CI-SEQUENCES AND ALMOST COMPLETE INTERSECTIONS

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Abstract. We study the Hilbert function and the graded Betti numbers of almost complete intersection artinian algebras. We show that that every Hilbert function of a complete intersection artinian algebra is the Hilbert function of an almost complete intersection algebra. In codimension 3 we focus on almost complete intersection artinian algebras whose Hilbert function coincides with that of a complete intersection defined by 3 forms of the same degree. We classify all the possible graded Betti numbers of such algebras and we specify what cancellations are allowed in a minimal graded free resolution.

1. Introduction

A well-studied and important numerical invariant of a standard graded algebra is the Hilbert function. It gives the dimension of the graded components of the algebra. Macaulay in [Mac] characterized the numerical sequences that occur as the Hilbert function of some standard graded algebra.

On the other hand a method to study the structure of a standard graded algebra is to find its graded free resolution. It determines the graded Betti numbers of the algebra, which are a more refined numerical invariant. Indeed the graded Betti numbers of the algebra determine the Hilbert function.

Thus a classical problem is to compute all the possible graded Betti numbers compatible with an assigned Hilbert function. When H is the Hilbert function of a Cohen-Macaulay standard graded algebra of codimension 2 this problem was solved in [Ca], but the situation becomes more and more complicated when we treat codimensions greater than 2. In this case some sporadic result is available. For instance, in codimension 3, all the possible graded Betti numbers were found in [CV] and [RZ2], for Gorenstein artinian algebras with an assigned Hilbert function.

In this paper we focus our attention on CI-sequences (i.e Hilbert functions of complete intersection artinian algebras) and on almost complete intersection artinian algebras (i.e. artinian algebras R/I of codimension c, where R is a polynomial ring in c variables and I is an ideal with exactly $c+1$ minimal generators).

Almost complete intersection ideals were extensively studied since they have one more generator with respect a complete intersection ideal with the same codimension and also since they are directly linked to Gorenstein

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ideals. In particular, in codimension 3, using the remarkable structure theorem of Buchsbaum and Eisenbud (see [BE]), the resolution and the graded Betti numbers of almost complete intersection artinian algebras are well understood (see [RZ1], [RZ4]and [Z]).

In section 3 we show that every CI-sequence is the Hilbert function of an almost complete intersection algebra, which we can actually take generated by monomials.

In section 4, in Theorems 4.6, 4.7, 4.8, we find all the possible graded Betti numbers of the almost complete intersection artinian algebras whose Hilbert function is $H_{\mathrm{CI}(a,a,a)}$ i.e. the Hilbert function of a standard graded complete intersection artinian algebra defined by an ideal minimally generated by 3 forms of degree a. We need to use a careful liaison argument to guarantee that the linked Gorenstein ideal contain a regular sequence of prescribed type. We specify also what types of consecutive cancellations are allowed in the resolutions (see theorems quoted above and Corollary 4.10). Moreover we compute the maximum number of minimal second syzygies admissible for such algebras (see Corollary 4.9).

2. Preliminary facts

Throughout the paper k will be a infinite field and $R := k[x_1, \dots, x_c]$, $c \geq 3$, will be the standard graded polynomial k-algebra. We consider only homogeneous ideals.

Let $A = R/I$ be a standard graded artinian k-algebra, where I is a homogeneous ideal of R. Recall that A is said to be an *almost complete intersection algebra* if I is minimally generated by $c + 1$ elements (in this case I is said to be an almost complete intersection ideal). Every almost complete intersection ideal is directly linked to a Gorenstein ideal in a complete intersection.

Gorenstein ideals of codimension 3 are well understood, because of the structure theorem of Buchsbaum and Eisenbud (see [BE]). The possible graded Betti numbers for Gorenstein ideals of codimension 3 are described in the following theorem.

