Stability Properties of the Impulsive Goodwin's Oscillator in 1-cycle

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Abstract— **The Impulsive Goodwin's Oscillator (IGO) is a mathematical model of a hybrid closed-loop system. It arises by closing a special kind of continuous linear positive time-invariant system with impulsive feedback, which employs both amplitude and frequency pulse modulation. The structure of IGO precludes the existence of equilibria, and all its solutions are oscillatory. With its origin in mathematical biology, the IGO also presents a control paradigm useful in a wide range of applications, in particular dosing of chemicals and medicines. Since the pulse modulation feedback mechanism introduces significant nonlinearity and non-smoothness in the closed-loop dynamics, conventional controller design methods fail to apply. However, the hybrid dynamics of IGO reduce to a nonlinear, time-invariant discrete-time system, exhibiting a one-to-one correspondence between periodic solutions of the original IGO and those of the discrete-time system. The paper proposes a design approach that leverages the linearization of the equivalent discrete-time dynamics in the vicinity of a fixed point. A simple and efficient local stability condition of the 1-cycle in terms of the characteristics of the amplitude and frequency modulation functions is obtained.**

I. INTRODUCTION

Most of the research in control theory deals with the problem of steering a dynamical system to and stabilizing it at an equilibrium point. Yet, there is an increasing interest in oscillatory behaviors that are ubiquitous in physics, chemistry, biology, economics, engineering, and medicine [1]. Modeling and analysis of periodic and non-periodic oscillations is therefore a timely topic in nonlinear dynamics with rich applications in science and technology.

A periodic oscillation describes a repeating (cyclic) process. A standard example of the control actions that are performed repeatedly according to a schedule is taking prescription drugs. Under stationary conditions, taking the right dose at the right time usually works well. When the therapeutic effect is not sufficient, either the drug dose has to be increased or the interdose interval has to be reduced. Both regiment adjustments elevate the drug concentration in the organism and lead to a higher effect according to the dose-response relationship. Similar control mechanisms that manipulate both the timing and magnitude of discrete actions appear also outside medicine, e.g. in mechanical systems with impacts [2] or in pest management [3].

A fast (relative to the plant dynamics) control action can be approximated by an impulse, or the Dirac δ-function. A well-developed framework to handle a continuous system with impulsive output feedback is pulse-modulated control with amplitude and frequency modulation [4]. It was successfully utilized in the Impulsive Goodwin's Oscillator (IGO) devised to model pulsatile endocrine regulation [5], [6]. The IGO has found application in modeling biological data pertaining to feedback (non-basal) testosterone regulation in the male [7] and the multi-peak phenomenon in *levodopa*, a drug to treat Parkinson's disease [8].

Since the inception of the IGO, the research focus was primarily on discerning the complex dynamical phenomena exhibited by the model, namely periodic solutions of high multiplicity, chaos [9], and entrainment of oscillations to an exogenous periodic signal [10]. These studies can be characterized as *analysis* of the IGO's rich dynamics.

More recently, the *design* of the pulse-modulated feedback of the IGO to sustain a desired periodic solution was addressed. Two problems were solved for a given continuous plant and with respect to a so-called 1-cycle, a periodic solution that is distinguished by a single firing of the feedback in the least period. First, in [11], the problem of obtaining a stable 1-cycle with a given period and weight of the impulsive control sequence is solved. Second, output corridor control of the continuous plant, i.e. the problem of keeping the output within a predefined interval of values, was worked out in [12]. In both cases, applications to dosing of chemicals and drugs were envisioned.

This paper deals with stability analysis of the 1-cycle in the IGO that is indispensable to obtain a sustained periodic solution. A simple and efficient local stability condition for the 1-cycle in the form of a linear inequality is obtained. It allows to restrict the characteristics of the amplitude and frequency modulation functions so that the designed closed-loop solution is orbitally stable. Numerical experiments show that, in fact, almost all solutions of the nonlinear IGO system are attracted to a stable 1-cycle, if it exists [9].

