VARIATIONAL PRINCIPLES AND APPLICATIONS TO SYMMETRIC PDES

JAVIER FALCÓ AND DANIEL ISERT

Abstract. In this paper, we explore various equivalences of Ekeland's variational principle within the framework of group-invariant mappings. We introduce and analyze several key theorems, including the Drop theorem, the Petal theorem, Caristi-Kirk fixed-point theorem, and Takahashi's theorem, all of them within this context. Moreover, we extend the classical Drop theorem and Petal theorem to a more generalized setting. We also demonstrate the practical significance of these findings through numerous applications to diverse areas of mathematics. In particular, in the context of partial differential equations, we explore their implications on the solution of the Plateau problem, and in control theory. We also extend the classical Pontyargin maximum principle.

Contents

1. Introduction

Since the publication of the Ekeland's variational principle, [6], this result has proved to be one of the most powerful tools in analysis due to its wide-range of applications to functional analysis, fixed point theory, and partial differential equations, among others. Recently, the theory of group invariant mappings has gained a lot of strength inside the field of functional analysis, in its most general way, see for instance [5] and [1]. In 2024 we proved a group invariant version of the Ekeland's variational principle, see [9], and studied several applications of this result.

2020 *Mathematics Subject Classification.* 58E30,46B20,46B99.

Key words and phrases. Group invariant, partial differential equation, fixed point, variational principle, Banach space.

The first author was supported by grant PID2021-122126NB-C33 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe".

The purpose of this paper is to deepen the study of the Ekeland's variational principle by giving some equivalences of this result, such as the Drop theorem, the Petal theorem, the Caristi-Kirk fixed point theorem, and the Takahasi's theorem, all of them in their group invariant version. Furthermore, in our study of the Drop and the Petal theorem, by using group invariant techniques, we obtain generalizations of the classical ones. In this latter results, instead of searching for a group invariant point, we search for group invariant sets on the solution, and what's more surprisingly about them, they may not be equivalent to the group invariant Ekeland's variational principle.

As we mentioned earlier, one of the main applications of Ekeland's variational principle is to solve problems in partial differential equations (PDE). Continuing in this line, we will present here some new applications of these results, such as solving symmetric PDEs.

We will begin this manuscript by presenting all the notation and definitions we will require throughout the paper, as well as some useful results that will be used later on. Then, we will move to the core of the paper, where we study the equivalences of the group invariant Ekeland's variational principle. In particular we present the group invariant version of the Drop theorem, the Petal theorem, the Caristi-Kirk fixed point theorem, and the Takahasi's theorem. To conclude we will present applications in the fields of geometric analysis and PDEs.

2. Preliminaries

Let us begin by establishing some notation. We will denote a metric space by (M, d), a normed space by $(X, \|\cdot\|)$, and the space of linear and continuous mappings from X to X by $\mathcal{L}(X)$. Hereinafter, $G \subseteq \mathcal{L}(X)$ is reserved to denote a topological group of invertible bounded linear isometries with the relative topology of $\mathcal{L}(X)$. Also, g is reserved to denote an element of the group G.

To start, let us present the three notions of G-invariance. Let X, Y be two normed spaces and let G be a topological group. We say that:

(1) A point $x \in X$ is G-invariant, or invariant under the action of G if $g(x) = x$ for all $g \in G$.
(2) A set $K \subset X$ is G-invariant if for every $g \in G$, $g(K) = K$.
(3) A mapping $f\colon X \to Y$ is G-invariant if for every $x \in X$ and every $g \in G$ we have that
$$f(g(x)) = f(x).$$

In particular, a metric $d\colon X \times X \to [0, +\infty[$ is G-invariant if
$$d(g(x), g(y)) = d(x, y) \quad \forall\, x, y \in X.$$

Note that, if the distance is translation invariant, this is equivalent to assume that $d(x, 0) = d(g(x), 0)$ for all $x \in X$ and all $g \in G$.

We will always assume that the distance of the metric spaces is G-invariant.

As usual, we will denote by X_G, B_X, S_X the set of all group invariant points of X, the open unit ball of X, and the unit sphere of X respectively. We will always assume that the norm of the space X, $\|\cdot\|$, is G-invariant.

A fundamental definition of this paper is that of G-symmetrization point. Let us recall here what we understand by G-symmetric point. Let X be a Banach space and G be a compact topological group acting on X. If $x \in X$ we define the symmetrization point of x with respect to G, or the G-symmetrization point, as

$$\overline{x} = \int_G g(x) d\mu(g),$$

where μ is the Haar measure and the integral is the Bochner integral.

In [9, Theorem 12] we saw that Ekeland's G-invariant version only holds for convex functions with respect to the group G. Recall that if X is a Banach space and G be a compact topological group acting on X, a function $\varphi\colon X \to \mathbb{R}$ is convex with respect to G given that

$$\varphi\left(\int_G g(x) d\mu(g)\right) \leq \int_G \varphi(g(x)) d\mu(g) \quad \forall\, x \in X.$$

And we say that φ is linear with respect to G given that

$$\varphi\left(\int_G g(x) d\mu(g)\right) = \int_G \varphi(g(x)) d\mu(g) \quad \forall\, x \in X.$$

Then, a metric $d\colon X \times X \to [0, +\infty[$ is convex with respect to G if

$$d(\overline{x}, \overline{y}) \leq \int_G d(g(x), g(y)) d\mu(g).$$

Remark 1. *Observe that the definition of a metric d being convex with respect to the group is quite natural. Observe that, if d were to be invariant by translations, then this is equivalent to the fact that the norm associated with the metric d is convex with respect to G, which is always true. Indeed,*

$$d(\overline{x} - \overline{y}, 0) = d(\overline{x}, \overline{y}) \leq \int_G d(g(x), g(y)) d\mu(g) = \int_G d(g(x-y), 0) d\mu(g),$$

hence

$$\|\overline{x} - \overline{y}\| = d(\overline{x} - \overline{y}, 0) \leq \int_G \|g(x-y)\| d\mu(g) = \|x - y\|.$$

In section 3 we are going to work with the petal of two points, and the drop of a set. Let us present here these two objects, and study some interesting G-invariant properties related to them. Let (M, d) be a metric space, $a, b \in M$ and $\gamma > 0$, the petal of a and b is defined as

$$P_\gamma(a, b) = \{x \in M \ \mid\ \gamma d(a, x) + d(b, x) \leq d(a, b)\}.$$

Let $(X, \|\cdot\|)$ be a normed vector space, $a \in X$ and $B \subset X$ be a convex subset. The drop of a and B is defined as

$$D(a, B) = \{a + t(b - a) \ \mid\ t \in [0, 1],\, b \in B\}.$$

Let us observe the following properties of the drop and the petal when we consider a group G. The proof of these results is a direct consequence of the G-invariance of the distance, but we include them here for the sake of completeness.

