

# Construction of Gross-Neveu model using Polchinski flow equation

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## Abstract

The Gross-Neveu model is a quantum field theory model of Dirac fermions in two dimensions with a quartic interaction term. Like Yang-Mills theory in four dimensions, the model is renormalizable (but not super-renormalizable) and asymptotically free (i.e. its short-distance behaviour is governed by the free theory). We give a new construction of the massive Euclidean Gross-Neveu model in infinite volume based on the renormalization group flow equation. The construction does not involve cluster expansion or discretization of phase-space. We express the Schwinger functions of the Gross-Neveu model in terms of the effective potential and construct the effective potential by solving the flow equation using the Banach fixed point theorem. Since we use crucially the fact that fermionic fields can be represented as bounded operators our construction does not extend to models including bosons. However, it is applicable to other asymptotically free purely fermionic theories such as the symplectic fermion model.

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## 1 Introduction

We give a new construction of the massive Gross-Neveu model on the plane based entirely on the Polchinski flow equation. The Gross-Neveu model is one of the simplest barely-renormalizable (i.e. scaling critical) and asymptotically free models of quantum field theory. It is defined in terms of  $2N \in \mathbb{N}_+$  Dirac fields

$$\psi \equiv (\bar{\psi}, \underline{\psi}) \equiv ((\bar{\psi}^1, \dots, \bar{\psi}^N), (\underline{\psi}^1, \dots, \underline{\psi}^N)).$$

For every  $n \in \{1, \dots, N\}$  the fields  $\bar{\psi}^n \equiv (\bar{\psi}^{n,\alpha})_{\alpha \in \{1,2\}}$ ,  $\underline{\psi}^n \equiv (\underline{\psi}^{n,\alpha})_{\alpha \in \{1,2\}}$  transform under rotations of the plane as two-component spinors. The fields take values in a Grassmann algebra and, in particular, they all anti-commute. In what follows, we use the notation  $\bar{\psi} \equiv \psi^-$ ,  $\underline{\psi} \equiv \psi^+$ ,  $\psi \equiv (\psi^\sigma)_{\sigma \in \mathbb{G}}$ , where  $\mathbb{G} := \{-, +\} \times \{1, \dots, N\} \times \{1, 2\}$ , and

$$\bar{\psi} \cdot \underline{\psi} = \sum_{n=1}^N \sum_{\alpha=1}^2 \bar{\psi}^{n,\alpha} \underline{\psi}^{n,\alpha}, \quad \bar{\psi} \cdot \not{\partial} \underline{\psi} = \sum_{n=1}^N \sum_{\alpha_1, \alpha_2=1}^2 \sum_{j=1}^2 \bar{\psi}^{n,\alpha_1} \gamma_j^{\alpha_1, \alpha_2} \partial_j \underline{\psi}^{n,\alpha_2},$$

where  $\gamma_j = (\gamma_j^{\alpha_1, \alpha_2})_{\alpha_1, \alpha_2 \in \{1,2\}}$ ,  $j \in \{1, 2\}$ , are the Pauli matrices and  $\partial_j$ ,  $j \in \{1, 2\}$ , denote the derivatives with respect to the Cartesian coordinates of the plane. We first introduce a Gross-Neveu model with cutoffs  $\tau, \varepsilon \in (0, 1]$  that is defined on a two-dimensional torus  $\mathbb{T}_\tau^2$  of size  $1/\tau$  in terms of Dirac fields containing only the Fourier modes with frequencies less than  $1/\varepsilon$ . Subsequently, we study the limit  $\tau, \varepsilon \searrow 0$ . The free part of the action takes the form

$$A_\tau(\psi) = \int_{\mathbb{T}_\tau^2} \bar{\psi}(x) \cdot \underline{\psi}(x) \, dx + \int_{\mathbb{T}_\tau^2} \bar{\psi}(x) \cdot (\not{\partial} \underline{\psi})(x) \, dx. \quad (1.1)$$

The first term on the RHS of the above equality is called the mass term whereas the second one is called the kinetic term. The interaction potential is given by

$$U_{\tau, \varepsilon}(\psi) = \int_{\mathbb{T}_\tau^2} 1/g_{\tau, \varepsilon} (\bar{\psi}(x) \cdot \underline{\psi}(x))^2 \, dx + \int_{\mathbb{T}_\tau^2} r_{\tau, \varepsilon} \bar{\psi}(x) \cdot \underline{\psi}(x) \, dx. \quad (1.2)$$

The parameters  $1/g_{\tau, \varepsilon}, r_{\tau, \varepsilon} \in \mathbb{R}$  are called the bare coupling constant and the mass counterterm, respectively. The Grassmann measure of the Gross-Neveu model with the infrared

cutoff  $\tau \in (0, 1]$  and the ultraviolet cutoff  $\varepsilon \in (0, 1]$  is defined by the formula

$$\mu_{\tau,\varepsilon}(F) := \frac{\int F(\vartheta_\varepsilon * \psi_{\tau,\varepsilon}) \exp(-A_\tau(\psi_{\tau,\varepsilon}) + U_{\tau,\varepsilon}(\vartheta_\varepsilon * \psi_{\tau,\varepsilon})) d\psi_{\tau,\varepsilon}}{\int \exp(-A_\tau(\psi_{\tau,\varepsilon}) + U_{\tau,\varepsilon}(\vartheta_\varepsilon * \psi_{\tau,\varepsilon})) d\psi_{\tau,\varepsilon}} \quad (1.3)$$

for functionals of polynomial type  $F$  on Grassmann-valued Schwartz distributions on  $\mathbb{R}^2$ . We use the notation  $\int F(\psi_{\tau,\varepsilon}) d\psi_{\tau,\varepsilon}$  for the so-called Berezin integral [Ber87]. The Berezin integral can be viewed as a fermionic analog of the Lebesgue integral, even though it is not an integral in the usual sense. The Grassmann field  $\psi_{\tau,\varepsilon} \equiv (\psi_{\tau,\varepsilon}^\sigma)_{\sigma \in \mathbb{G}}$  on  $\mathbb{R}^2$  appearing in the above formula is periodic with period  $1/\tau$  and contains only Fourier modes with frequencies smaller than  $1/\varepsilon$ . Since working with the sharp cutoff in momentum space would result in slow decay of correlations in position space we convolve the Grassmann field  $\psi_{\tau,\varepsilon}$  with a Schwartz function  $\vartheta_\varepsilon \in \mathcal{S}(\mathbb{R}^2)$  supported in Fourier space inside the ball of radius  $1/\varepsilon$ . We denote by  $\mathcal{T}^{\mathbb{K}}$  the set of tuples  $(V^k)_{k \in \mathbb{K}}$  such that  $V^k \in \mathcal{T}$  for all  $k \in \mathbb{K}$ . Now we are ready to state our main result.

**Theorem 1.1.** *Let  $N \in \{2, 3, \dots\}$ . There exist  $\lambda_* \in (0, 1]$  and a choice of parameters  $(g_{\tau,\varepsilon}, r_{\tau,\varepsilon})_{\tau,\varepsilon \in (0,1]}$  such that for all  $\lambda \in (0, \lambda_*]$  the following statements are true.*

(A) *For every  $m \in \mathbb{N}_+$  and  $\varphi_1, \dots, \varphi_m \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$  the limit*

$$\langle S^m, \varphi_1 \otimes \dots \otimes \varphi_m \rangle := \lim_{\tau, \varepsilon \searrow 0} \int \psi(\varphi_1) \dots \psi(\varphi_m) \mu_{\tau,\varepsilon}(d\psi)$$

*exists and defines the  $m$ -point Schwinger function  $S^m \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{G}^m}$ , where  $\mu_{\tau,\varepsilon}$  is the Grassmann measure of the Gross-Neveu model with cutoffs  $\tau, \varepsilon \in (0, 1]$  defined in terms of the parameters  $g_{\tau,\varepsilon}, r_{\tau,\varepsilon} \in \mathbb{R}$ .*

(B) *For every  $m \in \mathbb{N}_+$  the distribution  $S^m \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{G}^m}$  is invariant under Euclidean transformations of the plane.*

(C) *For  $m \in \mathbb{N}_+$  let  $T^m \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{G}^m}$  be the  $m$ -point truncated Schwinger function associated to the hierarchy  $(S^m)_{m \in \mathbb{N}_+}$ . For every  $m \in \mathbb{N}_+$  there exist a collection  $(\hat{T}^{m,a,\sigma})_{a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m}$  of finite Borel measures on  $\mathbb{R}^{2(m-1)}$  such that*

$$\langle T^m, \varphi \rangle = \sum_{a \in \mathbb{A}^m} \sum_{\sigma \in \mathbb{G}^m} \int_{\mathbb{R}^{2m}} \hat{T}^{m,a,\sigma}(dy_1, \dots, dy_{m-1}) (\partial^a \varphi^\sigma)(x, x + y_1, \dots, x + y_{m-1}) dx$$

*for all  $\varphi \in \mathcal{S}(\mathbb{R}^{2m})^{\mathbb{G}^m}$ , where  $\mathbb{A} := \{0, 1, 2\}^2$ .*

(D) *The equalities*

$$\begin{aligned} \sum_{\sigma \in \mathbb{G}^2} \psi^{\sigma_1} \psi^{\sigma_2} \int_{\mathbb{R}^2} \hat{T}^{2,0,\sigma}(dy) &= \underline{\psi} \cdot \bar{\psi}, \\ \sum_{\sigma \in \mathbb{G}^4} \psi^{\sigma_1} \dots \psi^{\sigma_4} \int_{\mathbb{R}^6} \hat{T}^{4,0,\sigma}(dy_1, dy_2, dy_3) &= \lambda (\underline{\psi} \cdot \bar{\psi})^2 \end{aligned}$$

*hold for all Grassmann-valued  $\psi \equiv (\psi^\sigma)_{\sigma \in \mathbb{G}} \equiv (\bar{\psi}^{\alpha,\varsigma}, \underline{\psi}^{\alpha,\varsigma})_{\alpha \in \{1,2\}, \varsigma \in \{1, \dots, N\}}$ .*

(E) The truncated Schwinger functions decay stretched exponentially, more precisely

$$\lim_{|x| \rightarrow \infty} \exp(|x|^{1/2}) |\langle T^{m+n}, \varphi_x \otimes \psi \rangle| = 0$$

for all  $m, n \in \mathbb{N}_+$  and  $\varphi \in C_c^\infty(\mathbb{R}^{2m})^{\mathbb{G}^m}$ ,  $\psi \in C_c^\infty(\mathbb{R}^{2n})^{\mathbb{G}^n}$ , where for  $x \in \mathbb{R}^2$  we define  $\varphi_x \in C_c^\infty(\mathbb{R}^{2m})^{\mathbb{G}^m}$  by the equality  $\varphi_x(y_1, \dots, y_m) := \varphi(y_1 - x, \dots, y_m - x)$  for all  $y_1, \dots, y_m \in \mathbb{R}^2$ .

*Proof.* The theorem is an immediate consequence of Theorem 15.4.  $\square$

*Remark 1.2.* The precise definition of the Berezin integral and the Gross-Neveu measure with cutoffs can be found in Sec. 4. The notion of invariance under Euclidean transformations used in Item (B) of the above theorem is defined in Sec. 8.

*Remark 1.3.* The equalities stated in Item (D) of the above theorem should be viewed as renormalization conditions that fix implicitly the bare coupling constant  $g_{\tau, \varepsilon}$  and the mass counterterm  $r_{\tau, \varepsilon}$  appearing in the expression for the measure  $\mu_{\tau, \varepsilon}$ . We parameterize the Gross-Neveu models in terms of numerical constants linked directly to the Schwinger functions instead of using the parameters  $g_{\tau, \varepsilon}$  and  $r_{\tau, \varepsilon}$ . Note that Item (D) implies in particular that the truncated four-point Schwinger function is non-zero. As a result, the construction yields a non-trivial quantum field theory.

*Remark 1.4.* The result stated in Item (E) of the above theorem is not optimal. Actually, the truncated Schwinger functions decay exponentially. However, the exponential decay does not follow immediately from the construction presented in the paper and its proof requires a separate argument.

### Method of the proof

Our construction of the Gross-Neveu model, like the previous constructions [GK85a, FMRS86], utilizes the Wilsonian renormalization group theory [Wil71]. The fundamental object of this theory is the so-called effective potential  $U_{\tau, \varepsilon; t}$  depending on the spatial scale  $t \in [0, 1]$  and the cutoffs  $\tau, \varepsilon \in (0, 1]$ . The effective potential  $U_{\tau, \varepsilon; t}$  at the spatial scale  $t = 0$  coincides with the interaction potential  $U_{\tau, \varepsilon}$  defined by Eq. (1.2). The goal is to construct the effective potential  $U_{\tau, \varepsilon; t}$  at the spatial scale  $t = 1$ , which coincides with the generating functional of the so-called connected amputated Schwinger functions. To this end, one usually solves a certain equation that relates the effective potentials at different scales. The previous constructions [GK85a, FMRS86] of the Gross-Neveu model used the renormalization group transformation that relates the effective potentials  $U_{\tau, \varepsilon; t}$  at different discrete values of the scale parameter  $t \in \{L^{-k} \mid k \in \mathbb{N}_0\} \subset (0, 1]$ , where  $L \in (1, \infty)$  is sufficiently big. In contrast, the approach of this paper is based on the so-called Polchinski flow equation [WK74, Pol84] that is a certain differential equation in the scale parameter  $t \in (0, 1]$ .

It was recognized long time ago that the flow equation is a very powerful tool in perturbative quantum field theory. In fact, there is a short and general proof of perturbative renormalizability of QFT models based on the flow equation [Pol84, Kop07, Mul03, Sal99].

The renormalization problem is solved using a simple inductive argument and amounts to imposing appropriate boundary conditions for the flow equation. In particular, the proof avoids the problem of overlapping divergences appearing in all approaches based on the Feynman diagrammatical representations such as the BPHZ approach. However, the applicability of the Polchinski flow equation in non-perturbative constructions is limited. For bosonic theories it is generally believed that the flow equation is not useful non-perturbatively as it does not allow to take advantage of the boundedness from above of the interaction potential. The exception is the sine-Gordon model, which up to the second threshold was constructed [BK87, BB21, GM24] with the use of the Polchinski equation. In quantum field theory models with bosons the boundedness from above of the effective potential is usually necessary to ensure that the interacting measure has finite total mass. This is the so-called large field problem. It is possible to address this problem in the approach based on the renormalization group transformation but the problem seems intractable in the approach based on the Polchinski equation. As an aside, let us mention that the large field problem can be solved in the framework for singular stochastic PDEs developed in [Duc21, Duc22] based on a certain flow equation that plays an analogous role to the Polchinski equation. In fact, this framework was used recently to construct the bosonic fractional  $\Phi_3^4$  model in full super-renormalizable regime [DGR23].

Since fermionic fields can be represented by bounded operators the large field problem should be absent in purely fermionic models of quantum field theory and a construction of such models based on the Polchinski flow equation should be feasible. Such a construction is not straight-forward because the effective potential solving the Polchinski equation is necessarily a highly non-local functional, which in particular is generically not of polynomial type. A progress in this direction has recently been made by De Vecchi, Fresta, Gubinelli in [DFG22], where a new approach to super-renormalizable fermionic theories was developed based on a certain forward-backward stochastic differential equation (FBSDE). The main advantage of the FBSDE approach is the fact that in this approach one only needs to construct an effective potential that satisfies the flow equation up to a sufficiently small error term. In particular, for super-renormalizable models it is always possible to construct a suitable effective potential in the space of functionals of polynomial type using a certain recursive procedure. In principle the FBSDE approach can be also useful for barely-renormalizable models. However, it is clear that in the case of such models a suitable approximate solution of the Polchinski equation cannot be a functional of polynomial type. Consequently, allowing for an error term in the Polchinski equation does not provide an obvious benefit.

The main obstacle in constructing an exact solution of the Polchinski equation was the lack of a suitable norm in the space of functionals that is compatible with this equation. At technical level, the problem seems to be related to the fact that the scale decomposition of the free fermionic field, which behaves like a Grassmann cylindrical Wiener process, is not Lipschitz continuous in the scale parameter but only  $1/2$  Hölder continuous. We refer the reader to [SW00] and Sec. 2 for more details and to [BW99, BW88] for an earlier unsuccessful

attempt at proving bounds for fermionic correlations using the flow equation. The important contribution of the paper is a novel norm in the space of functionals in which the flow equation can be solved. The norm is defined with the use of the filtered non-commutative probability space constructed in [DFG22]. Using the new norm we rewrite the mild form of the Polchinski equation as a fixed point problem posed in a certain complete metric space and solve it using the Banach theorem. Finally, we express the Schwinger functions in terms of the effective potential  $U_{\tau,\varepsilon;t}$  at the spatial scale  $t = 1$  and prove their convergence as  $\tau, \varepsilon \searrow 0$ .

*Remark 1.5.* The sign of the parameter  $\lambda$  that appears in Item (D) of Theorem 1.1 plays a very important role in our construction. The Gross-Neveu model with  $\lambda > 0$  and  $N \in \{2, 3, \dots\}$  that we construct is asymptotically free – at high energies the Schwinger functions of the model are well approximated by the Schwinger functions of the free theory. We refer the reader to [GK89] for a discussion of the significance of the asymptotic freedom in constructions of barely renormalizable models of quantum field theory. For  $\lambda < 0$  and  $N \in \{2, 3, \dots\}$  the Gross-Neveu model is not expected to be asymptotically free and most likely does not exist non-perturbatively. For  $N = 1$  the Gross-Neveu model coincides with the so-called Thirring model. Even though the Thirring model is not asymptotically free it has some special properties that allow its non-perturbative construction [BFM07].

*Remark 1.6.* One of the manifestations of the asymptotic freedom of the Gross-Neveu model is the fact that the bare coupling constant  $1/g_{\tau,\varepsilon} > 0$  fixed by the renormalization condition stated in Item (D) of Theorem 1.1 vanishes logarithmically in the limit  $\varepsilon \searrow 0$ . Thus, naively one could expect that the construction yields a free theory. However, as we pointed out in Remark 1.3 the hierarchy of the constructed Schwinger functions is non-trivial. The underlying intuitive reason for the non-triviality is the fact that in the limit  $\varepsilon \searrow 0$  the fields become genuine distributions and consequently the quartic interaction term involving a pointwise products of the fields is very singular. See [Hai24] for a related phenomenon in the context of subcritical singular stochastic PDEs.

### Possible generalizations

The method developed in this work is applicable to other purely fermionic renormalizable models of quantum theory such as for example the symplectic fermion model with  $N \neq 8$ . The symplectic fermion model is a fermionic analog of the bosonic  $\Phi^4$  model in four-dimensions. The model is barely renormalizable. It describes  $N \in \{4, 5, \dots\}$  scalar fermionic fields. The kinetic part of the action of the model contains the Laplacian and the quartic interaction potential is invariant under symplectic transformation of the fields. See [GMR21] for a precise definition of the model. The assumption  $N \neq 8$  is probably not essential and is related to the fact that the construction presented in the paper requires that the one-loop beta-function  $\beta_2$  is positive. Our construction can be also applied to super-renormalizable models such as the Gross-Neveu model or symplectic fermion model with modified propagators that are less singular at the origin than the standard propagators. Even though our construction simplifies drastically in the super-renormalizable regime the FBSDE approach developed in [DFG22], which was discussed briefly above, seems advantageous as it allows to avoid the use compli-

cated norms in the space of functionals. Let us also mention that the method developed in this paper should be applicable to the sine-Gordon model of quantum field theory up to the second threshold.

### Overview of the literature

The Gross-Neveu model appeared for the first time in the work [MW73] by Mitter and Weisz who investigated the flow of renormalization group transformations in this model. In the paper [GN74] Gross and Neveu introduced the same model as a toy model of Yang-Mills theory in four dimensions. They studied the dynamical mass generation in a model with the chiral symmetry. They presented an argument suggesting that despite the fact that the action of such a model does not contain a mass term, which is prohibited by the chiral symmetry, the truncated Schwinger functions decay exponentially, that is the model is massive. Note that in this paper we construct the version of the Gross-Neveu model with a mass term.

The first mathematical construction of the Gross-Neveu model was given by Gawędzki and Kupiainen in [GK85a]. Another construction was given shortly thereafter in [FMRS86] by Feldman, Magnen, Rivasseau and Seneor. Both constructions rely on the discrete renormalization group method and the estimates for fermionic correlations derived with the use of the Gram-Hadamard inequality and the cluster expansion. In [DR00] Disertori and Rivasseau constructed the Gross-Neveu model by proving convergence of appropriately rearranged perturbation theory. Let us also mention the recent paper [DY23] by Dimock and Yuan who studied the flow of the renormalization group transformations in the massless Gross-Neveu model on a torus and established uniform boundedness of the partition function of the model in the UV cutoff.

The above-mentioned works concerned the Gross-Neveu model with a mass term on the plane or the model with the chiral symmetry on the unit torus. The problem of the dynamical mass generation was addressed in [KMR95], where it was proved that the two-point Schwinger functions of the chiral model with a fixed UV cutoff falls off exponentially. Some properties of the Gross-Neveu model related to the particle interpretation and the asymptotic completeness were established in [IM87, IM88a, IM88b]. The super-renormalizable Gross-Neveu model with less singular propagator was studied in [SW02]. A construction of the Gross-Neveu model with a more singular propagator, which is perturbatively non-renormalizable, was given in [GK85b].

Let us discuss some related results. The two-dimensional Yukawa model was constructed by Lesniewski in [Les87] using the technique developed in [GK85a]. The important contribution of [Les87] is a new elegant proof of bounds for fermionic connected correlations based on the Brydges-Battle-Federbush formula. An alternative approach to bounds for fermionic correlations was given by Salmhofer and Wiczerkowski in [SW00]. The infrared stable non-Gaussian fixed point of the renormalization group transformation in the symplectic fermion model with weakly relevant quartic interaction was constructed in [GMR21]. Let us mention once again the stochastic quantization approach to fermionic models based on a FBSDE developed in [DFG22], which was used to construct the symplectic fermion model with  $N = 4$  in infinite volume in full super-renormalizable regime. First steps towards the construction

of fermionic quantum field theories using the parabolic stochastic quantization method were made in [ABDG20, CHP23, DFGG23], where the framework of non-commutative probability suitable for fermionic stochastic analysis was developed and local well-posedness for the parabolic stochastic quantization equation of the two-dimensional Yukawa model was proved. We refer the reader to [FKT00] for a general overview of fermionic functional integrals with applications to constructive quantum field theory.

## 2 Strategy of the proof

**Definition 2.1.** We fix  $N \in \{2, 3, \dots\}$  and a small parameter  $\kappa := 1/1000$ .

**Definition 2.2.** Let  $\beta_2 := 2(N - 1)/\pi$ . For  $t, \lambda \in (0, 1]$  we define  $\lambda_t := (\lambda^{-1} - \beta_2 \log t)^{-1}$ . For  $t = 0$  we set  $\lambda_t = 0$ .