Theorem 2.1. *Let* $(d_1, \dots, d_{2n+1})$, $d_1 \leq \dots \leq d_{2n+1}$, *be a sequence of positive integers. It is the sequence of the degrees of the minimal generators of a Gorenstein ideal of codimension* 3 *iff*

1) $\vartheta = \frac{1}{n} \sum_{i=1}^{2n+1} d_i$ *is an integer;*
2) $\vartheta > d_i + d_{2n+3-i}$ *for* $2 \leq i \leq n$ *(Gaeta conditions).*

Proof. This is the main result in [Di]. □

Moreover a description of the possible graded Betti numbers for Gorenstein ideals of codimension 3 relatively to the Hilbert function can be found in [RZ2].

Let $\delta = (d_1, \dots, d_{2n+1})$, $d_1 \leq \dots \leq d_{2n+1}$, be the sequence of the degrees of the minimal generators of a Gorenstein ideal of codimension 3.

We set again $\vartheta = \frac{1}{n}\sum_{i=1}^{2n+1} d_i$. Then we can define an alternating matrix $\mathrm{Alt}(\delta) = (a_{ij})$, in the ring $S = k[\{x_{ij}\}]$, in the following way

$$a_{ij} = \begin{cases} x_{ij}^{\vartheta - d_i - d_j} & \text{if } i < j \text{ and } \vartheta - d_i - d_j > 0 \\ 0 & \text{if } i < j \text{ and } \vartheta - d_i - d_j \leq 0. \end{cases}$$

Let $I \subset S$ be the ideal generated by the submaximal pfaffians of A. Then an artinian reduction of S/I is a graded Gorenstein artinian algebra with sequence of the degrees of the minimal generators δ.

Let $\mathrm{Gor}(\delta)$ be the set of all Gorenstein ideals $I \subset k[x_1, x_2, x_3]$, whose sequence of the degrees of the minimal generators is δ.

Moreover we define

$$\mathrm{CI}_\delta = \{(a_1, a_2, a_3) \in (\mathbb{Z}^+)^3 \mid a_1 \leq a_2 \leq a_3 \text{ and } \exists I \in \mathrm{Gor}(\delta) \text{ containing a regular sequence of degrees } (a_1, a_2, a_3)\}.$$

To study ideals linked to Gorenstein ideals in a complete intersection, in codimension 3, we need the following result.

Theorem 2.2. *The set* CI_δ *has a unique minimal element.*

Proof. See Theorem 3.6 in [RZ3], where this minimal element is explicitly computed. □

We will denote the only minimal element of CI_δ with $\min(\delta)$.

Let H be the Hilbert function of a standard graded artinian algebra. A classical problem in Algebraic Geometry is to classify all the possible graded Betti numbers of the standard graded artinian algebras whose Hilbert function is H. This problem is completely settled in codimension 2 (see [Ca]). It is an open problem, in general, in codimension greater than two.

Definition 2.3. Let $I \subset k[x_1, \ldots, x_c]$ be a homogeneous ideal and let $F_\bullet$ be a graded minimal free resolution of R/I.

Let us suppose that there is i such that $F_i = L_i \oplus R(-a)$ and $F_{i+1} = L_{i+1} \oplus R(-a)$. We say that the module $R(-a)$ is *cancellable* if there exists a homogeneous ideal J such that the graded minimal free resolution of R/J is $G_\bullet$, where $G_i = L_i$, $G_{i+1} = L_{i+1}$ and $G_n = F_n$ otherwise.

Let us suppose that there is i such that $F_i = L_i \oplus R(-a) \oplus R(-b)$ and $F_{i+1} = L_{i+1} \oplus R(-a) \oplus R(-b)$. We say that the module $R(-a) \oplus R(-b)$ is a *cancellable couple* if there exists a homogeneous ideal J such that the graded minimal free resolution of R/J is $G_\bullet$, where $G_i = L_i$, $G_{i+1} = L_{i+1}$ and $G_n = F_n$ otherwise.

Remark 2.4. Note that, in the situation of Definition 2.3, R/I and R/J have the same Hilbert function. By [Pa], I and J are connected by a sequence of deformations over $\mathbb{A}^1_k$. So it is very important to study what modules in a graded minimal free resolution are cancellable. See also [Pe] for a discussion about cancellations in graded free resolutions.

