MODULI SPACES OF UNTWISTED WILD RIEMANN SURFACES

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Abstract. We construct moduli stacks of wild Riemann surfaces in the (pure) untwisted case, for any complex reductive structure group, and we define the corresponding (pure) wild mapping class groups.

Contents

1. Introduction

1.1. **Wild Riemann surfaces.** Wild (families of) Riemann surfaces, first introduced by Boalch [Boa14a], are central objects for the study of irregular isomonodromy. Classically, for a pointed compact Riemann surface $(\Sigma, \underline{a})$, where $\underline{a} = (a_1, \ldots, a_m) \in \Sigma^m$ is an ordered m-tuple of distinct points, one can look at the character variety $\mathcal{M}_{Betti}$ parametrising representations of $\pi_1(\Sigma \smallsetminus \underline{a})$ valued in a complex reductive algebraic group G. When $(\Sigma, \underline{a})$ varies in a holomorphic family over a connected complex manifold B, one obtains a family of character varieties over B, and an action of the fundamental group of B on $\mathcal{M}_{Betti}$. Fixing the genus g of the Riemann surface and the number m of marked points and considering the resulting moduli stack of pointed Riemann surfaces, one is led to the action of mapping class groups on character varieties, subject of much research (cf. [Wen11], [Pal14], [GPW20], [GLX21] among many others).

Recall that representations $\pi_1(\Sigma \smallsetminus \underline{a}) \to G$ correspond to G-bundles on Σ with a meromorphic connection having regular singularities at $\underline{a}$. In a series of works [Boa01], [Boa02], [Boa07], [Boa12], [Boa14a], Boalch gradually extended the above theory to the more general setting of irregular singularities. The starting point was [Boa01], which studies generic meromorphic connections on trivial vector bundles on the projective line, and shows that the Jimbo—Miwa–Ueno isomonodromic deformation equations [JMU81] arise from irregular isomonodromy connections, generalising the non-abelian Gauss–Manin connections (cf. the introduction and Theorem 7.1 in [Boa01]).

This provided the first motivation for the general definition of wild Riemann surfaces and wild character varieties, given in [Boa14a]. More precisely, in *loc. cit.* on the one hand Boalch introduced wild character varieties, generalising the spaces $\mathcal{M}_{Betti}$; on the other hand, he defined admissible families of wild Riemann surfaces over a base B, i.e., pointed families of Riemann surfaces over B together with an *admissible family of irregular types* (whose definition will be recalled below). The main result of [Boa14a] (Theorem 10.2) asserts that, given an admissible family of wild Riemann surfaces over a base B, the resulting wild character varieties assemble into a local system of Poisson varieties over B, with a complete flat Ehresmann connection (the irregular isomonodromy connection). In particular, one obtains an

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action of the fundamental group of B on the relevant wild character variety. As envisaged in [Boa18, §3] and [Boa14b, §8] (to which we refer the reader for more background and references), constructing moduli stacks of (admissible families of) wild Riemann surfaces would enable to define *wild mapping class groups* and study their action on wild character varieties.

The main aim of this document is to construct such stacks and define wild mapping class groups. We first recall the definitions of irregular types and wild Riemann surfaces, following [DRT, §1].

1.1.1. *Irregular types.* Let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan sub-algebra of a complex reductive Lie algebra; fix integers $p \geq 1$ and $g \geq 0$. Given a compact Riemann surface Σ of genus g and a point $a \in \Sigma$ we consider the completed local ring $\hat{\mathcal{O}}_{\Sigma,a}$ of Σ at a, with maximal ideal $\hat{\mathfrak{m}}_{\Sigma,a}$. We denote by $\mathscr{T}^{\leq p}_{\Sigma,a}$ the quotient $\hat{\mathfrak{m}}^{-p}_{\Sigma,a}/\hat{\mathcal{O}}_{\Sigma,a}$; it is the $\mathbf{C}$-vector space of germs of meromorphic functions at a with pole order at most p, up to homolorphic functions. An (untwisted) irregular type with pole order bounded by p at a is an element $Q \in \mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}^{\leq p}_{\Sigma,a}$ [Boa14a, Definition 7.1].

1.1.2. *Families of wild Riemann surfaces.* Let B be a complex manifold and $m \geq 1$ an integer. A B-family of wild Riemann surfaces of genus g with m marked points, and with pole orders at most p, is a triple $(\pi \colon \Sigma \to B, \underline{a}, \underline{Q})$ where $\pi \colon \Sigma \to B$ is a holomorphic family of compact Riemann surfaces of genus g, $\underline{a} = (a_1, \dots, a_m)$ is an m-tuple of non-intersecting holomorphic sections of π, and $\underline{Q} = (Q_1, \dots, Q_m)$ is an m-tuple of families of irregular types $Q_i(b) \in \mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}^{\leq p}_{\pi^{-1}(b),a_i(b)}$, smoothly varying with $b \in B$ (cf. [Boa14a, Definition 10.1], [DRT, Definition 1.1] and Definition 2.4.1 below).

1.2. **Aim of the text.** For every complex manifold B, we consider the groupoid $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}(B)$ of B-families of wild Riemann surfaces with m marked points and pole orders at most p (morphisms being isomorphism commuting with the projection to B, respecting the sections and the irregular types). In the body of the text, we will first show that the assignment $B \mapsto \mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}(B)$ is an analytic stack, and the natural map $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m} \to \mathsf{M}_{g,m}$ to the stack of m-pointed genus g compact Riemann surfaces is (representable by) a vector bundle. Subsequently, we will study the substacks of $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}$ obtained fixing the order of the poles of the irregular types after evaluation at the roots of $\mathfrak{g}$. The fundamental groups of these substacks will be the (global) wild mapping class groups whose definition is one of the main aims of this text.

1.3. **General notation and terminology.** The following notation and conventions will be in force throughout the document, unless otherwise stated.

- Fix a reductive complex Lie algebra $\mathfrak{g}$ with root system Φ, and a Cartan subalgebra $\mathfrak{t}$. Let r be the dimension of $\mathfrak{t}$ as a complex vector space.
- Fix integers $p, m \geq 1$ and $g \geq 0$.
- The category of complex manifolds is denoted by $\mathrm{Man}_{\mathbf{C}}$. For every complex manifold B, the tensor product $\mathfrak{t} \otimes_{\mathbf{C}} \Gamma(B, \mathcal{O}_B)$ is identified naturally with the set of holomorphic functions $f \colon B \to \mathfrak{t}$. Hence, the contravariant functor from $\mathrm{Man}_{\mathbf{C}}$ to Sets

$$B \mapsto \mathfrak{t} \otimes_{\mathbf{C}} \Gamma(B, \mathcal{O}_B)$$

is representable by $\mathfrak{t}$.

Further terminology and background on stacks can be found in the appendix.

2. Wild Riemann surfaces and their moduli

In this section, we will first recall the definition of family of wild Riemann surfaces over a complex manifold B; then we will define and study their moduli space. The letter B will always denote a complex manifold.

2.1. **Families of Riemann surfaces and their sections.** Let us start by defining families of (compact) Riemann surfaces of genus g over a complex manifold B; we will refer to them as Riemann surfaces over B.

2.1.1. **Definition.** A Riemann surface over B of genus g is a proper holomorphic submersion $\pi\colon \Sigma \to B$ of complex manifolds, such that all the fibres are connected Riemann surfaces of genus g.

2.1.2. *Local description of sections.* If $\sigma\colon B \to \Sigma$ is a section of a Riemann surface over B, then the differential of σ is injective at every point $b \in B$. As a consequence of the implicit function theorem, one shows (cf. [Huy05, Corollary 1.1.12]) that, in suitable local charts around b and $\sigma(b)$, the map σ has the form $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, 0)$. In particular $\sigma(B)$ is an effective Cartier divisor: the sheaf of $\mathcal{O}_\Sigma$-ideals $\mathcal{I}_\sigma$ consisting of functions vanishing on $\sigma(B)$ is locally free of rank one. It sits in a short exact sequence

$$0 \to \mathcal{I}_\sigma \to \mathcal{O}_\Sigma \to \sigma_* \mathcal{O}_B \to 0.$$

Furthermore, letting $\mathcal{O}_\Sigma(\sigma(B)) = \mathcal{H}om_{\mathcal{O}_\Sigma}(\mathcal{I}_\sigma, \mathcal{O}_\Sigma)$, the evaluation map $\mathcal{I}_\sigma \otimes_{\mathcal{O}_\Sigma} \mathcal{O}_\Sigma(\sigma(B)) \to \mathcal{O}_\Sigma$ is an isomorphism.

For an integer $k \geq 1$, let $\mathcal{I}_\sigma^k$ be k-fold product of the ideal sheaf $\mathcal{I}_\sigma$; it is isomorphic to the k-fold tensor product of $\mathcal{I}_\sigma$ (over $\mathcal{O}_\Sigma$), because the latter is locally free. The inverse of $\mathcal{I}_\sigma^k$, denoted by $\mathcal{O}_\Sigma(k\sigma(B))$, is the k-fold tensor product of $\mathcal{O}_\Sigma(\sigma(B))$.

2.2. **Sheaves of tails of meromorphic functions with bounded pole along a section.** Fix an integer $k \geq 1$, and a Riemann surface $\pi : \Sigma \to B$ over B with a section σ. Boalch introduced the notion of irregular type on Σ with a pole at σ of order at most k, cf. [Boa14a, §10]. The aim of this section is to reformulate the definition in sheaf-theoretic language, which is convenient for the purposes of this article. In particular, this reformulation allows to give a simple proof of Proposition 2.3.2 below, on which our subsequent arguments are based.

