A consistent kinetic Fokker-Planck model for gas mixtures

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Abstract: Rarefied gas dynamics is usually described by the Boltzmann equation. Unfortunately, the expense of evaluating this operator can be very prohibitive. This made it worthwhile to look for approximations that convey essentially an equivalent amount of physical information. One widely known approximative collision operator is the Bathnagar-Gross-Krook (BGK) operator. However, recently, the Foker-Planck approximation has become increasingly popular. Nevertheless, the modeling of gas mixtures in the context of the kinetic Fokker-Planck equation has so far only been addressed in a very few papers. In this paper, we propose a general multi-species Fokker-Planck model. We prove consistency of our model: conservation properties, positivity of all temperatures, H-Theorem and the shape of equilibrium as Maxwell distributions with the same mean velocity and temperature. Moreover, we derive the usual macroscopic equations.

1 Introduction

Rarefied gas dynamics for gas mixtures consisting of N species is usually described by the Boltzmann equation with binary interactions as in [12], chapter 6.2

$$\partial_t f_i + v \cdot \nabla_x f_i = Q_{ii}(f_i, f_i) + \sum_{j=1, j\neq i}^{N} Q_{ij}(f_i, f_j)$$

Here, $f_i(x, v, t) > 0, \; i = 1, ...N$ is the distribution function of species i where $x \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$ are the phase space variables in dimension $d \geq 1$ and $t \geq 0$ the time. The collision operator on the right-hand side consists of a term Q_{ii} describing the interactions of particles of the species with itself and a sum of collision operators $Q_{ij}, i \neq j$ describing the interactions of particles of the species i with particles of species j. Unfortunately, the expense of evaluating the Boltzmann operator can be prohibitive. Therefore one looks for approximations that convey essentially an equivalent amount of physical information. The most widely known of such an approximation is the Bhatnagar-Gross-Krook

(BGK) model [3]. The purpose of the BGK relaxation operator is to provide an approximation of the Boltzmann collision operator that is more computationally tractable, but still maintains important structural properties. In particular, it has the same collision invariants as the Boltzmann operator (which lead to conservation of the number of particles, momentum, and energy) and it satisfies an H-Theorem. In the multi-species case, these requirements are not as straightforward to satisfy, but it can be done. There are many BGK models for gas mixtures proposed in the literature [23, 30, 18, 16, 34, 25, 20, 11, 1], many of which satisfy these basic requirements and, in addition, are able to match some prescribed relaxation rates and/or transport coefficients that come from more complicated physics models or from experiment. Many of these approaches have been extended to accommodate ellipsoid statistical (ES-BGK) models, polyatomic molecules, chemical reactions, velocity dependent collision frequencies or quantum gases; see for example [9, 10, 26, 36, 19, 27, 28, 4, 5, 29, 24]. Many of them have the following structural similarity. Just like the Boltzmann equation for gas mixtures contains a sum of collision terms on the right-hand side, they also have a sum of BGK-type interaction terms in the relaxation operator.

$$Q_i^{BGK}(f_1, ... f_N) = \nu_{11} n_1 (M_1 - f_1) + \sum_{i=1, i\neq j}^{N} \nu_{ij} n_i (M_{ij} - f_i) \tag{1}$$

Here, M_i and M_{ij} denote locally Maxwell distribution functions of the form

$$M_i(x,v,t) = \frac{n_i}{\sqrt{2\pi \frac{T_i}{m_i}}^3} \exp(- \frac{|v-u_i|^2}{2 \frac{T_i}{m_i}}), \;\; i = 1,..,N \tag{2}$$

$$M_{ij}(x,v,t) = \frac{n_{ij}}{\sqrt{2\pi \frac{T_{ij}}{m_j}}^d} \exp(- \frac{|v-u_{ij}|^2}{2 \frac{T_{ij}}{m_i}}), \;\; i \neq j, \;\; i,j = 1,...N \tag{3}$$

where $\nu_{ij} n_i$ denote the collision frequencies of species i with species j. The parameters ν_{ij} are assumed to be positive and only depend on x and t. The parameters $n_i, n_{ij}, u_i, u_{ij}, T_i, T_{ij}$ are determined such that we have the following conservation properties: conservation of mass, momentum and energy of the individual species in interaction with the species itself:

1. $\int Q_{ii}^{BGK}(f_i, f_i) dv = 0 \quad \text{for} \quad i = 1, ..., N,$
2. $\int m_i v Q_{ii}^{BGK}(f_i, f_i) dv = 0 \quad \text{for} \quad i = 1, ..., N$
3. $\int m_i |v|^2 Q_{ii}^{BGK}(f_i, f_i) dv = 0 \quad \text{for} \quad i = 1, ..., N.$

Conservation of total mass, momentum and energy

1. $\int Q_{ij}^{BGK}(f_i, f_j) dv = 0$ for $i, j = 1, 2,$
2. $\int (m_1 v Q_{12}^{BGK}(f_1, f_2) + m_2 v Q_{21}^{BGK}(f_2, f_1)) dv = 0,$
3. $\int (m_1 |v|^2 Q_{12}^{BGK}(f_1, f_2) + m_2 |v|^2 Q_{21}^{BGK}(f_2, f_1)) dv = 0.$

For this, we relate the distribution functions to macroscopic quantities by mean-values of f_i

$$\int f_i(v) \begin{pmatrix} 1 \\ v \\ m_i|v-u_i|^2 \end{pmatrix} dv =: \begin{pmatrix} n_i \\ n_i u_i \\ 3 n_i T_i \end{pmatrix}, \tag{4}$$

where n_i is the number density, u_i the mean velocity and T_i the temperature which is related to the pressure p_i by $p_i = n_i T_i$. Note that in this paper we shall write T_i instead of $k_B T_i$, where k_B is Boltzmann's constant. A general BGK model for gas mixtures which contains most of the BGK models for gas mixtures in the literature is provided in [25]. We will introduce this model briefly in the following. If we assume

$$n_{12} = n_1 \quad \text{and} \quad n_{21} = n_2 \tag{5}$$

in (3), we have conservation of the number of particles, see Theorem 2.1 in [25]. If we further assume that u_{12} is a linear combination of u_1 and u_2

$$u_{12} = \delta u_1 + (1-\delta) u_2, \quad \delta \in \mathbb{R}, \tag{6}$$

then we have conservation of total momentum provided that

$$u_{21} = u_2 - \frac{m_1}{m_2} \varepsilon (1-\delta)(u_2 - u_1), \tag{7}$$

see Theorem 2.2 in [25]. If we further assume that T_{12} is of the following form