The rest of the paper is composed as follows. In Section II, the IGO model is briefly introduced for the reader's convenience. In this section, the dynamics of the IGO are discussed and a discrete map propagating the state vector of the continuous part of the model through the firings of the pulse-modulated feedback is given. The latter is used to derive an explicit expression for the 1-cycle. In Section III, the main result of the paper is formulated yielding a linear inequality that provides a necessary and sufficient condition of a 1-cycle with given parameters. Finally, a numerical example is given in Section IV to illustrate the developed theory.

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II. IMPULSIVE GOODWIN'S OSCILLATOR

Consider a third-order linear time-invariant system

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t). \tag{1}$$

Here matrices A, C are as follows:

$$A = \begin{bmatrix} -a_1 & 0 & 0 \\ g_1 & -a_2 & 0 \\ 0 & g_2 & -a_3 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \tag{2}$$

where $a_1, a_2, a_3 > 0$ are *distinct* constants, and $g_1, g_2 > 0$ are positive gains. The scalar function y is the measured output, and the state variables are $x = [x_1, x_2, x_3]^\top$. It follows that the matrix A is Hurwitz stable and Metzler.

Impulsive feedback

Continuous-time system (1) is controlled by a pulse-modulated feedback where the impulse weights and their timing are determined by the continuous plant output $y(t)$:

$$\begin{aligned} &x(t_n^+) = x(t_n^-) + \lambda_n B, \quad t_{n+1} = t_n + T_n, \\ &T_n = \Phi(y(t_n)),\ \lambda_n = F(y(t_n)), \quad B = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top, \end{aligned} \tag{3}$$

where $n = 0, 1, \ldots$. The minus and plus in a superscript in (3) denote the left-sided and a right-sided limit, respectively. Notice that the jumps in $x(t)$ lead only to discontinuities in $x_1(t)$, whereas $x_2(t)$, and $y(t) = x_3(t)$ remain continuous. The instants t_n are called (impulse) firing times and λ_n represents the corresponding impulse weight.

In theory of pulse-modulated systems [4], $F(\cdot)$ is called the amplitude modulation function and $\Phi(\cdot)$ is referred to as the frequency modulation function. The modulation functions are assumed to be continuous and monotonic, $F(\cdot)$ be *non-increasing*, and $\Phi(\cdot)$ be *non-decreasing*.

These monotonicity assumptions imply that controller (3) implements a negative feedback from the continuous output to the amplitude and frequency of the pulses. Namely, an increased value of $y(t_n)$ results in a decreased lower (or unchanged) weight λ_n for the next impulse fired at t_{n+1}. Furthermore, the interval between the impulses can only increase, which means that the sequence of pulses becomes sparser. This feedback mechanism prevents the controlled output from diverging. Notably, the negative feedback action is implemented by means of positive signals only.

To explicitly restrict the domain where the solutions of closed-loop system (1), (3) ultimately evolve, boundedness of the modulation functions is required

$$\Phi_1 \le \Phi(\cdot) \le \Phi_2, \quad 0 < F_1 \le F(\cdot) \le F_2, \tag{4}$$

where Φ_1, Φ_2, F_1, F_2 are positive constants. Under these limitations, all solutions eventually arrive at an invariant 3-dimensional box, which can be computed explicitly [9].

Definition 1: The IGO is a hybrid system arising as a feedback interconnection of the continuous LTI block in (1) and the impulsive feedback in (3).

The class of design problems that captures our interest involves guaranteeing certain desired properties of the IGO through the selection of modulation functions $F(\cdot)$, $\Phi(\cdot)$. These functions serve as the designer's degrees of freedom in the impulsive controller tuning.