Proposition 2. *Let (M, d) be a metric space and $G \subseteq \mathcal{L}(M)$ be a topological group of isometries acting on M. Let $\gamma > 0$, then*

$$g(P_\gamma(a, b)) = P_\gamma(g(a), g(b)) \quad \forall\, a, b \in M, \text{ and } \forall\, g \in G.$$

Proof. Let $x \in P_\gamma(a,b)$. Observe that

$$d(g(a), g(b)) \geq \gamma d(g(a), g(x)) + d(g(x), g(b)) \quad \forall\, g \in G.$$

Since G is compact, and d is G-invariant, this is equivalent to

$$d(g^{-1}(g(a)), g^{-1}(g(b))) \geq \gamma d(g^{-1}(g(a)), g^{-1}(g(x))) + d(g^{-1}(g(x)), g^{-1}(g(b))).$$

But observe that this is the same as

$$d(a,b) \geq \gamma d(a,x) + d(x,b),$$

so $g(x) \in P_\gamma(g(a), g(b))$. This shows that $g(P_\gamma(a,b)) \subseteq P_\gamma(g(a), g(b))$.

Take now $x \in P_\gamma(g(a), g(b))$, we want to see that $g^{-1}(x) \in P_\gamma(a,b)$. Notice that

$$d(a,b) \geq \gamma d(a, g^{-1}(x)) + d(b, g^{-1}(x)),$$

if, and only if,

$$d(g(a), g(b)) \geq \gamma d(g(a), g(g^{-1}(x))) + d(g(b), g(g^{-1}(x))),$$

by the G-invariance of d. But this last inequality reads as follows:

$$d(g(a), g(b)) \geq \gamma d(g(a), x) + d(g(b), x).$$

This shows the other inclusion and concludes the proof. □

Proposition 3. *Let $(X, \|\cdot\|)$ be a normed vector space, $G \subseteq \mathcal{L}(X)$ be a topological group of isometries acting on X, $a \in X$ and $B \subset X$ be a convex subset. Then:*

$$g(D(a,B)) = D(g(a), g(B)).$$

Proof. Let $x \in D(x_0, B) = \operatorname{conv}(x_0, B)$. Then, for some $b \in B$, and $t > 0$:

$$x = tx_0 + (1-t)b.$$

Now, by the linearity of g we know that

$$g(x) = tg(x_0) + (1-t)g(b) \in \operatorname{conv}(g(x_0), g(B)) = D(g(x_0), g(B)).$$

This shows that $g(D(a,B)) \subseteq D(g(a), g(B))$.

Now take $x \in D(g(x_0), g(B)) = \operatorname{conv}(g(x_0), g(B))$, then for some $z \in g(B)$, and for $t > 0$ it is clear that

$$x = tg(x_0) + (1-t)z.$$

Since $z \in g(B)$, there exists $b \in B$ such that $z = g(b)$. Then, by linearity of g,

$$x = g(tx_0 + (1-t)b) \in g(\operatorname{conv}(x_0, B)) = g(D(x_0, B)).$$

This concludes the proof. □

The following G-invariant properties answer some of the natural questions that rise in our scenario of G-invariant mappings. The proof of this results is straightforward.

Property 4. *Let (M,d) be a metric space and $G \subseteq \mathcal{L}(M)$ be a topological group of isometries acting on M. Then, $a, b \in M$ are G-invariant if and only if $P_\gamma(a,b)$ is G-invariant for every $\gamma > 0$.*

Proposition 5. *Let $(X, \|\cdot\|)$ be a metric space, $G \subseteq \mathcal{L}(X)$ be a topological group of isometries acting on X, and $A, B \subseteq X$ be two G-invariant subsets. Then conv(A,B) is G-invariant.*

As a consequence of this result, we obtain the following.

Corollary 6. *Let $(X, \|\cdot\|)$ be a normed space and $G \subseteq \mathcal{L}(X)$ be a topological group of isometries acting on X. Assume that $x_0 \in X$, $B \subset X$ is a convex subset, and that both are G-invariant. Then $D(x_0, B)$ is G-invariant.*

We are going to show now that the previous conditions are indeed required. Through the following examples we will be working with the group $G = \{Id, \sigma\} \subseteq \mathbb{R}^2$, where $\sigma \colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $\sigma(x, y) = (y, x)$.

Example 7. *In $\mathbb{R}^2$ take $b = (0, 0)$, which is clearly G-invariant, and $a = (2, 0)$, which is not G-invariant. Then, the petal $P_\gamma(a, b)$ is not G-invariant. Also, take $B = B((0, 3), 1)$ the open ball centered at $(0, 3)$ with radius 1, which is clearly not G-invariant. Then, the drop $D(b, B)$ is not G-invariant.*

However if we consider a non G-invariant point a, we cannot say anything about the drop.

Example 8. *In $\mathbb{R}^2$ suppose $x = (0, 1)$ which is not G-invariant, and take $B = B((0, 0), 2)$ the ball centered at $(0, 0)$ with radius 2, which is clearly G-invariant, by G-invariance of the norm. Then $D(x_0, B) = B = B((0, 0), 2)$ which is G-invariant as before. Now, suppose $x = (5, 0)$ which is not G-invariant, and take $B = B((0, 0), 2)$ the ball centered at $(0, 0)$ with radius 2, which is clearly G-invariant. Since $g(x_0) = (0, 5)$, we can see that $x_0 \in D(x_0, B)$, but $x_0 \notin D(g(x_0), B)$. So, $D(x_0, B)$ is not G-invariant.*

Through the paper, we will need to perform some operations on group invariant sets. These operations are compatible with the G-invariance of the sets. We would like to conclude this section, by recalling some of these results. The proofs of these results are straightforward.

Property 9. *Let (M, d) be a metric space and $G \subseteq \mathcal{L}(M)$ be a topological group of isometries acting on M. If $A, B \subseteq M$ are two G-invariant subsets, then $A \cap B$ is again G-invariant and $(A \cap B)_G = A_G \cap B_G$.*

Property 10. *Let X be a Banach space and $G \subseteq \mathcal{L}(X)$ be a topological group of isometries acting on X. If $A, B \subseteq X$ are two G-invariant subsets, then $A \backslash B$ is again G-invariant and $(A \backslash B)_G = A_G \backslash B_G$.*

3. Equivalence of the Drop theorem, Petal theorem, and Ekeland's variational principle

The purpose of this section is to present a proof of the equivalence of the group invariant version of the Drop theorem (Theorem 13), the Petal theorem (Theorem 12), and the Ekeland's variational principle (Theorem 11). To prove these equivalences, we are going to use the following version of the Ekeland's variational principle, based on [9, Theorem 12].

Observe that [9, Theorem 12] is only stated for normed spaces, but a slight revision of the proof shows us the following metric version of the Theorem.

Theorem 11. *Let (M, d) be a complete metric space and $G \subseteq \mathcal{L}(M)$ be a compact topological group of isometries acting on M. Suppose d is G-invariant, and let $f \colon M \to \mathbb{R} \cup \{+\infty\}$ be a mapping that is lower semiconinuous, bounded below, proper, G-invariant and convex with respect to G. Then for all $\gamma > 0$ and every*

G-invariant point $x_0 \in M$, there exists a G-invariant point $a \in M$ such that

$$f(a) < f(x) + \gamma d(x,a) \quad \forall\, x \in M,\, x \neq a, \tag{1}$$

$$f(a) \leq f(x_0) - \gamma d(a, x_0). \tag{2}$$

Proof. Consider the subspace $S = \{x \in M \mid f(x) + \gamma d(x, x_0) \leq f(x_0)\}$, which is closed, and in [9, Theorem 12] take $f_{|S}$ which satisfies all the conditions of the Theorem. Then, we know that there exists a G-invariant point $a \in S$ such that

$$f(a) < f(x) + \gamma d(a,x) \quad \forall\, a \neq x \in M.$$

But, since $a \in S$ we have that

$$f(a) \leq f(x_0) - \gamma d(a, x_0).$$

□

To continue, let us present the G-invariant version of the Petal theorem and Drop theorem.