3. Almost Complete Intersection sequences

In this section we show that if a sequence occurs as Hilbert function of a complete intersection artinian algebra then it is also the Hilbert function of a monomial almost complete intersection artinian algebra.

Definition 3.1. A sequence of positive integers H is called a CI-*sequence* if it is the Hilbert function of standard graded artinian complete intersection algebra. H is called an ACI-*sequence* if it is the Hilbert function of standard graded artinian almost complete intersection algebra.

Since the Koszul complex is a graded minimal free resolution for the standard graded artinian complete intersection algebras, it is easy to describe the CI-sequences.

Proposition 3.2. *In any codimension, every* CI-*sequence is an* ACI-*sequence.*

Proof. Let $H = H_{\mathrm{CI}(a_1,\ldots,a_r)}$ be a CI-sequence, with $2 \le a_1 \le \ldots \le a_r$. Let $a_r + 1 \le h \le a_r + a_1 - 1$ and let us consider the ideal

$$I_Q = (x_1^{a_1}, \ldots, x_{r-1}^{a_{r-1}}, x_r^h, x_1^{a_1+a_r-h} x_2^{a_2-a_1} \cdots x_{r-1}^{a_{r-1}-a_{r-2}} x_r^{h-a_{r-1}}).$$

We would like to show that $H_Q = H$.

We set $I_Z = (x_1^{a_1}, \ldots, x_{r-1}^{a_{r-1}}, x_r^h)$. It is the ideal of a complete intersection and $I_Z \subset I_Q$. Let $I_G = I_Z : I_Q$. We have

$$I_G = (x_1^{h-a_r}, x_2^{a_1}, \ldots, x_{r-1}^{a_{r-2}}, x_r^{a_{r-1}}),$$

hence I_G is the ideal of a complete intersection of type $(h - a_r, a_1, \ldots, a_{r-1})$. Therefore, if we set $\vartheta_Z = h + \sum_{i=1}^{r-1} a_i$, we have

$$H_Q(n) = H_Z(n) - H_G(\vartheta_Z - n).$$

The Hilbert function of R/I_G is equal to the Hilbert function of $R/I_{G'}$, where

$$I_{G'} = (x_1^{a_1}, \ldots, x_{r-1}^{a_{r-1}}, x_r^{h-a_r}).$$

But $I_Z \subset I_{G'}$ and we set $I_W = I_Z : I_{G'}$. Of course $I_W = (x_1^{a_1}, \ldots, x_{r-1}^{a_{r-1}}, x_r^{a_r})$, hence

$$H_W(n) = H_Z(n) - H_{G'}(\vartheta_Z - n) = H_Z(n) - H_G(\vartheta_Z - n) = H_Q(n),$$

so $H_Q = H_W = H_{\mathrm{CI}(a_1,\ldots,a_r)}$. □

Remark 3.3. Note that there are Gorenstein sequences that are not *ACI*-sequences. For instance $H = (1, 3, 1)$ is a Gorenstein sequence, but it is not an *ACI*-sequence. Indeed if a standard graded artinian algebra R/I has Hilbert function equal to H, then I is forced to have 5 minimal generators in degree 2.

4. Graded Betti numbers

In this section we compute all the possible graded Betti numbers of the almost complete intersection artinian algebras whose Hilbert function is $H_{\mathrm{CI}(a,a,a)}$.

If A_Q is an artinian algebra, we denote by t_Q the number of the last minimal syzygies of A_Q.

Let A_Q be an almost complete intersection artinian algebra with Hilbert function $H_{\mathrm{CI}(a,a,a)}$, $a \ge 2$. Then I_Q has 4 generators of degrees a, a, a, h, with $a + 1 \le h \le 3a - 2$. Moreover, since $\Delta^3 H_Q(2a) = 3$, I_Q has at least 3 minimal first syzygies of degree $2a$, so a graded minimal free resolution of A_Q is of the type

$$0 \to G_3 \oplus R(-3a) \to G_3 \oplus R(-2a)^3 \oplus R(-h) \to R(-a)^3 \oplus R(-h) \to R,$$

with $\operatorname{rank} G_3 = t_Q - 1$.