2.2.1. Tensoring the exact sequence $0 \to \mathcal{I}_\sigma^k \to \mathcal{O}_\Sigma \to \mathcal{O}_\Sigma/\mathcal{I}_\sigma^k \to 0$ with $\mathcal{O}_\Sigma(k\sigma(B))$ - which preserves exactness because $\mathcal{O}_\Sigma(k\sigma(B))$ is locally free - we obtain a short exact sequence

$$0 \to \mathcal{O}_\Sigma \to \mathcal{O}_\Sigma(k\sigma(B)) \to \mathscr{T}_{\Sigma,\sigma}^{\leq k} \to 0$$

whose cokernel $\mathscr{T}_{\Sigma,\sigma}^{\leq k}$ will be called the sheaf of tails of meromorphic functions on Σ with poles along $\sigma(B)$, of order bounded by k.

2.2.2. For example, if Σ is a Riemann surface and $\sigma : \{*\} \to \Sigma$ is the map with image a point $a \in \Sigma$, let $\mathfrak{m}_{\Sigma,a}$ be the maximal ideal of the local ring $\mathcal{O}_{\Sigma,a}$. Then $\mathscr{T}_{\Sigma,a}^{\leq k}$ is the skyscraper sheaf at a with stalk $\mathfrak{m}_{\Sigma,a}^{-k}/\mathcal{O}_{\Sigma,a} \simeq \hat{\mathfrak{m}}_{\Sigma,a}^{-k}/\hat{\mathcal{O}}_{\Sigma,a}$. For a family $\Sigma \to B$ with a section σ, choosing charts around a point $b \in B$ and $\sigma(b) \in \Sigma$ as in §2.1.2, meromorphic functions of the form $\frac{f}{z_{n+1}^k}$ with f holomorphic give rise to sections of $\mathscr{T}_{\Sigma,\sigma}^{\leq k}$ (in a neighbourhood of $\sigma(b)$). In general, sections of $\mathscr{T}_{\Sigma,\sigma}^{\leq k}$ can be written in this form only locally.

2.2.3. *Properties of $\mathscr{T}_{\Sigma,\sigma}^{\leq k}$.* For every $x \in \Sigma \smallsetminus \sigma(B)$ there is an open neighbourhood U of x which does not intersect $\sigma(B)$, hence $\mathcal{I}_\sigma|U = \mathcal{O}_\Sigma|U$. It follows that $\mathscr{T}_{\Sigma,\sigma}^{\leq k}$ is a sheaf supported on $\sigma(B)$ and killed by I_σ^k, so it is the pushforward of a unique sheaf, denoted by $\mathscr{T}_\sigma^{\leq k}$, on the complex analytic space $(\sigma(B), \mathcal{O}_\Sigma/I_\sigma^k)$. In particular, even though $\mathscr{T}_{\Sigma,\sigma}^{\leq k}$ was defined as a quotient of a sheaf of meromorphic functions on the whole Σ, it only carries information about germs of such functions along $\sigma(B)$. Precisely, if $V \subset U \subset \Sigma$ are two opens having the same intersection with $\sigma(B)$, then $\mathscr{T}_{\Sigma,\sigma}^{\leq k}(U) = \mathscr{T}_{\Sigma,\sigma}^{\leq k}(V)$.

We will denote by $B^{(k)}$ the analytic space $(\sigma(B), \mathcal{O}_\Sigma/I_\sigma^k)$, and by $\pi^{(k)}\colon B^{(k)} \to B$ the composition of π and the inclusion of $B^{(k)}$ in Σ.

2.2.4. **Lemma.** *The pushforward $\pi_* \mathscr{T}_{\Sigma,\sigma}^{\leq k}$ is a locally free sheaf of rank k on B.*

Proof. The sheaf $\pi_* \mathscr{T}^{\leq k}_{\Sigma,\sigma}$ coincides with the pushforward of $\mathscr{T}^{\leq k}_{\sigma}$ via the map $\pi^{(k)} \colon B^{(k)} \to B$. Choosing local coordinates around a point of $\sigma(B)$ as in §2.1.2, we see that we can write locally $\mathscr{T}^{\leq k}_{\Sigma,\sigma} = z_{n+1}^{-k}\mathcal{O}_\Sigma/\mathcal{O}_\Sigma$, hence $\mathscr{T}^{\leq k}_{\sigma}$ is locally free of rank one. On the other hand, we claim that $\pi^{(k)}_* \mathcal{O}_{B^{(k)}}$ is a locally free $\mathcal{O}_B$-module of rank k. To prove this, we may, and will, replace B by an open neighbourhood of any of its points, and work with the same local coordinates as before. The section $\sigma : B \to \Sigma$ factors through $B^{(k)}$, and it induces the natural projection $\mathcal{O}_\Sigma/z_{n+1}^k \mathcal{O}_\Sigma \to \mathcal{O}_\Sigma/z_{n+1}\mathcal{O}_\Sigma \simeq \mathcal{O}_B$. The map $\pi^{(k)}$ induces on global sections a section of this projection; hence, the claim follows from Lemma 2.2.5 below. $\square$

2.2.5. **Lemma.** *Let A be a ring, $f \in A$ a nonzerodivisor, $k \geq 1$ an integer and $s \colon A/(f) \to A/(f^k)$ a section of the projection $q \colon A/(f^k) \to A/(f)$. The map s endows $A/(f^k)$ with the structure of a free $A/(f)$-module of rank k with basis $1, f, \dots, f^{k-1}$.*

Proof. By induction on k; for $k = 1$ the claim is clear, so let us suppose that $k \geq 2$. Take $a \in A$ and let $\alpha_0 = q(a)$. Then $q(a - s(\alpha_0)) = q(a) - q \circ s \circ q(a) = 0$, so we can write $a = s(\alpha_0) + f a_1$ for some $a_1 \in A/(f^k)$. Assume that $\alpha_0' \in A/(f), a_1' \in A/(f^k)$ also satisfy $a = s(\alpha_0') + f a_1'$. Then $s(\alpha_0 - \alpha_0') = f(a_1' - a_1)$; taking the image via q we find that $\alpha_0 = \alpha_0'$, hence $f(a_1' - a_1) = 0 \in A/(f^k)$. As f is not a zero divisor, we deduce that $a_1' - a_1 \in (f^{k-1})$. Therefore, we have proved that α_0 is unique, and a_1 is unique in $A/(f^{k-1})$, which implies the statement by induction. $\square$

2.2.6. *Functoriality.* Let us set $\mathscr{T}^{\leq k}_{|B} = \pi_* \mathscr{T}^{\leq k}_{\Sigma,\sigma}$. Let $\varphi \colon B' \to B$ be a holomorphic map, $\pi' \colon \Sigma' \to B'$ the base change of π, and $\sigma' \colon B' \to \Sigma'$ the pullback of σ. We have a commutative diagram

$$\begin{tikzcd}
\Sigma' \arrow[r, "\psi"] \arrow[dd, bend left=60, "\pi'"] & \Sigma \arrow[dd, bend left=60, "\pi"] \\
B'^{(k)} \arrow[u] & B^{(k)} \arrow[u] \\
B' \arrow[u] \arrow[uu, bend left=60, "\sigma'"] \arrow[r, "\varphi"] & B \arrow[u] \arrow[uu, bend left=60, "\sigma"]
\end{tikzcd}$$

and $\sigma'(B') \subset \Sigma'$ is cut out by the ideal sheaf image of $\psi^* \mathcal{I}_\sigma$; in other words, we have $\sigma'_* \mathcal{O}_{B'} = \mathcal{O}_{\Sigma'}/\psi^* \mathcal{I}_\sigma = \psi^* \sigma_* \mathcal{O}_B$.

2.2.7. **Lemma.** *The following assertions hold true.*

(1) $\psi^* \mathcal{I}_\sigma = \mathcal{I}_{\sigma'}$.
(2) $\psi^* \mathcal{O}_\Sigma(\sigma(B)) = \mathcal{O}_{\Sigma'}(\sigma'(B'))$.
(3) $\psi^* \mathscr{T}^{\leq k}_{\Sigma,\sigma} = \mathscr{T}^{\leq k}_{\Sigma',\sigma'}$.

Proof. (1) Pulling back the exact sequence $0 \to \mathcal{I}_\sigma \to \mathcal{O}_\Sigma \to \sigma_* \mathcal{O}_B \to 0$ via ψ, and using that $\psi^* \sigma_* \mathcal{O}_B = \sigma'_* \mathcal{O}_{B'}$ we get an exact sequence

$$\psi^* \mathcal{I}_\sigma \to \mathcal{O}_{\Sigma'} \to \sigma'_* \mathcal{O}_{B'} \to 0.$$

Therefore we obtain a surjection $\psi^* \mathcal{I}_\sigma \to \mathcal{I}_{\sigma'}$ of invertible $\mathcal{O}_{\Sigma'}$-modules, which is necessarily an isomorphism.

(2) Pulling back the identity $\mathcal{I}_\sigma \otimes_{\mathcal{O}_\Sigma} \mathcal{O}_\Sigma(\sigma(B)) \simeq \mathcal{O}_\Sigma$ and using (1) we deduce that $\psi^*(\mathcal{O}_\Sigma(\sigma(B)))$ is the inverse of $\mathcal{I}_{\sigma'}$, hence it coincides with $\mathcal{O}_{\Sigma'}(\sigma'(B'))$.