Theorem 12 (G-invariant Petal theorem)**.** *Let (M,d) be a metric space, $G \subseteq \mathcal{L}(M)$ be a compact topological group of isometries acting on M, and $C \subset M$ a complete G-invariant subset. Assume that d is G-invariant and convex with respect to G. Let $x_0 \in C_G$, $b \in (M \backslash C)_G$, $r \leq d(b,C)$ and $s = d(b,x_0)$. Then, for all $\gamma > 0$, there exists a G-invariant point $a \in C \cap P_\gamma(x_0,b)$ such that $C \cap P_\gamma(a,b) = \{a\}$.*

Theorem 13 (G-invariant Drop theorem)**.** *Let $(X, \|\cdot\|)$ be a normed space, $G \subseteq \mathcal{L}(X)$ be a compact topological group of isometries acting on X, and $C \subset X$ a complete G-invariant subset. Assume that $x_0 \in C_G$, and $B = \overline{B(b,r)}$, where $b \in X_G$ and $r < d(b,C)$. Then there exists a G-invariant point $a \in C \cap D(x_0,B)$ such that $C \cap D(a,B) = \{a\}$.*

Let us now present the main goal of this manuscript, that is, to show that the previous three theorems are equivalent.

Theorem 11 ⇒ Theorem 12. Define the function

$$\begin{array}{rccc} f\colon & M & \to & \mathbb{R} \\ & x & \mapsto & d(x,b), \end{array}$$

which is continuous, bounded below by r and G-invariant. Note that since d is convex with respect to G, so is f.

Applying now Ekeland's variational principle we know that there exists $a \in S_G$ such that

$$f(a) < f(x) + \gamma d(a,x) \quad \forall\, x \in M,\ x \neq a \tag{3}$$

$$f(a) < f(x_0) - \gamma d(a, x_0). \tag{4}$$

By (3) we know that for every $x \in M \backslash \{a\}$ we have that $x \notin P_\gamma(a,b)$, this meaning that

$$M \cap P_\gamma(a,b) = \{a\}\,.$$

This concludes the proof. Observe also, that by (4),

$$d(a,b) < d(x_0,b) - \gamma d(x_0,a),$$

hence

$$\gamma d(x_0,a) < -d(a,b) + d(x_0,b) = s - d(a,b).$$

By hypothesis we know that $r < d(b, M) < d(b, a)$, thus

$$d(x_0, a) < \frac{s - r}{\gamma}.$$

$\square$

Theorem 12 ⇒ Theorem 13. Let $B = B(b, r)$, and consider $X = C \cap D(x_0; B)$ which is a complete and G-invariant subspace. Define $d = d(b, C)$ and $\gamma = \frac{d-r}{d+r}$. By the Petal's theorem there exists a G-invariant point, say a, such that $\{a\} = X \cap P_\gamma(a, b)$.

Observe now that, since $t = d(a, b) \geq d > r$, it is clear that

$$\frac{d - r}{d + r} \leq \frac{t - r}{t + r},$$

so $D(a, B) \subseteq P_\gamma(a, b)$ for $t > r$. Moreover, since $a \in D(x_0, B)$, then $D(a, B) \subseteq D(x_0, B)$. Therefore

$$D(a, B) \cap C \subseteq D(a, B) \cap (D(x_0, B) \cap C) \subseteq P_\gamma(a, b) \cap X = \{a\}.$$

$\square$

Remark 14. *Note that to prove the last implication, it is not required the condition of d being convex with respect to G.*

In order to prove the last implication we will use the following lemma, whose proof can be found in [11, Lemma 2.3].

Lemma 15. *Let X be a normed vector space, $B = \overline{B((0, h), r)} \subseteq X \times \mathbb{R}$ with radius $r \in]0, h[$ and the norm $\|(x, r)\| = \max(\|x\|, r)$. Then, the cone $K = \mathbb{R}_+ B$ generated by B is given by*

$$K = \left\{(x, t) \in X \times \mathbb{R} \mid t \geq r^{-1}(h - r)\|x\|\right\}.$$

Now we can move to the proof of the last implication.

Theorem 13 ⇒ Theorem 11. We start replacing d by $d' = \min(\delta, d)$ where $\delta = \frac{1}{\gamma}(f(x_0) - \inf f(M) + 1)$. Observe that the two conditions of Ekeland's variational principle will still hold if we consider the distance d' instead of the distance d. Let F be the normed vector space of the continuous functions in M with the supremum norm. Then, (M, d) can be isometrically embedded in F via the mapping

$$\begin{array}{ccc} (M, d) & \rightarrow & F \\ x & \mapsto & d_x(y) = d(x, y), \end{array}$$

and $M \subseteq F$ is complete. Define $E = F \times \mathbb{R}$ with the norm $\|(x, t)\| = \max(\|x\|, |t|)$.

Without loss of generality we may assume that $x_0 = 0$ and $f(x_0) = 0$. If this were not the case, we could achieve this by translating via the mapping $x \mapsto f(x) - f(x_0)$. We define now $\psi = -f$ which is G-invariant, and observe that $\psi(x_0) = f(x_0) = 0$. Take $m = \sup\{\psi(x) \mid x \in M\}$, $r > \frac{m}{\gamma}$, $h = \gamma r + r > m + r$, and define finally $B = B((0, h), r) = B(0, r) \times [h - r, h + r]$ and $K = \mathbb{R}_+ B$. For given $(x, t) \in B$ we have that

$$t \geq h - r > m,$$

therefore $(x,t) \notin C = \mathrm{Hipo}(\psi) = \{(x,t) \in M \times \mathbb{R} \mid t \le \psi(x)\}$. Observe also that $(0,0) \in Hipo(\psi)$. Hence, by Theorem 13, there exists a G-invariant point $(a,\alpha) \in C \cap D((0,0),B)$ such that

$$\{(a,\alpha)\} = C \cap D((a,\alpha),B).$$

Notice that

$$D((0,0),B) = \{(0,0) + t((0,0)+b) \mid t \in [0,1],\, b \in B\} =$$

$$= \{tb \mid t \in [0,1],\, b \in B\} = [0,1] \cdot B = [0,1] \cdot B((0,h),r).$$

Since $(a,\alpha) \in D((0,0),B)$, in particular $a \in B(0,r)$ and $(a,h) \in B \subseteq D((0,0),B)$. Then, by convexity

$$(a,t) = \beta(a,\alpha) + (1-\beta)(a,h) \quad \text{for } \beta \in [0,1],$$

thus $(a,t) \in D((0,0),B)$. Observe now that it can not happen that $\alpha < \psi(a)$, otherwise $(a,\alpha) \in B$ and $(a,\alpha) \in \mathrm{int}(C)$, a contradiction. Therefore, $\alpha = \psi(a)$.