The discrete-time representation and 1-cycles

The hybrid dynamics of the IGO can be reduced to the discrete-time equation by noticing that the sequence of state vectors $X_n = x(t_n^-)$ obeys the recurrence formula [6]

$$\begin{aligned} X_{n+1} &= Q(X_n), \\ Q(\xi) &\triangleq \mathrm{e}^{A\Phi(C\xi)} \left(\xi + F(C\xi)B\right). \end{aligned} \tag{5}$$

Since the plant is autonomous in between the impulsive feedback firings, the continuous state trajectory on the interval (t_n, t_{n+1}) is uniquely defined by X_n as

$$x(t) = \mathrm{e}^{(t-t_n)A}(X_n + \lambda_n B), \quad t \in (t_n, t_{n+1}). \tag{6}$$

In this sense, the properties of the IGO (being a hybrid dynamical system) are completely determined by the properties of impulse-to-impulse map Q, defined in (5).

As known [9], discrete-time system (5) and, therefore, the IGO, can exhibit a wide range of periodic and non-periodic oscillation, including deterministic chaos. In this study, only the simplest periodic solution of (1), (3) with one firing of the impulsive feedback in the least solution period is treated. It is termed 1-cycle, see e.g. [13], and, by definition, corresponds to the *periodic* instants of pulses $t_{n+1} = t_n + T$, $T > 0$ and the *constant* sequence of amplitudes $X_n = X(t_n^-) \equiv X$, where X is the fixed point of map Q:

$$X = Q(X), \tag{7}$$

The characteristics of the 1-cycle, i.e. the (least) period and the impulse weight, are then defined by the fixed point as $T = \Phi(y_0)$, $\lambda = F(y_0)$, $y_0 = CX$.

As previously demonstrated in [11], the solution to nonlinear equations (7) can be analytically expressed using the parameters of plant (1) and the characteristics of the 1-cycle λ, T. This explicit solution of (7) is conveniently formulated in terms of divided differences (DDs). The first DD of a function $h : \mathbb{R} \to \mathbb{R}$ is defined as

$$h[x_1, x_2] \triangleq \frac{h(x_1) - h(x_2)}{x_1 - x_2}, \quad \forall x_1 \neq x_2,$$

and higher-order divided DDs are introduced recursively by

$$h[x_0, \ldots, x_k] = \frac{h[x_1, \ldots, x_k] - h[x_0, \ldots, x_{k-1}]}{x_k - x_0}.$$

For the sake of simplicity, only pairwise distinct sets of values $x_0, \ldots, x_k$ are considered here.

Denote $\mu(x) \triangleq \frac{1}{\mathrm{e}^{-x} - 1}$. Using the standard definition of an analytic function on matrices [14], $\mu(M) = (\mathrm{e}^{-M} - I)^{-1}$ for an arbitrary matrix M with non-zero eigenvalues. The special structure of matrix A allows to compute $\mu(TA)$, $T > 0$, through DDs of the function μ by means of the Opitz formula [14], [15] and leading to the following proposition.

Proposition 1: [11] If IGO (1), (3) exhibits a 1-cycle of the period T with the weight λ, then the fixed point satisfying (7) is uniquely determined as

$$X = \lambda\mu(TA)B = \lambda \begin{bmatrix} \mu(-a_1 T) \\ g_1 \mu[-a_1 T, -a_2 T] \\ g_1 g_2 \mu[-a_1 T, -a_2 T, -a_3 T] \end{bmatrix}. \tag{8}$$

The availability of an analytic expression for X ensures one-to-one map between the pair (λ, T) and the fixed point. This fact has enabled the *design* of the IGO whose 1-cycles have predefined parameters λ, T [11], [12], [16]. In order to guarantee the existence of such a 1-cycle, one has to find the (nonlinear) modulation functions Φ, F such that

$$\lambda = F(y_0), \quad T = \Phi(y_0), \text{ where}$$
$$y_0 \triangleq CX = \lambda g_1 g_2 \mu[-a_1 T, -a_2 T, -a_3 T].$$

The problem is, however, that the resulting 1-cycle can turn out (orbitally) *unstable* and thus fail to pertain in the face of perturbation.

In the next section, the main result of this paper is presented, offering a simple analytic stability criterion.