Let $A_Q = R/I_Q$ be an almost complete intersection artinian algebra of codimension 3. Then there is a fundamental numerical invariant d^*, (which is the degree of one of the 4 minimal generators of I_Q) that can be computed directly by the graded Betti numbers of A_Q (see Theorem 4.1 and Proposition 4.3 in [Z]). We will compute d^*, in our framework, in Corollary 4.3.

Proposition 4.1. *Let $A_Q = R/I_Q$ be an almost complete intersection artinian algebra with Hilbert function $H_{\mathrm{CI}(a,a,a)}$. Then A_Q has a minimal first syzygy of degree $a+h$ iff t is even.*

Proof. We can perform a liaison of I_Q in a complete intersection ideal I_Z of type (a,a,h). Then $I_G = I_Z : I_Q$ is a Gorenstein ideal, such that $H_G = H_{\mathrm{CI}(h-a,a,a)}$. We set $I_Z = (f_a, g_a, f_h)$, with $\deg f_a = \deg g_a = a$ and $\deg f_h = h$.

Let us suppose that A_Q has a minimal first syzygy of degree $a+h$. Then $a+h \leq 3a-1$, hence $a+1 \leq h \leq 2a-1$. Consequently A_Q has a resolution of type

$$\begin{aligned} &0 \to G_3' \oplus R(-(a+h)) \oplus R(-3a) \to \\ &\quad \to G_3' \oplus R(-2a)^3 \oplus R(-(a+h)) \oplus R(-h) \to R(-a)^3 \oplus R(-h) \to R, \end{aligned}$$

with $\operatorname{rank} G_3' = t-2$. Using mapping cone, we can get a resolution (not necessarily minimal) of I_G. In particular, since $\vartheta_Z = 2a+h$, we obtain for I_G generators of degrees $h-a, a, a, a, h$ and other $t-2$ generators of degrees greater than a. Among these generators, the only ones that can eventually be non-minimal for I_G are f_a, g_a, f_h. But $H_G = H_{\mathrm{CI}(h-a,a,a)}$, so $\vartheta_G = a+h$; therefore f_h is not minimal for I_G, since, otherwise, I_G should have a minimal first syzygy of degree a. This is impossible since I_G has only one minimal generator of degree smaller than a.

Consequently I_G has one generator of degree $h-a$, three generators of degree a and other $t-2$ generators of degrees greater than a. But, again, $H_G = H_{\mathrm{CI}(h-a,a,a)}$, so I_G has exactly 2 minimal generators of degree a, therefore I_G has $t+3$ minimal generators. Consequently $t+3$ is an odd number, i.e. t is even.

If t is even and $d^* = a$, then, for the structure of first syzygies of an almost complete intersection artinian algebra of codimension 3, I_Q has minimal first syzygies of degrees $a+h, 2a, 2a$.

If t is even and $d^* = h$, then, for the same reason, I_Q has three minimal first syzygies of degrees $a+h$. $\square$

Proposition 4.2. *Let $A_Q = R/I_Q$ be an almost complete intersection artinian algebra with Hilbert function $H_{\mathrm{CI}(a,a,a)}$. Let us suppose that t is even. Then A_Q has only one minimal first syzygy of degree $a+h$.*

Proof. Let r be the number of minimal first syzygies of I_Q of degree $a+h$. By Proposition 4.1, $u \geq 1$. Moreover I_Q has exactly u minimal second syzygies of degree $a+h$.

We link again I_Q in a complete intersection ideal I_Z of type a, a, h. Let $I_G = I_Z : I_Q$. I_G has only two minimal generators of degree a. Since t is even, only one of the three generators of I_Z is minimal for I_G, so I_G has $u+1=2$ minimal generators of degree a. Hence $u=1$. $\square$

Corollary 4.3. *Let $A_Q = R/I_Q$ be an almost complete intersection artinian algebra with Hilbert function $H_{\mathrm{CI}(a,a,a)}$. Let (a,a,a,h) be the vector of the minimal generators degrees of I_Q, $a+1 \leq h \leq 3a-2$.*

1) *If t_Q is even then $d^* = a$.*
2) *If t_Q is odd then $d^* = h$.*

Proof. If t_Q is even then, by Proposition 4.2, A_Q has only one minimal first syzygy of degree $a+h$. Hence $d^* \neq h$, i.e. $d^* = a$.