(3) The k-fold tensor product of the isomorphism in (2) yields an isomorphism $\psi^* \mathcal{O}_\Sigma(k\sigma(B)) = \mathcal{O}_{\Sigma'}(k\sigma'(B'))$, which implies the claim by right-exactness of pullback. $\square$

By adjunction we have a natural transformation $\mathrm{Id} \to \psi_* \psi^*$, inducing $\pi_* \to \pi_* \psi_* \psi^*$, so $\pi_* \to \varphi_* \pi'_* \psi^*$ and finally $\varphi^* \pi_* \to \pi'_* \psi^*$. Applying these functors to $\mathscr{T}^{\leq k}_{\Sigma,\sigma}$ and using the previous lemma we get a map $\varphi^* \mathscr{T}^{\leq k}_{|B} \to \mathscr{T}^{\leq k}_{|B'}$.

2.2.8. **Lemma.** *The above map $\varphi^* \mathscr{T}^{\leq k}_{|B} \to \mathscr{T}^{\leq k}_{|B'}$ is an isomorphism.*

Proof. By [Sta18, Tag 01HQ] and Lemma 2.2.7 there is a unique map $\gamma\colon B'^{(k)} \to B^{(k)}$ making the top square of the diagram in §2.2.6 Cartesian. We claim that the following diagram is Cartesian as well:

$$\begin{array}{ccc} B'^{(k)} & \xrightarrow{\gamma} & B^{(k)} \\ \big\downarrow{\scriptstyle \pi'^{(k)}} & & \big\downarrow{\scriptstyle \pi^{(k)}} \\ B' & \xrightarrow{\varphi} & B. \end{array}$$

Indeed

$$B^{(k)} \times_B B' = B^{(k)} \times_\Sigma (\Sigma \times_B B') = B^{(k)} \times_\Sigma \Sigma' = B'^{(k)}.$$

Now $\mathscr{T}_\sigma^{\leq k}$ is the pullback of $\mathscr{T}_{\Sigma,\sigma}^{\leq k}$ (cf. the proof of [Sta18, Tag 08KS]), hence using Lemma 2.2.7 we deduce that $\gamma^*\mathscr{T}_\sigma^{\leq k} = \mathscr{T}_{\sigma'}^{\leq k}$. The lemma follows from the natural isomorphism $\varphi^*\pi_*^{(k)}(\mathscr{T}_\sigma^{\leq k}) \simeq \pi_*'^{(k)}\gamma^*(\mathscr{T}_\sigma^{\leq k})$, which is the base change result [BS76, Theorem 3.4, p. 116]. ☐

2.3. **Irregular types.** Let $\pi\colon \Sigma \to B$ be a Riemann surface over B with a section σ. An *irregular type* on B with pole at σ of order at most k is a global section of the sheaf $\mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}_{\Sigma,\sigma}^{\leq k}$.

2.3.1. *Remark.* Let us spell out the definition of irregular type. Recall that $\mathscr{T}_{\Sigma,\sigma}^{\leq k} = \mathcal{O}_\Sigma(k\sigma(B))/\mathcal{O}_\Sigma$. If $(e_1, \ldots, e_r)$ is a basis of $\mathfrak{t}$ as a complex vector space, then every global section Q of $\mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}_{\Sigma,\sigma}^{\leq k}$ can be written uniquely as $\sum_{j=1}^r Q^{(j)} e_j$, with $Q^{(j)}$ a section of $\mathcal{O}_\Sigma(k\sigma(B))/\mathcal{O}_\Sigma$. Such a section is a meromorphic function defined locally (around each point of $\sigma(B)$, cf. §2.2.2) up to holomorphic terms, and with pole along $\sigma(B)$ of order at most k. Therefore, irregular types can be thought of (locally) as meromorphic $\mathfrak{t}$-valued functions with pole along $\sigma(B)$ of order at most k, up to holomorphic terms.

If $B' \to B$ is a holomorphic map then, by adjunction and Lemma 2.2.7(3), we obtain a map

$$\mathscr{T}_{\Sigma,\sigma}^{\leq k} \to \psi_*\psi^*\mathscr{T}_{\Sigma,\sigma}^{\leq k} \to \psi_*\mathscr{T}_{\Sigma',\sigma'}^{\leq k}$$

inducing a map on global sections $\Gamma(\Sigma, \mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}_{\Sigma,\sigma}^{\leq k}) \to \Gamma(\Sigma', \mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}_{\Sigma',\sigma'}^{\leq k})$. Similarly, for a map $B'' \to B'$ of manifolds over B we have a map $\Gamma(\Sigma', \mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}_{\Sigma',\sigma'}^{\leq k}) \to \Gamma(\Sigma'', \mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}_{\Sigma'',\sigma''}^{\leq k})$. We obtain a contravariant functor $\mathscr{IT}_{B,\sigma}^{\mathfrak{t},\leq k}$ from complex manifolds over B to sets sending $B' \to B$ to $\Gamma(\Sigma', \mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}_{\Sigma',\sigma'}^{\leq k})$.

2.3.2. **Proposition.** *Let $\pi\colon \Sigma \to B$ be a Riemann surface over B of genus g, $\sigma\colon B \to \Sigma$ a section and $k \geq 1$ an integer. The functor $\mathscr{IT}_{B,\sigma}^{\mathfrak{t},\leq k}$ is representable by a vector bundle of rank rk on B.*

Proof. For $\varphi\colon B' \to B$, we have

$$\Gamma(\Sigma', \mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}_{\Sigma',\sigma'}^{\leq k}) = \Gamma(B', \mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}_{|B'}^{\leq k}) = \Gamma(B', \mathfrak{t} \otimes_{\mathbf{C}} \varphi^*\mathscr{T}_{|B}^{\leq k}),$$

where the last identification follows from Lemma 2.2.8. Hence, the functor $\mathscr{IT}_{B,\sigma}^{\mathfrak{t},\leq k}$ is the functor of global sections of $\mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}_{|B}^{\leq k}$, which is a locally free sheaf of rank rk over B by Lemma 2.2.4. The vector bundle over B attached to this locally free sheaf represents $\mathscr{IT}_{B,\sigma}^{\mathfrak{t},\leq k}$ (see §4.1.1). ☐

2.3.3. We call the vector bundle in the previous proposition the bundle of irregular types on B with pole at σ of order at most k, and we still denote it by $\mathscr{IT}_{B,\sigma}^{\mathfrak{t},\leq k}$. We also denote by $\mathscr{T}_{B,\sigma}^{\leq k}$ the vector bundle of rank k over B representing the functor sending $B' \to B$ to $\Gamma(\Sigma', \mathscr{T}_{\Sigma',\sigma'}^{\leq k})$. If $(\sigma_1, \ldots, \sigma_m)$ are non-intersecting sections, we denote by $\mathscr{IT}_{B,\sigma_1,\ldots,\sigma_m}^{\mathfrak{t},\leq k}$ the fibre product of the bundles $\mathscr{IT}_{B,\sigma_i}^{\mathfrak{t},\leq k}$ over B.

2.3.4. *The root stratification.* We will stratify the space $\mathscr{IT}_{B,\sigma}^{\mathfrak{t},\leq k}$ using the following inputs.

(1) For $j \leq k$ we have natural injections $\mathscr{T}_{\Sigma',\sigma'}^{\leq j} \to \mathscr{T}_{\Sigma',\sigma'}^{\leq k}$ for every $B' \to B$ - apply the snake lemma to

$$0 \to \mathcal{O}_{\Sigma'} \to \mathcal{O}_{\Sigma'}(j\sigma(B')) \to \mathscr{T}_{\Sigma',\sigma'}^{\leq j} \to 0$$
$$0 \to \mathcal{O}_{\Sigma'} \to \mathcal{O}_{\Sigma'}(k\sigma(B')) \to \mathscr{T}_{\Sigma',\sigma'}^{\leq k} \to 0.$$

Hence we obtain a morphism $\mathscr{T}_{B,\sigma}^{\leq j} \to \mathscr{T}_{B,\sigma}^{\leq k}$ identifying the source with a sub-vector bundle of the target (similarly, we have a morphism $\mathscr{IT}_{B,\sigma}^{\mathfrak{t},\leq j} \to \mathscr{IT}_{B,\sigma}^{\mathfrak{t},\leq k}$).

(2) A root $\alpha \in \Phi$ induces natural surjections of sheaves $\mathfrak{t} \otimes \mathscr{T}^{\leq k}_{\Sigma',\sigma'} \to \mathscr{T}^{\leq k}_{\Sigma',\sigma'}$ for every $B' \to B$, hence a submersion $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k}_{B,\sigma} \to \mathscr{T}^{\leq k}_{B,\sigma}$ of vector bundles over B.

Given a collection of integers $\mathbf{d} = (d_\alpha)_{\alpha \in \Phi}$ with $d_\alpha \leq k$ for every $\alpha \in \Phi$, we denote by $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k,\leq \mathbf{d}}_{B,\sigma}$ the intersection of the fibre products $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k}_{B,\sigma} \times_{\mathscr{T}^{\leq k}_{B,\sigma}} \mathscr{T}^{\leq d_\alpha}_{B,\sigma} \subset \mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k}_{B,\sigma}$ as $\alpha \in \Phi$ varies. This is a closed submanifold of $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k}_{B,\sigma}$ (we will justify in §2.3.5 why it is a manifold); by construction, it is the moduli space of irregular types at σ with pole order bounded by k, and such that after evaluation at each $\alpha \in \Phi$ the pole order is bounded by d_α.