*Remark 2.3.* We identify functions and distributions on the torus  $\mathbb{T}_\tau^2$  with periodic functions and distributions on  $\mathbb{R}^2$  with period  $1/\tau$ .

In this section we describe in informal terms the main steps of the proof of Theorem 1.1 highlighting the most important ideas. We use the framework of non-commutative probability. A non-commutative probability space is the pair  $(\mathcal{F}, \mathbf{E})$  consisting of a unital Banach subalgebra  $\mathcal{F}$  of the algebra of operators acting on a separable Hilbert space and a continuous normalized linear functional  $\mathcal{F} \rightarrow \mathbb{C}$ . Recall that the Grassmann measure (1.3) of the Gross-Neveu model is defined with the use of the Berezin integral, which, like the Lebesgue integral, is only well-defined in finite dimension. For this reason, it is advantageous to use as a reference measure a certain Gaussian Grassmann measure, called the free field measure with cutoffs  $\tau, \varepsilon \in (0, 1]$ , defined by the formula

$$\nu_{\tau, \varepsilon}(F) := \frac{\int F(\vartheta_\varepsilon * \psi_{\tau, \varepsilon}) \exp(-A_\tau(\psi_{\tau, \varepsilon})) d\psi_{\tau, \varepsilon}}{\int \exp(-A_\tau(\psi_{\tau, \varepsilon})) d\psi_{\tau, \varepsilon}}$$

for functionals  $F$  of polynomial type. Note that the RHS of the above equality involves only the free part of the action  $A_\tau(\psi_{\tau, \varepsilon})$ , which is quadratic in the field  $\psi_{\tau, \varepsilon}$ . We stress that the above formula for the measure  $\nu_{\tau, \varepsilon}$  is meaningful only if  $\tau, \varepsilon \in (0, 1]$ . We would like to construct a collection of anti-commuting random variables  $(\Psi_{\tau, \varepsilon})_{\tau, \varepsilon \in [0, 1]}$  in a non-commutative probability space such that  $\nu_{\tau, \varepsilon}(F) = \mathbf{E}(F(\Psi_{\tau, \varepsilon}))$  for all  $\tau, \varepsilon \in (0, 1]$ . To this end, we use the so-called Osterwalder-Schrader construction [OS72, OS73]. For all  $\tau, \varepsilon \in [0, 1]$  we define  $\Psi_{\tau, \varepsilon} \in \mathcal{S}'(\mathbb{R}^2, \mathcal{B}(\mathcal{H}))^\mathbb{G}$  as a certain linear combination of creation and annihilation operators acting in a fermionic Fock space  $\mathcal{H}$  with the vacuum state  $\Omega \in \mathcal{H}$ . Let  $\mathcal{B}(\mathcal{H})$  be the algebra of bounded operators on  $\mathcal{H}$ . The unital algebra  $\mathcal{F}$  is defined as the Banach subalgebra of  $\mathcal{B}(\mathcal{H})$  generated by  $\langle \Psi_{\tau, \varepsilon}, \varphi \rangle$  with  $\varphi \in L^2(\mathbb{R}^2)^\mathbb{G}$ . The expected value  $\mathbf{E} : \mathcal{F} \rightarrow \mathbb{C}$  is defined by the formula  $\mathbf{E}F(\Psi_{\tau, \varepsilon}) = (\Omega, F(\Psi_{\tau, \varepsilon})\Omega)_\mathcal{H}$ , where  $(\bullet, \bullet)_\mathcal{H}$  denotes the scalar product in  $\mathcal{H}$ . The fields  $\Psi_{\tau, \varepsilon}$  are defined in such a way that they all anti-commute,

$$\mathbf{E}\langle \Psi_{\tau, \varepsilon}, \phi \rangle = 0, \quad \mathbf{E}\langle \Psi_{\tau, \varepsilon}, \phi \rangle \langle \Psi_{\tau, \varepsilon}, \eta \rangle = \langle \phi, G_{\tau, \varepsilon} * \eta \rangle$$



and higher moments of  $\Psi_{\tau,\varepsilon}$  can be expressed in terms of the covariance using an analog of the formula valid for commuting Gaussian random variables. If  $\tau \in (0, 1]$ , then  $\Psi_{\tau,\varepsilon}$  is periodic with period  $1/\tau$ . The propagator  $G_{\tau,\varepsilon}$  with the IR cutoff  $\tau \in (0, 1]$  and the UV cutoff  $\varepsilon \in (0, 1]$  is defined as a periodization of  $G_\varepsilon$  with the period  $1/\tau$ , the propagator  $G_\varepsilon$  with the UV cutoff  $\varepsilon \in (0, 1]$  is defined by  $G_\varepsilon := \vartheta_\varepsilon * G * \vartheta_\varepsilon$  and the propagator  $G$  is the inverse of the differential operator

$$\frac{1}{2} \begin{pmatrix} \not{D} + 1 & 0 \\ 0 & \not{D}^t - 1 \end{pmatrix}$$

appearing in the free part of the action (1.1). The family of functions  $\vartheta_\varepsilon \in \mathcal{S}(\mathbb{R}^2)$  parameterized by  $\varepsilon \in (0, 1]$  is chosen in such a way that it converges to the Dirac delta at the origin as  $\varepsilon \searrow 0$  and for all  $\varepsilon \in (0, 1]$  the Fourier transform of  $\vartheta_\varepsilon$  is supported in a ball of radius  $1/\varepsilon$ . Moreover, if  $\varepsilon = 1$ , then  $\vartheta_\varepsilon = 0$  and we define  $\vartheta_\varepsilon$  with  $\varepsilon = 0$  to be the Dirac delta at the origin. If  $\tau = 0$ , then  $G_{\tau,\varepsilon} := G_\varepsilon$  and if  $\varepsilon = 0$ , then  $G_\varepsilon := G$ . We call  $\Psi_{\tau,\varepsilon}$  the free field with cutoffs  $\tau, \varepsilon \in (0, 1]$ . Using  $\Psi_{\tau,\varepsilon}$  one rewrites the formula (1.3) in the following way

$$\mu_{\tau,\varepsilon}(F) = \frac{\mathbf{E}(F(\Psi_{\tau,\varepsilon})e^{U_{\tau,\varepsilon}(\Psi_{\tau,\varepsilon})})}{\mathbf{E}(e^{U_{\tau,\varepsilon}(\Psi_{\tau,\varepsilon})})}. \quad (2.1)$$

Note that for all  $\varepsilon \in (0, 1]$  the free field  $\Psi_{\tau,\varepsilon}$  belongs to  $C^\infty(\mathbb{R}^2, \mathcal{B}(\mathcal{H}))^\mathbb{G}$ . Consequently, the RHS of the above equality is well-defined for all  $\tau, \varepsilon \in (0, 1]$  (provided the denominator is not zero). Since in the limit  $\varepsilon \searrow 0$  the free field  $\Psi_{\tau,\varepsilon}$  is not a function over  $\mathbb{R}^2$  but only a Schwartz distribution the pointwise products in the expression (1.2) for the potential  $U_{\tau,\varepsilon}$  become meaningless. In the limit  $\tau \searrow 0$  the expression (1.2) for the potential  $U_{\tau,\varepsilon}$  is ill-defined due to unbounded domain of integration. Thus, the expression (2.1) for the measure  $\mu_{\tau,\varepsilon}$  becomes singular in the limit  $\varepsilon \searrow 0$  or  $\tau \searrow 0$ . Because of the presence of the unit mass term in the free part of the action the proof of the existence of the limit  $\tau \searrow 0$  is relatively easy. The limit  $\varepsilon \searrow 0$  is quite subtle as it only exists if the parameters  $g_{\tau,\varepsilon}$  and  $r_{\tau,\varepsilon}$  of the potential (1.2) diverge at particular rate as  $\varepsilon \searrow 0$ .

In order to study the limit  $\varepsilon \searrow 0$  of  $\mu_{\tau,\varepsilon}(F)$  we use the renormalization group theory [Wil71, Pol84]. To this end, we introduce a certain scale decomposition of the propagator  $G_{\tau,\varepsilon}$  and the free field  $\Psi_{\tau,\varepsilon}$ . The scale decomposition  $[0, 1] \ni t \mapsto G_{\varepsilon;t} \in L^1(\mathbb{R}^2)^{\mathbb{G}^2}$  of the propagator  $G_\varepsilon$  is defined by  $G_{\varepsilon;t} := \vartheta_t * G_{\tau,\varepsilon} * \vartheta_t$  for  $t \in [0, 1]$ . Then  $G_{\tau,\varepsilon;t}$  is defined as the periodization of  $G_{\varepsilon;t}$ . The parameter  $t \in [0, 1]$  plays the role of an artificial UV cutoff. Note that due to the properties of the function  $\vartheta_t$  for every  $t \in (0, 1]$  and  $\tau, \varepsilon \in [0, 1]$  the propagator  $G_{\tau,\varepsilon;t}$  is smooth and essentially constant at spatial scales smaller than  $t$  and it captures the behavior of  $G_{\tau,\varepsilon}$  at spatial scales larger than  $t$ . The scale decomposition  $[0, 1] \mapsto \Psi_{\tau,\varepsilon;t}$  of the free field  $\Psi_{\tau,\varepsilon}$  is defined in a non-commutative probability space containing the probability space of the free field as a collection of anti-commuting random variables such that for all  $\tau, \varepsilon \in [0, 1]$  the following conditions are satisfied:

- (1)  $\Psi_{\tau,\varepsilon;t}$  is Gaussian with mean zero and covariance  $G_{\tau,\varepsilon;t}$  for all  $t \in [0, 1]$ ,
- (2)  $\Psi_{\tau,\varepsilon;t} \in C^\infty(\mathbb{R}^2, \mathcal{B}(\mathcal{H}))$  for all  $t \in (0, 1]$ ,

- (3)  $\Psi_{\tau,\varepsilon;t}$  is essentially constant at spatial scales smaller than  $t \in (0, 1]$ ,
- (4)  $\Psi_{\tau,\varepsilon;t}$  captures the behavior of  $\Psi_{\tau,\varepsilon}$  at spatial scales larger than  $t \in (0, 1]$ ,
- (5)  $t \mapsto \Psi_{\tau,\varepsilon;t}$  has independent increments.

Observe that the condition (1) implies, in particular, that  $\text{Law}(\Psi_{\tau,\varepsilon;t=0}) = \text{Law}(\Psi_{\tau,\varepsilon})$  and  $\Psi_{\tau,\varepsilon;t=1} = 0$ . A scale decomposition satisfying the conditions (1)–(4) could be defined by the formula  $\Psi_{\tau,\varepsilon;t} = \vartheta_t * \Psi_{\tau,\varepsilon}$  for all  $t \in [0, 1]$ . However, with this definition, the process  $t \mapsto \Psi_{\tau,\varepsilon;t}$  does not have independent increments, a property that plays a crucial role in the whole construction. In order to define a scale decomposition of the free field satisfying all conditions (1)–(5) we follow the strategy proposed in [DFG22]. To this end, we use the so-called Fermionic white noise  $\xi \in \mathcal{S}'(\mathbb{R}^2 \times [0, 1], \mathcal{B}(\mathcal{H}))^\mathbb{G}$  on spacetime  $\mathbb{R}^2 \times [0, 1]$ . The Fermionic white noise is a anti-commuting Gaussian random variable in some non-commutative probability space  $(\mathcal{F}, \mathbf{E})$  defined using a variant of the Osterwalder-Schrader construction described above. The Hilbert space  $\mathcal{H}$  is a conveniently chosen Fermionic Fock space, the expected value  $\mathbf{E}$  is defined as above,  $\xi$  is expressed as a certain linear combination of creation and annihilation operators such that  $\xi$  has mean zero and its covariance coincides with the Dirac delta on the diagonal in  $(\mathbb{R}^2 \times [0, 1])^2$  and  $\mathcal{F}$  is the sub-algebra of  $\mathcal{B}(\mathcal{H})$  generated by  $\langle \xi, f \rangle$  with  $f \in L^2(\mathbb{R}^2 \times [0, 1])^\mathbb{G}$ . We also introduce propagators  $G_{\tau,\varepsilon;t}^\pm$  such that for all  $t \in (0, 1]$  the Fourier transform of  $G_{\tau,\varepsilon;t}^\pm$  is contained in a shell of radius of order  $1/t$  and  $G_{\tau,\varepsilon;t}^+ * G_{\tau,\varepsilon;t}^- := \partial_t G_{\tau,\varepsilon;t}$ . The scale decomposition  $[0, 1] \mapsto \Psi_{\tau,\varepsilon;t} \equiv (\bar{\Psi}_{\tau,\varepsilon;t}, \underline{\Psi}_{\tau,\varepsilon;t})$  of the free field  $\Psi_{\tau,\varepsilon} \equiv (\bar{\Psi}_{\tau,\varepsilon}, \underline{\Psi}_{\tau,\varepsilon})$  satisfying the conditions (1)–(5) is defined by the formula

$$\bar{\Psi}_{\tau,\varepsilon;t} := \int_{\mathbb{T}_\tau^2 \times [t, 1]} G_{\tau,\varepsilon;s}^-(\bullet - y) \xi(dy, ds), \quad \underline{\Psi}_{\tau,\varepsilon;t} := \int_{\mathbb{T}_\tau^2 \times [t, 1]} G_{\tau,\varepsilon;s}^+(\bullet - y) \xi(dy, ds).$$

The free field is defined to be  $\Psi_{\tau,\varepsilon} := \Psi_{\tau,\varepsilon;0}$ . For  $t, s \in [0, 1]$  such that  $s \leq t$  the increment  $\Psi_{\tau,\varepsilon;t,s}$  of the scale decomposition of the free field is defined by  $\Psi_{\tau,\varepsilon;t,s} := \Psi_{\tau,\varepsilon;s} - \Psi_{\tau,\varepsilon;t}$  and the algebra  $\mathcal{F}_{t,s}$  is defined as the Banach subalgebra of  $\mathcal{F} \subset \mathcal{B}(\mathcal{H})$  generated by  $\xi(f)$ , where  $f \in L^2(\mathbb{R}^2 \times [0, 1])$  is s.t.  $\text{supp } f \subset \mathbb{R}^2 \times [s, t]$ . Note that  $\Psi_{\tau,\varepsilon;t,s}$  is supported in the Fourier space in a shell of radii of order  $1/t$  and  $1/s$  and depends only on the white noise in the time interval  $[s, t]$ . Hence,  $\langle \Psi_{\tau,\varepsilon;t,s}, \varphi \rangle \in \mathcal{F}_{t,s}$  for all  $\varphi \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G}$ . Observe also that for all  $t \in (0, 1]$  the free field can be decomposed into the low and high frequency part  $\Psi_{\tau,\varepsilon} = \Psi_{\tau,\varepsilon;1,t} + \Psi_{\tau,\varepsilon;t,0}$ , where  $\Psi_{\tau,\varepsilon;1,t}$  contains only the Fourier modes with frequencies in the ball with radius of order  $1/t$  and  $\Psi_{\tau,\varepsilon;t,0}$  contains only the Fourier modes with frequencies outside the ball with radius of order  $1/t$ . Finally, let us mention that in the non-commutative probability space  $(\mathcal{F}, \mathbf{E})$  of the white noise it is possible to construct [DFG22] the conditional expectation  $\mathbf{E}_t : \mathcal{F} \rightarrow \mathcal{F}_{1,t}$  given the algebra  $\mathcal{F}_{1,t}$  that has the usual properties of the conditional expectation. In intuitive terms,  $\mathbf{E}_t$  integrates out the high frequency part  $\Psi_{\tau,\varepsilon;t,0}$  of the free field  $\Psi_{\tau,\varepsilon} = \Psi_{\tau,\varepsilon;1,t} + \Psi_{\tau,\varepsilon;t,0}$  and acts trivially on the low frequency part  $\Psi_{\tau,\varepsilon;1,t}$ .

Now let us introduce the effective potential and the Polchinski flow equation that play a central role in our construction. To this end, fix  $t \in (0, 1]$  and let  $F$  be a functional of polynomial type such that  $F(\Psi_{\tau,\varepsilon}) = F(\Psi_{\tau,\varepsilon;1,t})$ , that is  $F$  depends only on the low frequency

part of the free field. For such functional the formula (2.1) for the interacting measure  $\mu_{\tau,\varepsilon}$  can be rewritten in the form

$$\mu_{\tau,\varepsilon}(F) \propto \mathbf{E} \mathbf{E}_t(F(\Psi_{\tau,\varepsilon;1,t})e^{U_{\tau,\varepsilon}(\Psi_{\tau,\varepsilon})}) = \mathbf{E}(F(\Psi_{\tau,\varepsilon;1,t})\mathbf{E}_te^{U_{\tau,\varepsilon}(\Psi_{\tau,\varepsilon;1,t}+\Psi_{\tau,\varepsilon;t,0})}), \quad (2.2)$$

where we used the tower property of the conditional expectation as well as the fact that  $F(\Psi_{\tau,\varepsilon;1,t}) \in \mathcal{F}_{1,t}$ . Suppose that a functional  $U_{\tau,\varepsilon;t}$  satisfies the following equality

$$\exp(U_{\tau,\varepsilon;t}(\phi)) := \mathbf{E} \exp(U_{\tau,\varepsilon}(\phi + \Psi_{\tau,\varepsilon;t,0})) \quad (2.3)$$

for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)$  valued in the odd part of some Grassmann algebra  $\mathcal{G}$  independent of the white noise  $\xi(f)$ . The choice of the Grassmann algebra  $\mathcal{G}$  does not play an important role. However,  $\mathcal{G}$  has to have at least as many generators as the Grassmann algebra of the free field with cutoffs  $\tau, \varepsilon \in (0, 1]$ . For this reason, it is convenient to assume that  $\mathcal{G}$  is infinite-dimensional. We call a functional as above an effective potential at spatial scale  $t \in (0, 1]$ . Using an effective potential and Eq. (2.2) we conclude that

$$\mu_{\tau,\varepsilon}(F) \propto \mathbf{E}(F(\Psi_{\tau,\varepsilon;1,t})e^{U_{\tau,\varepsilon;t}(\Psi_{\tau,\varepsilon;1,t})})$$

for all functionals  $F$  as above. Note that the RHS of the above equality involves only the low frequency part  $\Psi_{\tau,\varepsilon;1,t}$  of the free field, which is smooth for all  $\tau, \varepsilon \in [0, 1]$ . Consequently, the above formula remains meaningful without cutoffs provided an effective potential  $U_{\tau,\varepsilon;t}$  is well-defined. This suggests that the family of effective potentials  $(U_{\tau,\varepsilon;t})_{t \in (0,1]}$  determines the measure  $\mu_{\tau,\varepsilon}$  completely and one can prove the existence of the limit  $\tau, \varepsilon \searrow 0$  of  $\mu_{\tau,\varepsilon}$  by studying the limit  $\tau, \varepsilon \searrow 0$  of  $(U_{\tau,\varepsilon;t})_{t \in (0,1]}$ . Actually, using the translational invariance of the Berezin integral one shows the following formula

$$\mu_{\tau,\varepsilon}(\exp(\langle \bullet, \varphi \rangle)) = \exp(\langle \varphi, G_{\tau,\varepsilon} * \varphi \rangle / 2 + U_{\tau,\varepsilon;1}(G_{\tau,\varepsilon} * \varphi) - U_{\tau,\varepsilon;1}(0)) \quad (2.4)$$

for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)$  valued in the odd part of  $\mathcal{G}$ , which provides a direct link between an effective potential  $U_{\tau,\varepsilon;t}$  at the spatial scale  $t = 1$  and a generating functional of the Schwinger functions. The upshot is that in order to construct the Schwinger functions and prove their convergence as  $\tau, \varepsilon \searrow 0$  it is enough to construct the family of effective potentials and prove its convergence as  $\tau, \varepsilon \searrow 0$ . The basic idea of the renormalization group theory is to derive a certain equation that relates the effective potentials at different scales and then use it to construct the effective potential. In the discrete renormalization group method one studies an equation that relates the effective potential at different discrete values of the scale parameter. Our construction uses instead the so-called Polchinski flow equation, which is a Hamilton-Jacobi-Bellman type equation for the function  $(t, \phi) \mapsto U_{\tau,\varepsilon;t}(\phi)$ . The mild form of this equation is given by

$$\begin{aligned} U_{\tau,\varepsilon;t}(\phi) &= \mathbf{E} U_{\tau,\varepsilon}(\Psi_{\tau,\varepsilon;t,0} + \phi) \\ &+ \frac{1}{2} \int_0^t \mathbf{E} \langle D_\phi U_{\tau,\varepsilon;s}(\Psi_{\tau,\varepsilon;t,s} + \phi), \dot{G}_{\varepsilon;s} * D_\phi U_{\tau,\varepsilon;s}(\Psi_{\tau,\varepsilon;t,s} + \phi) \rangle_\tau ds \end{aligned} \quad (2.5)$$

for all  $t \in [0, 1]$  and  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G}$  valued in the odd part of  $\mathcal{G}$ . In the above formula,  $\dot{G}_{\varepsilon;t} := \partial_t G_{\varepsilon;t}$  and  $D_\phi F$  stands for the functional derivative of  $F$ . The flow equation (2.5) is well-posed provided  $\tau, \varepsilon \in (0, 1]$ . Note that the first term on the RHS of Eq. (2.5) involves the original potential  $U_{\tau,\varepsilon}$  defined by Eq. (1.2), which depends on the parameters  $g_{\tau,\varepsilon}, r_{\tau,\varepsilon} \in \mathbb{R}$ . Recall that these parameters are not given. Our task is to find  $(g_{\tau,\varepsilon}, r_{\tau,\varepsilon})_{\tau,\varepsilon \in (0,1]}$  such that the Schwinger functions converge as  $\tau, \varepsilon \searrow 0$ . This is the so-called renormalization problem.

