $a \vee b$ to be any finite generalized prefix code such that $aS \cap bS = (a \vee b)S$. The proof is straightforward and can be found in [20, lemma 3.24].

Lemma 4.5. *Let S be a finitely aligned k-monoid. Then the intersection of any two right ideals generated by a finite generalized prefix code is either empty or generated by a finite generalized prefix code.*

We can now define our second group associated with a finitely aligned k-monoid. The following definition arose from unpublished work of John Fountain. A finitely generated right ideal of S is said to be *projective* if it is generated by a finite generalized prefix code. Recall that in free monoids all right ideals are generated by prefix codes. Let S be a finitely aligned k-monoid. Define $\mathsf{P}(S)$ to be all the isomorphisms between the finitely generated right ideals generated by generalized prefix codes. By Lemma 4.5, $\mathsf{P}(S)$ is an inverse monoid. This leads to our second group associated with S:

$$\boxed{\mathscr{G}'(S) \cong \mathsf{P}(S)^e/\sigma.}$$

This group is constructed from the inverse monoid of all isomorphisms between the right ideals generated by the finite maximal generalized prefix codes.

We potentially have two groups associated with a finitely aligned k-monoid. We shall now prove that for a natural class of k-monoids, the two groups we have defined are, in fact, isomorphic.

We need one further assumption on our k-monoids. We shall assume that the associated monoid homomorphism δ is surjective. This assumption has an important consequence. Let $\mathbf{m} \in \mathbb{N}^k$ be any element. Define $C_{\mathbf{m}}$ to be all the elements a of S such that $\delta(a) = \mathbf{m}$. Because δ is surjective, the set $C_{\mathbf{m}}$ is non-empty. Suppose that $a, b \in C_{\mathbf{m}}$ are comparable. Then $au = bv$ where $u, v \in S$. But a and b have the same size and so $a = b$. Thus $C_{\mathbf{m}}$ is a generalized prefix code. We now show that it is maximal. Let a be any element of S. Then we can find an element u such that $\delta(au) \geq \mathbf{m}$. It follows by the (UFP) that $au = bv$ where $\delta(b) = \mathbf{m}$. We have therefore proved the following.

Lemma 4.6. *Let S be a k-monoid the associated monoid homomorphism of which is surjective. Then for each $\mathbf{m} \in \mathbb{N}^k$ we have that $C_{\mathbf{m}}$ is a maximal generalized prefix code.*

The proof of the following is key.

Lemma 4.7. *Let S be a k-monoid in which δ is surjective. Then every essential, finitely generated right ideal of S contains a right ideal generated by a finite maximal generalized prifix code.*

Proof. Let XS be an essential right ideal where X is finite. Let $\mathbf{m}$ be the joint of the sizes of the elements of X. We prove that $C_{\mathbf{m}}S \subseteq XS$. Let $a \in C_{\mathbf{m}}$. Since XS is essential, there exist elements $u, v \in S$ such that $au = xv$ for some $x \in X$. By assumption, $\delta(a) \geq \delta(x)$. Thus by Lemma 3.5, we have that $a = xt$ for some $t \in S$. We have proved that $C_{\mathbf{m}} \subseteq XS$. $\square$

It can now be proved that that each element of $\mathsf{R}(S)^e$ extends an element of $\mathsf{P}(S)^e$ [18, Lemma 7.9]. This implies that the groups $\mathscr{G}'(S)$ and $\mathscr{G}(S)$ are isomorphic. We now summarize what we have found.

Theorem 4.8. *Let S be a finitely aligned k-monoid the asspociated monoid homomorphism of which is surjective. Then there is a group $\mathscr{G}(S)$ associated with S which can be constructed either as $\mathsf{R}(S)^e/\sigma$ or as $\mathsf{P}(S)^e/\sigma$.*

Our group is therefore intimately connected with the structure of the finite, maximal generalized prefix codes on S.

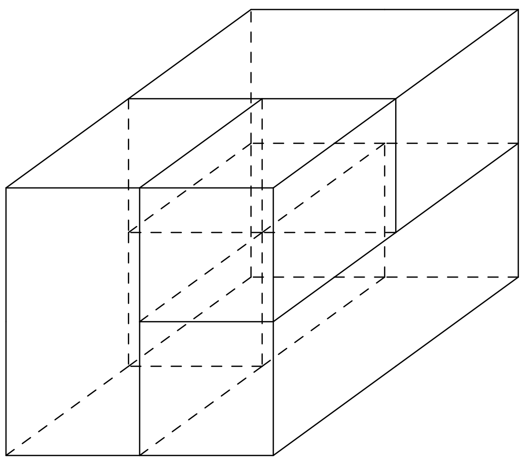
We can now construct some groups using the above theorem. Let $A_1, \ldots, A_k$ be k free monoids on non-empty alphabets. Then $A_1^* \times \ldots \times A_k^*$ is a k-monoid in which the associated monoid homomorphism is surjective and which is finitely aligned. Accordingly, we may construct the group $\mathscr{G}(A_1^* \times \ldots \times A_k^*)$. When $k = 1$, we are back to the Thompson groups contructed in [14, 15]. When $A = A_1$ contains 2 elements, and we take the n-fold direct product $A^* \times \ldots \times A^*$ then the group $\mathscr{G}(A^* \times \ldots \times A^*)$ is the group nV, the higher dimensional Thompson group of Matt Brin [3]. See [18] for details.