We write $\mathbf{d}' = (d'_\alpha)_{\alpha \in \Phi} < \mathbf{d} = (d_\alpha)_{\alpha \in \Phi}$ if $d'_\alpha \leq d_\alpha$ for every $\alpha \in \Phi$, and at least one of the inequalities is strict. We define

$$\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k,\mathbf{d}}_{B,\sigma} = \mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k,\leq \mathbf{d}}_{B,\sigma} \smallsetminus \cup_{\mathbf{d}'<\mathbf{d}} \mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k,\leq \mathbf{d}'}_{B,\sigma} \subset \mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k}_{B,\sigma}.$$

By construction, $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k,\mathbf{d}}_{B,\sigma}$ is an open submanifold of $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k,\leq \mathbf{d}}$, and a locally closed submanifold of $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k}_{B,\sigma}$. It parametrises irregular types at σ with pole order bounded by k, and such that after evaluation at $\alpha \in \Phi$ the pole order is (everywhere) equal to d_α.

2.3.5. *Local structure of root strata.* Let us describe $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k,\mathbf{d}}_{B,\sigma}$ locally on B. Given $b \in B$, we choose small enough open polydiscs $U \ni b$ and $V \ni \sigma(b)$ with charts as in §2.1.2. Restricted to U, the sheaf $\mathscr{T}^{\leq k}_{|B}$ is free over $\mathcal{O}_U$, with basis $(z_{n+1}^{-1}, \ldots, z_{n+1}^{-k})$; we may, and will, use this basis to identify $\mathscr{T}^{\leq k}_{|B}$ restricted to U with $\mathcal{O}_U^k$. Concretely, the point is that a meromorphic function on V with pole of order at most k along $\sigma(B)$ can be written uniquely, up to holomorphic terms, as a sum $f_1 z_{n+1}^{-1} + \ldots + f_k z_{n+1}^{-k}$, where $f_1, \ldots, f_k$ are holomorphic functions of $z_1, \ldots, z_n$, which are identified with functions on U.

Thanks to the above identification, we see that the functor of points of the pullback of $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k}_{B,\sigma}$ to U is representable by $\mathfrak{t}^k \times U$. The pullback of $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k,\leq \mathbf{d}}_{B,\sigma}$ to U is representable by the intersection of the manifolds

$$\left(\prod_{1 \leq i \leq d_\alpha} \mathfrak{t} \times \prod_{d_\alpha < i \leq k} \ker(\alpha) \right) \times U, \qquad \alpha \in \Phi.$$

In other words, we can write the pullback to U of $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k,\leq \mathbf{d}}_{B,\sigma}$ as the product of U and

$$\prod_{i=1}^{k} \left(\cap_{d_\alpha < i} \ker(\alpha) \right) \subset \mathfrak{t}^k.$$

Each of the spaces in the above product is an intersection of hyperplanes through the origin in $\mathfrak{t}$, cut out by the equations $\alpha = 0$ for α such that $d_\alpha < i$. Hence $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k,\leq \mathbf{d}}_{B,\sigma}$ is a manifold, and the same is true for the open $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k,\mathbf{d}}_{B,\sigma} \subset \mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k,\leq \mathbf{d}}_{B,\sigma}$ (which, explicitly, is locally a product of intersections of hyperplane complements).

2.4. **Stacks of wild Riemann surfaces.** In this section we will define stacks of wild Riemann surfaces (of fixed genus, number of marked points, and maximal pole order of the irregular types). We start by introducing (families of) wild Riemann surfaces. Later on we will ask, as in [Boa14a, Definition 10.1], such families to be *admissible*; this will lead to a stratification of the stack of wild Riemann surfaces, mirroring the stratification on $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq k}_{B,\sigma}$ described above.

2.4.1. **Definition.** A wild Riemann surface over a complex manifold B of genus g, with m marked points and pole orders bounded by p, is a triple $(\pi\colon \Sigma \to B, \underline{a}, \underline{Q})$ consisting of

(1) a Riemann surface $\pi\colon \Sigma \to B$ of genus g in the sense of Definition 2.1.1;
(2) an m-tuple $\underline{a} = (a_1, \ldots, a_m)$ of mutually non-intersecting sections $a_i\colon B \to \Sigma$ of π;
(3) an m-tuple $\underline{Q} = (Q_1, \ldots, Q_m)$ of irregular types $Q_i \in \Gamma(\Sigma, \mathfrak{t} \otimes_{\mathbf{C}} \mathscr{T}^{\leq p}_{\Sigma,a_i})$.

2.4.2. *Definition of* $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}$. For every complex manifold B, the groupoid $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}(B)$ is by definition the category whose objects are wild Riemann surfaces over B of genus g, with m marked points and pole orders bounded by p. An isomorphism from $(\pi_1\colon \Sigma_1 \to B, \underline{a}_1, \underline{Q}_1)$ to $(\pi_2\colon \Sigma_2 \to B, \underline{a}_2, \underline{Q}_2)$ is an isomorphism $\psi : \Sigma_1 \to \Sigma_2$ such that $\pi_2 \circ \psi = \pi_1$, the pullback of $\underline{a}_2$ is $\underline{a}_1$ (in other words, $\psi \circ a_{1,i} = a_{2,i}$ for $1 \leq i \leq m$) and the pullback of $\underline{Q}_2$ is $\underline{Q}_1$. As above we see that any holomorphic map $B' \to B$ induces a functor from $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}(B)$ to $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}(B')$. Forgetting irregular types we get functorial maps $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}(B) \to \mathsf{M}_{g,m}(B)$ for every manifold B, where $\mathsf{M}_{g,m}(B)$ is the groupoid of genus g Riemann surfaces over B with m mutually disjoint sections.

2.4.3. *Definition of* $\mathsf{WM}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m}$. Let us now define admissible families wild Riemann surfaces over a manifold B, following [Boa14a, Definition 10.1]. Fix integers $0 \leq d_{\alpha,i} \leq p$ for each $\alpha \in \Phi$ and $1 \leq i \leq m$, and let $\underline{\mathbf{d}} = (\mathbf{d}_1, \ldots, \mathbf{d}_m)$ with $\mathbf{d}_i = (d_{\alpha,i})_{\alpha\in\Phi}$.

If $(\pi\colon \Sigma \to B, \underline{a}, \underline{Q})$ is a wild Riemann surface as above, then for every $\alpha \in \Phi$ and every $1 \leq i \leq m$ we can look at $\alpha \circ Q_i \in \Gamma(\Sigma, \mathscr{T}^{\leq p}_{\Sigma,a_i})$. Recall that we have injections of sheaves $\mathscr{T}^{\leq d_{\alpha,i}}_{\Sigma,a_i} \to \mathscr{T}^{\leq p}_{\Sigma,a_i}$ (cf. §2.3.4). We will use the following terminology.

- We say that $\alpha \circ Q_i$ has order at most $d_{\alpha,i}$ if it belongs to $\Gamma(\Sigma, \mathscr{T}^{\leq d_{\alpha,i}}_{\Sigma,a_i})$.
- We say that $\alpha \circ Q_i$ has order $d_{\alpha,i}$ if it belongs to $\Gamma(\Sigma, \mathscr{T}^{\leq d_{\alpha,i}}_{\Sigma,a_i})$ and, for every $b \in B$, its pullback to the fibre $\Sigma_b = \pi^{-1}(b)$ does not belong to $\Gamma(\Sigma_b, \mathscr{T}^{\leq d_{\alpha,i}-1}_{\Sigma_b,a_{i,b}})$.
- We say that Q_i has root order at most $\mathbf{d}_i$ (resp. root order $\mathbf{d}_i$) if $\alpha \circ Q_i$ has order at most (resp. equal to) $d_{\alpha,i}$ for every $\alpha \in \Phi$. We say that $\underline{Q}$ has root order at most $\underline{\mathbf{d}}$ (resp. root order $\underline{\mathbf{d}}$) if Q_i has root order at most $\mathbf{d}_i$ (resp. order $\mathbf{d}_i$) for all $1 \leq i \leq m$.
- We say that $(\pi\colon \Sigma \to B, \underline{a}, \underline{Q})$ is admissible of root order $\underline{\mathbf{d}}$ if $\underline{Q}$ has root order $\underline{\mathbf{d}}$. The groupoid of admissible wild Riemann surfaces (over B, of genus g, with m marked points and pole order bounded by p) of root order $\underline{\mathbf{d}}$ is denoted by $\mathsf{WM}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m}(B)$.

2.4.4. *Remark.* Let us rephrase more concretely the condition that $\alpha \circ Q_i$ has order $d_{\alpha,i}$. For every $b \in B$, pulling back $\alpha \circ Q_i \in \Gamma(\Sigma, \mathscr{T}^{\leq p}_{\Sigma,a_i})$ to the fibre Σ_b we obtain an element $(\alpha \circ Q_i)_b \in \Gamma(\Sigma_b, \mathscr{T}^{\leq p}_{\Sigma_b,a_{i(b)}}) \simeq z^{-p}\mathcal{O}_{\Sigma_b,a_{i(b)}}/\mathcal{O}_{\Sigma_b,a_{i(b)}}$, where z is a local coordinate on Σ_b around $a_{i(b)}$. Then $\alpha \circ Q_i$ has order $d_{\alpha,i}$ if and only if for every $b \in B$ the pole order of $(\alpha \circ Q_i)_b$ is $d_{\alpha,i}$.