$$T_{12} = \alpha T_1 + (1-\alpha) T_2 + \gamma |u_1 - u_2|^2, \quad 0 \leq \alpha \leq 1, \gamma \geq 0, \tag{8}$$

then we have conservation of total energy provided that

$$\begin{aligned} T_{21} = \left[\frac{1}{d} \varepsilon m_1 (1-\delta) \left(\frac{m_1}{m_2} \varepsilon (\delta - 1) + \delta + 1 \right) - \varepsilon \gamma \right] |u_1 - u_2|^2 \\ + \varepsilon (1-\alpha) T_1 + (1 - \varepsilon(1-\alpha)) T_2, \end{aligned} \tag{9}$$

see Theorem 2.3 in [25]. In order to ensure the positivity of all temperatures, the parameters δ and γ are restricted to

$$0 \leq \gamma \leq \frac{m_1}{d} (1-\delta) \left[(1 + \frac{m_1}{m_2} \varepsilon) \delta + 1 - \frac{m_1}{m_2} \varepsilon \right], \tag{10}$$

and

$$\frac{\frac{m_1}{m_2} \varepsilon - 1}{1 + \frac{m_1}{m_2} \varepsilon} \leq \delta \leq 1, \tag{11}$$

see Theorem 2.5 in [25].

Moreover, it can be shown that this model satisfies an H-Theorem, see Theorem 2.4 in [25], meaning that we have the following inequality

$$\sum_{i=1,..N} \int Q_{ij}^{BGK}(f_i, f_j) \log f_i dv \leq 0$$

with equality if and only if f_i, f_j are Maxwell distributions with the same mean velocity and temperature.

In the following, we will briefly motivate the meaning and possible choices of the free parameters α, δ, γ, for more details see [32]. One possibility is that one can choose the parameters such that one can generate special cases in the literature [23, 30, 2, 16, 34, 13, 18, 11, 20]. For instance if one chooses $\varepsilon = 1$, $\delta = \frac{m_1}{m_1+m_2}$, $\alpha = \frac{m_1^2+m_2^2}{(m_1+m_2)^2}$ and $\gamma = \frac{m_1 m_2}{(m_1+m_2)^2}\frac{m_2}{d}$, one obtains the model by Hamel in [30]. In [19] also such relaxation parameters are used to fix in the continuum limit Fick‘s law for diffusion velocities and Newton's law for viscous stress in the relevant set of Navier-Stokes equations.

Another possibility is to choose the parameters in a way such that the macroscopic exchange terms of momentum and energy can be matched in a certain way for example that they coincide with the ones for the Boltzmann equation. For this, we first present the macroscopic equations with exchange terms of the BGK model (1). If we multiply the BGK model for gas mixtures by $1, m_j v, m_j \frac{|v|^2}{2}$ and integrate with respect to v, we obtain the following macroscopic conservation laws

$$\begin{aligned}
\partial_t n_1 + \nabla_x \cdot (n_1 u_1) &= 0, \\
\partial_t n_2 + \nabla_x \cdot (n_2 u_2) &= 0, \\
\partial_t (m_1 n_1 u_1) + \nabla_x \cdot \int m_1 v \otimes v f_1(v) dv + \nabla_x \cdot (m_1 n_1 u_1 \otimes u_1) &= f_{m_{1,2}}, \\
\partial_t (m_2 n_2 u_2) + \nabla_x \cdot \mathbb{P}_2 + \nabla_x \cdot (m_2 n_2 u_2 \otimes u_2) &= f_{m_{2,1}}, \\
\partial_t \left(\frac{m_1}{2} n_1 |u_1|^2 + \frac{3}{2} n_1 T_1 \right) + \nabla_x \cdot \int m_1 |v|^2 v f_1(v) dv &= F_{E_{1,2}}, \\
\partial_t \left(\frac{m_2}{2} n_2 |u_2|^2 + \frac{3}{2} n_2 T_2 \right) + \nabla_x \cdot \int m_2 |v|^2 v f_2(v) dv &= F_{E_{2,1}},
\end{aligned}$$

with exchange terms $f_{m_{i,j}}$ and $F_{E_{i,j}}$ given by

$$\begin{aligned}
f_{m_{1,2}} &= -f_{m_{2,1}} = m_1 \nu_{12} n_1 n_2 (1-\delta)(u_2 - u_1), \\
F_{m_{1,2}} &= -F_{m_{2,1}} \\
&= \left[\nu_{12} \frac{1}{2} n_1 n_2 m_1 (\delta - 1)(u_1 + u_2 + \delta(u_1 - u_2)) + \frac{1}{2} \nu_{12} n_1 n_2 \gamma (u_1 - u_2) \right] \cdot (u_1 - u_2) \\
&+ \frac{3}{2} \varepsilon \nu_{21} n_1 n_2 (1-\alpha)(T_2 - T_1).
\end{aligned}$$

Here, one can observe a physical meaning of α and δ. We see that α and δ show up in the exchange terms of momentum and energy as parameters in front of the relaxation of u_1 towards u_2 and T_1 towards T_2. So they determine, together with the collision frequencies, the speed of relaxation of the mean velocities and the temperatures to a common value.

Here now, as it is done in section 4.1 in [20] or section 4 in [32] and compare the relaxation rates in the space-homogeneous case to the relaxation rates for the space-homogeneous Boltzmann equation. In [20], they find values for ν_{kj} such that either the relaxation rate for the mean velocities or the relaxation for the temperatures coincides with the corresponding rate of the Boltzmann equation. But using the free parameters α, δ and γ one is able to match both of the relaxation rates at the same time.

Another approximate model is offered by a Fokker-Planck collision term

$$Q_i^{FP}(f_1, ...f_N) = c_{ii} \text{div}(\nabla_v (\frac{T_i}{m_i} f_i) + (v - u_i) f_i) + \sum_{j=1}^N c_{ij} \ \text{div}(\nabla_v (\frac{T_{ij}}{m_i} f_i) + (v - u_{ij}) f_i) \tag{12}$$

The quantity c_{ij} is a friction constant, see [37] for a motivation of the one species case. Concerning the literature on multi-species Fokker Planck models, there are less results than in the BGK case, but the interest in multi-species Fokker-Planck models has been increased more and more recently. Models for gas mixtures can be found in [15, 7, 17, 21] for different choices of u_{ij}, T_{ij}. We will discuss this later in section 2.3. The diffusion limit of a kinetic Fokker-Planck system for charged particles towards the Nernst-Planck equations was proved in [38]. Furthermore, in [15, 22], the limit of vanishing electron-ion mass ratios for non-homogeneous kinetic Fokker-Planck systems was investigated. In [7], the authors provide the first existence analysis of a multi-species Fokker-Planck system of the shape above. The works [17, 21] provide an extended Fokker-Planck model for hard-spheres gas mixtures with to be able to also capture correct diffusion coefficients, mixture viscosity and heat conductivity coefficients in the hydrodynamic regime of the Navier-Stokes equations.