III. STABILITY OF 1-CYCLE

The 1-cycle in closed-loop system (1), (3) corresponding to the fixed point X is known to be (locally exponentially) orbitally stable [6], [9] if only only if

$$Q'(X) = \mathrm{e}^{A\Phi(y_0)}\,(I + F'(y_0)BC) + \Phi'(y_0)AXC,$$

is a Schur stable matrix[1]. As pointed out in [11, Proposition 3], the Jacobian can be written as

$$Q'(X) = \mathrm{e}^{A\Phi(y_0)} + \begin{bmatrix} J & D \end{bmatrix} \begin{bmatrix} F'(y_0) \\ \Phi'(y_0) \end{bmatrix} C, \qquad (9)$$

where $J = \mathrm{e}^{AT} B > 0, D = AX < 0$.

Since plant (1) is Hurwitz, stability of the 1-cycle is always guaranteed for zero slopes of the modulation functions, for instance, when $F(y) = \text{const}$, $\Phi(y) = \text{const}$. However, this essentially eliminates the output feedback, at least in the vicinity of the fixed point. To improve the convergence to the stationary solution under perturbation, the spectral radius of the Jacobian has to be minimized.

Insufficiency of standard stabilization methods

The right-hand side of (9) has apparent similarity to the problem of stabilization of a discrete time-invariant linear system by a static output feedback, see e.g. [17]. This is the problem of finding such gain matrix K_d that the system

$$x_d(t+1) = A_d x_d(t) + B_d u_d(t),$$
$$y_d(t) = C_d x_d(t),$$

is (asymptotically) stabilized by the control law $u_d(t) = K_d y_d(t)$. Equivalently, one is looking for a K_d that makes the matrix $A_d + B_d K_d C_d$ Schur-stable. For the reasons described above, the largest possible set of such controllers is sought, despite the existence of the trivial solution $K_d \equiv 0$. Although the static output feedback stabilization problem appears to be simple, a complete characterization of the gains solving it is missing. For instance, the pole placement problems via static feedback are usually considered in the situation where the total number of scalar entries $\dim y_d \dim u_d$ in K_d is not less than the state dimension $\dim x_d$ [18], which inequality is, obviously, violated in the present case (cf. $\dim x_d = 3, \dim u_d = 2, \dim y_d = 1$).

Another idea suggested by the similarity between the problem of stabilizing the fixed point of a 1-cycle in (1), (3) and solving the static output feedback problem is to reformulate the stability condition as a system of bilinear matrix inequalities (BMI)

$$(A_\Phi + WKE)^\top P(A_\Phi + WKE) - P < 0, \quad P > 0, \quad (10)$$

where

$$A_\Phi = \mathrm{e}^{A\Phi(y_0)}, W = \begin{bmatrix} J & D \end{bmatrix}, K = \begin{bmatrix} F'(y_0) & \Phi'(y_0) \end{bmatrix}^\top,$$

and P, K are the decision variables. Again, one may notice that the inequality is feasible since it is always satisfied for $K = 0$ and some P. However, the non-convexity of (10) makes it difficult to find the optimal (with respect to some performance index) solution; all known methods, in general, return only local solutions [19].

Main result: a linear stability condition

Although the expression for the Jacobian matrix is complicated, a necessary and sufficient analytic criteria for $Q'(X)$ being Schur stable can be obtained in terms of a *linear inequality* in the slopes $F'(y_0), \Phi'(y_0)$. This makes the result here quite different from the standard Schur stability criteria, such as the Schur-Cohn stability test, the Jury criterion, and the Liénard-Chipart criterion (all of them lead lead to nonlinear stability conditions, see, e.g., [11, Lemma 1]). The next theorem is our main result.