2.4.5. **Proposition.** *The assignment* $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}\colon \mathrm{Man}_{\mathbf{C}} \to$ Groupoids *sending* B *to* $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}(B)$ *is a stack.*

Proof. Let us verify that the properties defining a stack (cf. §4.1.4) are satisfied.

(1) Objects glue. Let $(B_i)_{i\in I}$ be an open cover of a manifold B. Suppose we are given objects $(\pi_i\colon \Sigma_i \to B_i, \underline{a}_i, \underline{Q}_i)$ of $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}(B_i)$ for each i, together with isomorphisms of their restrictions to $B_i \cap B_j$ satisfying the cocycle condition on triple intersections. We need to show that there is $(\pi\colon \Sigma \to B, \underline{a}, \underline{Q})$ pulling back to $(\pi_i\colon \Sigma_i \to B_i, \underline{a}_i, \underline{Q}_i)$ over each B_i. Since $\mathsf{M}_{g,m}$ is a stack, we do get $(\pi\colon \Sigma \to B, \underline{a})$ with the desired property. Now, each irregular irregular type $\underline{Q}_i$ is a collection of sections of the pullbacks to B_i of $\mathfrak{t} \otimes (\pi_* \mathscr{T}^{\leq p}_{\Sigma,a_i}), 1 \leq i \leq m$. Such sections glue to a section on B because $\pi_*\mathscr{T}^{\leq p}_{\Sigma,a_i}$ is a sheaf on B.

(2) Isomorphisms glue. Isomorphisms of objects in $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}(B)$ are by definition isomorphisms of Riemann surfaces over B (respecting the extra structures), which glue. ☐

We can now state the main result of this section.

2.4.6. **Theorem.** *Given integers* $g \geq 0, m \geq 1, p \geq 1$, *and given* $\underline{\mathbf{d}} = (d_{\alpha,i})_{1\leq i\leq m,\alpha\in\Phi}$ *with* $0 \leq d_{\alpha,i} \leq p$, *the following assertions hold true.*

(1) The stack $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m}$ *is an analytic stack.*

(2) The map $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m} \to \mathsf{M}_{g,m}$ *is representable by vector bundles. More precisely, let* B *be a complex manifold and* $B \to \mathsf{M}_{g,m}$ *a map corresponding to an* m*-pointed genus* g *Riemann surface* $(\Sigma \to B, \underline{a})$, *where* $\underline{a} = (a_1, \ldots, a_m)$. *The fibre product* $\mathsf{WM}^{\mathfrak{t},\leq p}_{g,m} \times_{\mathsf{M}_{g,m}} B$ *is isomorphic to* $\mathscr{I}\mathscr{T}^{\mathfrak{t},\leq p}_{B,a_1,\ldots,a_m}$.

(3) *The assignment* $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}\colon \mathrm{Man}_{\mathbf{C}} \to \mathrm{Groupoids}$ *sending* B *to* $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}(B)$ *is an analytic stack.*

(4) *The map* $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}} \to \mathsf{M}_{g,m}$ *is representable by manifolds which are locally products of hyperplane complements in affine spaces; in particular, it is a submersion. The fibre product of* $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}$ *with a point over* $\mathsf{M}_{g,m}$ *is a product of* m *products of hyperplane complements in complex vector spaces as defined in [DRT, Definition 1.3].*

Proof. We know from the above proposition that $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p}$ is a stack, which implies that the same is true for $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}$.

Let us now prove the second point. Let $B \to \mathsf{M}_{g,m}$ be a manifold as in the statement, and let B' be a manifold. By definition (cf. §4.1.6), the groupoid $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p} \times_{\mathsf{M}_{g,m}} B(B')$ has the following description.

- Objects are triples $(\varphi\colon B' \to B, (\pi'\colon \Sigma' \to B', \underline{a}', \underline{Q}'), \tau)$, where $\tau\colon \Sigma \times_{B,\varphi} B' \to \Sigma'$ is an isomorphism commuting with the maps to B' and respecting marked points.
- Isomorphisms between $(\varphi\colon B' \to B, (\pi'_1\colon \Sigma'_1 \to B', \underline{a}'_1, \underline{Q}'_1), \tau_1)$ and $(\varphi\colon B' \to B, (\pi'_2\colon \Sigma'_2 \to B', \underline{a}'_2, \underline{Q}'_2), \tau_2)$ are isomorphisms $\psi\colon \Sigma'_1 \to \Sigma'_2$ commuting with the structure maps to B', respecting marked points and irregular types, and such that $\psi \circ \tau_1 = \tau_2$.

Therefore, we see that every object of $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p} \times_{\mathsf{M}_{g,m}} B(B')$ is isomorphic to one of the form $(\varphi\colon B' \to B, (\Sigma \times_B B' \to B', \varphi^*(\underline{a}), \underline{Q}'), \mathrm{Id})$, where $\underline{Q}'$ is a collection of irregular types on $\Sigma \times_B B'$; furthermore, objects of this form have no automorphisms. In other words, the groupoid $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p} \times_{\mathsf{M}_{g,m}} B(B')$ is equivalent to $\mathscr{IT}_{B,a_1,\ldots,a_m}^{\mathfrak{t},\leq p}(B)$.

Let us now deduce that $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p}$ is an analytic stack. Choose a surjective submersion $U \to \mathsf{M}_{g,m}$ from a complex manifold U. Then the fibre product $V = U \times_{\mathsf{M}_{g,m}} \mathsf{WM}_{g,m}^{\mathfrak{t},\leq p}$ is a manifold, and the map $V \to \mathsf{WM}_{g,m}^{\mathfrak{t},\leq p}$ is a surjective submersion, hence $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p}$ is an analytic stack. The same argument shows that the fact that $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}$ is analytic follows from point (4).

To prove (4), take a map $B \to \mathsf{M}_{g,m}$ corresponding to an m-pointed genus g Riemann surface $(\Sigma \to B, \underline{a})$. The fibre product $B \times_{\mathsf{M}_{g,m}} \mathsf{WM}_{g,m}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}$ is representable by the product (over B) of the spaces representing $\mathscr{IT}_{B,a_i}^{\mathfrak{t},\leq p,\mathbf{d}_i}$, for $1 \leq i \leq m$. We have seen in §2.3.5 that each $\mathscr{IT}_{B,a_i}^{\mathfrak{t},\leq p,\leq \mathbf{d}_i}$ is representable by a manifold over B which, locally on B, is an intersection of the manifolds

$$B \times \left(\prod_{1 \leq j \leq d_{\alpha,i}} \mathfrak{t} \times \prod_{d_{\alpha,i} < j \leq p} \ker(\alpha) \right), \qquad \alpha \in \Phi\,. \tag{2.4.6.1}$$

Therefore, the functor $\mathscr{IT}_{B,a_i}^{\mathfrak{t},\leq p,\mathbf{d}_i}\colon \mathrm{Man}_{\mathbf{C}} \to$ Sets of irregular types at a_i with root order $\mathbf{d}_i$ is representable by a manifold which, locally on B, is the open submanifold of the intersection of the manifolds in (2.4.6.1) whose j-th component, for $1 \leq j \leq p$, is the product of B and

$$\bigcap_{d_{\alpha,i} < j} \ker(\alpha) \cap \bigcap_{d_{\alpha,i} = j} (\mathfrak{t} \smallsetminus \ker(\alpha))\,.$$

The above manifold coincides with the one defined in [DRT, Definition 1.3], hence the proof of (4) is complete. □

2.4.7. **Corollary.** *If* $\mathsf{M}_{g,m}$ *is (representable by) a manifold then* $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}$ *is a manifold as well, and the map* $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}} \to \mathsf{M}_{g,m}$ *is a holomorphic submersion.*

3. Wild mapping class groups

In this section we define and study the fundamental groups of the stacks of admissible (families of) wild Riemann surfaces, called wild mapping class groups.

3.1. **Definition and basic properties.** Thanks to Theorem 2.4.6 we know that $\mathsf{WM}_{g,m}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}$ is an analytic stack. To such an object, once a base point is chosen, one can attach a well-defined (topological) fundamental group, cf. Definition 4.2.5.

3.1.1. **Definition.** Given integers $g \geq 0, m \geq 1$ and $\underline{\mathbf{d}} = (d_{\alpha,i})_{1\leq i\leq m, \alpha\in\Phi}$ with $d_{\alpha,i} \geq 0$, set $p = \max(\underline{\mathbf{d}})$. The wild mapping class group $\Gamma^{\mathfrak{t},\underline{\mathbf{d}}}_{m,g}$ is

$$\Gamma^{\mathfrak{t},\underline{\mathbf{d}}}_{m,g} = \pi_1(\mathsf{WM}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m}, x)$$

for any point x.

3.1.2. *Remark.* Since the stacks $\mathsf{WM}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m}$ are path connected (because the same is true for $\mathsf{M}_{g,m}$ and for the fibres of the morphism $\mathsf{WM}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m} \to \mathsf{M}_{g,m}$), the group $\Gamma^{\underline{\mathbf{d}}}_{m,g}$ is well defined up to isomorphism.