Applying now Lemma 15 we know that

$$K = \mathbb{R}_+ B = \left\{(x,t) \in F \times \mathbb{R} \mid t \ge r^{-1}(h-r)\|x\|\right\}.$$

Since $(a,\alpha) \in B$ and $\alpha = \psi(a)$:

$$\psi(a) \ge r^{-1}(h-r)\|a\| = \gamma\|a\| = \gamma d(a,x_0),$$

from where we deduce (2), taking into account that $\psi = -f$ and $f(x_0) = 0$, i.e.,

$$f(a) \le f(x_0) - \gamma d(a,x_0).$$

Let now $(x,t) \in (a,\alpha) + K$, where $x \in M \setminus \{a\}$ and $t \le m$. It is clear that $t - \alpha \ge \gamma\|x-a\| > 0$, so we can write

$$(x-a, t-\alpha) = s(z, h-r-\alpha)$$

where $z = s^{-1}(x-a)$, $s = \frac{t-\alpha}{h-r-\alpha}$. Observe that, since

$$t - \alpha \le m - \alpha < \gamma r - \alpha = h - \alpha - r,$$

$$h - r - \alpha \ge h - r - m > 0,$$

it is clear that $s \in]0,1[$. Moreover, K is a convex cone, so

$$(a+z, h-r) = (a,\alpha) + (z, h-r-\alpha) \in K,$$

thus $(a+z,h-r) \in K \cap (E \times \{h-r\}) \subseteq B$, and by convexity of $D((a,\alpha),B)$ we have that

$$(x,t) = (a,\alpha) + s((a+z,h-r) - (a,\alpha)) \in D((a,\alpha),B).$$

In particular $(x,t) \notin C$. Since for all $x \in M$, $\psi(x) \le m$, then $(x,\psi(x)) \notin (a,\alpha)+K$, therefore

$$\psi(x) - \psi(a) < \gamma\|x-a\|.$$

Thus

$$f(a) < f(x) + \gamma\|x-a\|.$$

□

We would like now to focus on a different version of group invariant Drop theorem and Petal's theorem, where instead of looking for group invariant points in the solution, we search for group invariant sets. Let us give a previous definition.

Definition 16. *Let (M, d) be a metric space and G be a compact topological group of isometries acting on M. For a point $x \in E$ we define*

$$s_G(x) := \inf\{d(x, g(x)) \mid g \in G \text{ and } g(x) \neq x\}.$$

The following result is a slight modification of the Petal theorem that allows us to extend the classical Petal theorem to what we call the flower theorem.

Proposition 17. *Let (M, d) be a metric space, $G \subseteq \mathcal{L}(M)$ be a compact topological group of isometries acting on M, and $C \subset M$ a complete G-invariant subset of M. Let $x_0 \in C_G$, $b \in M \backslash C$. Then, for every $\gamma > 0$, there exists $a \in C \cap P_\gamma(x_0, b)$ such that*

$$C \cap P_\gamma(g(a), g(b)) = \{g(a)\} \text{ for every } g \in G.$$

Furthermore, for every $g, g' \in G$ with $d(g(b), g'(b)) > 2d(b, C)$ we have that

$$P_\gamma(g(a), g(b)) \cap P_\gamma(g'(a), g'(b)) = \emptyset.$$

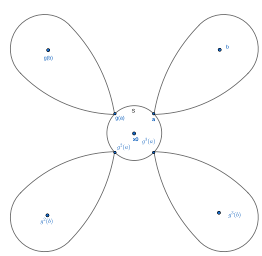

FIGURE 1. Flower theorem statement.

We can also obtain an analogous generalization for the Drop theorem.

Proposition 18. *Let $(E, \|\cdot\|)$ be a normed space, $G \subseteq \mathcal{L}(E)$ be a compact topological group of isometries acting on E, and $C \subset E$ be a complete G-invariant subset of E. Let $x_0 \in C_G$, $b \in E \backslash C$. Then, there exists $a \in C \cap D(x_0, b)$ such that*

$$C \cap D(g(a), g(b)) = \{g(a)\} \text{ for every } g \in G.$$

Furthermore, for every $g, g' \in G$ with $d(g(b), g'(b)) > 2d(b, C)$ we have that

$$D(g(a), g(b)) \cap D(g'(a), g'(b)) = \emptyset$$

for every $g, g' \in G$ with $g(b) \neq g'(b)$.

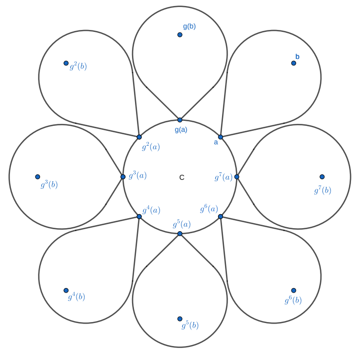

FIGURE 2. Generalized Drop theorem statement.

4. CARISTI-KIRK, EKELAND AND TAKAHASI'S THEOREM

In this section we want to show the equivalence between the Ekeland's variational principle, the Caristi-Kirk fixed point theorem, and the Takahasi's theorem, all of them in the group invariant setting. Let's first do some assumptions.

Let (M, d) be a complete metric space and $G \subseteq \mathcal{L}(M)$ be a compact topological group of isometries acting on M, so that d is G-invariant. Let $f\colon M \times M \to \mathbb{R} \cup \{+\infty\}$ be a function which is lower semicontinuous, G-invariant and convex with respect to G in the second coordinate that also satisfies

$$f(x,x) = 0 \quad \forall\, x \in M, \tag{5}$$

$$f(x,y) \le f(x,z) + f(z,y) \quad \forall\, x,y,z \in M. \tag{6}$$

Assume that there exists $x_0 \in M_G$ such that

$$\inf_{x \in M} f(x_0, x) > -\infty, \tag{7}$$

and define the set

$$S_0 = \{x \in M \mid f(x_0,x) + d(x_0,x) \le 0\}\,. \tag{8}$$

Note that by the G-invariance of f, d and x_0, the set S_0 is also G-invariant.

The assumptions on f and the existence of x_0 will hold through this section.

In order to show the previously mentioned equivalences we will use the following auxiliary result.

Theorem 19. *Let $U \subseteq M$ be G-invariant satisfying that*

$$\forall\, y \in S_0 \backslash U,\ \exists\, x \in M_G \text{ such that } x \neq \overline{y} \text{ and } f(\overline{y}, x) + d(\overline{y}, x) \le 0. \tag{9}$$

Then, there exists $\hat{x} \in (S_0 \cap U)_G$.

Proof. Let us construct recursively a sequence of G-invariant points $x_n \in M$. Consider the initial point x_0 as the one given in assumption (7), so

$$\inf_{x \in M} f(x_0, x) > -\infty.$$

Given x_n define the set

$$S_n = \{x \in M \mid f(x_n,x) + d(x_n,x) \le 0\}\,,$$

and the number

$$\gamma_n = \inf_{x \in S_n} f(x_n, x).$$

Notice that for $n = 0$, the set S_0 is the one given in equation (8).

Clearly the set S_n is G-invariant since x_n is G-invariant, d is G-invariant and f is G-invariant with respect to the second coordinate. Also by assumption (5) it is clear that $S_n \neq \emptyset$, since $x_n \in S_n$, and $\gamma_n \leq 0$.