The aim of this paper is to present a general multi-species Fokker-Planck model with collision terms of the shape (12) with free parameters α, δ, γ similar as this is done in [25] for the multi-species BGK model. More concrete, we want to characterize for which choice of the parameters u_{ij}, T_{ij} the conservation properties are satisfied, all temperatures are positive and we have an H-Theorem. The models [15, 7, 17, 21] can be shown to be a special case of this model presented here. This provides the possibility to create different exchange terms of momentum and energy in the macroscopic equations.

The outline of this paper is as follows: in section 2 we present the model for two species and prove conservation properties of this model in section 2.1, positivity of all temperatures in 2.2 and an H-Theorem in section 2.4. In section 2.3 we derive macroscopic equations and discuss several special cases in the literature [15, 7, 17, 21].

2 General multi-species Fokker-Planck model

In this section, we present a general multi-species Fokker-Planck model and consider for which choice of u_{ij}, T_{ij}, we have conservation properties, an entropy inequality, the expected shape in equilibrium (Maxwell distribution with common mean velocity and temperature) and positivity of all temperatures. For simplicity, we present this model for two-species, but everything can be extended to a general number of N species, since we made the assumption of only considering binary interactions. So in the rest of the paper, we consider the following system of Fokker-Planck equations

$$\partial_t f_1 + v \cdot \nabla_x f_1 = c_{11} n_1 \text{ div}(\nabla_v (\frac{T_1}{m_1} f_1) + (v - u_1) f_1) + c_{12} n_2 \text{div}(\nabla_v (\frac{T_{12}}{m_1} f_1) + (v - u_{12}) f_1)$$
$$\partial_t f_1 + v \cdot \nabla_x f_1 = c_{22} n_2 \text{ div}(\nabla_v (\frac{T_2}{m_2} f_2) + (v - u_2) f_2) + c_{21} n_1 \text{div}(\nabla_v (\frac{T_{21}}{m_2} f_2) + (v - u_{21}) f_2)$$

To be flexible in choosing the relation ship between the constants c_{12}, c_{21}, we now assume the relationship

$$c_{12} = \varepsilon \ c_{21}, \varepsilon \leq 1 \text{ and } \varepsilon \frac{m_1}{m_2} \leq 1 \tag{13}$$

Note, that the assumption on ε covers the two common cases in the literature for ε which are $\varepsilon = \frac{m_2}{m_1}$ and $\varepsilon = 1$ if the notation of 1 and 2 is chosen in a suitable way.

2.1 Conservation properties

This section shows how the macroscopic quantities u_{ij}, T_{ij} in the interspecies Maxwell distributions have to be chosen in order to ensure the macroscopic conservation properties. We note that the mass is automatically conserved.

Theorem 1 (Conservation of total momentum)**.** *Assume the condition* (13) *for the collision frequencies and that u_{12} is a linear combination of u_1 and u_2*

$$u_{12} = \delta u_1 + (1 - \delta) u_2, \quad \delta \in \mathbb{R}. \tag{14}$$

Then we have conservation of total momentum

$$\int m_1 v [Q_{11}^{FP}(f_1, f_1) + Q_{12}^{FP}(f_1, f_2)] dv + \int m_2 v [Q_{22}^{FP}(f_2, f_2) + Q_{21}^{FP}(f_2, f_1)] dv = 0,$$

provided that

$$u_{21} = u_2 - (1 - \delta) \varepsilon \frac{m_1}{m_2} (u_2 - u_1). \tag{15}$$

Proof. The flux of momentum of species 1 is given by

$$\begin{aligned} f_{m_{1,2}} := & m_1 c_{11} n_1 \int v \, \mathrm{div}(\nabla_v (\frac{T_1}{m_1} f_1) + (v-u_1) f_1) dv \\ & + c_{12} m_1 n_2 \int v \, \mathrm{div}(\nabla_v (\frac{T_{12}}{m_1} f_1) + (v-u_{12}) f_1) dv \\ & = -m_1 c_{12} n_2 \int (\nabla_v (\frac{T_{12}}{m_1} f_1) + (v-u_{12}) f_1) dv \\ & = m_1 c_{12} n_1 n_2 (u_{12} - u_1) = m_1 c_{12} n_1 n_2 (1-\delta)(u_2 - u_1). \end{aligned} \tag{16}$$

The flux of momentum of species 2 is given by

$$f_{m_{2,1}} = m_2 c_{21} n_2 n_1 (u_{21} - u_2). \tag{17}$$

In order to get conservation of momentum we therefore need

$$m_1 c_{12} n_1 n_2 (1-\delta)(u_2 - u_1) + m_2 c_{21} n_1 n_2 (u_{21} - u_2) = 0,$$

which holds provided u_{21} satisfies (15) □

Remark 1. *If we write $\tilde{\delta} = 1 - \frac{m_1}{m_2}\varepsilon(1-\delta)$ we obtain a similar structure for u_{21} as for u_{12}*

$$u_{21} = \tilde{\delta} u_2 + (1-\tilde{\delta}) u_1.$$

Theorem 2 (Conservation of total energy)**.** *Assume conditions* (14) *and* (15) *and assume that T_{12} is of the following form*

$$T_{12} = \alpha T_1 + (1-\alpha) T_2 + \gamma |u_1 - u_2|^2, \quad 0 \leq \alpha \leq 1, \gamma \geq 0. \tag{18}$$

Then we have conservation of total energy

$$\int \frac{m_1}{2} |v|^2 (Q_{11}^{FP}(f_1, f_1) + Q_{12}^{FP}(f_1, f_2)) dv + \int \frac{m_2}{2} |v|^2 (Q_{22}^{FP}(f_2, f_2) + Q_{21}^{FP}(f_2, f_1)) dv = 0,$$

provided that

$$T_{21} = \left[\frac{1}{d}\varepsilon m_1 (1-\delta) - \varepsilon\gamma\right] |u_1 - u_2|^2 + \varepsilon(1-\alpha) T_1 + (1 - \varepsilon(1-\alpha)) T_2. \tag{19}$$