3.1.3. *An explicit presentation of* $\mathsf{WM}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m}$*.* Recall that we can write $\mathsf{M}_{g,m} = [T_{g,m}/\Gamma_{g,m}]$, where the Teichmüller space $T_{g,m}$ is a contractible complex manifold (see §4.1.9). By Theorem 2.4.6 the fibre product $\widetilde{WT}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m} = \mathsf{WM}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m} \times_{\mathsf{M}_{g,m}} T_{g,m}$ is a complex manifold, and it is a locally trivial fibration over $T_{g,m}$ whose fibres are products of spaces of irregular types of root order $\mathbf{d}_i$ at a point, denoted by $\mathscr{IT}^{\mathfrak{t},\leq p,\mathbf{d}_i}$, for $1 \leq i \leq m$. In fact, since $T_{g,m}$ is contractible the fibration $\widetilde{WT}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m} \to T_{g,m}$ is topologically trivial: there is a homeomorphism

$$T_{g,m} \times \prod_{i=1}^{m} \mathscr{IT}^{\mathfrak{t},\leq p,\mathbf{d}_i} \simeq \widetilde{WT}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m}.$$

The top row of the following cartesian diagram provides a chart of $\mathsf{WM}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m}$:

$$\begin{tikzcd}
\widetilde{WT}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m} \arrow[r] \arrow[d] & \mathsf{WM}^{\mathfrak{t},\leq p,\underline{\mathbf{d}}}_{g,m} \arrow[d] \\
T_{g,m} \arrow[r] & \mathsf{M}_{g,m}.
\end{tikzcd}$$

3.1.4. *Local versus global wild mapping class group.* Let us denote the fundamental group of $\prod_{i=1}^{m} \mathscr{IT}^{\mathfrak{t},\leq p,\mathbf{d}_i}$ - the "local" wild mapping class group - by $\mathrm{L}\Gamma^{\underline{\mathbf{d}}}$. The homotopy exact sequence attached to the right vertical map above yields a short exact sequence

$$1 \to \mathrm{L}\Gamma^{\underline{\mathbf{d}}} \to \Gamma^{\mathfrak{t},\underline{\mathbf{d}}}_{m,g} \to \Gamma_{g,m} \to 1$$

expressing the wild mapping class group as an extension of the usual mapping class group by the local wild mapping class group.

3.2. **Example:** $\mathsf{WM}^{\mathfrak{t},\leq p}_{1,1}$**.** We have $\mathsf{M}_{1,1} = [\mathbf{H}/\mathrm{SL}_2(\mathbf{Z})]$, where $\mathbf{H} = \{\tau \in \mathbf{C} \mid \mathrm{Im}(\tau) > 0\}$ and $\mathrm{SL}_2(\mathbf{Z})$ acts on it via Möbius transformations. Over $\mathbf{H}$ we have an elliptic curve (i.e. family of compact connected pointed Riemann surfaces of genus one)

$$\mathcal{E} = (\mathbf{H} \times \mathbf{C})/\mathbf{Z}^2, \ \sigma\colon \mathbf{H} \to \mathcal{E}$$

where $(a, b) \in \mathbf{Z}^2$ acts sending (τ, z) to $(\tau, z + a\tau + b)$, and $\sigma(\tau) = (\tau, 0)$. Therefore, the fibre $\mathcal{E}_\tau$ of $\mathcal{E}$ over $\tau \in \mathbf{H}$ is the torus $\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ with the marked point 0. The coordinate z is a local coordinate around each $\sigma(\tau)$, hence irregular types for $\mathcal{E}$ at σ (with pole order at most p) can be written as

$$\sum_{j=1}^{p} z^{-j} A_j(\tau) \tag{3.2.0.1}$$

where each $A_i\colon \mathbf{H} \to \mathfrak{t}$ is a holomorphic function. Therefore, we see that $\widetilde{WT}^{\mathfrak{t},\leq p}_{1,1}$ is isomorphic to $\mathbf{H} \times \mathfrak{t}^p$.

Division by $c\tau+d$ sends $\mathbf{Z}+\tau\mathbf{Z} = (c\tau+d)\mathbf{Z}+(a\tau+b)\mathbf{Z}$ to $\mathbf{Z}+\frac{a\tau+b}{c\tau+d}\mathbf{Z}$. If $\frac{a\tau+b}{c\tau+d} = \tau$, then $z \mapsto \frac{z}{c\tau+d}$ induces an automorphism of $\mathcal{E}_\tau$. Its action on irregular types can be explicitly written down using (3.2.0.1). For instance, take $\tau = i$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\sum_{j=1}^{p} z^{-j} A_j(\tau)$ is sent to $\sum_{j=1}^{p} i^j z^{-j} A_j(\tau)$; in particular we see that, if $A_j(i)$ is non-zero for some j not divisible by 4, then the previous automorphism of $\mathcal{E}_i$ does not induce an automorphism of the relevant wild Riemann surface.

Generalising the above argument, one sees that points of $\mathsf{WM}_{1,1}^{\mathrm{t},\leq p}$ have finitely many automorphisms (as the same is true for the elliptic curves $\mathcal{E}_\tau$), and that their automorphism groups are all trivial if for each τ there is j not divisible by 2 or 3 such that $A_j(\tau)$ is non-zero.

4. Appendix: background on analytic stacks

In this appendix we recall and give references for some generalities on stacks and their fundamental group which are needed in the text. The material in this section is not new, and is included for the reader's convenience; for the same reason we try to provide some motivation for the concept of (analytic) stack. Needless to say, this is not meant to be an introduction to the subject, for which we refer the reader to the references given below.

4.1. Stacks.

We collect here the background material on stacks that is used in the text. We will focus on analytic stacks, and our main references will be [Noo05, BN06, Hei05].

4.1.1. *Motivation: the moduli space of sections of a locally free sheaf.* Let us start by showing the classical theory of moduli spaces in action from a modern viewpoint, in a simple example of interest in this document. Fix a complex manifold X and a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules of finite rank n. For every complex manifold B with a map $\varphi\colon B \to X$ we can consider the global sections $\Gamma(B, \varphi^*\mathcal{E})$ of the pullback of $\mathcal{E}$ to B. We would like to construct the moduli space of all such global sections, i.e. a complex manifold $V(\mathcal{E}) \xrightarrow{\pi} X$ such that there are natural bijections, for every $\varphi\colon B \to X$,

$$\Gamma(B, \varphi^*\mathcal{E}) \xrightarrow{\sim} \{\sigma\colon B \to V(\mathcal{E}) \mid \pi \circ \sigma = \varphi\}.$$

We claim that the vector bundle $\pi\colon V(\mathcal{E}) \to X$ attached to the locally free sheaf $\mathcal{E}$ satisfies this property. Indeed, recall that holomorphic sections of π correspond to elements of $\Gamma(X, \mathcal{E})$. Now for every $\varphi\colon B \to X$ we can consider the pullback $\varphi^*V(\mathcal{E}) \to B$; it is the subset of $B \times V(\mathcal{E})$ consisting of points (b, v) such that $\varphi(b) = \pi(v)$, and the inclusion $\varphi^*V(\mathcal{E}) \subset B \times V(\mathcal{E})$ is a holomorphic immersion. It follows that maps $\sigma\colon B \to V(\mathcal{E})$ such that $\pi \circ \sigma = \varphi$ correspond bijectively to holomorphic sections $B \to \varphi^*V(\mathcal{E})$ of the projection $\varphi^*V(\mathcal{E}) \to B$. But $\varphi^*V(\mathcal{E})$ is the vector bundle attached to $\varphi^*(\mathcal{E})$, hence such sections correspond to elements of $\Gamma(B, \varphi^*(\mathcal{E}))$.

4.1.2. *Complex manifolds as sheaves.* In §4.1.1, we have described the vector bundle $V(\mathcal{E})$ attached to a locally free sheaf $\mathcal{E}$ in terms of the holomorphic maps from an arbitrary manifold B to it: holomorphic maps $B \to V(\mathcal{E})$ are in natural bijection with couples (φ, s) consisting of a holomorphic map $\varphi\colon B \to X$ and an element $s \in \Gamma(B, \varphi^*\mathcal{E})$. In general, for every complex manifold X, we can look at the functor $h(X)\colon \mathrm{Man}_{\mathbf{C}}^{op} \to \mathrm{Sets}$ sending B to the set $\mathrm{Hom}(B, X)$ of holomorphic maps from B to X. A holomorphic map $X \to Y$ induces a natural transformation of functors $h(X) \to h(Y)$. This gives rise to the Yoneda embedding

$$\begin{aligned} \mathrm{Man}_{\mathbf{C}} &\to \mathrm{Functors}(\mathrm{Man}_{\mathbf{C}}^{\mathrm{op}}, \mathrm{Sets}) \\ X &\mapsto h(X). \end{aligned}$$

By Yoneda's lemma, the functor $X \mapsto h(X)$ is fully faithful; in other words, we can see the category of complex manifolds as a subcategory of the category $\mathrm{Functors}(\mathrm{Man}_{\mathbf{C}}^{op}, \mathrm{Sets})$. Constructing a (fine) moduli space amounts to formulating a moduli problem, i.e., writing down a functor $M\colon \mathrm{Man}_{\mathbf{C}}^{\mathrm{op}} \to \mathrm{Sets}$, and proving that M is isomorphic to $h(X)$ for some complex manifold X. If such an X exists, it is unique up to isomorphism, and we say that the moduli problem M is representable by X.