For $n \geq 1$, suppose x_{n-1} is known and G-invariant, and $\gamma_{n-1} > -\infty$. Set a G-invariant point $x_n \in S_{n-1}$ such that

$$f(x_{n-1}, x_n) \leq \gamma_{n-1} + \frac{1}{n}. \tag{10}$$

Under this assumptions we are going to show that $S_n \subseteq S_{n-1}$. Let $x \in S_n$, by assumption (5) and the fact that $x_n \in S_n$ and $x_n \in S_{n-1}$ it is clear that

$$f(x_{n-1}, x) + d(x_{n-1}, x) \leq f(x_{n-1}, x_n) + d(x_{n-1}, x_n) + f(x_n, x) + d(x_n, x) \leq 0,$$

so, indeed $x \in S_{n-1}$. Applying now (5) and (10) we see that $S_n \subseteq S_{n-1}$, and,

$$\begin{aligned} \gamma_n = \inf_{x \in S_n} f(x_n, x) &\geq \inf_{x \in S_n} \left(f(x_{n-1}, x) - f(x_{n-1}, x_n)\right) \\ &\geq \inf_{x \in S_{n-1}} \left(f(x_{n-1}, x) - f(x_{n-1}, x_n)\right) \\ &= \gamma_{n-1} - f(x_{n-1}, x_n) \geq -\frac{1}{n}. \end{aligned}$$

Then, if $x \in S_n$

$$d(x_n, x) \leq -f(x_n, x) \leq -\gamma_n \leq \frac{1}{n}.$$

Thus, $\operatorname{diam}(S_n) \to 0$. Moreover for every $k \geq n$ it is clear that $x_k \in S_k \subseteq S_n$. In particular,

$$d(x_k, x_n) \leq \frac{1}{n}.$$

Hence $\{x_n\}$ is a Cauchy sequence of group invariant points. Therefore, since M_G is closed, there exists a group invariant point, say $\hat{x}$, which is the limit of the sequence. Since $\operatorname{diam}(S_n) \to 0$, it is clear that

$$\bigcap_{n=0}^{+\infty} S_n = \{\hat{x}\}.$$

We claim that $\hat{x} \in U$. By contradiction, if $\hat{x} \notin U$, we know by hypothesis that there exists $x \in M$ such that $x \neq \hat{x}$ and

$$f(\hat{x}, x) + d(\hat{x}, x) \leq 0.$$

Also, since $\hat{x} \in \bigcap_{n=0}^{+\infty} S_n$

$$f(x_n, \hat{x}) + d(x_n, \hat{x}) \leq 0 \quad \forall\, n \geq 0.$$

Now, applying (5), we obtain

$$f(x_n, x) + d(x_n, x) \leq 0 \quad \forall\, n \geq 0,$$

this meaning that $x \in \bigcap_{n=0}^{+\infty} S_n$. But this would be a contradiction with the fact that $\bigcap_{n=0}^{+\infty} S_n = \{\hat{x}\}$. So $\hat{x} \in U$. $\square$

To conclude this section we present the group invariant equivalences of Theorem 19, that are the group invariant generalizations of Ekeland's theorem, Takahashi's theorem, and Caristi-Kirk fixed point theorem respectively.

Theorem 20. *Let (M,d) be a complete metric space and $G \subseteq \mathcal{L}(M)$ be a compact topological group of isometries acting on M. Then, the following results are equivalent:*

(i) Let $U \subseteq M$ be G-invariant satisfying that

$$\forall\, y \in S_0 \backslash U,\ \exists\, x \in M_G \text{ with } x \neq \overline{y} \text{ and } f(\overline{y}, x) + d(\overline{y}, x) \leq 0.$$

Then, there exists $\hat{x} \in (S_0 \cap U)_G$.

(ii) There exists $\hat{x} \in S_0$ such that $\hat{x}$ is G-invariant, and $f(\hat{x}, x) + d(\hat{x}, x) > 0$ for all $x \in M$, $x \neq \hat{x}$.

(iii) Suppose $\forall\, y \in S_0$ with $\inf_{x \in M} f(\overline{y}, x) < 0$, there exists

$$x \in M_G \text{ with } x \neq \overline{y} \text{ and } f(\overline{y}, x) + d(\overline{y}, x) \leq 0.$$

Then, there exists $\hat{x} \in (S_0)_G$ such that $f(\hat{x}, x) \geq 0$ for all $x \in M_G$.

(iv) Let $T\colon M \to M$ be a multivalued mapping such that for every $y \in S_0$ there exists

$$x \in (T(y))_G \text{ with } f(\overline{y}, x) + d(\overline{y}, x) \leq 0.$$

Then, there exists $\hat{x} \in (S_0)_G$ such that $\hat{x} \in T(\hat{x})$.

Proof. $(ii) \Rightarrow (i)$

We know that there exists some G-invariant point $\hat{x} \in S_0$ such that

$$f(\hat{x}, x) + d(\hat{x}, x) > 0 \quad \forall\, x \neq \hat{x}.$$

In particular $\hat{x} \in U$, hence $\hat{x} \in S_0 \cap U$.

$(i) \Rightarrow (ii)$

Take $y \in M$ and define

$$\Gamma(y) = \{x \in M \mid x \neq \overline{y},\ \ f(\overline{y}, x) + d(\overline{y}, x) \leq 0\}.$$

Define now $U = \{y \in M \mid \Gamma(y) = \emptyset\}$. Then, if $y \notin U$, by definition, there exists some x such that $x \in \Gamma(y)$. Applying now (i) there exists some G-invariant point $\hat{x} \in S_0 \cap U$. So $\Gamma(\hat{x}) = \emptyset$, therefore

$$f(\hat{x}, x) + d(\hat{x}, x) > 0 \quad \forall\, x \neq \hat{x}.$$

$(iii) \Rightarrow (i)$

We proceed by contradiction. Assume that $y \notin U$ for every $y \in S_0$. Then, by hypothesis, there exists some $x \in M_G$ such that $x \neq \overline{y}$ and

$$f(\overline{y}, x) + d(\overline{y}, x) \leq 0.$$

From this we can deduce that $\inf\{f(\overline{y}, x) \mid x \in M\} < 0$. Therefore, applying (iii), there exists some G-invariant point $\hat{x} \in S_0$ such that $f(\hat{x}, x) \geq 0$ for every $x \in M_G$. This implies that

$$f(\hat{x}, x) + d(\hat{x}, x) \geq 0 \quad \forall\, \hat{x} \neq x \in M_G.$$

Thus a contradiction.

$(i) \Rightarrow (iii)$

Define

$$U = \left\{ y \in X \mid \inf_{x \in M} f(\overline{y}, x) \geq 0 \right\}.$$

Then the hypothesis of (i) follows from the hypothesis of (iii). Since (i) holds, there exists some G-invariant point $\hat{x} \in S_0 \cap U$. And since $\hat{x} \in U$, by definition,

$$\inf_{x \in M} f(\hat{x}, x) \geq 0.$$

In particular $f(\hat{x}, x) \geq 0$ for every $x \in M_G$.

$(iv) \Rightarrow (i)$

Define the multivalued map $T\colon M \to M$ as follows

$$T(y) = \{x \in M_G \mid x \neq \overline{y}\}.$$

Proceed by contradiction. Suppose $y \notin U$ for every $y \in S_0$. Then, the hypothesis of (iv) follows from (i). Then, there exists some G-invariant point $\hat{x} \in S_0$. But this is a contradiction with the definition of T.

$(i) \Rightarrow (iv)$

Define

$$U = \{y \in M \mid y \in T(y) \text{ and is } G-\text{invariant}\}.$$

By the hypothesis of (iv) we obtain the hypothesis of (i). And since (i) holds, we obtain a G-invariant point $\hat{x} \in S_0 \cap U$. In particular $\hat{x} \in U$, and by definition $\hat{x} \in T(\hat{x})$. □

5. Applications of the Ekeland variational principle

5.1. **Drop and generalized Drop theorem.** We are going to present a geometrical application of the Drop theorem dealing with the notion of the contingent cone to a subset C of a Banach space X. Recall that the contingent cone to C in $a \in C$ is

$$K_C(a) = \limsup_{t \to 0^+} t^{-1}(C - a).$$

So, $v \in K_C(a)$ if, and only if, $\liminf_{t \to 0^+} t^{-1} d(a + tv, C) = 0$.