A comment on functionals is in order. We identify functionals  $U$  with collections of kernels  $(U^{m,\sigma})_{m \in \mathbb{N}_0, \sigma \in \mathbb{G}^m}$  such that

$$U(\phi) = U^0 + \sum_{m \in \mathbb{N}_+} \sum_{\sigma \in \mathbb{G}^m} \langle U^{m,\sigma}, \phi^{\sigma_1} \otimes \dots \otimes \phi^{\sigma_m} \rangle_\tau \quad (2.6)$$

for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G}$  valued in the odd part of  $\mathcal{G}$ , where  $U^0 \in \mathbb{C}$  and  $U^{m,\sigma} \in \mathcal{S}'(\mathbb{T}_\tau^2)$  is antisymmetric for  $m \in \mathbb{N}_+, \sigma \in \mathbb{G}^m$ . We interpret the flow equation as an equation for the hierarchy of kernels  $(U^{m,\sigma})_{m \in \mathbb{N}_0, \sigma \in \mathbb{G}^m}$ . In order to study the infinite volume limit it is useful to consider instead the hierarchy of kernels  $(V^{m,\sigma})_{m \in \mathbb{N}_+, \sigma \in \mathbb{G}^m}$  such that  $V^{m,\sigma} \in \mathcal{S}'(\mathbb{R}^2)$  for all  $m \in \mathbb{N}_+, \sigma \in \mathbb{G}^m$  and such that  $V^{m,\sigma}$  is related to  $U^{m,\sigma}$  by the equality

$$\langle U^{m,\sigma}, \phi_1 \otimes \dots \otimes \phi_m \rangle_\tau = \langle V^{m,\sigma}, \chi_\tau \phi_1 \otimes \phi_2 \otimes \dots \otimes \phi_m \rangle \quad (2.7)$$

for all  $m \in \mathbb{N}_+, \sigma \in \mathbb{G}^m$  and  $\phi_1, \dots, \phi_m \in C^\infty(\mathbb{T}_\tau^2)$ , where  $\chi_\tau \in C_c^\infty(\mathbb{R}^2)$  is such that its periodization with period  $1/\tau$  coincides with the constant function equal to one. The pairing  $\langle \bullet, \bullet \rangle_\tau$  is the pairing between distributions and test functions on  $\mathbb{T}_\tau^2$ , whereas the pairing  $\langle \bullet, \bullet \rangle$  is the pairing between distributions and test functions on  $\mathbb{R}^2$ . By the translational invariance of  $V^{m,\sigma}$  and the periodicity of  $\phi_1, \dots, \phi_m$  the RHS of Eq. (2.7) is independent of the choice of the function  $\chi_\tau$ . In informal terms,  $U^{m,\sigma}$  is obtained by the periodization of  $V^{m,\sigma}$  in  $m-1$  variables. Because of the assumption that  $V^{m,\sigma}$  is antisymmetric the choice of these  $m-1$  variables is not important. Note that the hierarchy  $(V^{m,\sigma})_{m \in \mathbb{N}_+, \sigma \in \mathbb{G}^m}$  contains no information about  $U^0$ , which is redundant. It turns out that if the flow equation (2.5) is satisfied up to a constant, then it is possible to choose the constant such that the flow equation holds exactly. The advantage of working with the kernels  $(V^{m,\sigma})_{m \in \mathbb{N}_+, \sigma \in \mathbb{G}^m}$  rather than  $(U^{m,\sigma})_{m \in \mathbb{N}_0, \sigma \in \mathbb{G}^m}$  is that the kernels  $(V^{m,\sigma})_{m \in \mathbb{N}_+, \sigma \in \mathbb{G}^m}$  are not periodic and decay rapidly in directions transversal to the diagonal. Collections of kernels  $(V^{m,\sigma})_{m \in \mathbb{N}_+, \sigma \in \mathbb{G}^m}$  are identified with functionals  $V$  by the formula

$$V[\varphi] = \sum_{m \in \mathbb{N}_+} \sum_{\sigma \in \mathbb{G}^m} \langle V^{m,\sigma}, \varphi^{\sigma_1} \otimes \dots \otimes \varphi^{\sigma_m} \rangle \quad (2.8)$$

for all  $\varphi \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G}$  valued in the odd part of  $\mathcal{G}$ . In order to introduce a topology in the space of functionals  $V \equiv (V^{m,\sigma})_{m \in \mathbb{N}_+, \sigma \in \mathbb{G}^m}$  for all  $m \in \mathbb{N}_+, \sigma \in \mathbb{G}^m$  it is convenient to make the following ansatz

$$\langle V^{m,\sigma}, \varphi_1 \otimes \dots \otimes \varphi_m \rangle = \sum_{a \in \mathbb{A}^m} \langle V^{m,a,\sigma}, \partial^{a_1} \varphi_1 \otimes \dots \otimes \partial^{a_m} \varphi_m \rangle \quad (2.9)$$

for all  $\varphi_1, \dots, \varphi_m \in \mathcal{S}(\mathbb{R}^2)$ , where  $\mathbb{A} = \{0, 1, 2\}^2$  is the set of spatial multi-indices and for all  $m \in \mathbb{N}_+$ ,  $a \in \mathbb{A}^m$ ,  $\sigma \in \mathbb{G}^m$  the distributions  $V^{m,a,\sigma} \in \mathcal{S}'(\mathbb{R}^{2m})$  are measures such that the following norm

$$\|V^{m,a,\sigma}\|_{\mathcal{M}^m} = \sup_{x_1 \in \mathbb{R}^2} \int |V^{m,a,\sigma}(x_1, dx_2, \dots, dx_m)|$$

is finite. We also introduce the notation  $\mathbf{J}\varphi = (\partial^a \varphi^\sigma)_{a \in \mathbb{A}, \sigma \in \mathbb{G}} \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}$  for the jet extension of the function  $\varphi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$ . The representation (2.9) of  $V^{m,\sigma}$  in terms of  $(V^{m,a,\sigma})_{a \in \mathbb{A}^m}$  is of course not unique. The fact that the potential of the Gross-Neveu model  $U_{\tau,\varepsilon}$  does not involve derivatives suggests that one could set  $V^{m,a,\sigma} = 0$  unless  $a = 0$ . However, it turns out that another representation of  $V^{m,\sigma}$  in terms of  $(V^{m,a,\sigma})_{a \in \mathbb{A}^m}$  is more convenient to solve the renormalization problem. In what follows, we identify functionals  $V$  with collections of kernels  $(V^{m,a,\sigma})_{m \in \mathbb{N}_+, a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m}$  such that Eqs. (2.8) and (2.9) are satisfied. We also use the notation  $V^m$  for the collection of kernels  $(V^{m,a,\sigma})_{a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m}$ . For  $\tau, \varepsilon \in [0, 1]$  and  $t, s \in (0, 1]$  we introduce the following maps in the space of functionals

$$\begin{aligned} (\mathbf{A}_{\tau,\varepsilon;t,s} V)[\varphi] &:= \Pi_\circ V[\mathbf{J}\Psi_{\tau,\varepsilon;t\vee s,s} + \varphi], \\ \mathbf{B}_{\varepsilon;s}(V)[\varphi] &= \Pi_\circ \langle D_\varphi V[\varphi] \otimes D_\varphi V[\varphi], (\mathbf{J} \otimes \mathbf{J}) \dot{G}_{\varepsilon;s}(\bullet - \bullet) \rangle \end{aligned} \quad (2.10)$$

where  $\Pi_\circ V[\varphi] := V[\varphi] - V[0]$  and  $\varphi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}$  is an arbitrary Schwartz function valued in the odd part of  $\mathcal{G}$ . The above definitions should be interpreted as hierarchies of equations for the kernels  $V \equiv (V^{m,a,\sigma})_{m \in \mathbb{N}_+, a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m}$ . Note that the map  $\Pi_\circ$  ensures that the functionals  $(\mathbf{A}_{\tau,\varepsilon;t,s} V)[\varphi]$  and  $(\mathbf{B}_{\varepsilon;s}(V))[\varphi]$  vanish for  $\varphi = 0$  and are of the form (2.8). We stress that  $\mathbf{A}_{\tau,\varepsilon;t,s} V$  is a functional taking values in  $\mathcal{F}_{t,s} \subset \mathcal{B}(\mathcal{H})$ . We denote by  $\mathbf{A}_{\tau,\varepsilon;t,s}^{m,a,\sigma} V$  and  $\mathbf{B}_{\varepsilon;s}^{m,a,\sigma}(V)$  the kernels of the functionals  $\mathbf{A}_{\tau,\varepsilon;t,s} V$  and  $\mathbf{B}_{\varepsilon;s}(V)$ . The symbols  $\mathbf{A}_{\tau,\varepsilon;t,s}^m V$  and  $\mathbf{B}_{\varepsilon;s}^m(V)$  denote collections of kernels.

Now we shall rewrite the flow equation (2.5) in a form that is suitable for studying the infinite volume limit  $\tau \searrow 0$ . To this end, let the functional  $V_{\tau,\varepsilon} \equiv (V_{\tau,\varepsilon}^{m,a,\sigma})_{m \in \mathbb{N}_+, a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m}$  be related to the potential (1.2) of the Gross-Neveu model by Eqs. (2.6), (2.7) and (2.9) and let  $t \mapsto V_{\tau,\varepsilon;t} \equiv (V_{\tau,\varepsilon;t}^{m,a,\sigma})_{m \in \mathbb{N}_+, a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m}$  be a solution of the following flow equation

$$V_{\tau,\varepsilon;t}[\mathbf{J}\varphi] = \mathbf{E} \mathbf{A}_{\tau,\varepsilon;t,0} V_{\tau,\varepsilon}[\mathbf{J}\varphi] + \int_0^t \mathbf{E} \mathbf{A}_{\tau,\varepsilon;t,s} \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s})[\mathbf{J}\varphi] ds \quad (2.11)$$

for all  $\varphi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$  valued in the odd part of  $\mathcal{G}$ . The above equation is written using the notation introduced in the previous paragraph. For  $t \in [0, 1]$  we define the functional  $U_{\tau,\varepsilon;t} \equiv (U_{\tau,\varepsilon;t}^{m,\sigma})_{m \in \mathbb{N}_0, \sigma \in \mathbb{G}^m}$  related to  $V_{\tau,\varepsilon;t}$  by Eqs. (2.7) and (2.9). This leaves  $U_{\tau,\varepsilon;t}^0 \in \mathbb{C}$  unspecified but for a suitable choice of  $t \mapsto U_{\tau,\varepsilon;t}^0$  the function  $(t, \phi) \mapsto U_{\tau,\varepsilon;t}(\phi)$  satisfies the flow equation (2.5). Hence, we are led to study the flow equation (2.11). The advantage of Eq. (2.11) over Eq. (2.5) is that it is well-posed in the limit  $\tau \searrow 0$ . Let us stress that a solution  $t \mapsto V_{\tau,\varepsilon;t} \equiv (V_{\tau,\varepsilon;t}^{m,a,\sigma})_{m \in \mathbb{N}_+, a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m}$  of Eq. (2.11) is not unique. Observe that a solution of the following equation

$$V_{\tau,\varepsilon;t}[\varphi] = \mathbf{E} \mathbf{A}_{\tau,\varepsilon;t,0} V_{\tau,\varepsilon}[\varphi] + \int_0^t \mathbf{E} \mathbf{A}_{\tau,\varepsilon;t,s} \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s})[\varphi] ds \quad (2.12)$$

for all  $\varphi \in C^\infty(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}$  valued in the odd part of  $\mathcal{G}$  is also a solution of Eq. (2.11). However, the reverse implication is not true because of the non-uniqueness of the representation (2.9) of  $V^{m,\sigma}$  in terms of  $(V^{m,a,\sigma})_{a \in \mathbb{A}^m}$ . We shall take advantage of this non-uniqueness in order to solve the renormalization problem. In particular, a solution of Eq. (2.11) that we are going to construct will not satisfy Eq. (2.12).

Let us now investigate the renormalization problem. To this end, we have to study the limit  $\varepsilon \searrow 0$ . First note that since for  $\varepsilon = 0$  the increment  $\Psi_{\tau,\varepsilon;t,0}$  is not a function over  $\mathbb{R}^2$  but only a distribution and the functional  $V_{\tau,\varepsilon}$  involves pointwise products the first term on the RHS of Eq. (2.11) becomes singular in the limit  $\varepsilon \searrow 0$ . We hope that the second term becomes singular in this limit as well and for specific choices of the parameters  $g_{\tau,\varepsilon}, r_{\tau,\varepsilon} \in \mathbb{R}$  of the original potential the singularities of both terms cancel out. We would like to rewrite Eq. (2.11) in a form that exhibits the above-mentioned cancellation of singularities. To this end, we have to come up with an appropriate space of functionals. The norm in this space should control the dependence of the norm  $\|V_{\tau,\varepsilon;t}^{m,a,\sigma}\|_{\mathcal{M}^m}$  of the kernels of the functional  $V_{\tau,\varepsilon;t}$  on  $t \in (0,1]$  and  $m \in \mathbb{N}_+$ ,  $a \in \mathbb{A}^m$ ,  $\sigma \in \mathbb{G}^m$ . Let us first concentrate on the dependence on  $t \in (0,1]$ . Using the fact that the perturbative corrections to the kernels  $V_{\tau,\varepsilon;t}^{m,a,\sigma}$  can be expressed in terms of Feynman diagrams by naive power counting argument for all  $m \in \mathbb{N}_+$ ,  $a \in \mathbb{A}^m$ ,  $\sigma \in \mathbb{G}^m$  the following bound

$$\|V_{\tau,\varepsilon;t}^{m,a,\sigma}\|_{\mathcal{M}^m} \lesssim t^{m/2+|a|-2} \quad (2.13)$$

should hold uniformly in  $\tau, \varepsilon \in [0,1]$  and  $t \in (0,1]$  up to logarithmic corrections, which we ignore for the moment. Since  $V_{\tau,\varepsilon;t}^{m,a,\sigma} = 0$  if  $m \in \mathbb{N}_+ \setminus 2\mathbb{N}_+$  by the charge conjugation invariance one can restrict attention to kernels  $V_{\tau,\varepsilon;t}^{m,a,\sigma}$  with  $m \in 2\mathbb{N}_+$ . This suggests that for  $\varepsilon = 0$  the norm of the kernel

$$(V_{\tau,\varepsilon;t}^{2,0,\sigma})_{\sigma \in \mathbb{G}^2},$$

may diverge polynomially and the norms of the kernels

$$(V_{\tau,\varepsilon;t}^{4,0,\sigma})_{\sigma \in \mathbb{G}^4}, \quad (V_{\tau,\varepsilon;t}^{2,a,\sigma})_{a \in \mathbb{A}^2, |a|=1, \sigma \in \mathbb{G}^2}$$

may diverge logarithmically as  $t \searrow 0$ . We call the above kernels relevant and marginal, respectively. The remaining kernels vanish in the limit  $t \searrow 0$  and are called irrelevant. Note that the functional  $V_{\tau,\varepsilon}$ , which is closely related to the original potential  $U_{\tau,\varepsilon}$  defined by Eq. (1.2), contains only relevant and marginal kernels. Using properties of the maps  $\mathbf{A}_{\tau,\varepsilon;t,s}$  and  $\mathbf{B}_{\varepsilon;s}$  as well as the bound (2.13) one shows that for all  $m \in \mathbb{N}_+$ ,  $a \in \mathbb{A}^m$ ,  $\sigma \in \mathbb{G}^m$  the bound

$$\|\mathbf{E}\mathbf{A}_{\tau,\varepsilon;t,s}^{m,a,\sigma}\mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s})\|_{\mathcal{M}^m} \lesssim s^{m/2+|a|-3}$$

holds uniformly in  $\tau, \varepsilon \in [0,1]$  and  $t, s \in (0,1]$  up to logarithmic corrections. By the Minkowski inequality this implies the bound

$$\begin{aligned} \left\| \int_0^t \mathbf{E}\mathbf{A}_{\tau,\varepsilon;t,s}^{m,a,\sigma}\mathbf{B}_{\varepsilon;s}(V_s) \, ds \right\|_{\mathcal{M}^m} &\leq \int_0^t \left\| \mathbf{E}\mathbf{A}_{\tau,\varepsilon;t,s}^{m,a,\sigma}\mathbf{B}_{\varepsilon;s}(V_s) \right\|_{\mathcal{M}^m} \, ds \\ &\lesssim \int_0^t s^{m/2+|a|-3} \, ds \lesssim t^{m/2+|a|-2} \end{aligned} \quad (2.14)$$

uniform in  $\tau, \varepsilon \in [0, 1]$  and  $t \in (0, 1]$  up to logarithmic corrections unless  $m = 4$  and  $a = 0$ , or  $m = 2$  and  $|a| \leq 1$ . Consequently, for the irrelevant kernels the bound (2.13) is consistent with the flow equation (2.11). The relevant kernels require special treatment. Indeed, if  $m = 4$  and  $a = 0$ , or  $m = 2$  and  $|a| \leq 1$ , then the last estimate in (2.14) is false.

In order to address the above-mentioned problem with the estimates for the relevant kernels we make the following ansatz

$$V_{\tau, \varepsilon; t} = U(1/g_{\tau, \varepsilon; t}, r_{\tau, \varepsilon; t}, z_{\tau, \varepsilon; t}) + W_{\tau, \varepsilon; t}, \quad (2.15)$$

where  $g_{\tau, \varepsilon; t} \in (0, \infty)$ ,  $r_{\tau, \varepsilon; t}, z_{\tau, \varepsilon; t} \in \mathbb{R}$  are some parameters,  $W_{\tau, \varepsilon; t}$  is a functional and for all  $g, r, z \in \mathbb{R}$  the functional  $U(g, r, z)$  is defined by the equality

$$U(g, r, z)[\psi] := \int_{\mathbb{R}^2} (g(\bar{\psi}(x) \cdot \underline{\psi}(x))^2 + r \bar{\psi}(x) \cdot \underline{\psi}(x) + z \bar{\psi}(x) \cdot (\not{\partial} \underline{\psi})(x)) \, dx$$

for all  $\psi = (\psi^\sigma)_{\sigma \in \mathbb{G}} \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$  valued in the odd part of  $\mathcal{G}$ . Since the functional  $U(g, r, z)$  is a linear combination of quartic and quadratic terms it holds  $V_{\tau, \varepsilon; t}^{m, a, \sigma} = W_{\tau, \varepsilon; t}^{m, a, \sigma}$  unless  $m = 4$  and  $a = 0$ , or  $m = 2$  and  $|a| \leq 1$ .

In order to specify the relation between  $V_{\tau, \varepsilon; t}$  and  $(g_{\tau, \varepsilon; t}, r_{\tau, \varepsilon; t}, z_{\tau, \varepsilon; t}, W_{\tau, \varepsilon; t})$  we have to first introduce the notion of the local part of a kernel and the remainder. The local part of a collection of kernels  $V_{\tau, \varepsilon; t}^4 = (V^{4, a, \sigma})_{a \in \mathbb{G}^4, \sigma \in \mathbb{G}^4}$  is defined by the equality

$$\mathbf{L}V^4(\underline{\psi} \cdot \bar{\psi})^2 := \sum_{\sigma \in \mathbb{G}^4} \psi^{\sigma_1} \dots \psi^{\sigma_4} \int_{\mathbb{R}^6} V^{4, 0, \sigma}(x_1, dx_2, dx_3, dx_4)$$

for all Grassmann-valued numbers  $\psi \equiv (\psi^\sigma)_{\sigma \in \mathbb{G}} \equiv (\bar{\psi}^{\alpha, \varsigma}, \underline{\psi}^{\alpha, \varsigma})_{\alpha \in \{1, 2\}, \varsigma \in \{1, \dots, N\}}$ . Note that the RHS of the above formula does not depend on  $x_1 \in \mathbb{R}^2$  because of the translational invariance of the kernels of the functional  $V_{\tau, \varepsilon; t}$ . It turns out that given a collection of kernels  $V^4 = (V^{4, a, \sigma})_{a \in \mathbb{A}^4, \sigma \in \mathbb{G}^4}$  that posses certain symmetries there exists a collection of kernels  $\mathbf{R}V^4 = ((\mathbf{R}V^4)^{a, \sigma})_{a \in \mathbb{A}^4, \sigma \in \mathbb{G}^4}$  such that  $(\mathbf{R}V^4)^{a, \sigma} = 0$  if  $a = 0$  and

$$\begin{aligned} \mathbf{L}V^4 \int_{\mathbb{R}^2} (\underline{\psi}(x) \cdot \bar{\psi}(x))^2 \, dx + \sum_{a \in \mathbb{A}^4} \sum_{\sigma \in \mathbb{G}^4} \langle (\mathbf{R}V^4)^{a, \sigma}, \partial^{a_1} \psi^{\sigma_1} \otimes \dots \otimes \partial^{a_4} \psi^{\sigma_4} \rangle \\ = \sum_{a \in \mathbb{A}^4} \sum_{\sigma \in \mathbb{G}^4} \langle V^{4, a, \sigma}, \partial^{a_1} \psi^{\sigma_1} \otimes \dots \otimes \partial^{a_4} \psi^{\sigma_4} \rangle \end{aligned} \quad (2.16)$$

for all  $\psi \in \mathcal{S}(\mathbb{R})^{\mathbb{G}}$  valued in the odd part of  $\mathcal{G}$ . Similarly, given a collection of kernels  $V^2 = (V^{2, a, \sigma})_{a \in \mathbb{A}^2, \sigma \in \mathbb{G}^2}$  that posses certain symmetries there exist numbers  $\mathbf{L}V^2, \mathbf{L}_\partial V^2 \in \mathbb{R}$  and a collection of kernels  $\mathbf{R}V^2 = ((\mathbf{R}V^2)^{a, \sigma})_{a \in \mathbb{A}^2, \sigma \in \mathbb{G}^2}$  such that  $(\mathbf{R}V^2)^{a, \sigma} = 0$  if  $|a| \leq 1$  and

$$\begin{aligned} \mathbf{L}V^2 \int_{\mathbb{R}^2} \underline{\psi}(x) \cdot \bar{\psi}(x) \, dx + \mathbf{L}_\partial V^2 \int_{\mathbb{R}^2} \bar{\psi}(x) \cdot (\not{\partial} \underline{\psi})(x) \, dx \\ + \sum_{a \in \mathbb{A}^2} \sum_{\sigma \in \mathbb{G}^2} \langle (\mathbf{R}V^2)^{a, \sigma}, \partial^{a_1} \psi^{\sigma_1} \otimes \partial^{a_2} \psi^{\sigma_2} \rangle = \sum_{a \in \mathbb{A}^2} \sum_{\sigma \in \mathbb{G}^2} \langle V^{2, a, \sigma}, \partial^{a_1} \psi^{\sigma_1} \otimes \partial^{a_2} \psi^{\sigma_2} \rangle \end{aligned} \quad (2.17)$$

for all  $\psi \in \mathcal{S}(\mathbb{R})^{\mathbb{G}}$  valued in the odd part of  $\mathcal{G}$ . The identities (2.16) and (2.17) are consequences of the Taylor theorem and the form of the local terms

$$\int_{\mathbb{R}^2} \bar{\psi}(x) \cdot \underline{\psi}(x) dx, \quad \int_{\mathbb{R}^2} \bar{\psi}(x) \cdot (\not{\partial} \underline{\psi})(x) dx, \quad \int_{\mathbb{R}^2} (\bar{\psi}(x) \cdot \underline{\psi}(x))^2 dx$$

appearing in these identities is dictated by the symmetries of the Gross-Neveu model. The parameter  $1/g_{\tau,\varepsilon;t} \in (0, \infty)$  is called the effective coupling constant at spatial scale  $t \in (0, 1]$  and is defined so that it satisfies the equation

$$1/g_{\tau,\varepsilon;t} = \mathbf{LEA}_{\tau,\varepsilon;t,0}^4 V_{\tau,\varepsilon} + \int_0^t \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds. \quad (2.18)$$