Proof. Using the energy flux of species 1

$$\begin{aligned}
F_{E_{1,2}} &:= c_{11}n_1 \int \frac{m_1}{2}|v|^2 \ \mathrm{div}(\nabla_v(\frac{T_1}{m_1}f_1) + (v-u_1))f_1 dv \\
&+ c_{12}n_2 \int \frac{m_1}{2}|v|^2 \ \mathrm{div}(\nabla_v(\frac{T_{12}}{m_1}f_1) + (v-u_{12}))f_1 dv \\
&= -c_{12}n_2m_1 \int v \cdot (\nabla_v(\frac{T_{21}}{m_1}f_1) + (v-u_{12}))f_1 dv \\
&= c_{12}n_2m_1 d \int \frac{T_{12}}{m_1} f_1 dv - c_{12}n_2m_1 \int v \cdot (v-u_{12}) f_1 dv \\
&= c_{12}n_2dn_1T_{12} - c_{12}n_2m_1(dn_1\frac{T_1}{m_1} + n_1|u_1|^2 - n_1u_1 \cdot u_{12}) \\
&= c_{12}n_2dn_1T_{12} - c_{12}n_2n_1(dT_1 + m_1u_1 \cdot (u_1 - u_{12})) \\
&= c_{12}n_1n_2d(T_{12} - T_1) - c_{12}m_1n_1n_2u_1 \cdot (u_1 - u_{12}) \\
&= c_{12}n_1n_2(1-\alpha)(T_2 - T_1) - c_{12}m_1n_2n_1((1-\delta)u_1 \cdot (u_1 - u_2) + \gamma|u_1 - u_2|^2)
\end{aligned}$$

where we used (14) and (18). Analogously the energy flux of species 2 towards 1 is

$$\begin{aligned}
F_{E_{2,1}} &= c_{21}m_2n_1n_2(u_2 \cdot (u_{21} - u_2)) + d\ c_{21}n_1n_2(T_{21} - T_2) \\
&= c_{21}m_2n_1n_2(1-\delta)\frac{m_1}{m_2}\varepsilon(u_2 \cdot (u_1 - u_2)) + d\ c_{21}n_1n_2(T_{21} - T_2)
\end{aligned}$$

Here, we substituted u_{21} with (15). Adding these two terms, we see that the total energy is conserved provided that T_{21} is given by (19). □

Remark 2. *We have* $0 \leq 1 - \varepsilon(1-\alpha) \leq 1$ *and* $0 \leq \varepsilon(1-\alpha) \leq 1$*, so that in* (19) *the two terms with the temperatures are also a convex combination of* T_1 *and* T_2*.*

2.2 Positivity of the temperatures

Theorem 3. *Assume that* $f_1(x,v,t), f_2(x,v,t) > 0$*. Then all temperatures* T_1*,* T_2*,* T_{12} *given by* (18) *and* T_{21} *given by* (19) *are positive provided that*

$$0 \leq \gamma \leq \frac{m_1}{d}(1-\delta) \tag{20}$$

Proof. T_1 and T_2 are positive as integrals of positive functions. T_{12} is positive because by construction it is a convex combination of T_1 and T_2. For T_{21} we consider the coefficients in front of $|u_1-u_2|^2$, T_1 and T_2. The term in front of T_1 is positive by definition. The positivity of the term in front of T_2 is equivalent to the condition $\alpha \geq 1 - \frac{1}{\varepsilon}$, which is satisfied since $\varepsilon \leq 1$, the positivity of the term in front of $|u_1 - u_2|^2$ is equivalent to the condition (20). □

Remark 3. *According to the definition of* γ*,* γ *is a non-negative number, so the right-hand side of the inequality in* (20) *must be non-negative. This condition* *is equivalent to*

$$\delta \leq 1. \tag{21}$$

2.3 Macroscopic equations and exchange terms of momentum and energy

In this section, we deal with macroscopic equations, exchange terms of momentum and energy, and special cases in the literature. With a specific choice of the parameters we can generate special cases in the literature [15, 7, 17]. For instance, in [15] the mean mixture velocities and temperatures are chosen to be

$$u_{12} = u_{21} = \frac{u_1 + u_2}{2}; \quad T_{12} = T_{21} = \frac{m_2 T_1 + m_1 T_2}{m_1 + m_2} + \frac{m_1 m_2}{m_1 + m_2} \frac{1}{2d} |u_1 - u_2|^2$$

so we can generate this model by choosing

$$\alpha = \frac{m_2}{m_1 + m_2}, \quad \delta = \frac{1}{2}, \quad \gamma = \frac{1}{2d} \frac{m_1 m_2}{m_1 + m_2}$$

With the choice of

$$\alpha = \frac{c_{12} n_2}{c_{12} n_2 + c_{21} n_1}, \delta = \frac{c_{12} m_1 n_1}{c_{12} m_1 n_1 + c_{12} m_2 n_2}, \gamma = \frac{c_{12} m_1 n_1 c_{21} m_2 n_2}{d(c_{12} n_1 + c_{21} n_2)(c_{21} n_1 n_1 + c_{12} m_2 n_2)}$$

we can generate $u_{12}, u_{21}, T_{12}, T_{21}$ as in [7] given by

$$u_{12} = u_{21} = \frac{c_{21} m_1 n_1 u_1 + c_{12} m_2 n_2 u_2}{c_{12} m_2 n_2 + c_{21} m_1 n_1},$$
$$T_{12} = T_{21} = \frac{c_{21} n_1 T_1 + c_{12} n_2 T_2}{c_{12} n_2 + c_{21} n_1} + \frac{c_{12} m_1 n_1 c_{21} m_2 n_2}{d(c_{12} n_1 + c_{21} n_2)(c_{21} n_1 n_1 + c_{12} m_2 n_2)} |u_1 - u_2|^2$$

Another possibility for example is to choose as mixture velocity the velocity of the other species as it is done for example in [17] with $\delta = 0$ to have

$$u_{12} = u_2, u_{21} = u_1.$$

Another way to see the influence of the parameters is in the macroscopic exchange terms of momentum and energy. For this, we first present the macroscopic equations with exchange terms of the Fokker-Planck model (12). If we multiply the Fokker-Planck model for gas mixtures by $1, m_j v, m_j \frac{|v|^2}{2}$ and integrate with respect to v, we obtain the following macroscopic conservation laws

$$\begin{aligned}
\partial_t n_1 + \nabla_x \cdot (n_1 u_1) &= 0, \\
\partial_t n_2 + \nabla_x \cdot (n_2 u_2) &= 0, \\
\partial_t (m_1 n_1 u_1) + \nabla_x \cdot \int m_1 v \otimes v f_1(v) dv &= f_{m_{1,2}}, \\
\partial_t (m_2 n_2 u_2) + \nabla_x \cdot \int m_2 v \otimes v f_2(v) dv &= f_{m_{2,1}}, \\
\partial_t \left(\frac{m_1}{2} n_1 |u_1|^2 + \frac{3}{2} n_1 T_1 \right) + \nabla_x \cdot \int m_1 |v|^2 v f_1(v) dv &= F_{E_{1,2}}, \\
\partial_t \left(\frac{m_2}{2} n_2 |u_2|^2 + \frac{3}{2} n_2 T_2 \right) + \nabla_x \cdot \int m_2 |v|^2 v f_2(v) dv &= F_{E_{2,1}},
\end{aligned}$$