If t_Q is odd then A_Q has not first syzygies of degree $a+h$ by Proposition 4.1. So we get $d^* = h$. $\square$

Corollary 4.4. *Let $A_Q = R/I_Q$ be an almost complete intersection artinian algebra with Hilbert function $H_{\mathrm{CI}(a,a,a)}$. Let (a,a,a,h) be the vector of the minimal generators degrees of I_Q, $a+1 \leq h \leq 3a-2$.*

If $h \geq 2a$ then t_Q is odd.

Proof. If t_Q is even then by Proposition 4.1 A_Q has a minimal first syzygy of degree $a+h$. The highest second syzygy of A_Q has degree $3a$, so $a+h \leq 3a-1$, hence $a+1 \leq h \leq 2a-1$. $\square$

Proposition 4.5. *Let $A_Q = R/I_Q$ be an almost complete intersection artinian algebra with Hilbert function $H_{\mathrm{CI}(a,a,a)}$. Let $(a,a,a,a+1)$ be the vector of the minimal generators degrees of I_Q.*

Then $t_Q = 2$ and the graded minimal free resolution of A_Q is

$$0 \to R(-(2a+1)) \oplus R(-3a) \to R(-2a)^3 \oplus R(-(2a+1)) \oplus R(-(a+1)) \to \\ \to R(-a)^3 \oplus R(-(a+1)) \to R.$$

An example of this is the monomial ideal $I_Q = (x^a, y^{a+1}, z^a, x^{a-1}y)$, in $k[x,y,z]$.

Proof. We link I_Q in a complete intersection ideal I_Z of type $(a,a,a+1)$. Let $I_G = I_Z : I_Q$. It is a Gorenstein ideal with Hilbert function $H_{\mathrm{CI}(1,a,a)}$, so I_G is a complete intersection ideal of type $(1,a,a)$. So we have a liaison of a complete intersection $\mathrm{CI}(1,a,a)$ in a complete intersection $\mathrm{CI}(a,a,a+1)$.

We have two possibilities. In the liaison we can use two minimal generators of degree a or one minimal and one not minimal generator of degree a. In the first case we get a complete intersection $\mathrm{CI}(a,a,a)$. In the second case we get an almost complete intersection whose grade minimal free resolution is the one requested. $\square$

According to what we said, if $t = t_Q$ is even, the graded minimal free resolution of A_Q is of the type

$$0 \to G \oplus R(-(a+h)) \oplus R(-3a) \to \\ \to G \oplus R(-2a)^3 \oplus R(-(a+h)) \oplus R(-h) \to R(-a)^3 \oplus R(-h) \to R$$

$a \geq 2$, $a+1 \leq h \leq 2a-1$, where G is a graded free module, $\operatorname{rank} G = t-2$, $G \cong G^{\vee}(-d)$, and $d = 3a+h$. Therefore we have a decomposition

$G \cong L \oplus L^{\vee}(-d)$, where L is a graded free module, $\operatorname{rank} L = \frac{t-2}{2}$. So we obtain the graded minimal free resolution

$$\begin{aligned}0 \to L \oplus L^{\vee}(-(3a+h)) \oplus R(-(a+h)) \oplus R(-3a) \to \\ \to L \oplus L^{\vee}(-(3a+h)) \oplus R(-2a)^3 \oplus R(-(a+h)) \oplus R(-h) \to \\ \to R(-a)^3 \oplus R(-h) \to R.\end{aligned}$$

Analogously, if $t = t_Q$ is odd, the graded minimal free resolution of A_Q is of the type

$$\begin{aligned}0 \to L \oplus L^{\vee}(-(3a+h)) \oplus R(-3a) \to \\ \to L \oplus L^{\vee}(-(3a+h)) \oplus R(-2a)^3 \oplus R(-h) \to R(-a)^3 \oplus R(-h) \to R\end{aligned}$$

$a \geq 2$, $a+2 \leq h \leq 3a-2$, $\operatorname{rank} L = \frac{t-1}{2}$.