Theorem 1: Assume that $0 < a_1 < a_2 < a_3$. If $F'(z_0) \le 0$ and $\Phi'(z_0) \ge 0$, then the Jacobian matrix $Q'(X)$ is Schur stable if and only if

$$\det(-I - Q'(X)) < 0, \qquad (11)$$

or, equivalently, the following linear inequality holds

$$C(I + e^{\Phi(y_0)A})^{-1}\,(F'(y_0)J + \Phi'(y_0)D) > -1. \qquad (12)$$

Furthermore, the Jacobian matrix $Q'(X)$ always has a positive real eigenvalue, lying in the interval $[\mathrm{e}^{-a_3 T}, \mathrm{e}^{-a_1 T}]$. Hence, the spectral radius of $Q'(X)$ is not less than $\mathrm{e}^{-a_3 T}$.

The proof of Theorem 1 is given in Appendix.

Final remarks. Two insights can be gained from the analysis above. First, since $JF'(y_0) + D\Phi'(y_0) \le 0$, for all feasible values of $F'(y_0), \Phi'(y_0)$, the feedback stabilizing the fixed point in (9) is negative, in a well-defined sense. This is despite the fact that all the signals comprising closed-loop system (1), (3) are positive. Then, the IGO explains how negative feedback is implemented in nature (e.g. endocrine systems [20]) by impulsive regulation when negative signals are not available. Second, existing design methods for (discrete) linear time-invariant systems can be adopted to the framework of IGO design in 1-cycle by applying them to the fixed point instead of the equilibrium.

[1]Recall that a matrix is Schur stable, or Schur, if all its eigenvalues are less than 1 in modulus.

IV. Numerical Example

To illustrate the theoretical results of Section III, consider the pharmacokinetic-pharmodynamic model of the muscle relaxant *atracurium* used under general closed-loop anesthesia. The model originates from [21] and has been used for investigating the performance of the IGO as a feedback dosing algorithm in [12]. The linear part of the model is of third order (see (1) with the state matrix

$$A = \begin{bmatrix} -0.0374 & 0 & 0 \\ 0.0374 & -0.1496 & 0 \\ 0 & 0.0560 & -0.3740 \end{bmatrix}.$$

The fixed point $X^\top = \begin{bmatrix} 136.4461 & 44.9637 & 7.4309 \end{bmatrix}$ corresponds to the 1-cycle with the parameters $\lambda = 415.8412$, $T = 37.3834$. Then,

$$J = \begin{bmatrix} 0.4733 \\ 0.1410 \\ 0.0221 \end{bmatrix}, \quad D = \begin{bmatrix} -10.0829 \\ -2.5705 \\ -0.3633 \end{bmatrix},$$

and inequality (12) gives the stability condition for the 1-cycle

$$0.0138 \cdot F'(y_0) - 0.1933 \cdot \Phi'(y_0) > -1. \tag{13}$$

In Fig. 1, one can compare the numerical calculation of the spectral radius of $Q'(X)$ with the result of applying the stability criterion in (12). It can be seen that the inequality in (13) correctly describes the values of $F'(y_0)$ and $\Phi'(y_0)$ yielding a stable 1-cycle.

Notice that the slopes of the frequency and amplitude modulation functions control the convergence to the 1-cycle under perturbation, whereas the parameters of the stationary solution (defined by X) remain the same. Multiple numerical studies of the IGO's dynamics (see e.g. [9]) indicate that a stable 1-cycle attracts almost all biologically feasible (positive) solutions.

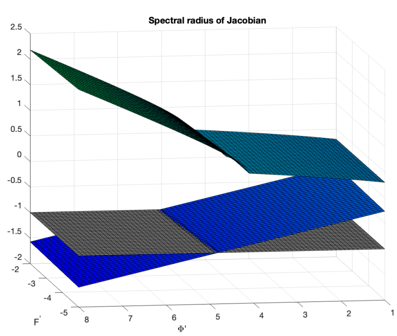

Fig. 1. Spectral radius of $Q'(X)$ and condition (12) as function of $\Phi'(y_0)$ and $F'(y_0)$. Green and light blue surface – spectral radius of the Jacobian. Dark blue surface – the left-hand side expression of inequality (12). Grey plane – stability border, i.e. -1.