with exchange terms $f_{m_{i,j}}$ and $F_{E_{i,j}}$ given by

$$\begin{aligned}
f_{m_{1,2}} &= -f_{m_{2,1}} = m_1 c_{12} n_1 n_2 (1-\delta)(u_2 - u_1), \\
F_{E_{1,2}} &= -F_{E_{2,1}} \\
&= c_{12} n_1 n_2 m_1 (1-\delta) u_1 \cdot (u_2 - u_1) + \gamma \frac{d}{m_1} |u_1 - u_2|^2 \\
&+ d c_{12} n_1 n_2 (1-\alpha)(T_2 - T_1).
\end{aligned} \tag{22}$$

2.4 H-theorem for mixtures

In this section we will prove an H-Theorem for the model (12). For this, we make the following additional assumptions on the free parameters. We make the stronger assumptions of (18), (20), (21).

$$\frac{\varepsilon}{1+\varepsilon} \leq \alpha \leq 1, \quad \frac{\varepsilon}{1+\varepsilon} \leq \delta \leq 1, \quad (1-\delta)^2 \frac{m_1}{d} \leq \gamma \leq (1-\delta) \frac{m_1}{d} \frac{\varepsilon}{1+\varepsilon} \tag{23}$$

Moreover, in order to simplify the notation we define the following quantities

$$\gamma_1 := (1-\delta)^2 \frac{m_1}{d}; \quad \gamma_2 := (1-\delta)^2 \frac{m_2}{d} \varepsilon^2 (\frac{m_1}{m_2})^2, \quad \tilde{\gamma} := \frac{m_1}{d} \varepsilon (1-\delta) - \varepsilon \gamma \tag{24}$$

and the temperatures

$$\bar{T}_{12} = \alpha T_1 + (1-\alpha) T_2; \quad \bar{T}_{21} = \varepsilon (1-\alpha) T_1 + (1 - \varepsilon(1-\alpha)) T_2 \tag{25}$$

We start with some lemmas which we will need later for the proof of the H-Theorem.

Lemma 4. *Let M_{12}, M_{21} the two Maxwell distributions given by* (3)*. Then we have*

$$
\begin{aligned}
&\frac{T_{12}}{m_1} c_{12} n_2 \int \frac{M_{12}^2}{f_1} \left(\frac{\nabla_v f_1 M_{12} - \nabla_v M_{12} f_1}{M_{12}^2} \right)^2 dv \\
&+ \frac{T_{21}}{m_2} c_{21} n_1 \int \frac{M_{21}^2}{f_2} \left(\frac{\nabla_v f_2 M_{21} - \nabla_v M_{21} f_2}{M_{21}^2} \right)^2 dv \\
&= \frac{T_{12}}{m_1} c_{12} n_2 \int \frac{|\nabla_v f_1|^2}{f_1} dv + \frac{T_{21}}{m_2} c_{21} n_1 \int \frac{|\nabla_v f_2|^2}{f_2} dv \\
&+ c_{12} n_2 n_1 d \frac{T_1 + \frac{m_1}{d}(1-\delta)^2 |u_1 - u_2|^2}{T_{12}} + c_{21} n_2 n_1 d \frac{T_2 + \frac{m_2}{d} \varepsilon^2 (\frac{m_1}{m_2})^2 (1-\delta)^2 |u_1 - u_2|^2}{T_{21}} \\
&- 2(1+\varepsilon) c_{21} n_1 n_2 d
\end{aligned}
$$

Proof. We can compute

$$
\begin{aligned}
&\frac{T_{12}}{m_1} c_{12} n_2 \int \frac{M_{12}^2}{f_1} \left(\frac{\nabla_v f_1 M_{12} - \nabla_v M_{12} f_1}{M_{12}^2} \right)^2 dv + \frac{T_{21}}{m_2} c_{21} n_1 \int \frac{M_{21}^2}{f_2} \left(\frac{\nabla_v f_2 M_{21} - \nabla_v M_{21} f_2}{M_{21}^2} \right)^2 dv \\
&= \frac{T_{12}}{m_1} c_{12} n_2 \int \frac{|\nabla_v f_1|^2}{f_1} dv + \frac{T_{21}}{m_2} c_{21} n_1 \int \frac{|\nabla_v f_2|^2}{f_2} dv \\
&+ c_{12} n_2 \int \frac{|v - u_{12}|^2}{T_{12}/m_1} f_1 dv + c_{21} n_1 \int \frac{|v - u_{21}|^2}{T_{21}/m_1} f_2 dv \\
&+ 2 c_{12} n_2 \int \nabla_v f_1 \cdot (v - u_{12}) dv + 2 c_{21} n_1 \int \nabla_v f_2 \cdot (v - u_{21}) dv \\
&= \frac{T_{12}}{m_1} c_{12} n_2 \int \frac{|\nabla_v f_1|^2}{f_1} dv + \frac{T_{21}}{m_2} c_{21} n_1 \int \frac{|\nabla_v f_2|^2}{f_2} dv \\
&+ c_{12} n_2 n_1 d \frac{T_1 + \frac{m_1}{d} |u_1 - u_{12}|^2}{T_{12}} + c_{21} n_2 n_1 d \frac{T_2 + \frac{m_2}{d} \varepsilon^2 (\frac{m_1}{m_2})^2 (1-\delta)^2 |u_2 - u_{21}|^2}{T_{21}} \\
&- 2(1+\varepsilon) c_{21} n_1 n_2 d \\
&= \frac{T_{12}}{m_1} c_{12} n_2 \int \frac{|\nabla_v f_1|^2}{f_1} dv + \frac{T_{21}}{m_2} c_{21} n_1 \int \frac{|\nabla_v f_2|^2}{f_2} dv \\
&+ c_{12} n_2 n_1 d \frac{T_1 + \frac{m_1}{d}(1-\delta)^2 |u_1 - u_2|^2}{T_{12}} + + c_{21} n_2 n_1 d \frac{T_2 + \frac{m_2}{d} \varepsilon^2 (\frac{m_1}{m_2})^2 (1-\delta)^2 |u_1 - u_2|^2}{T_{21}} \\
&- 2(1+\varepsilon) c_{21} n_1 n_2 d
\end{aligned}
$$

Here, we used $\nabla_v M_{12} = -\frac{v - u_{12}}{T_{12}/m_1} M_{12}$ and the relationship (14) and (15) for u_{12}, u_{21}. □