Now we prove that if we fix $H = H_{\mathrm{CI}(a,a,a)}$, h and the parity of t_Q there exists a maximal Betti sequence β of almost complete intersection artinian algebras compatible with this data and all the other Betti sequences can be obtained from β carrying out suitable cancellations. We will see that not all cancellations are allowed and we will specify which are.

Theorem 4.6. *Let* $H = H_{\mathrm{CI}(a,a,a)}$. *Let* $a+1 \leq h \leq 2a-1$. *We set*

$$F = \begin{cases} R(-(2a+1)) \oplus \ldots \oplus R(-(a+h)) & \text{if } h-a \text{ is odd} \\ R(-(2a+1)) \oplus \ldots \oplus R(-(a+h)) \oplus R(-(\frac{3a+h}{2})) & \text{if } h-a \text{ is even}\end{cases}$$

Then there exists an almost complete intersection artinian algebra with Hilbert function H *and graded minimal free resolution*

$$0 \to F \oplus R(-3a) \to R(-h) \oplus R(-2a)^3 \oplus F \to R(-a)^3 \oplus R(-h) \to R$$

Moreover let $\mathcal{A}_{H,h}$ *be the set of the almost complete intersection artinian algebras with Hilbert function* H, *generated in degrees* (a,a,a,h), $a+1 \leq h \leq 2a-1$, *and* t *even. Then*

1) *these graded Betti numbers are maximal among algebras in* $\mathcal{A}_{H,h}$;
2) *the only couples of cancellations allowed, among algebras in* $\mathcal{A}_{H,h}$, *are* $R(-i) \oplus R(-(3a+h-i))$, *for* $2a+1 \leq i \leq a+h-1$.

Proof. We set

$$\delta = \begin{cases} (h-a, a, a, a+1, \ldots, h-1) & \text{if } h-a \text{ is odd} \\ \left(h-a, a, a, a+1, \ldots, \frac{a+h}{2}, \frac{a+h}{2}, \ldots, h-1\right) & \text{if } h-a \text{ is even}\end{cases}$$

By Theorem 2.1 and by Theorem 3.6 in [RZ3], there exists a Gorenstein ideal I_G with sequence of the degrees of the minimal generators δ and containing a complete intersection ideal I_Z of type (a,a,h), with a generator of degree a minimal and a generator of degree a not minimal for I_G. The linked ideal $I_Q = I_Z : I_G$ has the requested graded minimal free resolution.

Let $A_Q = R/I_Q$ be an almost complete intersection artinian algebra with Hilbert function H, $a+1 \leq h \leq 2a-1$, and t even. We can link I_Q in a complete intersection ideal I_Z of type (a,a,h). Let $I_G = I_Z : I_Q$. Then $A_G = R/I_G$ is a Gorenstein algebra such that $H_G = H_{\mathrm{CI}(h-a,a,a)}$. We recall that the resolution of A_G is determined by the degrees of the minimal generators of I_G. The maximal Betti numbers relatively to the

Hilbert function were computed in [RZ2], Proposition 3.7 and Remark 3.8. Since $\Delta^2 H_{\mathrm{CI}(h-a,a,a)}(n) = -2$, for $a \leq n \leq h-1$, they are given by the resolution

$$\ldots \to R(-(h-a)) \oplus R(-a)^2 \oplus R(-(a+1))^2 \oplus \ldots \oplus R(-(h-1))^2 \to R.$$

But if I_G has two minimal generators in degree n, $a+1 \leq n \leq \left\lfloor \frac{a+h}{2} \right\rfloor$, by Theorem 4.4 in [RZ5] or also by Theorem 3.6 in [RZ3], it cannot contain a regular sequence of type (a,a,h). Therefore the maximal sequence of the degrees of the minimal generators for a Gorenstein ideal I_G, with $H_G = H_{\mathrm{CI}(h-a,a,a)}$ and containing a complete intersection ideal I_Z of type (a,a,h) is δ.