V. Conclusions

Stability of the 1-cycle in the Impulsive Goodwin's Oscillator (IGO) is examined. A linear inequality specifying the stability domain of the stationary solution in terms of the slopes of the frequency and amplitude modulation functions is derived. The result is instrumental in optimizing the convergence rate of perturbed solutions to the 1-cycle under stability guarantee, which topic is saved for future works. The IGO gives rise to a class of simple feedback controllers that implement administration of discrete doses to a continuous plant according to a desired schedule. The presence of the pulse-modulated feedback allows the IGO to manipulate both the doses and their timing to achieve the control goal.

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Appendix: Proof of Theorem 1

Throughout this section, all the assumptions of Theorem 1 are supposed to hold, in particular, $0 < a_1 < a_2 < a_3$. Consider the matrix

$$\begin{gathered} \mathcal{Q}(T,\xi,\eta) = \mathrm{e}^{AT} + (\xi J + \eta D)\, C, \\ D \triangleq A(\mathrm{e}^{-AT} - I)^{-1}B,\; J = \mathrm{e}^{AT}B, \end{gathered} \tag{14}$$

where A, B, C are matrices from (1).

We first prove a technical lemma.

Lemma 1: Suppose that $T > 0$, $\xi \leq 0$, and $\eta \geq 0$. Then matrix $\mathcal{Q} \triangleq \mathcal{Q}(T,\xi,\eta)$ has the following spectral properties:

1) $\mathcal{Q}$ has no eigenvalues on the interval $(\mathrm{e}^{-a_1 T}, \infty)$;
2) there exists a real eigenvalue $z_1 \in [\mathrm{e}^{-a_3 T}, \mathrm{e}^{-a_1 T}]$;
3) the product of two remaining eigenvalues $z_2 z_3$ does not exceed $e^{-(a_1+a_2)T} < 1$.
4) $\mathcal{Q}$ is not Schur stable if and only if z_2, z_3 are real and $\min(z_2, z_3) \leq -1$;

Proof: The characteristic polynomial $\chi(z) \triangleq \det(zI - \mathcal{Q})$, thanks to the Schur complement formula, is written as

$$\begin{aligned} \chi(z) &= \det(zI - \mathrm{e}^{AT}) w(z), \quad \text{where} \\ w(z) &\triangleq \frac{\chi(z)}{\det(zI - \mathrm{e}^{AT})} = \\ &= 1 - C(zI - \mathrm{e}^{AT})^{-1}(\xi J + \eta D). \end{aligned} \tag{15}$$

Notice that $w(z) \geq 1$ whenever z is real and $z > \mathrm{e}^{-a_1 T}$, because $1 - w(z)$ can be decomposed into the series

$$C(zI - e^{AT})^{-1}(\xi J + \eta D) = z^{-1} \sum_{k=0}^{\infty} z^{-k} C e^{kAT} (\xi J + \eta D),$$

whose coefficients are non-positive matrices, because $J > 0$ and $D < 0$ [11, Proposition 3]. Furthermore, from the triangular structure of A, the characteristic polynomial

$$\det(zI - \mathrm{e}^{AT}) = (z - \mathrm{e}^{-a_1 T})(z - \mathrm{e}^{-a_2 T})(z - \mathrm{e}^{-a_3 T}) \tag{16}$$

is positive for all $z > \mathrm{e}^{-a_1 T}$, which entails that $\chi(z) > 0$. This completes the proof of statement 1).

To prove statement 2), a more subtle argument is needed that requires us to compute the residual of rational function

$w(z)$ at $z = e^{-a_3 T_0}$. To this end, consider the diagonalization of matrix A. It can be checked that

$$S^{-1}AS = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix}, \text{ where}$$
$$S = \begin{bmatrix} 1 & 0 & 0 \\ \frac{g_1}{a_2-a_1} & 1 & 0 \\ \frac{g_1 g_2}{(a_2-a_1)(a_3-a_1)} & \frac{g_2}{a_3-a_2} & 1 \end{bmatrix} \text{ and}$$
$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{g_1}{a_2-a_1} & 1 & 0 \\ -\frac{g_1 g_2}{(a_3-a_2)(a_3-a_1)} & -\frac{g_2}{a_3-a_2} & 1 \end{bmatrix}.$$