The parameter  $r_{\tau,\varepsilon;t}$  satisfies the equation

$$r_{\tau,\varepsilon;t} = \mathbf{LEA}_{\tau,\varepsilon;0,t}^2 V_{\tau,\varepsilon} + \mathbf{LEA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;0} - 1/g_{\tau,\varepsilon;t}, 0, 0) + \int_0^t \mathbf{LEA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds. \quad (2.19)$$

The parameter  $z_{\tau,\varepsilon;t}$  satisfies the equation

$$z_{\tau,\varepsilon;t} = \int_0^t \mathbf{L}_{\partial} \mathbf{EA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds. \quad (2.20)$$

Finally, the functional  $W_{\tau,\varepsilon;t}$  satisfies the equations

$$W_{\tau,\varepsilon;t}^m = \int_0^t \mathbf{EA}_{\tau,\varepsilon;t,s}^m \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds, \quad m \in \mathbb{N}_+ \setminus \{2, 4\}, \quad (2.21)$$

$$W_{\tau,\varepsilon;t}^m = \int_0^t \mathbf{REA}_{\tau,\varepsilon;1,s}^m \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds - \mathbf{EC}_{\tau,\varepsilon;1,t}^m W_{\tau,\varepsilon;t}, \quad m \in \{2, 4\}, \quad (2.22)$$

where  $\mathbf{C}_{\tau,\varepsilon;1,t} V := \mathbf{A}_{\tau,\varepsilon;1,t} V - V$ . Note that  $\mathbf{C}_{\tau,\varepsilon;1,t}^m W_{\tau,\varepsilon;t}$  depends only on  $(W_{\tau,\varepsilon;t}^{m+k})_{k \in \mathbb{N}_+}$ . Hence, the RHS of Eq. (2.22) does not depend on  $W_{\tau,\varepsilon;t}^m$ . The functional  $V_{\tau,\varepsilon;s}$  appearing in Eqs. (2.18), (2.19), (2.20), (2.21), (2.22) is related to  $(g_{\tau,\varepsilon;s}, r_{\tau,\varepsilon;s}, z_{\tau,\varepsilon;s}, W_{\tau,\varepsilon;s})$  by Eq. (2.15). In this way we obtain a closed system of equations for  $(g_{\tau,\varepsilon;\bullet}, r_{\tau,\varepsilon;\bullet}, z_{\tau,\varepsilon;\bullet}, W_{\tau,\varepsilon;\bullet})$ . Using Eqs. (2.16) and (2.17) one proves that given a solution  $(g_{\tau,\varepsilon;\bullet}, r_{\tau,\varepsilon;\bullet}, z_{\tau,\varepsilon;\bullet}, W_{\tau,\varepsilon;\bullet})$  of this system of equations the functional  $V_{\tau,\varepsilon;\bullet}$  related to  $(g_{\tau,\varepsilon;\bullet}, r_{\tau,\varepsilon;\bullet}, z_{\tau,\varepsilon;\bullet}, W_{\tau,\varepsilon;\bullet})$  by Eq. (2.15) is a solution of the flow equation (2.11). Let us mention that this solution does not satisfy Eq. (2.12). The argument presented in the previous paragraph together with the fact that the relevant kernels of the functional  $\mathbf{REA}_{\tau,\varepsilon;1,s} \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s})$  vanish identically by the definition of the map  $\mathbf{R}$  suggest that the integrands in Eqs. (2.21), (2.22) are absolutely integrable also for  $\varepsilon = 0$ . Now let us study more closely Eq. (2.18) for the effective coupling constant. Noting that

$$\partial_t g_{\tau,\varepsilon;t} = -g_{\tau,\varepsilon;t}^2 \partial_t (1/g_{\tau,\varepsilon;t}) = -g_{\tau,\varepsilon;t}^2 \mathbf{LEA}_{\tau,\varepsilon;1,t}^4 \mathbf{B}_{\varepsilon;t}(V_{\tau,\varepsilon;t})$$



and using the fact that  $\mathbf{LEA}_{\tau,\varepsilon;t,0}^4 V_{\tau,\varepsilon} = 1/g_{\tau,\varepsilon}$ , where  $g_{\tau,\varepsilon}$  is the parameter of the potential (1.2), we obtain

$$g_{\tau,\varepsilon;t} = g_{\tau,\varepsilon} - \int_0^t g_{\tau,\varepsilon;s}^2 \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds.$$

We fix the parameter  $g_{\tau,\varepsilon} = g_{\tau,\varepsilon;t=0}$  implicitly by imposing the following renormalization condition

$$g_{\tau,\varepsilon;t=1} = 1/\lambda, \quad (2.23)$$

where  $\lambda \in (0, 1]$  is assumed to be sufficiently small. Taking into account the above boundary condition we rewrite the equation for  $g_{\tau,\varepsilon;t}$  in the following form

$$g_{\tau,\varepsilon;t} = 1/\lambda + \int_t^1 g_{\tau,\varepsilon;s}^2 \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds. \quad (2.24)$$

It turns out that the asymptotic behavior of

$$s \mapsto g_{\tau,\varepsilon;s}^2 \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) = g_{\tau,\varepsilon;s}^2 \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(U(1/g_{\tau,\varepsilon;s}, r_{\tau,\varepsilon;s}, z_{\tau,\varepsilon;s}) + W_{\tau,\varepsilon;s})$$

at  $s = 0$  coincides with the asymptotic behavior of

$$s \mapsto g_{\tau,\varepsilon;s}^2 \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(U(1/g_{\tau,\varepsilon;s}, 0, 0)) = \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0))$$

The above claim is justified a posteriori by assuming that the tuple  $(g_{\tau,\varepsilon;\bullet}, r_{\tau,\varepsilon;\bullet}, z_{\tau,\varepsilon;\bullet}, W_{\tau,\varepsilon;\bullet})$  belongs to the set  $\mathcal{Y}_{\tau,\varepsilon}$  defined below. Moreover, a direct computation yields

$$\mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0)) = \beta_2 s^{-1}, \quad \beta_2 = 2(N-1)/\pi,$$

for  $\tau = 0$  and  $\varepsilon = 0$ . Consequently, for  $\tau = 0$  and  $\varepsilon = 0$  the asymptotic behavior of  $t \mapsto 1/g_{\tau,\varepsilon;t}$  at  $t = 0$  coincides with the asymptotic behavior of  $t \mapsto \lambda_t$ , where  $\lambda_t$  was introduced in Def. 2.2. Note that the asymptotic behavior of the effective coupling constant  $1/g_{\tau,\varepsilon;t}$  is easiest to determine by studying Eq. (2.24) for its inverse  $g_{\tau,\varepsilon;t}$ . This is one of the reasons why we chose  $g_{\tau,\varepsilon;t}$  rather than the effective coupling constant  $1/g_{\tau,\varepsilon;t}$  to be one of the fundamental variables. Note that by the above argument the effective coupling constant  $1/g_{\tau,\varepsilon;t}$  with  $\varepsilon = 0$  vanishes logarithmically as  $t \searrow 0$ . Thus, at high energy the behavior of Schwinger functions of the Gross-Neveu model without the UV cutoff should not differ much from the behavior of the Schwinger functions of the free theory. This property is called in the literature the asymptotic freedom and plays a crucial role in our construction. Let us remark that our scale decomposition  $G_{\varepsilon;t}$  of the propagator  $G_\varepsilon$  satisfies the condition  $\dot{G}_{\varepsilon;t} = 0$  for  $t \in (0, \varepsilon]$ . As a result,  $\mathbf{B}_{\varepsilon;t}(\bullet) = 0$  and  $g_{\tau,\varepsilon;t} = g_{\tau,\varepsilon;\varepsilon}$  for  $t \in [0, \varepsilon]$ . Hence, the flow of the effective coupling constant in the model with the UV cutoff  $\varepsilon \in (0, 1]$  halts when the scale parameter  $t$  reaches the value  $\varepsilon$ . In particular, the model with the UV cutoff is not asymptotically free and the value of the parameter  $g_{\tau,\varepsilon} = g_{\tau,\varepsilon;0} = g_{\tau,\varepsilon;\varepsilon}$  of the potential (1.2) fixed by the renormalization

condition (2.23) vanishes logarithmically as  $\varepsilon \searrow 0$ . As a side remark, we mention that the assumption  $\lambda > 0$  is necessary for the asymptotic freedom of the model without the UV cutoff as for  $\lambda < 0$  the function  $t \mapsto \lambda_t$  blows up at the scale  $t = e^{1/(\lambda\beta_2)}$ , which is called in the literature the Landau pole. Because of the asymptotic freedom the logarithmic corrections, which were ignored in the bound (2.13), improve the asymptotic behavior of  $t \mapsto \|V_{\tau,\varepsilon;t}^{m,a,\sigma}\|_{\mathcal{M}^m}$  at  $t = 0$ . Actually, taking into account the logarithmic vanishing of the effective coupling constant a direct inspection of perturbative corrections yields the following bound

$$\|V_t^{m,a,\sigma}\|_{\mathcal{M}^m} \lesssim \lambda_{\varepsilon\sqrt{t}}^{(m/2-1)\vee 1} t^{m/2+|a|-2}$$

uniform in  $\tau, \varepsilon \in [0, 1]$  and  $t \in (0, 1]$  for all  $m \in \mathbb{N}_+$ ,  $a \in \mathbb{A}^m$ ,  $\sigma \in \mathbb{G}^m$ . Using the above bound one shows that

$$\|\mathbf{EA}_{\tau,\varepsilon;t,s}^{m,a,\sigma} \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s})\|_{\mathcal{M}^m} \lesssim \lambda_s^{(m/2-1)\vee 2} s^{m/2+|a|-3}$$

uniformly in  $\tau, \varepsilon \in [0, 1]$  and  $t, s \in (0, 1]$ . Hence, using Lemma 2.4 (A) we obtain

$$\begin{aligned} \left\| \int_0^t \mathbf{EA}_{\tau,\varepsilon;u,s}^{m,a,\sigma} \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds \right\|_{\mathcal{M}^m} &\lesssim \int_0^t \lambda_s^{(m/2-1)\vee 2} s^{m/2+|a|-3} ds \\ &\lesssim \lambda_t^{(m/2-1)\vee 2} t^{m/2+|a|-2} \end{aligned} \quad (2.25)$$

uniform in  $\tau, \varepsilon \in [0, 1]$  and  $t, u \in (0, 1]$  unless  $m = 4$  and  $a = 0$ , or  $m = 2$  and  $|a| \leq 1$ . By Lemma 2.4 (D) applied with  $\rho = 1$  we have

$$\left\| \int_0^t \mathbf{EA}_{\tau,\varepsilon;u,s}^{m,a,\sigma} \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds \right\|_{\mathcal{M}^m} \lesssim \int_0^t \lambda_s^2 s^{-1} ds \lesssim \lambda_t \quad (2.26)$$

uniformly in  $\tau, \varepsilon \in [0, 1]$  and  $t, u \in (0, 1]$  for  $m = 4$  and  $a = 0$ , or  $m = 2$  and  $|a| = 1$ . If  $m = 2$  and  $a = 0$ , then for  $\varepsilon = 0$  the function  $s \mapsto \mathbf{EA}_{\tau,\varepsilon;u,s}^{m,a,\sigma} \mathbf{B}_{\varepsilon;s}(V_s)$  is not integrable at  $s = 0$ . However, by Lemma 2.4 (C) applied with  $\rho = 2$ ,  $\varrho = -1$  and  $\lambda \in (0, 1]$  small enough we have

$$\left\| \int_t^1 \mathbf{EA}_{\tau,\varepsilon;u,s}^{m,a,\sigma} \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds \right\|_{\mathcal{M}^m} \lesssim \int_t^1 \lambda_s^2 s^{-2} ds \lesssim \lambda_t^2 t^{-1} \quad (2.27)$$

uniformly in  $\tau, \varepsilon \in [0, 1]$  and  $t, u \in (0, 1]$  for  $m = 2$  and  $a = 0$ . The bound (2.26) suggests that Eq. (2.20) for  $z_{\tau,\varepsilon;t}$  is well-posed for  $\varepsilon = 0$ . On the other hand, Eq. (2.19) for  $r_{\tau,\varepsilon;t}$  is not expected to be well-posed for  $\varepsilon = 0$ . To address the problem with Eq. (2.19) we impose the following renormalization condition

$$r_{\tau,\varepsilon;t=1} = 0$$

that fixes implicitly the mass counterterm  $r_{\tau,\varepsilon} = r_{\tau,\varepsilon;t=0}$  and rewrite this equation so that it involves the integral appearing on the LHS of the bound (2.27). Using Eq. (2.19) we obtain the following equation

$$r_{\tau,\varepsilon;t} = -\mathbf{LEA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;t}, 0, 0) - \int_t^1 \mathbf{LEA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds. \quad (2.28)$$

Let us remark that since we do not impose any non-trivial renormalization condition for the parameter  $z_{\tau,\varepsilon;t}$  introducing this parameter in the ansatz (2.15) is not necessary and is done mainly because it is convenient when proving estimates.

The claim is that the system of equations (2.24), (2.28), (2.20), (2.21), (2.22) for the tuple  $(g_{\tau,\varepsilon;\bullet}, r_{\tau,\varepsilon;\bullet}, z_{\tau,\varepsilon;\bullet}, W_{\tau,\varepsilon;\bullet})$  remains well-posed in the limit  $\tau, \varepsilon \searrow 0$ . Since we would like to construct a solution of this system of equations using the contraction principle we have to first find an appropriate complete metric space of tuples  $(g_{\tau,\varepsilon;\bullet}, r_{\tau,\varepsilon;\bullet}, z_{\tau,\varepsilon;\bullet}, W_{\tau,\varepsilon;\bullet})$ . We will define this metric space as a closed subset of a certain Banach space. The norm in the Banach space has to control the dependence of  $g_{\tau,\varepsilon;t}, r_{\tau,\varepsilon;t}, z_{\tau,\varepsilon;t} \in \mathbb{R}$  on  $t \in (0, 1]$  as well as the dependence of  $\|W_{\tau,\varepsilon;t}^{m,a,\sigma}\|_{\mathcal{M}^m}$  on  $t \in (0, 1]$  and  $m \in \mathbb{N}_+$ ,  $a \in \mathbb{A}^m$ ,  $\sigma \in \mathbb{G}^m$ . To this end, for  $\alpha, \beta \in [1, \infty)$  and  $\gamma \in [0, \infty)$  we introduce the Banach space of functionals  $s \mapsto W_s = (W_s^{m,a,\sigma})_{m \in \mathbb{N}_+, a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m}$  depending on the scale parameter  $s \in (0, 1]$  equipped with the following norm

$$\|W_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}} := \sup_{m \in \mathbb{N}_+} \alpha^m m^\beta \|W_\bullet^m\|_{\mathcal{V}^{m;\gamma}},$$

where

$$\|W_\bullet^m\|_{\mathcal{V}^{m;\gamma}} := \sup_{s \in (0,1]} \sum_{a \in \mathbb{A}^m} \sum_{\sigma \in \mathbb{G}^m} \lambda_s^{-\rho_{\gamma,\kappa}(m)} s^{2-m/2-|a|} \|w_s^m W_s^{m,a,\sigma}\|_{\mathcal{M}^m}.$$

The weight  $w_s^m$  depending only on the relative coordinates and growing stretched exponentially is needed to establish stretched exponential decay of truncated correlations as well as to prove the desired estimates for the map  $\mathbf{R}$  introduced above. The bound (2.25) suggest that the kernels of a solution  $W_{\tau,\varepsilon;\bullet}$  of Eqs. (2.21) and (2.22) should have finite norms  $\|W_{\tau,\varepsilon;\bullet}^m\|_{\mathcal{V}^{m;\gamma}}$  defined in terms of  $\rho_{\gamma,\kappa}(m) = (m/2 - 1) \vee 2$ . In order to give oneself some wiggle room when proving estimates we choose instead  $\rho_{\gamma,\kappa}(m) := \gamma + 2\kappa m$  with  $\gamma = 2 - 80\kappa$ , where  $\kappa = 1/1000$  was fixed in Def. 2.1. It turns out the kernels of the functional  $V_{\tau,\varepsilon;\bullet}$  related to  $(g_{\tau,\varepsilon;\bullet}, r_{\tau,\varepsilon;\bullet}, z_{\tau,\varepsilon;\bullet}, W_{\tau,\varepsilon;\bullet})$  by Eq. (2.15) have finite norms  $\|t \mapsto \theta(t - \varepsilon) V_{\tau,\varepsilon;t}^m\|_{\mathcal{V}^{m;\gamma}}$  with  $\gamma = 1 - 40\kappa$ . The presence of the function  $\theta(t - \varepsilon)$  in the above expression is needed because in the model with the UV cutoff the flow of the effective coupling constant  $1/g_{\tau,\varepsilon;t}$  halts at  $t = \varepsilon$  and consequently  $1/g_{\tau,\varepsilon;t}$  does not vanish in the limit  $t \searrow 0$  if  $\varepsilon \in (0, 1]$ . Recall that  $\dot{G}_{\varepsilon;t} = 0$  and  $\mathbf{B}_{\varepsilon;t}(\bullet) = 0$  if  $t \in (0, \varepsilon]$ . As a result, the RHS of Eqs. (2.24), (2.28), (2.20), (2.21), (2.22) depend only on  $t \mapsto \theta(t - \varepsilon) V_{\tau,\varepsilon;t}$ . The usefulness of the norm  $\|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$  comes from the following estimates

$$\|s \mapsto s \mathbf{B}_{\varepsilon;s}(V_s)\|_{\mathcal{V}^{\alpha,\beta-1;2\gamma}} \leq C \|s \mapsto V_s\|_{\mathcal{V}^{\alpha,\beta;\gamma}}^2 \quad (2.29)$$

and

$$\|t \mapsto \int_0^t \Pi_{>4} V_s / s \, ds\|_{\mathcal{V}^{\alpha,\beta;\gamma}} \leq C \|s \mapsto V_s\|_{\mathcal{V}^{\alpha,\beta-1;\gamma}} \quad (2.30)$$

valid for  $\alpha \in [1, \infty)$ ,  $\beta \in (2, \infty)$ ,  $\gamma \in [0, \infty)$ , where  $C \in (0, \infty)$  is a universal constant and given a functional  $V = (V^{m,a,\sigma})_{m \in \mathbb{N}_+, a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m}$  we define the functional  $\Pi_{>4} V$  by the equalities  $(\Pi_{>4} V)^{m,a,\sigma} = 0$  if  $m \leq 4$  and  $(\Pi_{>4} V)^{m,a,\sigma} = V^{m,a,\sigma}$  if  $m > 4$ . The presence of

the operator  $\Pi_{>4}$  in the second of the above estimates is related to the fact that the relevant kernels of  $s \mapsto \mathbf{B}_{\varepsilon;s}(V_s)$  are not absolutely integrable at  $s = 0$  and need a special treatment. Observe also that since the scale dependent functional  $s \mapsto V_s - \Pi_{>4}V_s$  has only finitely many non-zero kernels the parameters  $\alpha$  and  $\beta$  do not play any role when estimating its  $\|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$  norm. The estimates (2.29) and (2.30) imply in particular that

$$\|t \mapsto \int_0^t \Pi_{>4} \mathbf{B}_{\varepsilon;s}(V_s) ds\|_{\mathcal{V}^{\alpha,\beta;2\gamma}} \leq C \|s \mapsto V_s\|_{\mathcal{V}^{\alpha,\beta;\gamma}}^2. \quad (2.31)$$

The remarkable property of the above bound is the fact that the norms on both sides of this bound have the same parameter  $\alpha$  and  $\beta$ . Recall that the RHS of Eq. (2.21) involves  $\mathbf{E}\mathbf{A}_{\tau,\varepsilon;t,s} \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s})$ . In particular, to be able to solve the above system in the space  $\mathcal{V}^{\alpha,\beta;\gamma}$  of functionals  $W_{\tau,\varepsilon;\bullet}$  the validity of an estimate of the form

$$\|t \mapsto \int_0^t \Pi_{>4} \mathbf{E}\mathbf{A}_{\tau,\varepsilon;t,s} V_s / s ds\|_{\mathcal{V}^{\alpha,\beta;\gamma}} \leq C \|s \mapsto V_s\|_{\mathcal{V}^{\alpha,\beta-1;\gamma}} \quad (\text{false}) \quad (2.32)$$

with a universal constant  $C \in (0, \infty)$  appears to be crucial. Indeed, the bounds (2.29) and (2.32) imply the bound

$$\|t \mapsto \int_0^t \Pi_{>4} \mathbf{E}\mathbf{A}_{\tau,\varepsilon;t,s} \mathbf{B}_{\varepsilon;s}(V_s) ds\|_{\mathcal{V}^{\alpha,\beta;2\gamma}} \leq C \|s \mapsto V_s\|_{\mathcal{V}^{\alpha,\beta;\gamma}}^2 \quad (\text{false}) \quad (2.33)$$

with a universal constant  $C \in (0, \infty)$ , which would allow to control the  $\|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$  norm of the RHS of Eq. (2.21). Recall that the definition of the map  $\mathbf{A}_{\tau,\varepsilon;t,s}$  involves the increment  $\Psi_{\tau,\varepsilon;t,s}$  of the scale decomposition of the free field. Using the bound

$$|\mathbf{E}(\Psi_{\tau,\varepsilon;t,s}^{\sigma_1}(x_1) \dots \Psi_{\tau,\varepsilon;t,s}^{\sigma_k}(x_k))| \leq \sup_{\sigma \in \mathbb{G}} \|\Psi_{\tau,\varepsilon;t,s}^{\sigma}\|_{\mathcal{C}}^k$$

where  $\|\phi\|_{\mathcal{C}} := \sup_{x \in \mathbb{R}^2} \|\phi(x)\|_{\mathcal{B}(\mathcal{H})}$ , as well as a similar bound for  $\partial^a \Psi_{\tau,\varepsilon;t,s}^{\sigma}$  one shows, with some effort, that the bound (2.32) would be true for sufficiently small  $\lambda \in (0, 1]$  if the following bound

$$\|\partial^a \Psi_{\tau,\varepsilon;t,s}^{\sigma}\|_{\mathcal{C}} \leq c (s^{-1/2-|a|} - t^{-1/2-|a|}), \quad 0 < s \leq t \leq 1, \quad (\text{false}) \quad (2.34)$$

with a universal constant  $c \in (0, \infty)$  was true. The above estimate would imply in particular that the scale decomposition of the free field is locally Lipschitz continuous. However, the scale decomposition of the free field should have similar regularity in the scale parameter, which plays the role of time, to the Grassmann Brownian motion, which is only Hölder continuous with exponent  $1/2$ . In fact, the following bound

$$\|\partial^a \Psi_{\tau,\varepsilon;t,s}^{\sigma}\|_{\mathcal{C}} \leq c (s^{-1-2|a|} - t^{-1-2|a|})^{1/2}, \quad 0 < s \leq t \leq 1, \quad (2.35)$$

holds true with a universal constant  $c \in (0, \infty)$ . This indicates that the bound (2.32) is false. The above argument is obviously not conclusive. However, closer analysis reveals that it is unlikely that the estimate (2.33) is true. We refer the reader to [SW00] for related comments. As an aside, we mention that as observed in [SW00] the proof of estimates for fermionic

correlations given in [BW88, BW99] is not correct precisely because the scale decomposition of the free field is not Lipschitz continuous. This suggests that, in a sense, the difficulty we face is related to the fact that the scale decomposition of the free field is only  $1/2$  Hölder continuous. Note also that in the discrete renormalization group one studies scales  $t, s \in (0, 1]$  such that  $s = t/L$  with  $L \in (1, \infty)$  fixed. For scales  $t, s \in (0, 1]$  satisfying the above relation the estimates (2.34) and (2.35) with some constants  $c \in (0, \infty)$  are equivalent.