Generalized prefix codes turns out to be the most efficient way to describe the higher dimensional Thompson groups since the standard way using homeomorphisms of dyadic intervals does not work in dimension 3 and higher. See example below.

Example 4.9. The following is a generalized prefix code in $\{a_1, a_2\}^* \times \{b_1, b_2\}^* \times \{c_1, c_2\}^*$:

$$C = \{a_1b_1, a_2c_1, b_2c_2, a_1b_2c_1, a_2b_1c_2\}.$$

It can be viewed diagrammatically as follows:



See [19, Example 12.8] for more details.

We do not show this here, but the groups we have constructed also occur as the groups of units of Boolean inverse monoids [30]. To prove this requires us to generalize the right-infinite strings over an alphabet A to higher dimensions. How this is done is described in [18, 20].

5. Further generalizations

We may generalize k-monoids in the way described in [19] by allowing the group of units to be non-trivial. Let S be a monoid. We say that a monoid homomorphism $\lambda \colon S \to \mathbb{N}^k$ is a *size map* if $\lambda^{-1}(\mathbf{0})$ is precisely the group of units of S.

Definition. A monoid S is said to be a *generalized k-monoid* if there is a size map $\delta\colon S \to \mathbb{N}^k$ satisfying the *weak factorization property (WFP)*: if $\delta(x) = \mathbf{m} + \mathbf{n}$ then there exist elements x_1 and x_2 of S such that $x = x_1x_2$ where $\delta(x_1) = \mathbf{m}$ and $\delta(x_2) = \mathbf{n}$ and, furthermore, if x_1' and x_2' are any elements such that $x = x_1'x_2'$ where $\delta(x_1') = \mathbf{m}$ and $\delta(x_2') = \mathbf{n}$ then $x_1' = x_1g$ and $x_2' = g^{-1}x_2$ for some invertible element g.

Observe that k-monoids are simply the generalized k-monoids having a trvivial group of units. One of the goals of this section is to prove that self-similar group actions lead naturally to examples of generalized k-monoids [16, 21, 22]. We shall use the following concept. A *Levi monoid* is a monoid S equipped with a size map $\lambda\colon S \to \mathbb{N}$ which is equidivisible and which contains at least one non-invertible element [21]. We may assume, without loss of generality, that $a \in S$ is an atom if and only if $\lambda(a) = 1$ by [21, Proposition 2.8]. Every generalized 1-monoid is a Levi monoid by [19, Theorem 3.4 (2)]. We prove the converse.

Proposition 5.1. *Every Levi monoid is a generalized 1-monoid.*

Proof. Let S be a Levi monoid with size map $\lambda\colon S \to \mathbb{N}$ having the property that $a \in S$ is an atom if and only if $\lambda(a) = 1$. We prove that S is a generalized 1-monoid. Let $a \in S$ be any element. If a is invertible then $\lambda(a) = 0$. Suppose that $a = gh = g_1h_1$ where g, h, g_1, h_1 are call invertible. Put $k = g^{-1}g_1$, an invertible element. Then $gk = g_1$ and $k^{-1}h = h_1$. We may therefore suppose that a is not invertible. By [21, Lemma 2.6 (4)], we may write a as a product of $\lambda(a)$ atoms. Thus $a = a_1 \dots a_s$ where $s = \lambda(a)$. Suppose that $s = m+n$. Then we may factorize $a = bc$ where $b = a_1 \dots a_m$ and $c = a_{m+1} \dots a_s$. Suppose that $a = de$ where $\lambda(d) = m$ and $\lambda(e) = n$. Then we may write $d = d_1 \dots d_m$, all atoms, and $e = e_{m+1} \dots e_s$, also all atoms. Thus $a = d_1 \dots d_m e_{m+1} \dots e_s$. By [21, Lemma 2.11 (2)], there are invertible elements $g_1, \dots, g_{s-1}$ such that $a_1 = d_1g_1, \dots, a_m = g_{m-1}^{-1}d_mg_m$, and $a_{m+1} = g_m^{-1}e_s, \dots, a_s = g_{s-1}^{-1}e_s$. Thus $b = dg_m$ and $c = g_m^{-1}e$ where g_m is invertible. □

Recall that the left cancellative Levi monoids are precisely the left Rees monoids; these are the subject of [16] and are precisely the Zapp-Szép products of free monoids and groups and so they contain the self-similar group actions. We have therefore proved the following.