Denote for brevity $\bar{B} \triangleq S^{-1}B$, $\bar{C} \triangleq CS$, that is,

$$\bar{B} = \begin{bmatrix} 1 \\ -\frac{g_1}{a_2-a_1} \\ -\frac{g_1 g_2}{(a_3-a_2)(a_3-a_1)} \end{bmatrix}, \ \bar{C} = \begin{bmatrix} \frac{g_1 g_2}{(a_2-a_1)(a_3-a_1)} \\ \frac{g_2}{a_3-a_2} \\ 1 \end{bmatrix}^\top$$

Considering the function

$$\rho_z(s) = (zI - e^{Ts})^{-1}\left(\xi\, e^{Ts} + \eta(e^{-Ts} - 1)^{-1}s\right),$$

one obtains the relation

$$\rho_z(A) = S\rho_z(S^{-1}AS)S^{-1} = \\ = S\begin{bmatrix} \rho_z(-a_1) & 0 & 0 \\ 0 & \rho_z(-a_2) & 0 \\ 0 & 0 & \rho_z(-a_3) \end{bmatrix} S^{-1}$$

Hence, the rational function w from (15) can be written as

$$w(z) = 1 - \sum_{i=1}^{3} \bar{c}_i \bar{b}_i \rho_z(-a_i).$$

It can now be noticed that the residual of w at $z = e^{-a_3 T}$ is non-positive. Indeed, $\bar{c}_3 = 1 > 0, \bar{b}_3 < 0$, and hence

$$\lim_{z \to e^{-a_3 T}} \left(z - e^{-a_3 T}\right) w(z) = \\ = -\bar{c}_3 \bar{b}_3 \lim_{z \to e^{-a_3 T}} \left(z - e^{-a_3 T}\right) \rho_z(-a_3) = \\ = -\bar{c}_3 \bar{b}_3 \left(\xi\, e^{-a_3 T} - \eta a_3 (e^{a_3 T} - 1)^{-1}\right) \le 0.$$

Here we used the fact functions $\rho_z(e^{-a_1 T})$, $\rho_z(e^{-a_2 T})$ are analytic in z in a vicinity of $e^{-a_3 T}$, and also $\xi \le 0, \eta \ge 0$. On the other hand, recalling the definition of $w(z)$ and (16), the latter residual can be computed as

$$\lim_{z \to e^{-a_3 T}} \left(z - e^{-a_3 T}\right) w(z) = \\ = \frac{\det\left(e^{-a_3 T} I - \mathcal{Q}\right)}{(e^{-a_3 T} - e^{-a_2 T})(e^{-a_3 T} - e^{-a_1 T})},$$

entailing that

$$\det(zI - \mathcal{Q})|_{z = e^{-a_3 T}} \le 0.$$

At the same time, it has been already proven that

$$\det(zI - \mathcal{Q})|_{z = e^{-a_1 T}} \ge 0,$$

which implies that second statement.

To prove the remaining statements, it suffices to notice that

$$\det \mathcal{Q} = \det e^{AT}\left(1 + C\, e^{-AT}(\xi J + \eta D)\right) = \\ = \det e^{AT}\left(1 + \xi CB + \eta CA(I - e^{AT})^{-1}B\right)$$

Obviously, $CB = 0$. It can be shown that the function

$$z \mapsto \psi(z) \triangleq \frac{z}{1 - e^z}$$

is concave on the interval $z \in (-\infty, 0)$.