One of the novel ideas of this work is the use of the following norm

$$\|W_\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}} := \sup_{u \in [0,1]} \|s \mapsto \mathbf{A}_{\tau,\varepsilon;u \vee s,s} W_s\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$$

in the space of scale dependent functionals  $s \mapsto W_s = (W_s^{m,a,\sigma})_{m \in \mathbb{N}_+, a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m}$ . Since  $\mathbf{A}_{\tau,\varepsilon;s,s} V_s = V_s$  the norm  $\|\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}}$  is stronger than  $\|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$ . We stress that the norm  $\|\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}}$  depends on the cutoffs  $\tau, \varepsilon \in (0, 1]$ . We also note that the functional  $\mathbf{A}_{\tau,\varepsilon;u \vee s,s} V_s$ , which appears in the definition of the norm  $\|\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}}$ , takes values in  $\mathcal{F}_{u \vee s,s} \subset \mathcal{F} \subset \mathcal{B}(\mathcal{H})$ . For this reason we are forced to work with kernels that are measures valued in a Banach space  $\mathcal{B}(\mathcal{H})$ . The main advantage of the norm  $\|\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}}$  over  $\|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$  is the validity of the following estimate

$$\left\| t \mapsto \int_0^t \Pi_{>4} \mathbf{E} \mathbf{A}_{\tau,\varepsilon;t,s} \mathbf{B}_{\varepsilon;s}(V_s) ds \right\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;2\gamma}} \leq C \|s \mapsto V_s\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}}^2, \quad (2.36)$$

where  $C \in (0, \infty)$  is some universal constant. As we argued above an analogous estimate is almost certainly false if the norm  $\|\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}}$  is replaced by the more standard norm  $\|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$ . Let us stress that the lack of a norm in the space of scale dependent functionals for which the estimate of the form (2.36) holds true was the main obstacle in the construction of Grassmann measures of fermionic quantum field theories with the use of the Polchinski flow equation. Since the estimate (2.36) is of crucial importance let us sketch the main ideas behind its proof. First, observe that in order to prove the estimate (2.36) we have to control the  $\|\bullet\|_{\mathcal{V}_{\tau,\varepsilon}^{\alpha,\beta;2\gamma}}$  norm of the functional  $\mathbf{A}_{u,t} \mathbf{E} \mathbf{A}_{t,s} \mathbf{B}_s(V_s)$  in terms of the  $\|\bullet\|_{\mathcal{V}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}}$  norm of the functional  $\mathbf{A}_{u,s}^m V_s$ . Next, note that for  $0 < s \leq t \leq u \leq 1$  it holds

$$\begin{aligned} \mathbf{A}_{\tau,\varepsilon;u,s} V &= \mathbf{A}_{\tau,\varepsilon;u,t} \mathbf{A}_{\tau,\varepsilon;t,s} V, & \mathbf{A}_{\tau,\varepsilon;u,s} \mathbf{B}_{\varepsilon;s}(V) &= \mathbf{B}_{\varepsilon;s}(\mathbf{A}_{\tau,\varepsilon;u,s} V), \\ \mathbf{A}_{\tau,\varepsilon;u,t} \mathbf{E} \mathbf{A}_{\tau,\varepsilon;t,s} V &= \mathbf{A}_{\tau,\varepsilon;u,t} \mathbf{E}_t \mathbf{A}_{\tau,\varepsilon;t,s} V = \mathbf{E}_t \mathbf{A}_{\tau,\varepsilon;u,t} \mathbf{A}_{\tau,\varepsilon;t,s} V = \mathbf{E}_t \mathbf{A}_{\tau,\varepsilon;u,s} V. \end{aligned}$$

The identities in the first line above follow easily from the definition (2.10) of the maps  $\mathbf{A}_{\tau,\varepsilon;t,s}$  and  $\mathbf{B}_{\varepsilon;s}$ . The first equality in the second line is a consequence of the fact that the kernels of the functional  $\mathbf{A}_{\tau,\varepsilon;t,s} V$  take values in  $\mathcal{F}_{t,s}$  and on  $\mathcal{F}_{t,s} \subset \mathcal{F}_{t,0}$  the expected value  $\mathbf{E}$  coincides with the conditional expectation  $\mathbf{E}_t$ . To prove the second equality in the second line we use the fact that the conditional expectation  $\mathbf{E}_t$  act trivially on  $\mathcal{F}_{u,t} \subset \mathcal{F}_{1,t}$ . As a result, the following identity

$$\mathbf{A}_{\tau,\varepsilon;u,t} \mathbf{E} \mathbf{A}_{\tau,\varepsilon;t,s} \mathbf{B}_{\varepsilon;s}(V) = \mathbf{E}_t \mathbf{B}_{\varepsilon;s}(\mathbf{A}_{\tau,\varepsilon;u,s} V) \quad (2.37)$$

holds true. We also note that  $\|\mathbf{E}_t V\|_{\mathcal{M}^m} \leq \|V\|_{\mathcal{M}^m}$  and

$$\|u \mapsto \mathbf{E}_t V_u\|_{\mathcal{V}^{\alpha,\beta;\gamma}} \leq \|u \mapsto V_u\|_{\mathcal{V}^{\alpha,\beta;\gamma}}. \quad (2.38)$$

Consequently, ignoring the presence of the operator  $\Pi_{>4}$  in the estimate (2.31) we obtain

$$\begin{aligned} & \left\| t \mapsto \int_0^t \mathbf{E} \mathbf{A}_{\tau,\varepsilon;t,s} \mathbf{B}_{\varepsilon;s}(V_s) ds \right\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;2\gamma}} = \sup_{u \in (0,1]} \left\| t \mapsto \int_0^t \mathbf{A}_{\tau,\varepsilon;u,t} \mathbf{E} \mathbf{A}_{\tau,\varepsilon;t,s} \mathbf{B}_{\varepsilon;s}(V_s) ds \right\|_{\mathcal{V}^{\alpha,\beta;2\gamma}} \\ &= \sup_{u \in (0,1]} \left\| t \mapsto \int_0^t \mathbf{E}_t \mathbf{B}_{\varepsilon;s}(\mathbf{A}_{\tau,\varepsilon;u,s} V_s) ds \right\|_{\mathcal{V}^{\alpha,\beta;2\gamma}} \leq \sup_{u \in (0,1]} \left\| t \mapsto \int_0^t \mathbf{B}_{\varepsilon;s}(\mathbf{A}_{\tau,\varepsilon;u,s} V_s) ds \right\|_{\mathcal{V}^{\alpha,\beta;2\gamma}} \\ &\leq C \sup_{u \in (0,1]} \|s \mapsto \mathbf{A}_{\tau,\varepsilon;u,s} V_s\|_{\mathcal{V}^{\alpha,\beta;\gamma}}^2 = C \|s \mapsto V_s\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}}^2. \end{aligned}$$

The first and the last equality above follow from the definition of the norm  $\|\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}}$ . The second equality above follows from the identity (2.37). The first bound above follows from the estimate (2.38). Finally, to prove the second bound above we used the estimate (2.31) with the operator  $\Pi_{>4}$  omitted. Since the presence of the operator  $\Pi_{>4}$  in the estimate (2.31) is crucial the above reasoning is of course not correct. However, a more complicated argument not ignoring the presence of the operator  $\Pi_{>4}$  in the estimate (2.31) allows to establish rigorously the desired bound (2.36).

We are ready to define the complete metric space in which we will solve the system of equations (2.24), (2.28), (2.20), (2.21), (2.22). Let us first introduce the Banach space of tuples

$$X_{\bullet} \equiv (g_{\bullet}, r_{\bullet}, z_{\bullet}, W_{\bullet}) \in C((0,1], \mathbb{C}) \times C((0,1], \mathbb{C}) \times C((0,1], \mathbb{C}) \times \mathcal{W}_{\tau,\varepsilon}^{8,4;2-80\kappa} =: \mathcal{X}_{\tau,\varepsilon}$$

equipped with the norm

$$\|X_{\bullet}\| := \sup_{t \in (0,1]} \lambda_t^{\kappa_1+1} |g_t| + \sup_{t \in (0,1]} \lambda_t^{\kappa_2-1} t |r_t| + \sup_{t \in (0,1]} \lambda_t^{\kappa_3-1} |z_t| + \|W_{\bullet}\|_{\mathcal{W}_{\tau,\varepsilon}^{8,4;2-80\kappa}}.$$

where  $\kappa_1, \kappa_2, \kappa_3 \in (0,1)$  are certain small parameters. Next, for  $\tau, \varepsilon \in [0,1]$  we define the complete metric space

$$\mathcal{Y}_{\tau,\varepsilon} := \{X_{\bullet} \in \mathcal{X}_{\tau,\varepsilon} \mid \|X_{\bullet}\|_{\mathcal{X}_{\tau,\varepsilon}} \leq 1, \forall t \in (0,1] \operatorname{Im} g_t = \operatorname{Im} r_t = \operatorname{Im} z_t = 0, \lambda_{\varepsilon \vee t} g_t \geq \lambda^{\kappa}\}$$

and a map

$$\mathcal{Y}_{\tau,\varepsilon} \ni X_{\bullet} \mapsto \mathbf{X}_{\tau,\varepsilon;t}(X_{\bullet}) := (\mathbf{g}_{\tau,\varepsilon;t}(X_{\bullet}), \mathbf{r}_{\tau,\varepsilon;t}(X_{\bullet}), \mathbf{z}_{\tau,\varepsilon;t}(X_{\bullet}), \mathbf{W}_{\tau,\varepsilon;t}(X_{\bullet})), \quad t \in (0,1],$$

such that a fixed point of  $\mathbf{X}_{\tau,\varepsilon;\bullet}$  coincides with a solution of the system of equations (2.24), (2.28), (2.20), (2.21), (2.22). In particular, for all  $\tau, \varepsilon \in [0,1]$  it holds  $\mathbf{g}_{\tau,\varepsilon;1}(X_{\bullet}) = 1/\lambda$  and  $\mathbf{r}_{\tau,\varepsilon;1}(X_{\bullet}) = 0$ . We prove that there exists  $\lambda_{\star} \in (0,1]$  such that for all  $\lambda \in (0, \lambda_{\star}]$  and  $\tau, \varepsilon \in [0,1]$  the map  $\mathbf{X}_{\tau,\varepsilon;\bullet} : \mathcal{Y}_{\tau,\varepsilon} \rightarrow \mathcal{Y}_{\tau,\varepsilon}$  is well-defined and is a contraction. We denote by  $X_{\tau,\varepsilon;\bullet}$  the fixed point of  $\mathbf{X}_{\tau,\varepsilon;\bullet}$ . We omit  $\tau$  and  $\varepsilon$  if  $\tau = 0$  and  $\varepsilon = 0$ . In order to establish convergence of the Schwinger functions as  $\tau, \varepsilon \searrow 0$  we have to control  $X_{\bullet} - X_{\tau,\varepsilon;\bullet}$ . To this end, we introduce a certain norm  $\|\bullet\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}}$  in  $\mathcal{X} \cup \mathcal{X}_{\tau,\varepsilon}$  weaker than  $\|\bullet\|_{\mathcal{X}} \vee \|\bullet\|_{\mathcal{X}_{\tau,\varepsilon}}$  and prove that

$$\lim_{\tau,\varepsilon \searrow 0} \|X_{\bullet} - X_{\tau,\varepsilon;\bullet}\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}} = 0. \quad (2.39)$$

The norm  $\|(g_\bullet, r_\bullet, z_\bullet, W_\bullet)\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}}$  is defined in terms of a norm  $\|W_\bullet\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{2,3;2-80\kappa}}$  that has a form similar to the norm  $\|W_\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{2,3;2-80\kappa}}$  but involves a different weight. The use of a different weight having some decay at infinity is crucial for the existence of the infinite volume limit  $\tau \searrow 0$ . Let us also mention that the norm  $\|\bullet\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{2,3;2-80\kappa}}$  is stronger than a certain norm  $\|\bullet\|_{\tilde{\mathcal{Y}}^{2,3;2-80\kappa}}$  independent of  $\tau, \varepsilon \in (0, 1]$  whose definition is very similar to the definition of the norm  $\|\bullet\|_{\mathcal{Y}^{2,3;2-80\kappa}}$  but involves a different weight.

Finally, let us discuss the convergence of the Schwinger functions as  $\tau, \varepsilon \searrow 0$ . To this end, first recall that the fixed point  $X_{\tau,\varepsilon;\bullet} = (g_{\tau,\varepsilon;\bullet}, r_{\tau,\varepsilon;\bullet}, z_{\tau,\varepsilon;\bullet}, W_{\tau,\varepsilon;\bullet})$  of the map  $\mathbf{X}_{\tau,\varepsilon;\bullet}$  is related to a solution  $V_{\tau,\varepsilon;\bullet} = U(1/g_{\tau,\varepsilon;\bullet}, r_{\tau,\varepsilon;\bullet}, z_{\tau,\varepsilon;\bullet}) + W_{\tau,\varepsilon;\bullet}$  of the flow equation (2.11) and  $V_{\tau,\varepsilon;\bullet} = (V_{\tau,\varepsilon;\bullet}^{m,a,\sigma})_{m \in \mathbb{N}_+, a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m}$  is related by Eqs. (2.6), (2.7) and (2.9) to a solution  $U_{\tau,\varepsilon;\bullet}$  of the flow equation (2.5). We conclude that the functional  $U_{\tau,\varepsilon;t}$  is an effective potential at the scale  $t \in (0, 1]$ , that is  $U_{\tau,\varepsilon;t}$  satisfies Eq. (2.3), and  $U_{\tau,\varepsilon;t=1}$  is related by Eq. (2.4) to the generating functional of the Schwinger functions. The convergence of the Schwinger functions as  $\tau, \varepsilon \searrow 0$  follows now from the existence of the limit (2.39).

**Lemma 2.4.** *For all  $\rho, \varrho \in \mathbb{R}$  and  $\eta \in (0, 1)$  there exists  $C \in (0, \infty)$  such that the following statements are true for all  $\lambda \in (0, 1]$  and  $t \in (0, 1]$ .*

- (A) *If  $\rho \geq 0$  and  $\varrho > 0$ , then  $\int_0^t \lambda_s^\rho s^{\varrho-1} ds \leq \lambda_t^\rho t^\varrho / \varrho$ .*
- (B) *If  $\rho \geq 0$  and  $\varrho > 0$ , then  $\int_0^t \lambda_s^\rho s^{\varrho-1} (1 - s/t)^{-\eta} ds \leq C \lambda_t^\rho t^\varrho$ .*
- (C) *If  $\rho \geq 0$  and  $\varrho < -\beta_2 \rho \lambda$ , then  $\int_t^1 \lambda_s^\rho s^{\varrho-1} ds \leq C \lambda_t^\rho t^\varrho$ .*
- (D) *If  $\rho > 0$ , then  $\int_0^t \lambda_s^{\rho+1} s^{-1} ds \leq C \lambda_t^\rho$ .*
- (E) *If  $\rho < 0$ , then  $\int_t^1 \lambda_s^{\rho+1} s^{-1} ds \leq C \lambda_t^\rho$ .*

*Remark 2.5.* Recall that  $\lambda_t$  and  $\beta_2$  were introduced in Def. 2.2.

*Proof.* For  $\rho \in [0, \infty)$  and  $\varrho \in (0, \infty)$  we have

$$\int_0^t \lambda_s^\rho s^{\varrho-1} ds \leq \lambda_t^\rho \int_0^t s^{\varrho-1} ds = \lambda_t^\rho t^\varrho / \varrho$$

as well as

$$\int_0^t \lambda_s^\rho s^{\varrho-1} (1 - s/t)^{-\eta} ds \leq \lambda_t^\rho \int_0^t s^{\varrho-1} (1 - s/t)^{-\eta} ds \leq \lambda_t^\rho t^\varrho \int_0^1 s^{\varrho-1} (1 - s)^{-\eta} ds.$$

This proves the bounds (A) and (B). To prove the bound (C) we use the fact that for  $\rho \in [0, \infty)$  and  $\varrho \in (-\infty, -\beta_2 \rho \lambda)$  the following estimates

$$\begin{aligned} \int_t^1 \lambda_s^\rho s^{\varrho-1} ds &\leq (-\varrho - \beta_2 \rho \lambda)^{-1} \int_t^1 (-\varrho - \beta_2 \rho \lambda_s) \lambda_s^\rho s^{\varrho-1} ds \\ &\leq (-\varrho - \beta_2 \rho \lambda)^{-1} \int_t^1 \partial_s (-\lambda_s^\rho s^\varrho) ds \leq \lambda_t^\rho t^\varrho / (-\varrho - \beta_2 \rho \lambda) \end{aligned}$$

hold true. The bound (D) is a consequence of the equalities

$$\int_0^t \lambda_s^{\rho+1} s^{-1} ds = \beta_2^{-1} \int_0^t \lambda_s^{\rho-1} d\lambda_s = \lambda_t^\rho / (\beta_2 \rho)$$

valid for  $\rho \in (0, \infty)$  and the bound (E) is a consequence of the estimates

$$\int_t^1 \lambda_s^{\rho+1} s^{-1} ds = \beta_2^{-1} \int_t^1 \lambda_s^{\rho-1} d\lambda_s \leq \lambda_t^\rho / (-\beta_2 \rho)$$

valid for  $\rho \in (-\infty, 0)$ . This finishes the proof.  $\square$

The rest of this article is organised as follows. In Sec. 3 we define the spaces of distributions and functionals on a torus and introduce the notation used in the paper. Sec. 4 contains the definition of the Berezin integral and the Grassmann measure  $\mu_{\tau,\varepsilon}$  of the Gross-Neveu model with cutoffs. In Sec. 5 we define the scale decomposition  $G_{\tau,\varepsilon;t}$  of the propagator  $G_{\tau,\varepsilon}$  and discuss its properties. In Sec. 6 we introduce the filtered non-commutative probability space  $(\mathcal{F}, \mathbf{E})$  of the spacetime white noise  $\xi$  and define the conditional expected value  $\mathbf{E}_t$  as well as the scale decomposition of the free field  $\Psi_{\tau,\varepsilon;t}$ . In Sec. 7 we present the equation for an effective potential and examine its relation with the Polchinski equation. In Sec. 8 we discuss the symmetries of the Gross-Neveu model. In Sec. 9 we define the weights. Sec. 10 contains the definitions of the spaces  $\mathcal{V}^{m;\gamma}$ ,  $\mathcal{V}^{\alpha,\beta;\gamma}$ ,  $\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}$  of kernels and functionals in infinite volume. In Sec. 11 we introduce the decomposition of kernels into the local part and the remainder and discuss its properties. In Sec. 12 we establish bounds for the maps  $\mathbf{A}_{\tau,\varepsilon;t,s}$  and  $\mathbf{B}_{\varepsilon;t}$ . In Sec. 13 we introduce the space  $\mathcal{X}_{\tau,\varepsilon}$  and the map  $\mathbf{X}_{\tau,\varepsilon}$  and prove that the map  $\mathbf{X}_{\tau,\varepsilon}$  is a contraction. In Sec. 14 we prove that the fixed point of the map  $\mathbf{X}_{\tau,\varepsilon}$  yields a solution of the Polchinski equation. In Sec. 15 we show that the effective potential at unit scale is directly related to the generating functional of the Schwinger functions and prove Theorem 1.1 stated in the introduction. The core of the proof of the main result is contained in Sec. 13, 14 and 15.

### 3 Distributions and functionals on torus

In this section we introduce the basic notation we use throughout the paper. In particular, we define the spaces of distributions and functionals on a torus.

**Definition 3.1.** For  $m \in \mathbb{N}_+$  we denote by  $\mathcal{S}(\mathbb{R}^{2m})$  and  $\mathcal{S}'(\mathbb{R}^{2m})$  the space of Schwartz functions and distributions, respectively. We denote by  $\langle V, \phi \rangle \in \mathbb{C}$  the standard pairing between a distribution  $V \in \mathcal{S}'(\mathbb{R}^{2m})$  and a test function  $\phi \in \mathcal{S}(\mathbb{R}^{2m})$ . By  $*$  we mean the convolution in  $\mathcal{S}'(\mathbb{R}^{2m})$ . We use the following convention for the Fourier transform

$$(\mathbf{F}f)(p) := \int_{\mathbb{R}^{2m}} f(x) e^{-ip \cdot x} dx, \quad f(x) = \frac{1}{(2\pi)^{2m}} \int_{\mathbb{R}^{2m}} (\mathbf{F}f)(p) e^{ip \cdot x} dp.$$



**Definition 3.2.** Let  $\tau \in (0, 1]$ . We define  $\mathbb{T}_\tau^2 := (\mathbb{R}/\tau^{-1}\mathbb{Z})^2$ . Occasionally, we identify  $\mathbb{T}_\tau^2$  with the set  $(-1/(2\tau), 1/(2\tau))^2 \subset \mathbb{R}^2$ . For  $m \in \mathbb{N}_+$  we denote by  $\mathcal{S}'(\mathbb{T}_\tau^{2m})$  the subspace of  $\mathcal{S}'(\mathbb{R}^{2m})$  consisting of distributions that are  $1/\tau$  periodic. Given  $\varphi \in L^1(\mathbb{R}^2)$  we define  $\mathbf{P}_\tau \varphi \in L^1(\mathbb{T}_\tau^2)$  by the equality

$$\mathbf{P}_\tau \varphi = \sum_{n \in \mathbb{Z}^2} \varphi(\bullet + n/\tau).$$

For  $V \in \mathcal{S}'(\mathbb{T}_\tau^{2m})$  and  $\phi \in C^\infty(\mathbb{T}_\tau^{2m})$  we write

$$\langle V, \phi \rangle_\tau := \langle V, \chi_\tau^{\otimes m} \phi \rangle \in \mathbb{C},$$

where  $\chi_\tau \in C_c^\infty(\mathbb{R}^2)$  is such that  $\mathbf{P}_\tau \chi_\tau = 1$ . By  $*_\tau$  we mean the convolution in  $\mathcal{S}'(\mathbb{T}_\tau^{2m})$ . We use the following convention for the Fourier transform

$$(\mathbf{F}_\tau f)(p) := \int_{\mathbb{T}_\tau^{2m}} f(x) e^{-ip \cdot x} dx, \quad f(x) = \tau^{2m} \sum_{p \in (2\pi\tau\mathbb{Z})^{2m}} (\mathbf{F}_\tau f)(p) e^{ip \cdot x} dp.$$

*Remark 3.3.* Using periodicity of  $V \in \mathcal{S}'(\mathbb{T}_\tau^{2m})$  and  $\phi \in C^\infty(\mathbb{T}_\tau^{2m})$  one shows that the bracket  $\langle V, \phi \rangle_\tau$  does not depend on the choice of  $\chi_\tau$ .