The cancellations allowed are the ones that depend from cancellations of the linked Gorenstein. So it is enough to use again Proposition 3.7 and Remark 3.8 in [RZ2]. □

Theorem 4.7. *Let $H = H_{\mathrm{CI}(a,a,a)}$. Let $a+2 \leq h \leq 2a-1$. We set*

$$F = \begin{cases} R(-(2a+1)) \oplus \ldots \oplus R(-(a+h-1)) & \text{if } h-a \text{ is odd} \\ R(-(2a+1)) \oplus \ldots \oplus R(-(a+h-1)) \oplus R(-(\frac{3a+h}{2})) & \text{if } h-a \text{ is even} \end{cases}$$

Then there exists an almost complete intersection artinian algebra with Hilbert function H and graded minimal free resolution

$$0 \to F \oplus R(-3a) \to R(-h) \oplus R(-2a)^3 \oplus F \to R(-a)^3 \oplus R(-h) \to R$$

Moreover let $\mathcal{B}_{H,h}$ be the set of the almost complete intersection artinian algebras with Hilbert function H, generated in degrees (a,a,a,h), $a+2 \leq h \leq 2a-1$, and t odd. Then

1) *these graded Betti numbers are maximal among algebras in $\mathcal{B}_{H,h}$;*
2) *the only couples of cancellations allowed, among algebras in $\mathcal{B}_{H,h}$, are $R(-i) \oplus R(-(3a+h-i))$, for $2a+1 \leq i \leq a+h-1$ and $t \geq 5$.*

Proof. We set δ as in the proof of Theorem 4.6, but now we link using a complete intersection ideal I_Z of type (a,a,h), with two generators of degree a both minimal for I_G.

The rest of the proof runs analogously as in in the proof of Theorem 4.6. □

Theorem 4.8. *Let $H = H_{\mathrm{CI}(a,a,a)}$. Let $2a \leq h \leq 3a-2$. We set*

$$F = \begin{cases} R(-(h+1)) \oplus \ldots \oplus R(-(3a-1)) & \text{if } h-a \text{ is odd} \\ R(-(h+1)) \oplus \ldots \oplus R(-(3a-1)) \oplus R(-(\frac{3a+h}{2})) & \text{if } h-a \text{ is even} \end{cases}$$

Then there exists an almost complete intersection artinian algebra with Hilbert function H and graded minimal free resolution

$$0 \to F \oplus R(-3a) \to R(-h) \oplus R(-2a)^3 \oplus F \to R(-a)^3 \oplus R(-h) \to R$$

Moreover let $\mathcal{B}_{H,h}$ be the set of the almost complete intersection artinian algebras with Hilbert function H, generated in degrees (a,a,a,h), $2a \leq h \leq 3a-2$ (in this case t is necessarily odd). Then

1) *these graded Betti numbers are maximal among algebras in $\mathcal{B}_{H,h}$;*
2) *the only couples of cancellations allowed, among algebras in $\mathcal{B}_{H,h}$, are $R(-i) \oplus R(-(3a+h-i))$, for $h+1 \leq i \leq 3a-1$ and $t \geq 5$.*

Proof. We set

$$\delta = \begin{cases} (a, a, h-a, h-a+1, \ldots, 2a-1) & \text{if } h-a \text{ is odd} \\ \left(a, a, h-a, h-a+1, \ldots, \frac{a+h}{2}, \frac{a+h}{2}, \ldots, 2a-1\right) & \text{if } h-a \text{ is even} \end{cases}$$

Again we link using a complete intersection ideal I_Z of type (a, a, h), with two generators of degree a both minimal for I_G (in this case we are forced for the minimality).