One feature distinguishing functors of the form $h(X)$ from arbitrary functors is that the former are sheaves on $\mathrm{Man}_{\mathbf{C}}$. This means that, if $(B_i)_{i \in I}$ is an open cover of a manifold B, and $b_i \in h(X)(B_i)$ are elements agreeing on the intersections $B_i \cap B_j$, then there is a unique $b \in h(X)(B)$ restricting to b_i on B_i. This is because the elements b_i are holomorphic maps $B_i \to X$, which can be glued to a holomorphic map $B \to X$ if they agree on all the intersections $B_i \cap B_j$.

4.1.3. *A moduli problem which is not a sheaf.* Many moduli problems of interest turn out not to be representable, often because they are not sheaves on $\mathrm{Man}_{\mathbf{C}}$. For example, consider the functor $M_{1,1}^{/iso}$ sending B to the set of isomorphism classes of elliptic curves over B. These are couples $(\pi\colon \Sigma \to B, \sigma)$ consisting of a proper submersion π of complex manifolds whose fibres are one dimensional complex tori, and a holomorphic section $\sigma\colon B \to \Sigma$ of π; isomorphisms are biholomorphisms commuting with the structure map to B and with the sections. We claim that $M_{1,1}^{/iso}$ is not a sheaf on $\mathrm{Man}_{\mathbf{C}}$. Indeed, take $B = \mathbf{C} \smallsetminus \{0\}$ and consider the product $B \times \mathbf{P}^2(\mathbf{C})$, whose elements we denote by $(b, [x : y : z])$. Let $\Sigma \subset B \times \mathbf{P}^2(\mathbf{C})$ be the submanifold cut out by the equation $by^2z = x^3 - z^2x$. Let $\pi\colon \Sigma \to B$ be the map induced by projection of $B \times \mathbf{P}^2(\mathbf{C})$ on the first component, and σ be the map sending b to $(b, [0 : 1 : 0])$. We can write $B = B_1 \cup B_2$, where B_1 (resp. B_2) is the complement in $\mathbf{C}$ of the half-line $\mathbf{R}_{\geq 0}$ (resp. $\mathbf{R}_{\leq 0}$). Over each B_i we can choose a holomorphic square root function $r_i\colon B_i \to \mathbf{C}$. Then, the map sending $(b, [x : y : z])$ to $(b, [x : r_i(b)y : z])$ induces an isomorphism between Σ and the elliptic curve Σ_0 with equation $y^2z = x^3 - z^2x$ over B_i. However, there is no isomorphism between Σ and Σ_0 over the whole B - the monodromy representation attached to Σ is non-trivial, cf. [Lvo19, Proposition 3.3] - hence $M_{1,1}^{/iso}$ is not a sheaf.

4.1.4. *Stacks.* The problem in the above example comes from the following phenomenon: over each B_i we have an isomorphism $f_i\colon \Sigma \to \Sigma_0$, but over $B_1 \cap B_2$ the isomorphisms f_1, f_2 differ by a non-trivial automorphism of Σ_0 (sending y to $-y$). Hence, we see that the existence of non-trivial automorphisms of elliptic curves is intimately related to the fact that $M_{1,1}^{/iso}\colon \mathrm{Man}_{\mathbf{C}}^{op} \to \mathrm{Sets}$ is not a sheaf. Grothendieck's idea to handle this issue is to make automorphisms part of the picture: rather than looking at functors $M\colon \mathrm{Man}_{\mathbf{C}}^{op} \to \mathrm{Sets}$, one considers (2-)functors

$$\mathsf{M}\colon \mathrm{Man}_{\mathbf{C}}^{op} \to \mathrm{Groupoids}.$$

Explicitly, for every complex manifold B we have a groupoid $\mathsf{M}(B)$ - i.e., a category all of whose morphisms are isomorphisms - and for every holomorphic map $f\colon B' \to B$ we have a functor $\mathsf{M}(f)^*\colon \mathsf{M}(B) \to \mathsf{M}(B')$. Furthermore, for each $g\colon B'' \to B'$ we have a natural transformation $\tau_{f,g}\colon \mathsf{M}(g)^* \circ \mathsf{M}(f)^* \Rightarrow \mathsf{M}(f \circ g)^*$. Finally, the natural transformations $\tau_{f,g}$ are required to be associative whenever we have a chain of three morphisms $B''' \xrightarrow{h} B'' \xrightarrow{g} B' \xrightarrow{f} B$.

For instance, one replaces the functor $M_{1,1}^{/iso}$ from the previous section with $\mathsf{M}_{1,1}\colon \mathrm{Man}_{\mathbf{C}}^{op} \to \mathrm{Groupoids}$ sending B to the groupoid whose objects are elliptic curves over B, and morphisms are isomorphisms.

We say that $\mathsf{M}\colon \mathrm{Man}_{\mathbf{C}}^{op} \to \mathrm{Groupoids}$ is a stack if it satisfies the following two conditions (cf. [Hei05, Definition 1.1, Remark 1.2]); below the pullback of an object $s \in \mathsf{M}(B)$ to an open $B' \subset B$ is denoted by $s_{|B'}$.

(1) Objects glue: take an open cover $(B_i)_{i \in I}$ of a manifold B, objects $s_i \in \mathsf{M}(B_i)$ and isomorphims $\varphi_{ij}\colon s_{i|B_i \cap B_j} \to s_{j|B_i \cap B_j}$ satisfying the cocycle condition $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on triple intersections $B_i \cap B_j \cap B_k$. There exists $s \in \mathsf{M}(B)$ together with isomorphisms $\varphi_i\colon s_{|B_i} \to s_i$ such that $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$.
(2) Isomorphisms glue: take an open cover $(B_i)_{i \in I}$ of a manifold B, and two objects $s, s' \in \mathsf{M}(B)$. Given isomorphisms $\varphi_i\colon s_{|B_i} \to s'_{|B_i}$ which agree on the intersections $B_i \cap B_j$, there is a unique $\varphi\colon s \to s'$ restricting to φ_i on each B_i.

A morphism of stacks $F\colon \mathsf{M} \to \mathsf{N}$ is a "natural transformation of groupoid-valued functors": precisely, it consists of the datum of a functor $F_B\colon \mathsf{M}(B) \to \mathsf{N}(B)$ for each manifold B, together with equivalences $\mathsf{N}(f)^* \circ F_B \simeq F_{B'} \circ \mathsf{M}(f)^*$ for each $f\colon B' \to B$.

Considering every set as a groupoid whose only morphisms are the identity morphisms, sheaves on $\mathrm{Man}_{\mathbf{C}}$ (in particular, objects of $\mathrm{Man}_{\mathbf{C}}$ via the Yoneda embedding) become special examples of stacks. In this appendix, we will denote the stack associated with a manifold B by $\underline{B}$; in the body of the text we follow the common abuse of denoting the stack associated with a manifold B still by B.

4.1.5. *Remark.* There is a notion of morphism between two morphisms of stacks, which we will call transformation of morphisms. Namely, let $F_1, F_2 \colon \mathsf{M} \to \mathsf{N}$ be morphisms of stacks. By definition, for each B we have functors $F_{1,B}, F_{2,B} \colon \mathsf{M}(B) \to \mathsf{N}(B)$. A transformation from F_1 to F_2 is a collection of natural transformations from $F_{1,B}$ to $F_{2,B}$ for every complex manifold B, compatible with pullbacks (cf. [Hei05, Remark 3.1.4]; formally, stacks form a 2-category with invertible 2-morphisms).

4.1.6. *Atlases and analytic stacks.* To sum up, the condition defining a stack can be thought of as a "sheaf condition" for functors valued in groupoids. Much as arbitrary sheaves on $\mathrm{Man}_{\mathbf{C}}$ have little geometric significance, we must impose additional conditions on stacks in order to be able to do geometry with them. This is captured by the crucial notion of atlas, a "good enough" approximation of a stack M by a complex manifold X. More precisely, we want to require the existence of a surjective submersion $X \to \mathsf{M}$; let us recall the precise meaning of this expression.

Firstly, we recall the definition of fibre product $\mathsf{M}_1 \times_{\mathsf{N}} \mathsf{M}_2$ of two stacks $F_1 \colon \mathsf{M}_1 \to \mathsf{N}, F_2 \colon \mathsf{M}_2 \to \mathsf{N}$. For $B \in \mathrm{Man}_{\mathbf{C}}$, objects of $\mathsf{M}_1 \times_{\mathsf{N}} \mathsf{M}_2(B)$ are triples (s_1, s_2, φ) consisting of an object $s_1 \in \mathsf{M}_1(B)$, an object $s_2 \in \mathsf{M}_2(B)$, and an isomorphism $\varphi \colon F_{1,B}(s_1) \to F_{2,B}(s_2)$. A morphism from (s_1, s_2, φ) to (s_1', s_2', φ') is a couple (φ_1, φ_2) consisting of a morphism $\varphi_1 \colon s_1 \to s_1'$ and a morphism $\varphi_2 \colon s_2 \to s_2'$, such that $F_{2,B}(\varphi_2) \circ \varphi = \varphi' \circ F_{1,B}(\varphi_1)$ for every B. We say that a morphism of stacks $F \colon \mathsf{M} \to \mathsf{N}$ is representable if, for every map of stacks $\underline{B} \to \mathsf{N}$ from a manifold B, the fibre product $\underline{B} \times_{\mathsf{N}} \mathsf{M}$ is of the form $\underline{B'}$ for a manifold B'.