Lemma 5. *We assume the estimate for α in* (23)*. Then we have*

$$
\varepsilon T_1 \bar{T}_{21} + \bar{T}_{12} T_2 \leq (1+\varepsilon) \bar{T}_{12} \bar{T}_{21} \tag{26}
$$

Proof. If we insert the expressions for $\bar{T}_{12}, \bar{T}_{21}$ given by (25) we get that (26) is equivalent to

$$(1-\alpha)\varepsilon(\alpha-(1-\alpha)\varepsilon)(T_1-T_2)^2 \geq 0$$

This is true if $1 \geq \alpha \geq \frac{\varepsilon}{1+\varepsilon}$ which we assumed in (23). $\square$

Lemma 6. *We assume* (23). *Then we have*

$$\varepsilon\gamma_1\tilde{\gamma} + \gamma\gamma_2 \leq (1+\varepsilon)\gamma\tilde{\gamma}$$

Proof. If we insert the expressions for $\gamma_1, \gamma_2, \tilde{\gamma}$ given by (24), we obtain

$$\begin{aligned} -((1+\varepsilon)\gamma(-\varepsilon\gamma+(1-\delta)\varepsilon\frac{m_1}{d})) + (1-\delta)^2\varepsilon\frac{m_1}{d}(-\varepsilon\gamma+(1-\delta)\varepsilon\frac{m_1}{d}) \\ +(1-\delta)^2\gamma\varepsilon^2\frac{m_1}{d}\frac{m_1}{m_2} \leq 0 \end{aligned}$$

This inequality is true if we can prove separately

$$\begin{aligned} \varepsilon\gamma(-\varepsilon\gamma+(1-\delta)\varepsilon\frac{m_1}{d}) \geq (1-\delta)^2\varepsilon\frac{m_1}{d}(-\varepsilon\gamma+(1-\delta)\varepsilon\frac{m_1}{d}) \\ \varepsilon\gamma^2-(1-\delta)\varepsilon\frac{m_1}{d}\gamma+(1-\delta)^2\varepsilon^2\gamma\frac{m_1}{d}\frac{m_1}{m_2} \leq 0 \end{aligned} \tag{27}$$

We start with the first inequality. The factor $-\varepsilon\gamma+(1-\delta)\varepsilon\frac{m_1}{d}$ is non-negative, since this is the condition ensuring positivity of the temperatures (21). Therefore, we get that γ has to satisfy

$$\gamma \geq (1-\delta)^2\frac{m_1}{d}$$

as assumed in (23). This is possible and no restriction to the upper bound ensuring the positivity (21), since we assumed $0 \leq \delta \leq 1$.
Now, for the second inequality in (27), we divide by $\varepsilon\gamma$ to get

$$\gamma-(1-\delta)\frac{m_1}{d}+(1-\delta)^2\varepsilon\frac{m_1}{d}\frac{m_1}{m_2} \leq 0$$

which is satisfied if

$$\gamma \geq (1-\delta)\frac{m_1}{d}((1-\delta)\varepsilon\frac{m_1}{m_2}-1)$$

This is satisfied due to the estimate on γ in (23) and assumption (13). $\square$

Lemma 7. *We assume* (23). *Then, we have*

$$\begin{aligned} \varepsilon T_1\tilde{\gamma}|u_1-u_2|^2+\varepsilon\gamma_1|u_1-u_2|^2\bar{T}_{21}+\bar{T}_{12}\gamma_2|u_1-u_2|^2+\gamma|u_1-u_2|^2T_2 \\ \leq (1+\varepsilon)\bar{T}_{12}\tilde{\gamma}|u_1-u_2|^2+(1+\varepsilon)\gamma|u_1-u_2|^2\bar{T}_{21} \end{aligned}$$

Proof. We insert the expressions for $\bar{T}_{12}, \bar{T}_{21}$ given by (25) and get

$$\begin{aligned}
&\varepsilon T_1 \tilde{\gamma}|u_1-u_2|^2 + \varepsilon\gamma_1|u_1-u_2|^2(\varepsilon(1-\alpha)T_1 + (1-\varepsilon(1-\alpha))T_2) \\
&+ (\alpha T_1 + (1-\alpha)T_2)\gamma_2|u_1-u_2|^2 + \gamma|u_1-u_2|^2 T_2 \\
&\leq (1+\varepsilon)(\alpha T_1 + (1-\alpha)T_2)\tilde{\gamma}|u_1-u_2|^2 \\
&+ (1+\varepsilon)\gamma|u_1-u_2|^2(\varepsilon(1-\alpha)T_1 + (1-\varepsilon(1-\alpha))T_2)
\end{aligned}$$

We compare the coefficients in front of T_1 and T_2 and obtain the inequalities

$$\begin{aligned}
\varepsilon\tilde{\gamma} + \varepsilon\gamma_1\varepsilon(1-\alpha) + \alpha\gamma_2 &\leq (1+\varepsilon)\alpha\tilde{\gamma} + (1+\varepsilon)\gamma\varepsilon(1-\alpha) \\
\varepsilon\gamma_1(1-\varepsilon(1-\alpha)) + (1-\alpha)\gamma_2 + \gamma &\leq (1+\varepsilon)(1-\alpha)\tilde{\gamma} + (1+\varepsilon)(1-\varepsilon(1-\alpha))\gamma
\end{aligned}$$

We start with the first inequality. According to the definition of γ_1, γ_2 given by (24) and the lower bound on γ given by (21) and assumption (13), we have

$$\gamma_1 \leq \gamma \text{ and } \gamma_2 \leq \gamma$$

Additionally, we observe that

$$\tilde{\gamma} = \varepsilon(1-\delta)\frac{m_1}{d} - \varepsilon\gamma \geq \gamma$$

since we assumed the stricter upper bound on γ in (23). The stricter upper bound on γ is not a contradiction to the lower bound since we assumed $\delta \geq \frac{\varepsilon}{1+\varepsilon} in$(23). All in all, this leads to

$$\begin{aligned}
\varepsilon\gamma_1\varepsilon(1-\alpha) + \alpha\gamma_2 \leq (\varepsilon^2(1-\alpha)+\alpha)\gamma &= (\varepsilon^2(1-\alpha) + \varepsilon(1-\alpha))\gamma + (\alpha - \varepsilon(1-\alpha))\gamma \\
&\leq (\varepsilon^2(1-\alpha) + \varepsilon(1-\alpha))\gamma + (\alpha - \varepsilon(1-\alpha))\tilde{\gamma}
\end{aligned}$$

which corresponds to the first inequality. The last inequality is possible since we assumed $\alpha \geq \frac{\varepsilon}{1+\varepsilon}$ in (23) and therefore the coefficient $\alpha - \varepsilon(1-\alpha)$ is non-negative. In a similar way, one can prove the second inequality. □