Proposition 5.2. *Self-similar group actions give rise to generalized 1-monoids.*

In [19], self-similar group actions were generalized from groups acting on free monoids to groups acting on k-monoids.

Several infinite series of groups acting simply transitively on products of k trees were constructed in [26]. Each such group gives an explicit example of a generalized $(k-1)$-monoid.

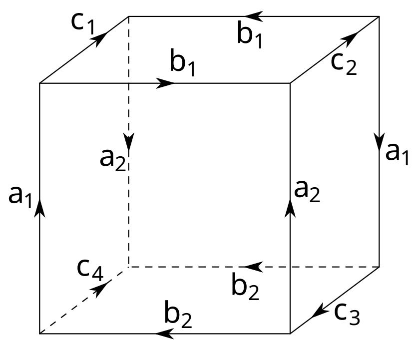
Example 5.3. We give here an example of a group acting simply transitively on a product of three trees, which will be used to construct a generalized 2-monoid. Hamiltonian quaternions can be used to get a (cubical) building of any dimension, for any set of odd primes. See [26] for details.

$a_1 = 1 + j + k, a_2 = 1 + j - k, a_3 = 1 - j - k, a_4 = 1 - j + k,$
$b_1 = 1 + 2i, b_2 = 1 + 2j, b_3 = 1 + 2k, b_4 = 1 - 2i, b_5 = 1 - 2j, b_6 = 1 - 2k,$
$c_1 = 1 + 2i + j + k, c_2 = 1 - 2i + j + k, c_3 = 1 + 2i - j + k, c_4 = 1 + 2i + j - k,$
$c_5 = 1 - 2i - j - k, c_6 = 1 + 2i - j - k, c_7 = 1 - 2i + j - k, c_8 = 1 - 2i - j + k.$

With this notation we have $a_i^{-1} = a_{i+2}$, $b_i^{-1} = b_{i+3}$, and $c_i^{-1} = c_{i+4}$, and the following group corresponds to the primes 3,5,7. This group acts on the product of three trees of valencies 4,6,8 with one orbit, and it is a fundamental group of the polyhedron P glued from squares decorated by the following relations.

$$\Gamma_{\{3,5,7\}} = \left\langle \begin{array}{c} a_1, a_2 \\ b_1, b_2, b_3 \\ c_1, c_2, c_3, c_4 \end{array} \,\middle|\, \begin{array}{c} a_1b_1a_4b_2,\ a_1b_2a_4b_4,\ a_1b_3a_2b_1, \\ a_1b_4a_2b_3,\ a_1b_5a_1b_6,\ a_2b_2a_2b_6 \\ a_1c_1a_2c_8,\ a_1c_2a_4c_4,\ a_1c_3a_2c_2,\ a_1c_4a_3c_3, \\ a_1c_5a_1c_6,\ a_1c_7a_4c_1,\ a_2c_1a_4c_6,\ a_2c_4a_2c_7 \\ b_1c_1b_5c_4,\ b_1c_2b_1c_5,\ b_1c_3b_6c_1, \\ b_1c_4b_3c_6,\ b_1c_6b_2c_3,\ b_1c_7b_1c_8, \\ b_2c_1b_3c_2,\ b_2c_2b_5c_5,\ b_2c_4b_5c_3, \\ b_2c_7b_6c_4,\ b_3c_1b_6c_6,\ b_3c_4b_6c_3 \end{array} \right\rangle.$$

In [12], it was shown that any group acting cocompactly on a product of k trees can be used to obtain a higher-rank graph with the number of vertices equal to the number of orbits. Thus a group acting simply transitively leads to a k-monoid. There are three alphabets, $A = \{a_1^{\pm}, a_2^{\pm}\}$, $B = \{b_1^{\pm}, b_2^{\pm}, b_3^{\pm}\}$, and $C = \{c_1^{\pm}, c_2^{\pm}, c_3^{\pm}, c_4^{\pm}\}$. The relations of $\Gamma_{\{3,5,7\}}$ containing the elements of alphabets of B and C only define a 2-monoid M. The geometric realization of the $(3, 5, 7)$ example consists of 24 cubes. The elements of the alphabet A act on M according to the cubes of P. See the following picture:



We do not know what the group is in this case.

The approach adopted in the example above can be applied to any of the groups constructed in [26]. It would be interesting to know which groups of actions on $(k-1)$-monoids arise.