Theorem 21. *Let X be a normed space and $G \subseteq \mathcal{L}(X)$ be a compact topological group of isometries acting on X. Assume that $C \subseteq X$ is a complete G-invariant subset. Let $x_0 \in C$ and $y \in X$ be G-invariant and such that the segment $[x_0, y]$ is not contained in C. Then, for each $\rho > 0$, there exists a G-invariant point $a \in C$ such that*

$$\|x_0 - a\| \leq \|x_0 - y\| + \rho, \text{ and } y \notin a + K_C(a).$$

Proof. Without loss of generality we may assume that $x_0 = 0$. Let $\lambda \geq 1$ such that $y = \lambda z$ with $z \in (E \backslash C)_G$. Let $r \in]0, \rho]$ with $r < d(z, C)$ and define $B = B(z, r)$. Then, by Theorem 13, we know that there exists a G-invariant point $a \in C \cap D(x_0, B)$ such that $\{a\} = C \cap D(a, B)$.

Let us write now $a = \alpha(z + b)$ with $\alpha \in [0, 1[$ and $\|b\| \leq r$, and define $t = \frac{\lambda - 1}{\lambda - \alpha}$. It is clear that $t \in [0, 1[$. Define now $w = ta + (1 - t)y$. Since $t, \alpha \in [0, 1[$ and $\|b\| \leq r$, we obtain that

$$\|w - z\| = \|t\alpha b + (t\alpha + (1 - t)\lambda - 1)z\| = t\alpha\|b\| < r.$$

Hence $w \in B$ and

$$(y - a) = (1 - t)^{-1}(w - a) \in \mathbb{R}_+(B \backslash \{a\}) \subseteq E \backslash K_C(a),$$

as $(a +]0, 1[(B - a)) \cap C = \emptyset$. Finally

$$\|x_0 - a\| \leq \operatorname{diam}(x_0, B) \leq \|x_0 - z\| + r \leq \|x_0 - y\| + \rho.$$

□

Remark 22. *Observe that the condition on y being G-invariant cannot be removed within this proof, since we define the point z so that $y = \lambda z$, and we use in this proof that z is G-invariant. If y were not G-invariant, then the only G-invariant point containing the line $y = \lambda z$ would be the constant zero.*

We can proceed similarly but, applying now Theorem 18 to obtain the following result.

Theorem 23. *Let X be a normed space and $G \subseteq \mathcal{L}(X)$ be a compact topological group of isometries acting on X. Assume that $C \subseteq X$ is a complete G-invariant subset. Let $x_0 \in C$ be G-invariant, and choose $y \in X$ such that the segment $[x_0, y]$ is not contained in C. Then, for each $\rho > 0$ and for each $g \in G$, there exists $g(a) \in C$ such that*

$$\|x_0 - g(a)\| \leq \|x_0 - g(y)\| + \rho, \text{ and } g(y) \notin g(a) + K_C(g(a)).$$

Moreover, if $s_G > 2d(z, C)$, where z is such that $z \in (E \backslash C)_G$ and $y = \lambda z$, then for every $g, g' \in G$, we have that

$$g(a) + K_C(g(a)) \cap g'(a) + K_C(g'(a)) = \emptyset.$$

5.2. **Partial differential equations.** Finally, we want to give some applications of the group invariant Ekeland's variational principle to the area of partial differential equations. Let us start fixing some notation. During this section $\Omega \subseteq \mathbb{R}^n$ is going to be an open-bounded subset of $\mathbb{R}^n$ with regular boundary. As usual, $W^{1,1}(\Omega)$ will be the Sobolev space of $L^1(\Omega)$, functions whose first weak derivatives are also in $L^1(\Omega)$, and $W_0^{1,1}(\Omega)$ will be the subset of functions of $W^{1,1}(\Omega)$ that vanish on the boundary of Ω. Also $W^{-1,q}(\Omega)$ will be the dual set of $W_0^{1,p}(\Omega)$ for $1 \leq p \leq \infty$, where q denotes the conjugate exponent of p, i.e, $\frac{1}{p} + \frac{1}{q} = 1$.

If $v \in W^{1,1}(\Omega)$, we will denote its weak derivative with respect to x_i by $\frac{\partial v}{\partial x_i}$, and ∇v is going to denote the gradient in the weak sense. If we do not say otherwise, all the derivatives that appear in this section should be understood in the sense of distributions.

Remark 24. *Observe that if a function, say f, is differentiable, then the weak derivative and the directional derivative coincide. This is the reason why we denote the weak derivative in the same way as the partial derivative.*

We will endow $W^{1,p}(\Omega)$ with the norm:

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p.$$

In all the applications we are going to study during this section, the group G will be a group of permutations on $\mathbb{R}^n$.

5.2.1. *Plateau problem.* Let us present a result that will be useful in Theorem 26, and whose proof can be found in [9, Corollary 15]

Lemma 25. *Let X be a Banach space and $G \subseteq \mathcal{L}(X)$ be a compact topological group of isometries acting on X. Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, Gâteaux differentiable, bounded below, G-invariant, convex with respect to G, and so that there exists constants $k, c > 0$ with*

$$f(x) \geq k\|x\| + c \quad \forall\, x \in X. \tag{11}$$

Then, the range of $\delta f(x)$ is dense in kB_G, where B_G is the closed unit ball in X_G^*.

The first result of this section guarantees that the perturbed Plateau problem, when perturbing by a G-invariant function, has a unique G-invariant solution.

Theorem 26. *Suppose Ω, and $v_0 \in W_0^{1,1}(\Omega)$ are G-invariant. Then, there exists in $W^{-1,\infty}(\Omega)$ a neighbourhood of the origin, and a dense subset $\mathcal{T}$ in this neighbourhood, such that, for every G-invariant $T \in \mathcal{T}$, the perturbed minimal hypersurface equation*

$$T = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\nabla v}{(1+|\nabla v|^2)^{\frac{1}{2}}},$$

$$v - v_0 \in W_0^{1,1}(\Omega),$$

and the perturbed Plateau's problem

$$\inf \left(\int_\Omega 1 + |\nabla v|^2 dx \right)^{\frac{1}{2}} - \langle T, v \rangle,$$

$$v - v_0 \in W_0^{1,1}(\Omega),$$

both have a unique G-invariant solution.

Proof. Define by

$$F(v) = \left(\int_\Omega 1 + |\nabla v + \nabla v_0|^2 dx \right)^{\frac{1}{2}},$$

which is the function to be minimized on $W_0^{1,1}(\Omega)$. It is known that this function is convex, continuous, and Gâteaux differentiable, with derivative

$$F'(v) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\nabla v - \nabla v_0}{(1+|\nabla v + \nabla v_0|^2)^{\frac{1}{2}}} \in W^{-1,\infty}(\Omega).$$

Observe that, since F is convex, in particular, is convex with respect to G.