Theorem 8 (H-theorem for mixture)**.** *Assume $f_1, f_2 > 0$. Assume the relationship between the collision frequencies* (13) *, the conditions for the interspecies Maxwellians* (5) *,* (14)*,* (15)*,* (18) *and* (19) *with $\alpha, \delta \neq 1$, the positivity of the temperatures* (20) *and the assumptions on the parameters* (23)*, then*

$$\int (\ln f_1)\ Q_{11}^{FP}(f_1,f_1) + (\ln f_1)\ Q_{12}^{FP}(f_1,f_2)dv + \int (\ln f_2)\ Q_{22}^{FP}(f_2,f_2) + (\ln f_2)\ Q_{21}^{FP}(f_2,f_1)dv \leq 0,$$

with equality if and only if f_1 and f_2 are Maxwell distributions with equal velocity and temperature.

Proof. The fact that $\int \ln f_k Q(f_k, f_k) dv \leq 0,\ k = 1, 2$ is shown in proofs of the H-theorem of the single Fokker-Planck-model, for example in [33]. In both cases we have equality if and only if $f_1 = M_1$ and $f_2 = M_2$.

Let us define

$$\begin{aligned}
I := &\int Q_{12}^{FP}(f_1, f_2) \ln f_1 dv + \int Q_{21}^{FP}(f_2, f_1) \ln f_2 dv \\
= &\int c_{12} n_1 \ \mathrm{div}(\nabla_v (\frac{T_{12}}{m_1} f_1) + (v - u_{12}) f_1) \ln f_1 dv \\
+ &\int c_{21} n_2 \mathrm{div}(\nabla_v (\frac{T_{21}}{m_2} f_2) + (v - u_{21}) f_2) \ln f_2 dv
\end{aligned}$$

Integration by parts leads to

$$\begin{aligned}
I = &- \int c_{12} n_1 (\nabla_v (\frac{T_{12}}{m_1} f_1) + (v - u_{12}) f_1) \frac{\nabla_v f_1}{f_1} dv \\
&- \int c_{21} n_2 (\nabla_v (\frac{T_{21}}{m_2} f_2) + (v - u_{21}) f_2) \frac{\nabla_v f_2}{f_2} dv \\
&= \int c_{12} n_1 \frac{T_{12}}{m_1} f_1 |\frac{\nabla_v f_1}{f_1}|^2 - \int c_{21} n_2 \frac{T_{21}}{m_2} f_2 |\frac{\nabla_v f_2}{f_2}|^2 dv \\
&- \int c_{12} n_2 (v - u_{12}) \cdot \nabla_v f_1 dv - \int c_{21} n_1 (v - u_{21}) \cdot \nabla_v f_2 dv \\
&= - \int c_{12} n_2 \frac{T_{12}}{m_1} f_1 |\frac{\nabla_v f_1}{f_1}|^2 dv - \int c_{21} n_2 \frac{T_{21}}{m_2} f_2 |\frac{\nabla_v f_2}{f_2}|^2 dv + c_{12} n_2 n_1 d + c_{21} n_1 n_2 d
\end{aligned}$$

By using the relationship (13), we obtain

$$I := - \int c_{12} n_2 \frac{T_{12}}{m_1} f_1 |\frac{\nabla_v f_1}{f_1}|^2) dv - \int c_{21} n_2 \frac{T_{21}}{m_2} f_2 |\frac{\nabla_v f_2}{f_2}|^2) dv + c_{21} n_2 n_1 d (1 + \varepsilon)$$

By using lemma 4, we can write this as

$$\begin{aligned}
I = &- \frac{T_{12}}{m_1} c_{12} n_2 \int \frac{M_{12}^2}{f_1} \left(\frac{\nabla_v f_1 M_{12} - \nabla_v M_{12} f_1}{M_{12}^2} \right)^2 dv \\
&- \frac{T_{21}}{m_2} c_{21} n_1 \int \frac{M_{21}^2}{f_2} \left(\frac{\nabla_v f_2 M_{21} - \nabla_v M_{21} f_2}{M_{21}^2} \right)^2 dv \\
&+ c_{12} n_2 n_1 d \frac{T_1 + \frac{m_1}{d}(1-\delta)^2 |u_1 - u_2|^2}{T_{12}} + c_{21} n_2 n_1 d \frac{T_2 + \frac{m_2}{d} \varepsilon^2 (\frac{m_1}{m_2})^2 (1-\delta)^2 |u_1 - u_2|^2}{T_{21}} \\
&- (1 + \varepsilon) c_{21} n_1 n_2 d
\end{aligned} \tag{28}$$

The first two terms are non-positive, so we get the claimed inequality if we can prove

$$
\begin{aligned}
&c_{12}n_2n_1d\left(T_1+\frac{m_1}{d}(1-\delta)^2|u_1-u_2|^2\right)T_{21}\\
&+c_{21}n_1n_2dT_{12}\left(T_2+\frac{m_2}{d}(1-\delta)^2\varepsilon^2(\frac{m_1}{m_2})^2|u_1-u_2|^2\right)\\
&\leq(1+\varepsilon)c_{21}n_1n_2dT_{12}T_{21}
\end{aligned}
$$

which is by using relationship (13) equivalent to

$$
\begin{aligned}
\varepsilon\left(T_1+\frac{m_1}{d}(1-\delta)^2|u_1-u_2|^2\right)T_{21}+T_{12}\left(T_2+\frac{m_2}{d}(1-\delta)^2\varepsilon^2(\frac{m_1}{m_2})^2|u_1-u_2|^2\right)\\
\leq(1+\varepsilon)T_{12}T_{21}
\end{aligned}
$$

With the notation introduced in (24) and (26), we can write this as

$$\gamma_1=\frac{m_1}{d}(1-\delta)^2,\quad \gamma_2=\frac{m_2}{d}(1-\delta)^2\varepsilon^2(\frac{m_1}{m_2})^2$$

and

$$T_{12}=:\bar{T}_{12}+\gamma|u_1-u_2|^2,\quad T_{21}=:\bar{T}_{21}+\tilde{\gamma}|u_1-u_2|^2$$