References

[1] J. Berstel, D. Perrin, *Theory of codes*, Academic Press, 1985.
[2] J.-C. Birget, The groups of Richard Thompson and complexity, *Int. J. Algebra Comput.* **14** (2004), 569–626.
[3] M. G. Brin, Higher dimensional Thompson groups, *Geometriae Dedicata* **108** (2004), 163–192.
[4] M. G. Brin, On the Zappa-Szép product, *Comm. Algebra* **33** (2005), 393–424.
[5] S. Carson, V. Gould, Right ideal Howson semigroups, *Semigroup Forum* **23** (2020), 62–85.
[6] V. A. R. Gould, Cohernt monoids, *J. Austral. Math. Soc.* **53** (1992), 166–182.

[7] K. R. Davidson, S. C. Power, D. Yang, Atomic representations of rank 2 graph algebras, *Journal of Functional Analysis* **255** (2008), 819–853.

[8] K. R. Davidson, D. Yang, Representations of higher rank graph algebras, *New York J. Math.* **15** (2009), 169–198.

[9] R. Hazlewood, I. Raeburn, A. Sims, S. B. G. Webster, Remarks on some fundamental results about higher-rank graphs and their C^*-algebras, *Proc. Edinb. Math. Soc.* **56** (2013), 575–597.

[10] A. Kumjian, D. Pask, Higher rank graph C^*-algebras, *New York J. Math.* **6** (2000), 1–20.

[11] G. Lallement, *Semigroups and combinatorial applications*, John Wiley, 1979.

[12] N. Larsen, A. Vdovina, *Higher dimensional digraphs from cube complexes and their spectral theory*, to appear in *Groups, Geometry, Dynamics.*

[13] M. V. Lawson, *Inverse semigroups*, World Scientific, 1998.

[14] M. V. Lawson, Orthogonal completions of the polycyclic monoids, *Communications in Algebra* **35** (2007), 1651–1660.

[15] M. V. Lawson, The polycyclic monoids P_n and the Thompson groups $V_{n,1}$, *Communications in Algebra* **35** (2007), 4068–4087.

[16] M. V. Lawson, A correspondence between a class of monoids and self-similar group actions I, *Semigroup Forum* **76** (2008), 489-517.

[17] M. V. Lawson, The polycyclic inverse monoids and the Thompson groups revisited, in (P. G. Romeo, A. R. Rajan eds) *Semigroups, categories and partial algebras ICSAA 2019*, Springer, Proc. in Maths and Stats, volume 345.

[18] M. V. Lawson, A. Vdovina, Higher dimensional generalizations of the Thompson groups, *Advances in Mathematics* **369** (2020), 107191.

[19] M. V. Lawson, A. Vdovina, A generalisation of higher-rank graphs, *Bull. Aust. Math. Soc.* **105** (2022), 257–266.

[20] M. V. Lawson, A. Sims, A. Vdovina, Higher dimensional generalizations of the Thompson groups via higher rank graphs, *J. Pure Appl. Algebra* **228** (2023), 107456.

[21] M. V. Lawson, A. Wallis, A correspondence between a class of monoids and self-similar group actions II, *Inter. J. Algebra Cpmput.* **25** (2015), 633–668.

[22] M. V. Lawson, A. Wallis, Levi categories and graphs of groups, *TAC* **32** (2017), 780–802.

[23] F. W. Levi, On semigroups, *Bull. Calcutta Math. Soc.* **36** (1944), 141–146.

[24] I. Raeburn, A. Sims, Product systems of graphs and the C^*-algebras of higher-rank graphs, *J. Operator Th.* **53** (2005), 399–429.

[25] G. Robertson, T. Steger, Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras, *J. reine Angew. Math.* **72** (1999), 115–144.

[26] N. Rungtanapirom, J. Stix, A. Vdovina, *Infinite series of quaternionic 1-vertex cube complexes, the doubling construction, and explicit cubical Ramanujan complexes. Internat. J. Algebra Comput.* **29** (2019), 951–1007.

[27] J. Spielberg, Groupoids and C^*-algebras for categories of paths, *Trans. Amer. Math. Soc.* **366** (2014), 5771–5819.

[28] J. Spielberg, Groupoids and C^*-algebras for left cancellative small categories, arXiv:1712:07720v2.

[29] J. Stix, A. Vdovina, *Simply transitive quaternionic lattices of rank 2 over* $\mathbb{F}_{q(t)}$ *and a non-classical fake quadric*, *Math. Proc. Cambridge Philos. Soc.* **163** (2017), 453–498.

[30] F. Wehrung, *Refinement monoids, equidecomposability types, and Boolean inverse semigroups*, Lecture Notes in Mathematics 2188, Springer, 2017.

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