The rest of the proof runs analogously as in the proof of Theorem 4.6. □

Corollary 4.9. *The maximum number of minimal second syzygies for an almost complete intersection artinian algebra with Hilbert function $H_{\mathrm{CI}(a,a,a)}$ is $t_a^{\max} = a+1$ if a is even, $t_a^{\max} = a$ if a is odd.*

Proof. Comparing the maximal graded Betti numbers that we state in Theorem 4.6, Theorem 4.7 and Theorem 4.8, we get this maximum number when $h = 2a$. □

Corollary 4.10. *Let $A_Q = R/I_Q$ be an almost complete intersection artinian algebra with Hilbert function $H_{\mathrm{CI}(a,a,a)}$, with I_Q generated in degrees a, a, a, h. Let us suppose that A_Q has a minimal first syzygy of degree $a+h$.*

Then the module $R(-(a+h))$ is cancellable in the minimal graded free resolution of A_Q iff $t_Q \geq 4$.

Proof. By Proposition 4.1, t_Q is even and by Proposition 4.2, A_Q has only one minimal first syzygy and only one minimal second syzygy of degree $a+h$. By Corollary 4.4, $a+1 \leq h \leq 2a-1$.

If $R(-(a+h))$ is cancellable then there exists an almost complete intersection artinian algebra A_W with $t_W = t_Q - 1$; since t_W is odd and A_W is not a complete intersection algebra, $t_W \geq 3$ i. e. $t_Q \geq 4$.

If $t_Q \geq 4$, then by Theorem 4.7 there exists an almost complete intersection artinian algebra with the same graded minimal free resolution of A_Q except for the module $R(-(a+h))$, that is therefore cancellable. □

Example 4.11. Let $a = 3$ and $h = 5$. By Theorem 4.6, the maximal resolution $F_\bullet$ with these data is

$$0 \to R(-7)^2 \oplus R(-8) \oplus R(-9) \to R(-5) \oplus R(-6)^3 \oplus R(-7)^2 \oplus R(-8) \to$$
$$\to R(-3)^3 \oplus R(-5) \to R$$

An ideal with this minimal graded free resolution can be realized in the following way. We set $\vartheta_Z = 2a + h = 11$. Let $v = (7, 7, 8, 9)$ be the vector of the degrees of the minimal second syzygies. We compute $v' = (11-9, 11-8, 11-7, 11-7) = (2, 3, 4, 4)$. We extend v' with a component equal to a. So we obtain the vector $u = (2, 3, 3, 4, 4)$. Now let us consider the alternating matrix $A = \mathrm{Alt}(2, 3, 3, 4, 4)$.

$$A = \begin{pmatrix} 0 & x_{12}^3 & x_{13}^3 & x_{14}^2 & x_{15}^2 \\ -x_{12}^3 & 0 & x_{23}^2 & x_{24} & x_{25} \\ -x_{13}^3 & -x_{23}^2 & 0 & x_{34} & x_{35} \\ -x_{14}^2 & -x_{24} & -x_{34} & 0 & 0 \\ -x_{15}^2 & -x_{25} & -x_{35} & 0 & 0 \end{pmatrix}.$$

Let p_i be the pfaffian obtained by A, by deleting the i-th row and i-th column. Let $p_{i,j,k}$ be the pfaffian obtained by A, by deleting the i-th, j-th and k-th row and i-th, j-th and k-th column.

Then the ideal $I_Q = (y_2p_1, p_2, y_1p_5, y_1y_2p_{1,2,5})$ is a perfect ideal in the ring $k[\{x_{ij}\}, y_1, y_2]$ with graded minimal free resolution $F_\bullet$. So it is enough to consider a artinian reduction of I_Q. The module $R(-8)$ (where $8 = a+h$) is cancellable. Indeed the ideal $I_W = (p_2, p_3, y_1p_5, y_1p_{2,3,5})$ has graded minimal free resolution

$$0 \to R(-7)^2 \oplus R(-9) \to R(-5) \oplus R(-6)^3 \oplus R(-7)^2 \to \\ \to R(-3)^3 \oplus R(-5) \to R.$$

Note that, in the previous resolution, $R(-7)^2$ trivially is not cancellable. Also $R(-7)$ is not cancellable by Theorem 4.7. So these graded Betti numbers are minimal for almost complete intersection algebras.

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