Then we get

$$
\begin{aligned}
\varepsilon(T_1+\gamma_1|u_1-u_2|^2)(\bar{T}_{21}+\tilde{\gamma}|u_1-u_2|^2)+(\bar{T}_{12}+\gamma|u_1-u_2|^2)(T_2+\gamma_2|u_1-u_2|^2)\\
\leq(1+\varepsilon)(\bar{T}_{12}+\gamma|u_1-u_2|^2)(\bar{T}_{21}+\tilde{\gamma}|u_1-u_2|^2)
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
\varepsilon T_1\bar{T}_{21}+\varepsilon T_1\tilde{\gamma}|u_1-u_2|^2+\varepsilon\gamma_1|u_1-u_2|^2\bar{T}_{21}+\varepsilon\gamma_1\tilde{\gamma}|u_1-u_2|^4+\bar{T}_{12}T_2\\
+\bar{T}_{12}\gamma_2|u_1-u_2|^2+\gamma|u_1-u_2|^2T_2+\gamma\gamma_2|u_1-u_2|^4\\
\leq(1+\varepsilon)(\bar{T}_{12}\bar{T}_{21}+\bar{T}_{12}\tilde{\gamma}|u_1-u_2|^2+\gamma|u_1-u_2|^2\bar{T}_{21}+\gamma\tilde{\gamma}|u_1-u_2|^4
\end{aligned}
$$

This is true if we have separately

$$\varepsilon T_1\bar{T}_{21}+\bar{T}_{12}T_2\leq(1+\varepsilon)\bar{T}_{12}\bar{T}_{21} \tag{29}$$

$$(\varepsilon\gamma_1\tilde{\gamma}+\gamma\gamma_2)|u_1-u_2|^4\leq(1+\varepsilon)\gamma\tilde{\gamma}|u_1-u_2|^4 \tag{30}$$

$$
\begin{aligned}
\varepsilon T_1\tilde{\gamma}|u_1-u_2|^2+\varepsilon\gamma_1|u_1-u_2|^2\bar{T}_{21}+\bar{T}_{12}\gamma_2|u_1-u_2|^2+\gamma|u_1-u_2|^2T_2\\
\leq(1+\varepsilon)\bar{T}_{12}\tilde{\gamma}|u_1-u_2|^2+(1+\varepsilon)\gamma|u_1-u_2|^2\bar{T}_{21}
\end{aligned} \tag{31}
$$

These three inequalities are satisfied according to lemmas 5, 6, 7. Then the last four terms in (28) can be estimated by zero from above. So we obtain

$$I \leq -\frac{T_{12}}{m_1} c_{12} n_2 \int \frac{M_{12}^2}{f_1} \left(\frac{\nabla_v f_1 M_{12} - \nabla_v M_{12} f_1}{M_{12}^2} \right)^2 dv$$

$$-\frac{T_{21}}{m_2} c_{21} n_1 \int \frac{M_{21}^2}{f_2} \left(\frac{\nabla_v f_2 M_{21} - \nabla_v M_{21} f_2}{M_{21}^2} \right)^2 dv$$

$$= -\frac{T_{12}}{m_1} c_{12} n_2 \int f_1 \left| \frac{M_{12}}{f_1} \nabla_v \left(\frac{f_1}{M_{12}} \right) \right|^2 dv - \frac{T_{21}}{m_2} c_{21} n_1 \int f_2 \left| \frac{M_{21}}{f_2} \nabla_v \left(\frac{f_2}{M_{21}} \right) \right|^2 dv$$

$$= -\frac{T_{12}}{m_1} c_{12} n_2 \int f_1 \left| \nabla_v \ln \frac{f_1}{M_{12}} \right|^2 dv - \frac{T_{21}}{m_2} c_{21} n_1 \int f_2 \left| \nabla_v \ln \frac{f_2}{M_{21}} \right|^2 dv \leq 0$$

with equality if and only if $f_1 = M_{12}$ and $f_2 = M_{21}$. This means the equality is characterized by two Maxwell distributions. In addition, if we compute the mean velocities of these expressions, we get in case of equality $u_1 = u_{12} = \delta u_1 + (1-\delta) u_2$ which leads to $u_1 = u_2$. Similar, for the temperatures, we obtain $T_1 = T_2$. □

Define the total entropy $H(f_1, f_2) = \int (f_1 \ln f_1 + f_2 \ln f_2) dv$. We can compute

$$\partial_t H(f_1, f_2) + \nabla_x \cdot \int (f_1 \ln f_1 + f_2 \ln f_2) v dv = S(f_1, f_2),$$

by multiplying the Fokker-Planck equation for the species 1 by $\ln f_1$, the Fokker-Planck equation for the species 2 by $\ln f_2$ and integrating the sum with respect to v.

Corollary 8.1 (Entropy inequality for mixtures)**.** *Assume $f_1, f_2 > 0$. Assume a fast enough decay of f_1, f_2 to zero for $v \to \infty$. Assume relationship* (13)*, the conditions* (5) , (14)*,* (15)*,* (18) *and* (19) *with $\alpha, \delta \neq 1$, the positivity of the temperatures* (20) *and the assumptions on the free parameters* (23)*, then we have the following entropy inequality*

$$\partial_t \left(\int f_1 \ln f_1 dv + \int f_2 \ln f_2 dv \right) + \nabla_x \cdot \left(\int v f_1 \ln f_1 dv + \int v f_2 \ln f_2 dv \right) \leq 0,$$

with equality if and only if f_1 and f_2 are Maxwell distributions with equal bulk velocity and temperature. Moreover at equilibrium the interspecies Maxwellians M_{12} and M_{21} satisfy $u_{12} = u_2 = u_1 = u_{21}$ and $T_{12} = T_2 = T_1 = T_{21}$.

We now explicitly specify the global equilibrium.

Theorem 9 (Equilibrium)**.** *Assume $f_1, f_2 > 0$. Assume relationship* (13) *, the conditions* (5)*,* (14)*,* (15)*,* (18) *and* (19)*and the positivity of the temperatures* (20)*. Then $Q_{11}^{FP}(f_1, f_1) + Q_{12}^{FP}(f_1, f_2) = 0$ and $Q_{22}^{FP}(f_2, f_2) + Q_{21}^{FP}(f_2, f_1) = 0$, if and only if f_1 and f_2 are Maxwell distributions with equal mean velocity and temperature.*

Proof. If $Q_{11}^{FP}(f_1, f_1) + Q_{12}^{FP}(f_1, f_2) = 0$ and $Q_{22}^{FP}(f_2, f_2) + Q_{21}^{FP}(f_2, f_1) = 0$, then $\ln f_1 \, Q_{11}^{FP}(f_1, f_1) + \ln f_1 \, Q_{12}^{FP}(f_1, f_2) + \ln f_2 \, Q_{22}^{FP}(f_2, f_2) + \ln f_2 \, Q_{21}^{FP}(f_2, f_1) = 0$ and so we have equality in the H-theorem □

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Declarations

- Competing Interests: Not applicable

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