*Remark 3.4.* Note that  $G * \phi = \mathbf{P}_\tau G *_\tau \phi \in C(\mathbb{T}_\tau^2)$  for all  $\tau \in (0, 1]$ ,  $G \in L^1(\mathbb{R}^2)$ ,  $\phi \in C(\mathbb{T}_\tau^2)$ .

*Remark 3.5.* For  $\tau = 0$  we identify  $\mathbb{T}_\tau^2$  with  $\mathbb{R}^2$ .

**Definition 3.6.** Let  $\mathbb{K}$  be a finite set and let  $\mathcal{T}$  be a topological space. We denote by  $\mathcal{T}^\mathbb{K}$  the set of maps  $\mathbb{K} \rightarrow \mathcal{T}$ . We identify elements  $V$  of  $\mathcal{T}^\mathbb{K}$  with tuples  $(V^k)_{k \in \mathbb{K}}$  and equip  $\mathcal{T}^\mathbb{K}$  with the product topology.

**Definition 3.7.** Fix  $\tau \in (0, 1]$  and a finite set  $\mathbb{K}$ . For  $m \in \mathbb{N}_+$ ,  $U \equiv (U^k)_{k \in \mathbb{K}^m} \in \mathcal{S}'(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m}$  and  $\varphi \equiv (\varphi^k)_{k \in \mathbb{K}^m} \in C^\infty(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m}$  we write  $\langle U, \varphi \rangle_\tau := \sum_{k \in \mathbb{K}^m} \langle U^k, \varphi^k \rangle_\tau$ . For  $m, n \in \mathbb{N}_+$ ,  $\varphi \in C^\infty(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m}$  and  $\psi \in C^\infty(\mathbb{T}_\tau^{2n})^{\mathbb{K}^n}$  we define  $\varphi \otimes \psi \in C^\infty(\mathbb{T}_\tau^{2(m+n)})^{\mathbb{K}^{m+n}}$  by the equality

$$(\varphi \otimes \psi)^{(k,l)} := \varphi^k \otimes \psi^l, \quad k \in \mathbb{K}^m, l \in \mathbb{K}^n.$$

For  $G \in L^1(\mathbb{R}^2)^{\mathbb{K}^2}$  and  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{K}$  we define  $G * \phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{K}$  by the equality

$$(G * \phi)^k = \sum_{l \in \mathbb{K}} G^{k,l} * \phi^l.$$

We define  $*_\tau$  in an analogous way.

**Definition 3.8.** Let  $\tau \in (0, 1]$ ,  $m \in \mathbb{N}_+$  and  $\mathbb{K}$  be a finite set. We say that  $U \in \mathcal{S}'(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m}$  is antisymmetric iff

$$\langle U, \varphi_1 \otimes \dots \otimes \varphi_m \rangle_\tau = (-1)^{\text{sgn}(\pi)} \langle U, \varphi_{\pi(1)} \otimes \dots \otimes \varphi_{\pi(m)} \rangle_\tau$$

for all  $\varphi_1, \dots, \varphi_m \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{K}$  and all permutations  $\pi \in \mathcal{P}_m$ .

**Definition 3.9** (Grassmann algebra). Let  $E$  be a vector space over  $\mathbb{C}$ . By definition the Grassmann algebra  $\mathcal{G}(E)$  is the exterior algebra of  $E$ . We define a unique grading  $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{G}^-$  such that  $E \subset \mathcal{G}^-$  and such that the exterior product satisfies the conditions  $\mathcal{G}^\pm \mathcal{G}^\pm \subset \mathcal{G}^+$  and  $\mathcal{G}^\pm \mathcal{G}^\mp \subset \mathcal{G}^-$ . We say that an algebra  $\mathcal{A}$  is a Grassmann algebra if  $\mathcal{A} = \mathcal{G}(E)$  for some vector space  $E$ . A Grassmann algebra  $\mathcal{G}(E)$  is finite/infinite dimensional if  $E$  is finite/infinite dimensional.

*Remark 3.10.* In what follows, the symbol  $\otimes_{\text{alg}}$  denotes the algebraic tensor product of vector spaces (the linear span of the set of simple tensors).

**Definition 3.11.** Let  $\tau \in (0, 1]$ ,  $\mathcal{G}$  be a Grassmann algebra and  $\mathbb{K}$  be a finite set. For  $m \in \mathbb{N}_+$ ,  $U \in \mathcal{S}'(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m} \otimes_{\text{alg}} \mathcal{G}$  and  $\phi \in C^\infty(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m} \otimes_{\text{alg}} \mathcal{G}$  we define  $\langle U, \phi \rangle_\tau \in \mathcal{G}$  by the equality

$$\langle U, \phi \rangle_\tau := \sum_{i=1}^k \sum_{j=1}^l \langle U_i, \phi_j \rangle_\tau g_i h_j \in \mathcal{G}, \quad U = \sum_{i=1}^k U_i \otimes g_i, \quad \phi = \sum_{j=1}^l \phi_j \otimes h_j,$$

for all  $k, l \in \mathbb{N}_+$ ,  $U_1, \dots, U_k \in \mathcal{S}'(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m}$ ,  $g_1, \dots, g_k \in \mathcal{G}$  and  $\phi_1, \dots, \phi_l \in C^\infty(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m}$ ,  $h_1, \dots, h_l \in \mathcal{G}$ . For  $m, n \in \mathbb{N}_+$  and  $\varphi \in C^\infty(\mathbb{T}_\tau^{2n})^{\mathbb{K}^n} \otimes_{\text{alg}} \mathcal{G}$ ,  $\psi \in C^\infty(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m} \otimes_{\text{alg}} \mathcal{G}$  we define  $\varphi \otimes \psi \in C^\infty(\mathbb{T}_\tau^{2(n+m)})^{\mathbb{K}^{n+m}} \otimes_{\text{alg}} \mathcal{G}$  by the equality

$$\varphi \otimes \psi := \sum_{i=1}^k \sum_{j=1}^l (\varphi_i \otimes \psi_j) \otimes g_i h_j, \quad \varphi = \sum_{i=1}^k \varphi_i \otimes g_i, \quad \psi = \sum_{j=1}^l \psi_j \otimes h_j,$$

for all  $k, l \in \mathbb{N}_+$ ,  $\varphi_1, \dots, \varphi_k \in C^\infty(\mathbb{T}_\tau^{2n})^{\mathbb{K}^n}$ ,  $g_1, \dots, g_k \in \mathcal{G}$  and  $\psi_1, \dots, \psi_l \in C^\infty(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m}$ ,  $h_1, \dots, h_l \in \mathcal{G}$ .

*Remark 3.12.* The bracket  $\langle U, \phi \rangle_\tau$  and the tensor product  $\varphi \otimes \psi$  do not depend on the choice of a representations of  $U, \phi, \varphi, \psi$  as sums of simple tensors.

*Remark 3.13.* Let  $m \in \mathbb{N}_+$  and  $E$  be a vector space of dimension grater or equal to  $m$ . An antisymmetric distribution  $U \in \mathcal{S}'(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m}$  is uniquely determined by the map

$$C^\infty(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m} \otimes_{\text{alg}} \mathcal{G}^- \ni \varphi \mapsto \langle U, \varphi^{\otimes m} \rangle_\tau \in \mathcal{G}, \quad (3.1)$$

where  $\mathcal{G} := \mathcal{G}(E)$ . Indeed, given  $\psi_1, \dots, \psi_m \in C^\infty(\mathbb{T}_\tau^2)^{\mathbb{K}}$  we choose  $\varphi = \sum_{j=1}^m \psi_j \otimes g_j$ , where  $g_1, \dots, g_m \in \mathcal{G}$  are such that  $g = g_1 \dots g_m \neq 0$ . Then  $m! \langle U, \psi_1 \otimes \dots \otimes \psi_m \rangle_\tau g = \langle U, \varphi^{\otimes m} \rangle_\tau$ . Note also that if  $\mathcal{G}$  is an infinite-dimensional Grassmann algebra, then for every  $m \in \mathbb{N}_+$  an antisymmetric distribution  $U \in \mathcal{S}'(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m}$  is uniquely determined by the map (3.1).

**Definition 3.14.** Let  $\tau \in (0, 1]$  and  $\mathbb{K}$  be a finite set. We call a functional a collection  $U = (U^m)_{m \in \mathbb{N}_0}$  such that  $U^0 \in \mathbb{C}$  and  $U^m \in \mathcal{S}'(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m}$  is antisymmetric for  $m \in \mathbb{N}_+$ . We denote the vector space of functionals by

$$\mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^{\mathbb{K}}) \subset \mathbb{C} \times \bigtimes_{m \in \mathbb{N}_+} \mathcal{S}'(\mathbb{T}_\tau^{2m})^{\mathbb{K}^m}$$

and endow it with the product topology. Let  $\mathcal{G}$  be an infinite-dimensional Grassmann algebra. For  $U \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{K})$  and  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{K} \otimes_{\text{alg}} \mathcal{G}^-$  we define

$$U(\phi) := U^0 + \sum_{m \in \mathbb{N}_+} \langle U^m, \phi^{\otimes m} \rangle_\tau \in \mathcal{G}. \quad (3.2)$$

For  $U \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{K})$ , an entire function  $f \in \mathbb{C} \rightarrow \mathbb{C}$  and  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{K} \otimes_{\text{alg}} \mathcal{G}^-$  we define  $f(U(\phi))$  by

$$f(U(\phi)) := \sum_{n \in \mathbb{N}_0} a_n U(\phi)^n, \quad f(z) = \sum_{n \in \mathbb{N}_0} a_n z^n, \quad z \in \mathbb{C}, \quad a_n \in \mathbb{C}, \quad n \in \mathbb{N}_0.$$

For  $k \in \mathbb{N}_0$ ,  $U \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{K})$  and  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{K} \otimes_{\text{alg}} \mathcal{G}^-$  the antisymmetric distribution

$$D_\phi^k U(\phi) \in \mathcal{S}'(\mathbb{T}_\tau^2)^{\mathbb{K}^k} \otimes_{\text{alg}} \mathcal{G}$$

is uniquely defined by the equality

$$\langle D_\phi^k U(\phi), \psi^{\otimes k} \rangle_\tau := \partial_u^k U(\phi + u\psi) \Big|_{u=0}$$

for all  $\phi, \psi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{K} \otimes_{\text{alg}} \mathcal{G}^-$ .

*Remark 3.15.* The assumption that  $\mathcal{G}$  is an infinite-dimensional Grassmann algebra is only needed to ensure that for all  $k \in \mathbb{N}_0$  the antisymmetric distribution  $D_\phi^k U(\phi)$  is uniquely defined, cf. Remark 3.13. Moreover, observe that given  $U \equiv (U^m)_{m \in \mathbb{N}_0} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{K})$  for all  $k \in \mathbb{N}_0$  it holds  $D_\phi^k U(\phi)|_{\phi=0} = k! U^k$ .

*Remark 3.16.* Note that for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{K} \otimes_{\text{alg}} \mathcal{G}^-$  there are only finitely many non-zero terms in the series on the RHS of Eq. (3.2) and in the series defining  $f(U(\phi))$ . Indeed, if  $\phi = \sum_{i=1}^n \phi_i \otimes g_i \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{K} \otimes_{\text{alg}} \mathcal{G}^-$ , then  $\phi^{\otimes m} = 0$  and  $U(\phi)^m = 0$  for  $m > n$ . Note that the unique functional  $W = (W^m)_{m \in \mathbb{N}_0} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{K})$  such that  $f(U(\phi)) = W(\phi)$  for  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{K} \otimes_{\text{alg}} \mathcal{G}^-$  is defined by the equations  $W^k := D_\phi^k f(U(\phi))|_{\phi=0}$  for all  $k \in \mathbb{N}_+$ .

*Remark 3.17.* As an example, let us consider a functional  $U \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2))$  of the form  $U(\phi) = \phi(x_1)\phi(x_2)$  for some fixed  $x_1, x_2 \in \mathbb{R}^2$ . Then  $D_\phi U(\phi) = -\phi(x_2)\delta_{x_1} + \phi(x_1)\delta_{x_2}$  and  $D_\phi^2 U(\phi) = \delta_{x_1} \otimes \delta_{x_2} - \delta_{x_2} \otimes \delta_{x_1}$ , where  $\delta_x \in \mathcal{S}'(\mathbb{T}_\tau^2)$  is the periodization of the Dirac delta at  $x \in \mathbb{R}^2$ . Furthermore, we have  $\langle D_\phi U(\phi), \psi \rangle_\tau = \psi(x_1)\phi(x_2) + \phi(x_1)\psi(x_2)$  as well as  $\langle D_\phi^2 U(\phi), \psi^{\otimes 2} \rangle_\tau = 2\psi(x_1)\psi(x_2)$ .

## 4 Gross-Neveu model with cutoffs

In this section we introduce the Grassmann algebra of the Gross-Neveu model and use it to define the Berezin integral and the Grassmann interacting measure  $\mu_{\tau, \varepsilon}$  of the Gross-Neveu model with cutoffs  $\tau, \varepsilon \in (0, 1]$ . We also state the conditions for the bump function  $\vartheta_\varepsilon$  used to implement the UV cutoff.

**Definition 4.1.** Let  $\mathbb{F} := \{1, \dots, N\} \times \{1, 2\}$ ,  $\mathbb{G} := \{-, +\} \times \mathbb{F}$ ,  $\mathbb{A} := \{0, 1, 2\}^2$ . We call  $\varsigma \in \mathbb{F}$  and  $\sigma \in \mathbb{G}$  spinor indices and  $a \in \mathbb{A}$  a spatial multi-index.

**Definition 4.2.** The so-called gamma matrices are defined by

$$\gamma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Gamma_j \equiv (\Gamma_j^{\varsigma_1, \varsigma_2})_{\varsigma_1, \varsigma_2 \in \mathbb{F}} := (\gamma_j)^{\oplus N}, \quad j \in \{1, 2\}.$$

*Remark 4.3.* Note that  $\gamma_1, \gamma_2$  are complex  $2 \times 2$  matrices and  $\Gamma_1, \Gamma_2$  are complex block diagonal  $2N \times 2N$  matrices. It holds

$$\gamma_2^t \gamma_2 = -1, \quad \gamma_2^t \gamma_1 \gamma_2 = \gamma_1^t, \quad \gamma_2^t \gamma_2 \gamma_2 = \gamma_2^t,$$

where  $A^t$  denotes the transposition of the matrix  $A$ . The matrices  $\Gamma_1, \Gamma_2$  satisfy the same identities.

**Definition 4.4.** For  $\phi = (\phi^\varsigma)_{\varsigma \in \mathbb{F}} \in \mathbb{C}^{\mathbb{F}}$  and  $\psi = (\psi^\varsigma)_{\varsigma \in \mathbb{F}} \in \mathbb{C}^{\mathbb{F}}$  we write  $\phi \cdot \psi = \sum_{\varsigma \in \mathbb{F}} \phi^\varsigma \psi^\varsigma$ .

**Definition 4.5.** Let  $\tau, \varepsilon \in (0, 1]$ ,  $\omega(p) := (|p|^2 + 1)^{1/2}$ ,  $\Lambda_{\tau, \varepsilon} := \{p \in (2\pi\tau\mathbb{Z})^2 \mid \varepsilon\omega(p) \leq 4\}$  and  $\mathcal{C}_{\tau, \varepsilon} := (\text{Span}\{x \mapsto e^{ip \cdot x} \mid p \in \Lambda_{\tau, \varepsilon}\})^{\mathbb{G}} \subset C^\infty(\mathbb{T}_\tau^2)^{\mathbb{G}}$ . The Grassmann algebra of the Gross-Neveu model  $\mathcal{G}_{\tau, \varepsilon}$  is the unital complex algebra whose generators

$$\{(\mathbf{F}_\tau \psi_{\tau, \varepsilon}^\sigma)(p) \mid p \in \Lambda_{\tau, \varepsilon}\} \quad (4.1)$$

satisfy the conditions

$$(\mathbf{F}_\tau \psi_{\tau, \varepsilon}^{\sigma_1})(p_1) (\mathbf{F}_\tau \psi_{\tau, \varepsilon}^{\sigma_2})(p_2) + (\mathbf{F}_\tau \psi_{\tau, \varepsilon}^{\sigma_2})(p_2) (\mathbf{F}_\tau \psi_{\tau, \varepsilon}^{\sigma_1})(p_1) = 0$$

for all  $p_1, p_2 \in \Lambda_{\tau, \varepsilon}$  and  $\sigma_1, \sigma_2 \in \mathbb{G}$ . We define a unique grading  $\mathcal{G}_{\tau, \varepsilon} = \mathcal{G}_{\tau, \varepsilon}^+ \oplus \mathcal{G}_{\tau, \varepsilon}^-$  such that  $(\mathbf{F}_\tau \psi_{\tau, \varepsilon}^\sigma)(p) \in \mathcal{G}_{\tau, \varepsilon}^-$  for all  $p \in \Lambda_{\tau, \varepsilon}$ ,  $\sigma \in \mathbb{G}$  and such that the product in  $\mathcal{G}_{\tau, \varepsilon}$  satisfies the conditions  $\mathcal{G}_{\tau, \varepsilon}^\pm \mathcal{G}_{\tau, \varepsilon}^\pm \subset \mathcal{G}_{\tau, \varepsilon}^+$  and  $\mathcal{G}_{\tau, \varepsilon}^\pm \mathcal{G}_{\tau, \varepsilon}^\mp \subset \mathcal{G}_{\tau, \varepsilon}^-$ . Moreover, we define the free Grassmann field

$$\psi_{\tau, \varepsilon} = (\psi_{\tau, \varepsilon}^\sigma)_{\sigma \in \mathbb{G}} \in \mathcal{C}_{\tau, \varepsilon} \otimes \mathcal{G}_{\tau, \varepsilon}^- \subset C^\infty(\mathbb{T}_\tau^2)^{\mathbb{G}} \otimes \mathcal{G}_{\tau, \varepsilon}^-$$

by the following equality

$$\psi_{\tau, \varepsilon}^\sigma(x) := \tau^2 \sum_{p \in \Lambda_{\tau, \varepsilon}} (\mathbf{F}_\tau \psi_{\tau, \varepsilon}^\sigma)(p) e^{ip \cdot x} dp \quad (4.2)$$

for all  $x \in \mathbb{R}^2$  and  $\sigma \in \mathbb{G}$ . For  $\varsigma \in \mathbb{F}$  we set  $\bar{\psi}_{\tau, \varepsilon}^\varsigma(x) := \psi_{\tau, \varepsilon}^{-, \varsigma}(x)$  and  $\psi_{\tau, \varepsilon}^\varsigma(x) := \psi_{\tau, \varepsilon}^{+, \varsigma}(x)$ .

*Remark 4.6.* For all  $\tau, \varepsilon \in (0, 1]$  it holds  $\mathcal{G}_{\tau, \varepsilon} = \mathcal{G}(E_{\tau, \varepsilon})$ , where the vector space  $E_{\tau, \varepsilon}$  coincides with the span of the set (4.1). In particular,  $\mathcal{G}_{\tau, \varepsilon}$  is a finite-dimensional Grassmann algebra in the sense of Def. 3.9.

*Remark 4.7.* Observe that for every  $\tau, \varepsilon \in (0, 1]$  it holds  $\psi_{\tau, \varepsilon}^{\otimes m} = 0$  for all  $m \in \mathbb{N}_+$  provided  $m > |\mathbb{G}| |\Lambda_{\tau, \varepsilon}|$ , where  $|S|$  denotes the number of elements of a set  $S$ .

**Definition 4.8.** Let  $\tau, \varepsilon \in (0, 1]$ . The Berezin integral is a linear map

$$\mathcal{G}_{\tau, \varepsilon} \ni g \mapsto \int g \, d\psi_{\tau, \varepsilon} \in \mathbb{C}$$

such that

$$\int g \, d\psi_{\tau, \varepsilon} = 1, \quad g = \prod_{p \in \Lambda_{\tau, \varepsilon}} \prod_{\varsigma \in \mathbb{F}} (\mathbf{F}_{\tau} \bar{\psi}_{\tau, \varepsilon}^{\varsigma})(p) (\mathbf{F}_{\tau} \psi_{\tau, \varepsilon}^{\varsigma})(p), \quad (4.3)$$

and such that it vanishes on any other monomial in the generators of  $\mathcal{G}_{\tau, \varepsilon}$ .

*Remark 4.9.* Note that the order in the product in Eq. (4.3) is unimportant because the elements of the set  $\{(\mathbf{F}_{\tau} \bar{\psi}_{\tau, \varepsilon}^{\varsigma})(p) (\mathbf{F}_{\tau} \psi_{\tau, \varepsilon}^{\varsigma})(p) \mid p \in \Lambda_{\tau, \varepsilon}, \varsigma \in \mathbb{F}\}$  commute. For a discussion of properties of the Berezin integral see for example [CSS13, Appendix A].

*Remark 4.10.* Let  $\mathcal{G}_1, \mathcal{G}_2$  be Grassmann algebras. We denote by  $\mathcal{G}_1 \otimes_{\text{alg}} \mathcal{G}_2$  the algebraic graded tensor product of graded algebras  $\mathcal{G}_1$  and  $\mathcal{G}_2$ . Note that  $\mathcal{G}_1 \otimes_{\text{alg}} \mathcal{G}_2$  is a Grassmann algebra. Given  $g_1 \in \mathcal{G}_1$  and  $g_2 \in \mathcal{G}_2$  we define  $g_1 + g_2 := g_1 \otimes 1 + 1 \otimes g_2 \in \mathcal{G}_1 \otimes_{\text{alg}} \mathcal{G}_2$ .

*Remark 4.11.* Let  $\mathcal{G}$  be a Grassmann algebra. We define the Berezin integral on  $\mathcal{G}_{\tau, \varepsilon} \otimes_{\text{alg}} \mathcal{G}$  as a linear map

$$\mathcal{G}_{\tau, \varepsilon} \otimes_{\text{alg}} \mathcal{G} \ni g \mapsto \int g \, d\psi_{\tau, \varepsilon} \in \mathcal{G}$$

such that

$$\int (g \otimes h) \, d\psi_{\tau, \varepsilon} := \left( \int g \, d\psi_{\tau, \varepsilon} \right) h$$

for all  $g \in \mathcal{G}_{\tau, \varepsilon}$  and  $h \in \mathcal{G}$ .