Now we want to check that F is G-invariant. Observe that, since v_0 is G-invariant, so is ∇v_0. Therefore,

$$\begin{aligned} F(g(v)) &= \left(\int_\Omega 1 + |\nabla g \circ v(x) + \nabla g \circ v_0(x)|^2 dx \right)^{\frac{1}{2}} \\ &= \left(\int_\Omega 1 + |\nabla v(g(x)) + \nabla v_0(g(x))|^2 dx \right)^{\frac{1}{2}} \\ &= \left(\int_\Omega 1 + |\nabla v(x) + \nabla v_0(x)|^2 dx \right)^{\frac{1}{2}} = F(v), \end{aligned}$$

since Ω is G-invariant.

Finally, we want to see that F satisfies equation (11) so we can apply Lemma 25. Observe that

$$\int_\Omega |\nabla v| dx - \int_\Omega |\nabla v_0| dx \leq \int_\Omega |\nabla v + \nabla v_0| dx = \|\nabla v + \nabla v_0\|_1 \leq \|\nabla v + \nabla v_0\|_2 =$$

$$= \left(\int_\Omega |\nabla v + \nabla v_0|^2 dx \right)^{\frac{1}{2}} \leq \left(\int_\Omega 1 + |\nabla v + \nabla v_0|^2 dx \right)^{\frac{1}{2}},$$

where we have used the triangle inequality and Hölder's inequality respectively. But now, by Poincaré's inequality, see [2, Corollary 9.19] we know that there exists a constant $C > 0$ such that

$$\|u\|_1 \leq C\|\nabla u\|_1 \quad \forall\, u \in W_0^{1,1}(\Omega).$$

In particular

$$\|\nabla u\|_1 \geq \frac{1}{C+1}\|u\|_{1,1} \quad \forall\, u \in W_0^{1,1}(\Omega).$$

Therefore

$$F(v) = \left(\int_\Omega 1 + |\nabla v + \nabla v_0|^2 dx\right)^{\frac{1}{2}} \geq \frac{1}{C+1}\|v\|_{1,1} - K,$$

where the constant $K = \|\nabla v_0\|_1$. Applying now Lemma 25 we deduce that there exists a dense subset $\mathcal{T}$ such that $\forall T \in \mathcal{T}$, $F'(v) = T$ has some solution $v \in W_0^{1,1}(\Omega)$. Finally, define for any $T \in W^{1,1}(\Omega)$

$$F_T(v) = F(v) - \langle T, v\rangle.$$

Then, for any $T \in \mathcal{T}$ there exists $v_T \in W_0^{1,1}(\Omega)$ such that $F_T'(v_T) = 0$. But since F_T is strictly convex, then v_T is the unique minimum of F_T in $W_0^{1,1}(\Omega)$. □

5.2.2. *General patial differential equations.* Let us start recalling the G-invariant version of the Palais-Smale minimizing sequences that can be found in [9, Corollary 14], and will be very useful for the proof of Theorem 28

Lemma 27. *Let X be a Banach space and $G \subseteq \mathcal{L}(X)$ be a compact topological group of isometries acting on X. Let $\varphi\colon X \to \mathbb{R}$ be Gâteaux differentiable, bounded below, G-invariant and convex with respect to the group. Then, there exists a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ such that*

(1) x_n is G-invariant for all $n \in \mathbb{N}$,
(2) $\varphi(x_n) \to \inf\{\varphi(x) \mid x \in X\}$,
(3) $\|\delta\varphi(x_n)\| \to 0$.

Using this lemma, we can obtain the following application.

Theorem 28. *Let $\Omega \subseteq \mathbb{R}^n$ be a G-invariant subset, and let $p \in]1, +\infty[$. Suppose $f\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a borelian function such that*

(i) $f(x,\xi) \geq 0$.
(ii) $\xi \mapsto f(x,\xi)$ is a C^1 function.
(iii) For given constants $a, b \geq 0$, f satisfies a growth condition: $|f_\xi'(x,\xi)| \leq a + b|\xi|^{p-1}$ for all $\xi \in \mathbb{R}^n$.

Suppose also that there exists a G-invariant function $v_0 \in W_0^{1,p}(\Omega)$ such that $\int_\Omega f(x, \nabla v_0(x))dx < +\infty$. Then, for all $\epsilon > 0$ there exists a G-invariant function $u_\epsilon \in W^{1,p}(\Omega)$ such that

(1) $\left\|\sum_{i=1}^n \frac{\partial}{\partial x_i}\frac{\partial f}{\partial \xi_i}(\cdot, \nabla u_\epsilon(\cdot))\right\|_{-1,q} \leq \epsilon.$

(2) $\int_\Omega \sum_{i=1}^n \left|\frac{\partial u_\epsilon}{\partial x_i}\right|^p dx = \alpha.$

Proof. Define the function

$$H(u) = \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p dx - \alpha \quad \forall\, u \in W_0^{1,p}(\Omega).$$

It is known that this function is a C^1 function on $W_0^{1,p}(\Omega)$, finite everywhere, with derivative

$$H'(u) = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right),$$

see [6, Lemma 4.2]. Observe now that H is G-invariant, since

$$\begin{aligned}
H(g^*(u)) &= \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\frac{\partial g^*(u)}{\partial x_i}(x)\right|^p dx - \alpha \\
&= \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}(g(x))\right|^p dx - \alpha \\
&= \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}(x)\right|^p dx - \alpha = H(u),
\end{aligned}$$

for all $u \in W_0^{1,p}(\Omega)$, and every $g \in G$. Now, it only remains to check that H is convex with respect to G. For given $u \in W_0^{1,p}(\Omega)$, observe that

$$H(\overline{u}) = \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\frac{\partial \overline{u}}{\partial x_i}(x)\right|^p dx - \alpha = \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\sum_{g\in G}\frac{1}{|G|}\frac{\partial g^*(u)}{\partial x_i}(x)\right|^p dx - \alpha.$$

Applying now the following inequality

$$|a+b|^p \le (|a|+|b|)^p \quad \text{for } p \ge 1,$$

it follows that

$$\begin{aligned}
\frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\sum_{g\in G}\frac{1}{|G|}\frac{\partial g^*(u)}{\partial x_i}(x)\right|^p dx - \alpha &\le \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\sum_{g\in G}\frac{1}{|G|}\left|\frac{\partial g^*(u)}{\partial x_i}(x)\right|\right|^p dx - \alpha \\
&\le \frac{1}{p}\int_\Omega \sum_{i=1}^n \sum_{g\in G}\frac{1}{|G|}\left|\frac{\partial g^*(u)}{\partial x_i}(x)\right|^p dx - \alpha \\
&= \sum_{g\in G}\frac{1}{|G|}\frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\frac{\partial g^*(u)}{\partial x_i}(x)\right|^p dx - \alpha \\
&= \sum_{g\in G}\frac{1}{|G|}H(g^*(u)) = H(u),
\end{aligned}$$

where we have used in the second inequality that the power function x^p is convex on $\mathbb{R}^+$ and Jensen's inequality.