Using the Opitz formula (see, e.g., Step 2 in the proof of [15, Lemma 11]), one obtains that

$$CA(I - e^{AT})^{-1}B = T^{-1}C\psi(TA)B = \\ = T^{-1}\psi[-a_1 T, -a_2 T, -a_3 T].$$

The generalized mean-value theorem [15, Lemma 10] entails now the existence of $\zeta \in (-a_3 T, -a_1 T)$ such tha $\psi[-a_1 T, -a_2 T, -a_3 T] = \psi''(\zeta)/2$. Thanks to the concavity of τ, one thus has $CA(I - e^{AT})^{-1}B \le 0$, whence

$$z_1 z_2 z_3 = \det \mathcal{Q} \le \det e^{AT} = e^{-(a_1 + a_2 + a_3)T}.$$

This implies statement 3) in virtue of $z_1 \ge e^{-a_3 T}$.

To prove statement 4), it suffices to notice that a pair of complex-conjugate eigenvalues $z_2 = z_3^*$ should have the modulus $|z_2| = |z_3| \le e^{-(a_1 + a_2)T/2} < 1$. Hence, if $\mathcal{Q}$ has one real and two complex-conjugate eigenvalues, it is automatically Schur stable. The only reason for being unstable is thus the existence of a *real* eigenvalue whose modulus is not less than 1. In view of statement 1), $\mathcal{Q}$ cannot have eigenvalue at 1. Hence, one of z_2, z_3 does not exceed -1 (in which case the remaining eigenvalue is, obviously, also real). ∎

Corollary 1: Let the assumption of Lemma 1 apply. Then, the following three statements are equivalent:

1) Matrix $\mathcal{Q} \triangleq \mathcal{Q}(T, \xi, \eta)$ is Schur stable;
2) The inequality holds as follows
$$\chi(-1) = \det(-I - \mathcal{Q}) < 0, \tag{17}$$
3) ξ, η obey the inequality
$$C(I + e^{TA})^{-1}(\xi J + \eta D) > -1. \tag{18}$$

Proof: To prove that conditions (17) and (18) are equivalent, it suffices to substitute $z = -1$ into (15) and notice that

$$\det(-I - e^{AT}) = -(1 + e^{-a_1 T})(1 + e^{-a_2 T})(1 + e^{-a_3 T})) < 0.$$

Hence, (18) holds (equivalently, $w(-1) > 0$) if and only if $\chi(1) < 0$, i.e., statements 2) and 3) are equivalent.

Obviously, 1) implies 2), because $\chi(z) \to -\infty$ when z is real and $z \to -\infty$. If one has $\chi(-1) \ge 0$, then matrix $\mathcal{Q}$ has an eigenvalue on $(-\infty, -1]$ and is thus not Schur stable.

To prove that 2) (and 3)) implies 1), consider a one-parameter family of matrices $\mathcal{Q}_\varepsilon \triangleq \mathcal{Q}(T, \varepsilon\xi, \varepsilon\eta)$ and the corresponding characteristic polynomials $\chi_\varepsilon(z) \triangleq \det(zI - \mathcal{Q}_\varepsilon)$. Notice that if (18) holds, then it remains valid by replacing ξ, η by $\varepsilon\xi, \varepsilon\eta$, where $\varepsilon \in [0, 1]$. Hence, $\chi_\varepsilon(-1) < 0$ for all $\varepsilon \in [0, 1]$. Obviously, χ_0 is a Schur polynomial and the coefficients of χ_ε continuously depend on ε. Hence, either χ_ε is Schur for all $\varepsilon \in [0,1]$, or there exists ε_0 such that χ_{ε_0} has a root on the unit circle $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$. The second alternative is, however, impossible: Lemma 1, applied to $\mathcal{Q}(T, \varepsilon_0\xi, \varepsilon_0\eta)$, states that the only possible eigenvalue on $\mathbb{S}$ is $z = -1$, whereas $\chi_{\varepsilon_0}(-1) < 0$. ■

The proof of Theorem 1

The proof is straightforward from Corollary 1. Applying the latter Corollary 1 to $T = \Phi(y_0)$, $\xi = F'(y_0)$, $\eta = \lambda\Phi'(y_0)$, one easily checks that $Q'(X) = \mathcal{Q}(T, \xi, \eta)$, where the matrix-valued function $\mathcal{Q}$ is defined in (14).■

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