**Lemma 4.12.** Let  $\tau, \varepsilon \in (0, 1]$  and  $\mathcal{G}$  be a Grassmann algebra. For all  $F \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{K})$  and  $\phi \in \mathcal{C}_{\tau, \varepsilon} \otimes \mathcal{G}^- \subset C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes \mathcal{G}^-$  it holds

$$\int F(\psi_{\tau, \varepsilon} + \phi) \, d\psi_{\tau, \varepsilon} = \int F(\psi_{\tau, \varepsilon}) \, d\psi_{\tau, \varepsilon},$$

where  $\psi_{\tau, \varepsilon} + \phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes (\mathcal{G}_{\tau, \varepsilon} \otimes \mathcal{G})^-$ .

*Proof.* The lemma follows from the invariance under translation of the Berezin integral, cf. [CSS13, Proposition A.12].  $\square$

**Definition 4.13.** Let  $\zeta_b \in (4/5, 1)$  and  $\vartheta \in C^\infty(\mathbb{R})$  be a function belonging to the Gevrey class of order  $1/\zeta_b$  such that  $\vartheta(t) = 1$  for  $t \in (-\infty, 1/2]$  and  $\vartheta(t) = 0$  for  $t \in [1, \infty)$ . For  $\varepsilon \in (0, 1]$  we define  $\vartheta_\varepsilon \in C^\infty(\mathbb{R}^2)$  by  $\mathbf{F}\vartheta_\varepsilon(p) = \vartheta(2\varepsilon\omega(p))^{1/2}$ .

*Remark 4.14.* Recall that the class of Gevrey functions of order  $s \in [1, \infty)$  on  $\mathbb{R}^d$  consists of  $\varphi \in C^\infty(\mathbb{R}^d)$  such that for some  $C \in (0, \infty)$  the bound

$$\sup_{p \in \mathbb{R}^d} |\partial^a \varphi(p)| \leq C^{1+|a|} (a!)^s$$

is satisfied for all  $a \in \mathbb{N}_0^d$ .

*Remark 4.15.* For an explicit construction of a function  $\vartheta$  satisfying the conditions stated in the above definition see [GMR21, App. A.1].

**Definition 4.16.** The free action functional  $A_\tau \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  and the interaction term  $U_{\tau,\varepsilon} \equiv U_\tau(g_{\tau,\varepsilon}, r_{\tau,\varepsilon}) \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  with cutoffs  $\tau, \varepsilon \in (0, 1]$  are defined uniquely by Eqs. (1.1) and (1.2) for all  $\psi \equiv (\bar{\psi}, \psi) \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^- \equiv (C^\infty(\mathbb{T}_\tau^2)^\mathbb{F} \times C^\infty(\mathbb{T}_\tau^2)^\mathbb{F}) \otimes_{\text{alg}} \mathcal{G}^-$ , where  $\mathcal{G}$  is an infinite-dimensional Grassmann algebra. The free and interacting measures with cutoffs  $\tau, \varepsilon \in (0, 1]$  are defined with the use of the Berezin integral introduced in Def. 4.8 by the equations

$$\nu_{\tau,\varepsilon}(F) := \frac{\int F(\vartheta_\varepsilon * \psi_{\tau,\varepsilon}) \exp(-A_\tau(\psi_{\tau,\varepsilon})) d\psi_{\tau,\varepsilon}}{\int \exp(-A_\tau(\psi_{\tau,\varepsilon})) d\psi_{\tau,\varepsilon}}$$

and

$$\mu_{\tau,\varepsilon}(F) := \frac{\int F(\vartheta_\varepsilon * \psi_{\tau,\varepsilon}) \exp(-A_\tau(\psi_{\tau,\varepsilon}) + U_{\tau,\varepsilon}(\vartheta_\varepsilon * \psi_{\tau,\varepsilon})) d\psi_{\tau,\varepsilon}}{\int \exp(-A_\tau(\psi_{\tau,\varepsilon}) + U_{\tau,\varepsilon}(\vartheta_\varepsilon * \psi_{\tau,\varepsilon})) d\psi_{\tau,\varepsilon}},$$

respectively, for all functionals  $F \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$ , where  $\psi_{\tau,\varepsilon} \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes \mathcal{G}_{\tau,\varepsilon}^-$  is defined by Eq. (4.2).

*Remark 4.17.* Note that by [CSS13, Proposition A.14] it holds

$$\int \exp(-A_\tau(\psi_{\tau,\varepsilon})) d\psi_{\tau,\varepsilon} = \prod_{p \in \Lambda_{\tau,\varepsilon}} \det((-\Xi^{\varsigma_1, \varsigma_2}(p))_{\varsigma_1, \varsigma_2 \in \mathbb{F}}) = \prod_{p \in \Lambda_{\tau,\varepsilon}} (-1)^N (1 + |p|^2)^N \neq 0,$$

where  $\Xi^{\varsigma_1, \varsigma_2}(p) := 1^{\varsigma_1, \varsigma_2} - \sum_{j \in \{1, 2\}} i\Gamma_j^{\varsigma_1, \varsigma_2} p^j$  for  $\varsigma_1, \varsigma_2 \in \mathbb{F}$ . As a result, the free field measure is well-defined for all  $\tau, \varepsilon \in (0, 1]$ . The interacting measure with cutoffs  $\tau, \varepsilon \in (0, 1]$  is well-defined provided

$$\frac{\int \exp(-A_\tau(\psi_{\tau,\varepsilon}) + U_{\tau,\varepsilon}(\vartheta_\varepsilon * \psi_{\tau,\varepsilon})) d\psi_{\tau,\varepsilon}}{\int \exp(-A_\tau(\psi_{\tau,\varepsilon})) d\psi_{\tau,\varepsilon}} = \nu_{\tau,\varepsilon}(\exp(U_{\tau,\varepsilon}(\bullet))) \neq 0.$$

This is proved in Remark 14.4 for a specific choice of parameters  $g_{\tau,\varepsilon}, r_{\tau,\varepsilon}$  fixed as in Theorem 1.1.

## 5 Scale decomposition of propagator

In this section we define the scale decomposition  $G_{\tau,\varepsilon;t}$  of the propagator  $G_{\tau,\varepsilon}$ . We also introduce auxiliary propagators  $G_{\varepsilon;t}^{\pm; \varsigma, \tilde{\varsigma}}$  that will be used in Sec. 6 to construct the scale decomposition  $\Psi_{\tau,\varepsilon;t}$  of the free field that satisfies the conditions discussed in Sec. 2.

**Definition 5.1.** For  $\varsigma_1, \varsigma_2 \in \mathbb{F}$  and  $p \in \mathbb{R}^2$  let  $\Xi^{\varsigma_1, \varsigma_2}(p) := 1^{\varsigma_1, \varsigma_2} - \sum_{j \in \{1, 2\}} i\Gamma_j^{\varsigma_1, \varsigma_2} p^j$  and  $\omega(p) := (|p|^2 + 1)^{1/2}$ . The free field covariance

$$G \equiv (G^\sigma)_{\sigma \in \mathbb{G}^2} \in L^1(\mathbb{R}^2)^{\mathbb{G}^2}$$

is defined by

$$G^\sigma(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} (\mathbf{F}G^\sigma)(p) e^{ip \cdot x} dp,$$

where

$$\begin{aligned} (\mathbf{F}G^{+, \varsigma_1, +, \varsigma_2})(p) &:= 0, & (\mathbf{F}G^{+, \varsigma_1, -, \varsigma_2})(p) &:= \Xi^{\varsigma_1, \varsigma_2}(p)/(\omega(p))^2, \\ (\mathbf{F}G^{-, \varsigma_1, -, \varsigma_2})(p) &:= 0, & (\mathbf{F}G^{-, \varsigma_1, +, \varsigma_2})(p) &:= -(\mathbf{F}G^{+, \varsigma_2, -, \varsigma_1})(-p) \end{aligned}$$

for all  $\varsigma_1, \varsigma_2 \in \mathbb{F}$ . For  $\varepsilon, t \in [0, 1]$  we define

$$G_\varepsilon \equiv (G_\varepsilon^\sigma)_{\sigma \in \mathbb{G}^2}, \quad G_{\varepsilon; t} \equiv (G_{\varepsilon; t}^\sigma)_{\sigma \in \mathbb{G}^2}, \quad \dot{G}_{\varepsilon; t} \equiv (\dot{G}_{\varepsilon; t}^\sigma)_{\sigma \in \mathbb{G}^2} \in L^1(\mathbb{R}^2)^{\mathbb{G}^2},$$

by the equalities

$$(\mathbf{F}G_\varepsilon^\sigma)(p) := \vartheta(2\varepsilon\omega(p)) (\mathbf{F}G^\sigma)(p), \quad (\mathbf{F}G_{\varepsilon; t}^\sigma)(p) := \vartheta(t\omega(p)) (\mathbf{F}G_\varepsilon^\sigma)(p), \quad \dot{G}_{\varepsilon; t}^\sigma := \partial_t G_{\varepsilon; t}^\sigma.$$

For  $\varepsilon \in [0, 1]$ ,  $t \in (0, 1]$  and  $\sigma \in \mathbb{G}^2$ ,  $\varsigma, \tilde{\varsigma} \in \mathbb{F}$  we define  $G_{\varepsilon; t}^{\pm; \varsigma, \tilde{\varsigma}} \in L^1(\mathbb{R}^2)$ , by the equalities

$$(\mathbf{F}G_{\varepsilon; t}^{+, \varsigma, \tilde{\varsigma}})(p) := (\mathbf{F}\dot{G}_{\varepsilon; t}^{+, \varsigma, -, \tilde{\varsigma}})(p), \quad (\mathbf{F}G_{\varepsilon; t}^{-, \varsigma, \tilde{\varsigma}})(p) := \vartheta(\varepsilon/2t) \vartheta(t\omega(p)/2) 1^{\varsigma, \tilde{\varsigma}}.$$

For  $\tau \in (0, 1]$  the periodizations of the kernels  $G^\sigma, G_\varepsilon^\sigma, G_{\varepsilon; t}^\sigma, \dot{G}_{\varepsilon; t}^\sigma, G_{\varepsilon; t}^{\pm; \varsigma, \tilde{\varsigma}}$  are defined with the use of the operator  $\mathbf{P}_\tau$  introduced in Def. 3.2 and are denoted by  $G_\tau^\sigma, G_{\tau, \varepsilon}^\sigma, G_{\tau, \varepsilon; t}^\sigma, \dot{G}_{\tau, \varepsilon; t}^\sigma, G_{\tau, \varepsilon; t}^{\pm; \varsigma, \tilde{\varsigma}}$ , respectively. We use the symbols  $\dot{G}_t^\sigma, G_t^{\pm; \varsigma, \tilde{\varsigma}}$  to denote  $\dot{G}_{\varepsilon; t}^\sigma, G_{\varepsilon; t}^{\pm; \varsigma, \tilde{\varsigma}}$  with  $\varepsilon = 0$ .

*Remark 5.2.* The kernels introduced in the above definition have the following properties.

- (A) If  $t \in (0, \varepsilon]$ , then  $\vartheta(2\varepsilon\omega(p))\vartheta(t\omega(p)) = \vartheta(2\varepsilon\omega(p))$  and  $G_{\varepsilon; t}^\sigma = G_\varepsilon^\sigma, \dot{G}_{\varepsilon; t}^\sigma = 0$ .
- (B) If  $t \in (0, \varepsilon/2]$ , then  $G_{\varepsilon; t}^{\pm; \varsigma, \tilde{\varsigma}} = 0$ . Moreover,  $\text{supp } \mathbf{F}_\tau G_{\tau, \varepsilon; t}^{\pm; \varsigma, \tilde{\varsigma}} \subset \Lambda_{\tau, \varepsilon}$  for all  $t \in (0, 1]$ .
- (C) If  $t \in [4\varepsilon, 1]$ , then  $\vartheta(2\varepsilon\omega(p))\dot{\vartheta}(t\omega(p)) = \dot{\vartheta}(t\omega(p))$  and  $\dot{G}_{\varepsilon; t}^\sigma = \dot{G}_t^\sigma, G_{\varepsilon; t}^{\pm; \varsigma, \tilde{\varsigma}} = G_t^{\pm; \varsigma, \tilde{\varsigma}}$ .
- (D) If  $t \in (0, 1/2)$ , then  $\dot{\vartheta}(t\omega(0)) = 0$  and  $\int_{\mathbb{R}^2} \dot{G}_{\varepsilon; t}^\sigma = 0$ .
- (E) Since  $\vartheta(\varepsilon/2t) \vartheta(t\omega(p)/2) = 1$  on the support of  $p \mapsto \vartheta(2\varepsilon\omega(p)) \dot{\vartheta}(t\omega(p))$ , it holds

$$\dot{G}_{\varepsilon; t}^{+, \varsigma, -, \tilde{\varsigma}} = \sum_{\hat{\varsigma} \in \mathbb{F}} G_{\varepsilon; t}^{+, \varsigma, \hat{\varsigma}} * G_{\varepsilon; t}^{-, \hat{\varsigma}, \tilde{\varsigma}}, \quad \dot{G}_{\tau, \varepsilon; t}^{+, \varsigma, -, \tilde{\varsigma}} = \sum_{\hat{\varsigma} \in \mathbb{F}} G_{\tau, \varepsilon; t}^{+, \varsigma, \hat{\varsigma}} *_\tau G_{\tau, \varepsilon; t}^{-, \hat{\varsigma}, \tilde{\varsigma}}.$$

Note also that  $\dot{G}_{\varepsilon; t}^{+, \varsigma, -, \tilde{\varsigma}}(x) = -\dot{G}_{\varepsilon; t}^{-, \tilde{\varsigma}, +, \varsigma}(-x)$  as well as  $\dot{G}_{\varepsilon; t}^{+, \varsigma, +, \tilde{\varsigma}} = 0$  and  $\dot{G}_{\varepsilon; t}^{-, \varsigma, -, \tilde{\varsigma}} = 0$ .

- (F) Since  $G_{\varepsilon; 1} = 0$  it holds  $G_\varepsilon = -\int_\varepsilon^1 \dot{G}_{\varepsilon; t} dt$  and  $G - G_\varepsilon = -\int_0^{4\varepsilon} (\dot{G}_t - \dot{G}_{\varepsilon; t}) dt$ .

**Lemma 5.3.** *For all  $a \in \mathbb{N}_0^2$  there exist  $\delta \in (0, 1)$  and  $C \in (0, \infty)$  such that for all  $\varepsilon \in [0, 1]$ ,  $t \in (0, 1]$ ,  $\varsigma_1, \varsigma_2 \in \mathbb{F}$  and  $x \in \mathbb{R}^2$  it holds*

$$\exp(\delta t^{-\zeta_b} |x|^{\zeta_b}) |\partial^a G_{\varepsilon; t}^{\pm; \varsigma_1, \varsigma_2}(x)| \leq C t^{-|a|-2},$$

where  $\zeta_b \in (4/5, 1)$  is the parameter introduced in Def. 4.13.

*Proof.* We first note the following identities

$$x^b \partial^a G_{\varepsilon; t}^{+, \varsigma_1, \varsigma_2}(x) = \frac{i^{|a|-|b|}}{(2\pi)^2} \int_{\mathbb{R}^2} \partial_p^b \left( p^a \vartheta(2\varepsilon\omega(p)) \dot{\vartheta}(t\omega(p)) \Xi^{\varsigma_1, \varsigma_2}(p)/\omega(p) \right) e^{ip \cdot x} dp \quad (5.1)$$

and

$$x^b \partial^a G_{\varepsilon; t}^{-, \varsigma_1, \varsigma_2}(x) = \frac{i^{|a|-|b|}}{(2\pi)^2} \int_{\mathbb{R}^2} \partial_p^b \left( p^a \vartheta(\varepsilon/2t) \vartheta(t\omega(p)/2) 1^{\varsigma_1, \varsigma_2} \right) e^{ip \cdot x} dp. \quad (5.2)$$

Note that the composition of a Gevrey function of order  $s \in [1, \infty)$  with an analytic function is a Gevrey function of order  $s \in [1, \infty)$ , cf. [Rod93, Prop. 1.4.6]. Consequently, since the function  $\vartheta$  is flat in the vicinity of the origin the function

$$\mathbb{R} \times \mathbb{R}^2 \ni (\varepsilon, p) \mapsto \vartheta((\varepsilon^2 + p^2)^{1/2}) \in \mathbb{R}$$

belongs to the Gevrey class of order  $1/\zeta_b$ . As a result, there exists  $C \in (0, \infty)$  such that for all  $\varepsilon \in [0, 1]$  and  $a \in \mathbb{N}_0^2$  it holds

$$\sup_{p \in \mathbb{R}^d} |\partial_p^a \vartheta(\varepsilon\omega(p))| = \varepsilon^{|a|} \sup_{p \in \mathbb{R}^d} |\partial_p^a \vartheta((\varepsilon^2 + p^2)^{1/2})| \leq C^{1+|a|} (a!)^{1/\zeta_b} \varepsilon^{|a|}.$$

By a similar argument, there exists  $C \in (0, \infty)$  such that for all  $t \in [0, 1]$  and  $a \in \mathbb{N}_0^2$  it holds

$$\begin{aligned} \sup_{p \in \mathbb{R}^d} |\partial_p^a \vartheta(t\omega(p)/2)| &= t^{|a|} \sup_{p \in \mathbb{R}^d} |\partial_p^a \vartheta((t^2 + p^2)^{1/2}/2)| \leq C^{1+|a|} (a!)^{1/\zeta_b} t^{|a|}, \\ \sup_{p \in \mathbb{R}^d} |\partial_p^a \dot{\vartheta}(t\omega(p))| &= t^{|a|} \sup_{p \in \mathbb{R}^d} |\partial_p^a \dot{\vartheta}((t^2 + p^2)^{1/2})| \leq C^{1+|a|} (a!)^{1/\zeta_b} t^{|a|}. \end{aligned}$$

Since the function  $p \mapsto \Xi^{\varsigma_1, \varsigma_2}(p)/\omega(p) \in \mathbb{R}$  is analytic in a strip of size one around the real axis there exists  $C \in (0, \infty)$  such that for all  $a \in \mathbb{N}_0^2$  it holds

$$\sup_{p \in \mathbb{R}^d} |\partial_p^a (\Xi^{\varsigma_1, \varsigma_2}(p)/\omega(p))| \leq C^{1+|a|} a!.$$

Noting that  $\vartheta(2\varepsilon\omega(p)) \dot{\vartheta}(t\omega(p))$  vanishes unless  $t\omega(p) \in (1/2, 1)$  and  $t > \varepsilon$  and  $\vartheta(t\omega(p)/2)$  vanishes unless  $t\omega(p) \in [0, 2]$  and using the general Leibniz rule we show there exists  $C \in (0, \infty)$  such that for all  $\varepsilon \in [0, 1]$ ,  $t \in (0, 1]$  it holds

$$\partial_p^b \left( p^a \vartheta(2\varepsilon\omega(p)) \dot{\vartheta}(t\omega(p)) \Xi^{\varsigma_1, \varsigma_2}(p)/\omega(p) \right) \leq C^{1+|b|} (b!)^{1/\zeta_b} t^{|b|-|a|} 1_{[0, 2]}(t\omega(p))$$

and

$$\partial_p^b (p^a \vartheta(\varepsilon/2t) \vartheta(t\omega(p)/2) 1^{\varsigma_1, \varsigma_2}) \leq C^{1+|b|} (b!)^{1/\zeta_b} t^{|b|-|a|} 1_{[0, 2]}(t\omega(p)).$$

Using Eqs. (5.1) and (5.2) we obtain that there exists  $C \in (0, \infty)$  such that for all  $\varepsilon \in [0, 1]$ ,  $t \in (0, 1]$  and  $k \in \mathbb{N}_0$  it holds

$$(|x|/t)^k |\partial^a G_{\varepsilon; t}^{\pm, \varsigma_1, \varsigma_2}(x)| \leq C^{1+k} (k!)^{1/\zeta_b} t^{-|a|-2}/2$$



We conclude that there exists  $C \in [e, \infty)$  such that for all  $\varepsilon \in [0, 1]$ ,  $t \in (0, 1]$  and  $l \in \mathbb{N}_+$  it holds

$$(|x|/t)^{\zeta_b l} |\partial^a G_{\varepsilon; t}^{\pm, \varsigma_1, \varsigma_2}(x)| \leq C^{1+k} (k!)^{1/\zeta_b} t^{-|a|-2}/2 \leq C^{1+3l/\zeta_b} l! t^{-|a|-2}/2$$

where  $k = \lceil \zeta_b l \rceil \leq l \wedge (\zeta_b l + 1)$ . The second of the above bounds follows from the estimates  $(k!)^{1/\zeta_b} \leq k^{k/\zeta_b} \leq l^{l+1/\zeta_b} \leq e^{2l/\zeta_b} l!$ . This implies that the bound stated in the lemma holds true for  $\delta = C^{-3/\zeta_b}/2$ .  $\square$

## 6 Scale decomposition of free field and coupling

In this section we introduce the scale decomposition of the free field  $\Psi_{\tau, \varepsilon; t}$  satisfying properties listed in Sec. 2. The scale decomposition of the free field is defined in the non-commutative probability space  $(\mathcal{F}, \mathbf{E})$  of the fermionic spacetime white noise  $\xi$ . We also introduce a filtration  $(\mathcal{F}_t)_{t \in [0, 1]}$  in  $\mathcal{F}$  as well as the conditional expected value  $\mathbf{E}_t$  that will play an essential role in the proof of the estimates stated in Sec. 12.

**Definition 6.1.** Let  $\mathfrak{h}$  be a Hilbert space. We denote by  $\Gamma_a(\mathfrak{h})$  the antisymmetric Fock space built on  $\mathfrak{h}$ . We denote by  $\mathbf{N}$  the number operator and by  $\mathbf{P} := (-1)^{\mathbf{N}}$  the parity operator.