4.1.7. *Remark.* Let us point out that the above notion of representability is very strong: if M, N are (stacks attached to) manifolds then we are asking in particular that all the fibres of F are manifolds, which is far from true for a general holomorphic map. A much better approach to the general theory consists in replacing $\mathrm{Man}_{\mathbf{C}}$ by the category of complex analytic spaces, which admits fibre products (cf. also [BN06, Remark 3.4]). However, the morphisms of stacks of interest in this document will satisfy the above condition. Hence, to make the text more accessible, we have chosen to work with the above definition.

4.1.8. **Definition.** A stack M is called analytic stack if there is a manifold X and a morphism of stacks $\underline{X} \to \mathsf{M}$ which is a surjective submersion, i.e. a representable map such that for every manifold B and every morphism $\underline{B} \to \mathsf{M}$ the projection $\underline{B} \times_{\mathsf{M}} \underline{X} \to \underline{B}$ is (induced by) a surjective holomorphic submersion of complex manifolds. Any manifold $\underline{X} \to \mathsf{M}$ with the previous property is called an atlas of the stack M.

4.1.9. *Moduli stacks of curves with marked points.* Let $g, m \geq 0$ be two nonnegative integers. The (2)-functor $\mathsf{M}_{g,m}$ sending a complex manifold B to the groupoid of families of compact Riemann surfaces of genus g over B with m (ordered) disjoint families of marked points is an analytic stack. More precisely, it is a quotient stack $[T_{g,m}/\Gamma_{g,m}]$, where $T_{g.m}$ is a Teichmüller space. This follows from the description of the functor of points of $T_{g,m}$ as a moduli space of Riemann surfaces with marked points and "Teichmüller structure" (cf. [Gro62, Théorème 3.1] for the case of $m = 0$).

4.2. **The fundamental group of an analytic stack.** In this section we give the definition of the (topological) fundamental group of an analytic stack, borrowed from [Noo05]. This is done in two steps: firstly, one defines the topological stack "underlying" an analytic stack; secondly, one extends the usual definition of fundamental group of a topological space to topological stacks. The first step rests on the dictionary between analytic stacks and (complex analytic) groupoids, which gives a concrete way to represent stacks in terms of "manifolds generators and relations".

4.2.1. *Presentation of analytic stacks via groupoids.* Let M be an analytic stack and $\underline{X} \to \mathsf{M}$ an atlas. The fibre product $\underline{X} \times_{\mathsf{M}} \underline{X}$ is of the form $\underline{R}$ for a manifold R endowed with two projections to X, which define the source and target morphism of a groupoid $X_\bullet$ (in $\mathrm{Man}_{\mathbf{C}}$). Conversely, each groupoid $R \rightrightarrows X$ gives rise to a quotient stack $[X/R]$ [Noo05, §3.2], [Hei05, §3]. Heuristically, the groupoid $X_\bullet$ can be thought of as a "presentation" of the stack M: the two projection maps describe which points of X should be identified to obtain M. However, it can happen that a point $x \in X$ gets identified with itself via more than one $r \in R$; this information is remembered by the stack M - and is related precisely to non-trivial automorphisms of the objects parametrised by M - but it would be lost looking at the quotient topological space X/R.

4.2.2. *Topological stack associated with an analytic stack.* Let M be an analytic stack; choose an atlas $\underline{X} \to \mathsf{M}$ and let $X_\bullet = R \rightrightarrows X$ be the resulting groupoid. Consider the topological spaces X^{top} and R^{top} underlying X and R, and the groupoid in topological spaces $X_\bullet^{top} = R^{top} \rightrightarrows X^{top}$; the associated stack on the category of topological spaces is denoted by M^{top}, and is called the topological stack attached to M (note that M^{top} is a topological stack in the sense of [Noo05, Definition 13.8], using as local fibrations those coming from [Noo05, Example 13.1.3]). The fact that M^{top} is well defined, i.e. it does not depend on the choice of the atlas, follows from the next lemma.

4.2.3. **Lemma.** *Let M be an analytic stack and let $\underline{X}_1 \to \mathsf{M}, \underline{X}_2 \to \mathsf{M}$ be two atlases. The quotient stacks $[X_1^{top}/R_1^{top}]$ and $[X_2^{top}/R_2^{top}]$ are equivalent.*

Proof. The lemma can be proved looking at an atlas refining X_1, X_2, as sketched in [Noo05, p. 79]. For completeness, let us give the details. The fibre product $\underline{X}_1 \times_{\mathsf{M}} \underline{X}_2$ is of the form $\underline{X}_3$ for a manifold X_3. The projection $X_3 \to X_1$ (resp. $X_3 \to X_2$) is a (locally trivial) $X_{2,\bullet}$-bundle (resp. $X_{1,\bullet}$-bundle) in the sense of [Hei05, §3], and the two actions defining the bundle structures commute. Hence, the same properties are true for the maps $X_3^{top} \to X_i^{top}$ for $i = 1, 2$, and the lemma follows from [Hei05, Lemma 3.2]. □

4.2.4. *The fundamental group.* A pointed analytic stack is a couple (M, x) consisting of an analytic stack M and a map $\underline{*} \to \mathsf{M}$, corresponding to an object $x \in \mathsf{M}(*)$ (the image of the identity morphism of $*$). We denote by (M^{top}, x) the associated pointed topological stack. Any pointed topological space (T, x) gives rise to a pointed topological stack $(\underline{T}, x)$. The following definition (coming from [Noo05, Definition 17.5]) is analogous to the usual definition of the fundamental group of a topological space; the subtlety and interest of the definition stem from the fact that the expressions "homotopy" and "map of pairs" have a more refined meaning in the world of stacks than in the context of topological spaces. This will be recalled below.

4.2.5. **Definition.** The fundamental group of a pointed analytic stack (M, x), denoted by $\pi_1(\mathsf{M}, x)$, is the set of homotopy classes of maps of pairs $(\underline{S}^1, 1) \to (\mathsf{M}^{top}, x)$, endowed with the group structure defined in [Noo05, pp. 59, 60].

4.2.6. *Remark.* Let us give some comments on the previous definition. The main point is that the notions of map of pairs and homotopy between continuous maps of topological spaces can be expressed via the commutativity of certain diagrams. However, as explained in Remark 4.1.5, in the world of stacks we have a notion of transformation between morphisms; to extend the classical definitions to the context of stacks, one includes a transformation from a map F_1 to a map F_2 whenever usually one would have asked the two maps to be equal.

For instance, a map $(\underline{S}^1, 1) \to (\mathsf{M}^{top}, x)$ consists by definition [Noo05, Definition 17.1] of the datum of a morphism $F\colon \underline{S}^1 \to \mathsf{M}^{top}$ together with a transformation between the composition $\underline{*} \xrightarrow{1} \underline{S}^1 \to \mathsf{M}^{top}$ and the map $\underline{*} \to \mathsf{M}^{top}$ corresponding to x. Such a transformation is induced by an isomorphism between $F(1)$ and x. In other words, we do not ask F to map 1 to x, but only to an object isomorphic to x, and we incorporate isomorphisms $F(1) \simeq x$ into our definition of "loop". In particular, we see that automorphisms of objects of $\mathsf{M}(*)$ contribute to the fundamental group of the stack, in a way explained more precisely in [Noo05, §17]. Finally, the notion of homotopy between maps of pairs is defined in [Noo05, Definition 17.2].

4.2.7. *Remark.* (1) If M is the analytic stack attached to a manifold X, then the transformations mentioned in the previous remark are always trivial, hence Definition 4.2.5 recovers the usual fundamental group of (X, x).
(2) There is an alternative way to define $\pi_1(\mathsf{M}^{top}, x)$ only involving fundamental groups of topological spaces: choose an atlas $X \to \mathsf{M}$ and consider the topological groupoid $X_\bullet^{top} = R^{top} \rightrightarrows X^{top}$. Consider the classifying space $BX_\bullet$ of $X_\bullet$, as defined in [Noo12, §4.1]. It comes with a natural map $BX_\bullet \to \mathsf{M}$ [Noo12, Proposition 6.1]; one defines $\pi_1(\mathsf{M}, x) = \pi_1(BX_\bullet, \tilde{x})$ for a point $\tilde{x}$ above x. The fact that this definition is independent of choices and coincides with the one given above follows from [Noo12, proofs of Theorem 10.5, Theorem 6.3].

4.2.8. *Example.* The approach described in the previous remark allows us to describe fundamental groups of quotient stacks, cf. [Noo12, Remark 4.3]. Firstly, if $\mathsf{M} = [*/G]$ for a (discrete) group G then $X = *$ and $BX_\bullet = EG/G$, where EG is the universal principal G-bundle (in particular, it is a simply connected space on which G acts freely), so $\pi_1(\mathsf{M}, *) = G$. More generally, if G acts on a topological space X and M is the quotient stack $[X/G]$ (i.e. it is attached to the groupoid $X_\bullet = G \times X \rightrightarrows X$ whose maps are the projection on X and the G-action) then $BX_\bullet$ is the quotient $X \times_G EG$ of $X \times EG$ by the antidiagonal action of G (Borel construction). In particular, if X is simply connected then $\pi_1([X/G], x) = G$ (no matter how big the stabilisers of points of X are). For instance, with the notation of §4.1.9, the fundamental group of $\mathsf{M}_{g,m}$ is $\Gamma_{g,m}$.

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