So we have shown that the inequality $H(\overline{u}) \le H(u)$ holds, therefore H is convex with respect to the group. Applying now Lemma 27, there exists a sequence of G-invariant points $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ such that

$$H(u_n) \to \inf\left\{H(v) \mid u \in W_0^{1,p}(\Omega)\right\}, \tag{12}$$

$$\|\partial H(u_n)\| \to 0. \tag{13}$$

From (13) one deduces directly (1). Observe that if we assume that H is bounded below by 0, then, from (12) we obtain (2). $\square$

5.2.3. *Control theory.* Let K be a G-invariant compact metrizable convex set, and consider the differential equation

$$\begin{cases} \frac{dx}{dt}(t) &= f(t,x(t),u(t)), \\ x(0) &= x_0, \end{cases} \tag{14}$$

where $x_0 \in \mathbb{R}^n$ and $f\colon [0,T] \times \mathbb{R}^n \times K \to \mathbb{R}^n$, for fixed $T \in \mathbb{R}$. Assume that the following properties are satisfied:

(i) f is continuous, G-invariant on the 3rd coordinate, and convex with respect to the group on the 3rd coordinate.
(ii) $\frac{\partial f}{\partial x_i}$ is well defined and is continuous for all $1 \le i \le n$.
(iii) There exists a $C > 0$ such that $\langle x, f(t,x,u)\rangle \le C(1+|x|^2)$.

We have the following result.

Theorem 29. *Suppose f satisfies the previous assumptions, and let $h\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Then, for all $\epsilon > 0$, there exists a G-invariant measurable control v, whose trajectory is y, such that*

$$\begin{cases} h(y(T)) \le \inf h(x(T)) + \epsilon, \\ \langle f(t,y(t),v(t)), p(t)\rangle \le \min_{u\in K} \langle f(t,y(t),u(t)), p(t)\rangle + \epsilon, \end{cases}$$

where p is the solution of the differential system

$$\begin{cases} \frac{dp_i}{dt}(t) = -\sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(t,y(t),v(t))p_j(t) \quad \forall 1 \le i \le n, \\ p(T) = h'(y(T)). \end{cases}$$

Proof. Devine V as the set of all G-invariant measurable controls $w\colon [0,T] \to K$, that is, all measurable controls $w\colon [0,T] \to K$ such that $g\big(w([0,T])\big) = w([0,T])$. Consider the following metric on V,

$$d(w_1,w_2) = \operatorname{meas}\{t \in [0,T] \mid w_1(t) \ne w_2(t)\}.$$

Then, V is a complete metric space, see [7, Lemma 10]. Now, consider the function

$$\begin{array}{cccc} F\colon & V & \to & \mathbb{R} \\ & u & \mapsto & h(x(T)), \end{array}$$

where x is the solution of the differential equation (14). It is well-known that F is continuous and bounded below, see [6, Lemma 7.3]. Let's see that F is G-invariant. Observe that we can express x as follows:

$$x(t) = x_0 + \int_0^t f(t,x(t),u(t))dt.$$

Hence, it is clear that x has some dependence on u, to highlight it, we are going to denote it by x_u. Therefore, by G-invariance of f, we have that

$$F(g^*(u)) = h(x_{g^*(u)}(t)) = h\left(x_0 + \int_0^t f(t,x(t),g(u(t)))dt\right) =$$

$$= h\left(x_0 + \int_0^t f(t,x(t),u(t))dt\right) = h(x_u(t)) = F(u).$$

Finally, we have to show that F is convex with respect to G. Again by convexity of f with respect to G, and using the convexity of K, it is clear that

$$\begin{aligned}
F\left(\frac{1}{|G|}\sum_{g\in G} g^*(u)\right) &= h\left(x_{\frac{1}{|G|}\sum_{g\in G} g^*(u)}(t)\right) \\
&= h\left(x_0 + \int_0^t f\left(t, x(t), \frac{1}{|G|}\sum_{g\in G} g(u(t))\right) dt\right) \\
&\leq h\left(x_0 + \frac{1}{|G|}\sum_{g\in G}\int_0^t f\left(t, x(t), g(u(t))\right) dt\right) \\
&= h\left(x_0 + \int_0^t f(t, x(t), u(t))dt\right) = h(x_u(t)) = F(u).
\end{aligned}$$

Applying Ekeland's variational principle, we know that there exists a G-invariant point $v \in V$ such that

$$F(v) \leq \inf_V F + \epsilon,$$

$$F(u) \geq F(v) - \epsilon d(u, v) \quad \forall\, u \in V.$$

From the first inequality it is clear that

$$g(v(T)) \leq \int g(x(T)) + \epsilon.$$

Taking now $t_0 > 0$ and $k_0 \in K$, define $u_\tau \in V$ for $\tau \geq 0$ as

$$u_\tau(t) = k_0 \quad \text{if } t \in [0, T] \cap]t_0 - \tau, t_0[,$$

$$u_\tau(t) = v(t) \quad \text{if } t \notin [0, T] \cap]t_0 - \tau, t_0[.$$

Clearly, if τ is sufficiently small, $d(u_\tau, v) = \tau$. Let us denote by x_τ the associated trajectory to u_τ, then $u_0 = v$ and $x_0 = y$, then taking u_τ in the second inequality of Ekeland, we obtain that

$$g(x_\tau(T)) - g(y(T)) \geq -\epsilon\tau \quad \forall\, \tau \geq 0.$$

Hence,

$$\frac{dg}{d\tau}(x_\tau(T))_{|\tau=0} \geq -\epsilon.$$

But it is known, see [7, Theorem 9], that

$$\frac{dg}{d\tau}(x_\tau(T))_{|\tau=0} = \langle f(t_0, y(t_0), k_0) - f(t_0, y(t_0), v(t_0)), p(t_0)\rangle.$$

Thus

$$\langle f(t_0, y(t_0), k_0) - f(t_0, y(t_0), v(t_0)), p(t_0)\rangle \geq -\epsilon,$$

which is the desired, since k_0 is any G-invariant point of K and t_0 is almost every point of $[0, T]$. $\square$

References

[1] R. Aron, J. Falcó, and M. Maestre, *Separation theorems for group invariant functionals*. In: Journal of geometric analysis, 28.1 (2018), pp. 303-315.

[2] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, 2010.

[3] D. Cohn, *Measure theory*, Birkhäuser, 2010.

[4] D. Cox, J. Little and D. Oshea, *Ideals, varieties and algorithms: an introduction to computational algebraic geometry and commutative algebra*, Springer Science & Buisness Media, 2013.

[5] S. Dantas, J. Falcó, and M. Jung, *Group invariant operators and some applications to norm-attaining theory*. In: Results in mathematics, 78.1 (2023), pp. 1-30.

[6] I. Ekeland, *On the variational principle*. In: Journal of mathematical analysis and applications, 47.2 (1974), pp. 324-353.

[7] I. Ekeland, *Nonconvex minimization problems*. In: Bulletin of the American Mathematical Society, 1.3 (1979), pp. 443-474.

[8] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach space theory: the basis for linear and nonlinear analysis*, Springer, 2011.

[9] J. Falcó, and D. Isert, *Group invariant variational principles*, to appear in RACSAM.

[10] P.M. Fitzpatrick, and H.L. Royden, *Real analysis*, China Machine Press, 2010.

[11] J. P. Penot, *The drop theorem, the petal theorem and Ekeland variational principle*. In: Nonlinear analysis theory, methods and applications, 10 (1986), pp. 813-822.

[12] W. Oettli, and M. Théra, *Equivalents of Ekeland's variational principle*. In: Bulletin of the Australian Mathematical Society, 48 (1993), pp. 385-392.

(Javier Falcó) Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain

Email address: `francisco.j.falco@uv.es`

(Daniel Isert) Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain

Email address: `daniel.isert@uv.es`