**Definition 6.2.** Let  $\mathcal{H} = \Gamma_a(L^2([0, 1] \times \mathbb{R}^2 \times \mathbb{F}))^{\otimes 2}$ . The creation and annihilation operators in  $\Gamma_a(L^2([0, 1] \times \mathbb{R}^2 \times \mathbb{F}))$  are denoted by  $a^*(f, \varsigma)$  and  $a(f, \varsigma)$ , where  $f \in L^2([0, 1] \times \mathbb{R}^2)$  and  $\varsigma \in \mathbb{F}$ . The vacuum state in  $\mathcal{H}$  is denoted by  $\Omega$ . Let  $\mathcal{B} := \mathcal{B}(\mathcal{H})$  be the Banach algebra of bounded operators on  $\mathcal{H}$ . The fermionic white noise is defined by

$$\begin{aligned} \xi^{-, \varsigma}(f) &\equiv \int_{[0, 1] \times \mathbb{R}^2} f(s, x) \xi^{-, \varsigma}(ds, dx) := (\mathbf{P} \otimes a^*(f, \varsigma) - a(f^c, \varsigma) \otimes \mathbf{1}), \\ \xi^{+, \varsigma}(f) &\equiv \int_{[0, 1] \times \mathbb{R}^2} f(s, x) \xi^{+, \varsigma}(ds, dx) := (a^*(f, \varsigma) \otimes \mathbf{1} + \mathbf{P} \otimes a(f^c, \varsigma)) \end{aligned}$$

for all  $f \in L^2([0, 1] \times \mathbb{R}^2)$ ,  $\varsigma \in \mathbb{F}$ , where  $f^c$  denotes the complex conjugate of the function  $f$ . We define  $\mathcal{F}$  to be the unital Banach subalgebra of  $\mathcal{B}$  generated by  $\xi^{\pm, \varsigma}(f)$ , where  $f \in L^2([0, 1] \times \mathbb{R}^2)$ ,  $\varsigma \in \mathbb{F}$ . We define a unique grading  $\mathcal{F} = \mathcal{F}^+ \oplus \mathcal{F}^-$  such that  $\xi^{\pm, \varsigma}(f) \in \mathcal{F}^-$  and such that the product in  $\mathcal{F}$  satisfies the conditions  $\mathcal{F}^{\pm} \mathcal{F}^{\pm} \subset \mathcal{F}^+$  and  $\mathcal{F}^{\pm} \mathcal{F}^{\mp} \subset \mathcal{F}^-$ . For  $s \in [0, 1]$  and  $t \in [s, 1]$  we define  $\mathcal{F}_{s, t}$  to be the unital Banach subalgebra of  $\mathcal{F}$  generated by  $\xi^{\pm, \varsigma}(f)$ , where  $f \in L^2([0, 1] \times \mathbb{R}^2)$ ,  $\varsigma \in \mathbb{F}$ ,  $\text{supp } f \subset [s, t] \times \mathbb{R}^2$ . For  $t \in [0, 1]$  we set  $\mathcal{F}_t := \mathcal{F}_{1, t}$ .

*Remark 6.3.* We note that that  $\mathcal{F}$  is an infinite-dimensional Grassmann algebra. One shows that elements of  $\mathcal{F}^-$  anti-commute with each other and elements of  $\mathcal{F}^+$  commute with all elements of  $\mathcal{F}$ .

*Remark 6.4.* It what follows, the following bound

$$\left\| \sum_{\varsigma \in \mathbb{F}} \xi^{\pm, \varsigma}(f^{\varsigma}) \right\|_{\mathcal{B}}^2 \leq 2 \sum_{\varsigma \in \mathbb{F}} \|f^{\varsigma}\|_{L^2([0, 1] \times \mathbb{R}^2)}^2$$

valid for all  $f = (f^\varsigma)_{\varsigma \in \mathbb{F}} \in L^2([0, 1] \times \mathbb{R}^2)^{\mathbb{F}}$  will play an important role. The above bounds is an immediate consequence of the identity

$$\|\sum_{\varsigma \in \mathbb{F}} a(f^\varsigma, \varsigma)\|_{\mathcal{B}} = \|\sum_{\varsigma \in \mathbb{F}} a^*(f^\varsigma, \varsigma)\|_{\mathcal{B}} = \sum_{\varsigma \in \mathbb{F}} \|f^\varsigma\|_{L^2([0, 1] \times \mathbb{R}^2)},$$

which follows easily from the anti-commutation relations of the fermionic creation and annihilation operators, for details see for example [BR97, Proposition 5.2.2].

**Definition 6.5** (Expected value). We define a continuous linear functional  $\mathbf{E} : \mathcal{F} \rightarrow \mathbb{C}$  by the equality  $\mathbf{E}F := (\Omega, F\Omega)$  for all  $F \in \mathcal{F}$ .

*Remark 6.6.* Note that it holds

$$\mathbf{E}(\xi^{\pm, \varsigma}(f) \xi^{\mp, \tilde{\varsigma}}(\tilde{f})) = \pm 1_{\varsigma, \tilde{\varsigma}} \langle f, \tilde{f} \rangle_{L^2([0, 1] \times \mathbb{R}^2)}, \quad \mathbf{E}(\xi^{\pm, \varsigma}(f) \xi^{\pm, \tilde{\varsigma}}(\tilde{f})) = 0$$

for all  $f, \tilde{f} \in L^2([0, 1] \times \mathbb{R}^2)$ ,  $\varsigma, \tilde{\varsigma} \in \mathbb{F}$ .

**Definition 6.7** (Conditional expected value). A family of continuous linear functionals  $(\mathbf{E}_t : \mathcal{F} \rightarrow \mathcal{F})_{t \in [0, 1]}$  is called the conditional expected value if

- (0)  $\mathbf{E}_t : \mathcal{F} \rightarrow \mathcal{F}_t$  for all  $t \in [0, 1]$ ,
- (1)  $\mathbf{E}_t F = \mathbf{E}F$  for all  $t \in [0, 1]$  and  $F \in \mathcal{F}_{0, t}$ ,
- (2)  $\mathbf{E}_t = \mathbf{E}_t \mathbf{E}_s : \mathcal{F} \rightarrow \mathcal{F}_t$  for  $s \in [0, 1]$  and  $t \in [s, 1]$ ,
- (3)  $\mathbf{E}_t(GFH) = G\mathbf{E}_t(F)H$  for all  $t \in [0, 1]$  and  $G, H \in \mathcal{F}_t$ ,  $F \in \mathcal{F}$ ,
- (4)  $\|\mathbf{E}_t F\|_{\mathcal{B}} \leq \|F\|_{\mathcal{B}}$  for all  $t \in [0, 1]$  and  $F \in \mathcal{F}$ .

*Remark 6.8.* For the construction of the conditional expected value see [DFG22, Appendix A].

*Remark 6.9.* Note that the condition (1) implies in particular that  $\mathbf{E}_1 = \mathbf{E}$ .

*Remark 6.10.* Let  $\mathcal{G}$  be a Grassmann algebra. We define  $\mathcal{F} \otimes_{\text{alg}} \mathcal{G}$  to be the algebraic graded tensor product of graded algebras  $\mathcal{F}$  and  $\mathcal{G}$ . We introduce the family of functionals  $(\mathbf{E}_t : \mathcal{F} \otimes_{\text{alg}} \mathcal{G} \rightarrow \mathcal{F}_t \otimes_{\text{alg}} \mathcal{G})_{t \in [0, 1]}$  defined uniquely by the equality  $\mathbf{E}_t(F \otimes g) = (\mathbf{E}_t F) \otimes g$  for all  $F \in \mathcal{F}$ ,  $g \in \mathcal{G}$  and  $t \in [0, 1]$ . The functional  $\mathbf{E} : \mathcal{F} \otimes_{\text{alg}} \mathcal{G} \rightarrow \mathcal{G}$  is defined in an analogous way.

**Definition 6.11.** Let  $\tau, \varepsilon \in [0, 1]$ ,  $s \in [0, 1]$  and  $t \in [s, 1]$ . We define operator-valued distributions

$$\Psi_{\tau, \varepsilon; t, s} \equiv (\Psi_{\tau, \varepsilon; t, s}^-, \Psi_{\tau, \varepsilon; t, s}^+) \equiv (\bar{\Psi}_{\tau, \varepsilon; t, s}, \Psi_{\tau, \varepsilon; t, s}) \equiv (\Psi_{\tau, \varepsilon; t, s}^\sigma)_{\sigma \in \mathbb{G}} \in \mathcal{S}'(\mathbb{T}_\tau^2, \mathcal{F}_{t, s} \cap \mathcal{F}^-)^{\mathbb{G}}$$

by the equation

$$\langle \Psi_{\tau, \varepsilon; t, s}^{\pm, \varsigma}, \varphi \rangle := \sum_{\tilde{\varsigma} \in \mathbb{F}} \int_{[s, t] \times \mathbb{T}_\tau^2} (G_{\tau, \varepsilon; u}^{\pm, \varsigma, \tilde{\varsigma}} * \varphi)(x) \xi^{\pm, \tilde{\varsigma}}(du, dx)$$

for all  $\varsigma \in \mathbb{F}$  and  $\varphi \in \mathcal{S}(\mathbb{R}^2)$ . We call  $\Psi_{\tau, \varepsilon} := \Psi_{\tau, \varepsilon; 1, 0}$  the free field with IR cutoff  $\tau \in [0, 1]$  and UV cutoff  $\varepsilon \in [0, 1]$ . We also define  $\Psi_{\tau, \varepsilon; s} := \Psi_{\tau, \varepsilon; 1, s}$ . We omit  $\tau$  and  $\varepsilon$  if  $\tau = 0$  and  $\varepsilon = 0$ .

*Remark 6.12.* Observe that by Remark 5.2 (B) for  $\tau, \varepsilon \in (0, 1]$  it holds

$$\Psi_{\tau, \varepsilon}^{\sigma}(x) := \tau^2 \sum_{p \in \Lambda_{\tau, \varepsilon}} (\mathbf{F}_{\tau} \Psi_{\tau, \varepsilon}^{\sigma})(p) e^{ip \cdot x}.$$

In particular,  $\Psi_{\tau, \varepsilon}^{\otimes m} = 0$  for all  $m \in \mathbb{N}_+$  provided  $m > |\mathbb{G}| |\Lambda_{\tau, \varepsilon}|$ , where  $|S|$  denotes the number of elements of a set  $S$ .

*Remark 6.13.* Let  $\mathcal{G}$  be a Grassmann algebra. For  $\phi \in C^{\infty}(\mathbb{T}_{\tau}^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}$  we define the paring  $\langle \Psi_{\tau, \varepsilon; t, s}, \phi \rangle_{\tau} \in \mathcal{F} \otimes_{\text{alg}} \mathcal{G}$  by the equality

$$\langle \Psi_{\tau, \varepsilon; t, s}, \phi \rangle_{\tau} := \sum_{i=1}^n \langle \Psi_{\tau, \varepsilon; t, s}, \phi_i \rangle_{\tau} \otimes g_i, \quad \phi = \sum_{i=1}^n \phi_i \otimes g_i,$$

for all  $n \in \mathbb{N}_+$ ,  $\phi_1, \dots, \phi_n \in C^{\infty}(\mathbb{T}_{\tau}^2)^{\mathbb{G}}$  and  $g_1, \dots, g_n \in \mathcal{G}$ . We identify  $A \in \mathcal{F}$  and  $g \in \mathcal{G}$  with

$$A \otimes 1 \in \mathcal{F} \otimes_{\text{alg}} \mathcal{G} \quad \text{and} \quad 1 \otimes g \in \mathcal{F} \otimes_{\text{alg}} \mathcal{G},$$

respectively, and use these identifications to make sense of  $A + g, Ag \in \mathcal{F} \otimes_{\text{alg}} \mathcal{G}$ .

*Remark 6.14.* Let  $\tau, \varepsilon \in [0, 1]$ ,  $s \in [0, 1]$  and  $t \in [s, 1]$  be such that  $\varepsilon > 0$  or  $s > 0$ . Using Remark 6.4, the fact that  $G_{\varepsilon; t}^{\pm; \zeta, \bar{\zeta}} = 0$  for  $t \in (0, \varepsilon/2]$  by Remark 5.2 (B) and Lemma 5.3 one shows that  $\Psi_{\tau, \varepsilon; t, s} \in C^{\infty}(\mathbb{T}_{\tau}^2, \mathcal{F}_{t, s})^{\mathbb{G}}$ . Note that it holds  $\mathbf{E} \Psi_{\tau, \varepsilon; t}^{\sigma}(x) = 0$  and

$$\mathbf{E}(\Psi_{\tau, \varepsilon; t}^{\sigma_1}(x) \Psi_{\tau, \varepsilon; s}^{\sigma_2}(y)) = G_{\tau, \varepsilon; t}^{\sigma_1, \sigma_2}(x - y), \quad (6.1)$$

where  $G_{\tau, \varepsilon; t} \in C^{\infty}(\mathbb{R}^2)^{\mathbb{G}^2}$  was introduced in Def. 5.1. Using the equality  $\Psi_{\tau, \varepsilon; t, s} = \Psi_{\tau, \varepsilon; s} - \Psi_{\tau, \varepsilon; t}$  one shows that it holds  $\mathbf{E} \Psi_{\tau, \varepsilon; t, s}^{\sigma}(x) = 0$  and

$$\begin{aligned} \mathbf{E}(\Psi_{\tau, \varepsilon; t, s}^{\sigma_1}(x) \Psi_{\tau, \varepsilon; t, s}^{\sigma_2}(y)) &= \mathbf{E}((\Psi_{\tau, \varepsilon; s}^{\sigma_1}(x) - \Psi_{\tau, \varepsilon; t}^{\sigma_1}(x)) (\Psi_{\tau, \varepsilon; s}^{\sigma_2}(y) - \Psi_{\tau, \varepsilon; t}^{\sigma_2}(y))) \\ &= (G_{\tau, \varepsilon; s}^{\sigma_1, \sigma_2} - G_{\tau, \varepsilon; t}^{\sigma_1, \sigma_2})(x - y). \end{aligned}$$

The expected values of products of more than two fields  $\Psi_{\tau, \varepsilon; t, s}$  can be expressed in terms of the covariance with the use of the generating functional

$$C^{\infty}(\mathbb{T}_{\tau}^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^{-} \ni \phi \mapsto \mathbf{E} \exp(\langle \Psi_{\tau, \varepsilon; t, s}, \phi \rangle_{\tau}) = \exp(-\langle \phi, (G_{\varepsilon; s} - G_{\varepsilon; t}) * \phi \rangle_{\tau} / 2) \in \mathcal{G}, \quad (6.2)$$

where  $\mathcal{G}$  is an infinite-dimensional Grassmann algebra. In order to prove (6.2) one decomposes  $\Psi_{\tau, \varepsilon; t, s}$  into a sum of two terms  $\Psi_{\tau, \varepsilon; t, s}^{(\pm)}$  involving only the creation and annihilation operators, applies the Baker–Campbell–Hausdorff formula

$$\exp(\langle \Psi_{\tau, \varepsilon; t, s}, \phi \rangle_{\tau}) = \exp(\langle \Psi_{\tau, \varepsilon; t, s}^{(+)} \rangle_{\tau}) \exp(\langle \Psi_{\tau, \varepsilon; t, s}^{(-)} \rangle_{\tau}) \exp(-[\langle \Psi_{\tau, \varepsilon; t, s}^{(+)} \rangle_{\tau}, \langle \Psi_{\tau, \varepsilon; t, s}^{(-)} \rangle_{\tau}] / 2),$$

where  $[\bullet, \bullet]$  denotes the commutator, and uses the fact that the vacuum state  $\Omega$  belongs to the intersection of the kernels of the annihilation operators.

**Lemma 6.15.** *Let  $\tau, \varepsilon \in (0, 1]$ . For all  $F \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  it holds*

$$\nu_{\tau, \varepsilon}(F) \equiv \frac{\int F(\vartheta_\varepsilon * \psi_{\tau, \varepsilon}) \exp(-A_\tau(\psi_{\tau, \varepsilon})) d\psi_{\tau, \varepsilon}}{\int \exp(-A_\tau(\psi_{\tau, \varepsilon})) d\psi_{\tau, \varepsilon}} = \mathbf{E}F(\Psi_{\tau, \varepsilon}).$$

*Remark 6.16.* Note that in particular it holds

$$\int \psi^{\sigma_1}(x) \psi^{\sigma_2}(y) \nu_{\tau, \varepsilon}(d\psi) = G_{\tau, \varepsilon}^{\sigma_1, \sigma_2}(x - y).$$

*Proof.* In view of Remarks 4.7 and 6.12 it suffices to prove the stated equality for functionals  $F = (F^m)_{m \in \mathbb{N}_0} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  such that  $F^m \neq 0$  only for finitely many  $m \in \mathbb{N}_0$ . In consequence, it is enough to prove the stated equality for functionals  $F = F_\eta$  for all  $\eta \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ , where the functional  $F_\eta \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  is defined by the equality  $F_\eta(\phi) := \exp(\langle \phi, \eta \rangle_\tau)$  for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . By [CSS13, Theorem A.16] it holds

$$\frac{\int F_\eta(\psi_{\tau, \varepsilon}) \exp(-A_\tau(\psi_{\tau, \varepsilon})) d\psi_{\tau, \varepsilon}}{\int \exp(-A_\tau(\psi_{\tau, \varepsilon})) d\psi_{\tau, \varepsilon}} = \exp(-\langle \eta, \tilde{G}_{\tau, \varepsilon} *_\tau \eta \rangle_\tau / 2),$$

where  $\mathbf{F}_\tau \tilde{G}_{\tau, \varepsilon}(p) := 1_{\Lambda_{\tau, \varepsilon}}(p) \mathbf{F}G(p)$ . Note that

$$F_\eta(\vartheta_\varepsilon * \psi_{\tau, \varepsilon}) = F_{\vartheta_\varepsilon * \eta}(\psi_{\tau, \varepsilon}).$$

Using the fact that  $\text{supp } \mathbf{F}_\tau \mathbf{P}_\tau \vartheta_\varepsilon \subset \Lambda_{\tau, \varepsilon}$  we show that

$$\vartheta_\varepsilon * \tilde{G}_{\tau, \varepsilon} * \vartheta_\varepsilon = \vartheta_\varepsilon * G_\tau * \vartheta_\varepsilon = G_{\tau, \varepsilon}$$

and

$$\frac{\int F_\eta(\vartheta_\varepsilon * \psi_{\tau, \varepsilon}) \exp(-A_\tau(\psi_{\tau, \varepsilon})) d\psi_{\tau, \varepsilon}}{\int \exp(-A_\tau(\psi_{\tau, \varepsilon})) d\psi_{\tau, \varepsilon}} = \exp(-\langle \eta, G_{\tau, \varepsilon} *_\tau \eta \rangle_\tau / 2).$$

On the other hand,

$$\mathbf{E}F_\eta(\Psi_{\tau, \varepsilon}) = \exp(-\langle \eta, G_{\tau, \varepsilon} *_\tau \eta \rangle_\tau / 2)$$

by Eq. (6.2) and Remark 3.4. This finishes the proof.  $\square$

**Lemma 6.17.** *Let  $\tau, \varepsilon \in (0, 1]$ ,  $s \in [0, 1]$  and  $t \in [s, 1]$ . For all  $F \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  and  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$  it holds*

$$\mathbf{E}F(\Psi_{\tau, \varepsilon; t, s} + \phi) = (\exp(\mathbf{D}_{\tau, \varepsilon; t, s} / 2) F)(\phi),$$

where the map  $\mathbf{D}_{\tau, \varepsilon; t, s}$  is defined by the equation

$$(\mathbf{D}_{\tau, \varepsilon; t, s} F)(\phi) := \langle \mathbf{D}_\phi^2 F(\phi), (G_{\tau, \varepsilon; s} - G_{\tau, \varepsilon; t})(\bullet - \bullet) \rangle_\tau$$

for all  $F \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  and  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ .

*Proof.* By the argument from the proof of Lemma 6.15 it suffices to prove the statement for all functionals of the form  $F = F_\eta$  for some  $\eta \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ , where  $F_\eta(\phi) := \exp(\langle \phi, \eta \rangle_\tau)$  for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . By direct computation we obtain

$$\mathbf{D}_{\tau,\varepsilon;t,s} F_\eta = \langle \eta, (G_{\tau,\varepsilon;s} - G_{\tau,\varepsilon;t}) *_\tau \eta \rangle_\tau F_\eta.$$

Since  $\langle \eta, (G_{\tau,\varepsilon;s} - G_{\tau,\varepsilon;t}) *_\tau \eta \rangle_\tau \in \mathcal{G}^+$  we arrive at

$$\begin{aligned} \exp(\mathbf{D}_{\tau,\varepsilon;t,s}/2) F_\eta &= \sum_{n=0}^{\infty} \frac{1}{2^n n!} \mathbf{D}_{\tau,\varepsilon;t,s}^n F_\eta = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \langle \eta, (G_{\tau,\varepsilon;s} - G_{\tau,\varepsilon;t}) *_\tau \eta \rangle_\tau^n F_\eta \\ &= \exp(-\langle \eta, (G_{\tau,\varepsilon;s} - G_{\tau,\varepsilon;t}) *_\tau \eta \rangle_\tau / 2) F_\eta. \end{aligned}$$

On the other hand,

$$\mathbf{E} F_\eta(\Psi_{\tau,\varepsilon;t,s} + \phi) = \mathbf{E} F_\eta(\Psi_{\tau,\varepsilon;t,s}) F_\eta(\phi) = \exp(-\langle \eta, (G_{\tau,\varepsilon;s} - G_{\tau,\varepsilon;t}) *_\tau \eta \rangle_\tau / 2) F_\eta(\phi)$$

by Eq. (6.2) and Remark 3.4. This finishes the proof.  $\square$

**Definition 6.18.** The vector space  $\mathcal{C}$  consists of continuous functions  $\phi : \mathbb{R}^2 \rightarrow \mathcal{B}$  such that the following norm

$$\|\phi\|_{\mathcal{C}} := \sup_{x \in \mathbb{R}^2} \|\phi(x)\|_{\mathcal{B}}$$

is finite. For  $\phi \in \mathcal{C}$  we define  $\|\phi\|_{\mathcal{C}} := \|\tilde{w} \phi\|_{\mathcal{C}}$ , where  $\tilde{w} \in C^\infty(\mathbb{R}^2)$  is defined by the equality  $\tilde{w}(x) = (1 + |x|)^{-1/2}$  for all  $x \in \mathbb{R}^2$ .

**Lemma 6.19.** *There exists  $\lambda_\star \in (0, 1]$  such that for all  $\lambda \in (0, \lambda_\star]$ ,  $\tau, \varepsilon \in [0, 1]$ ,  $t \in (0, 1]$ ,  $s \in (0, t]$  and  $a \in \mathbb{A} = \{0, 1, 2\}^2$ ,  $\sigma \in \mathbb{G}$  it holds*

$$\|\partial^a \Psi_{\tau,\varepsilon;t,s}^\sigma\|_{\mathcal{C}} \leq \lambda^{-\kappa} s^{-1/2-|a|}, \quad \|\tilde{w}(\partial^a \Psi_{t,s}^\sigma - \partial^a \Psi_{\tau,\varepsilon;t,s}^\sigma)\|_{\mathcal{C}} \leq \lambda^{-\kappa} \lambda_{\tau \vee \varepsilon}^\kappa \lambda_s^{-\kappa} s^{-1/2-|a|}.$$

*Proof.* Suppose that  $\sigma = (\pm, \varsigma)$  for some  $\varsigma \in \mathbb{F}$ . By Remark 6.4 we obtain

$$\|\partial^a \Psi_{\tau,\varepsilon;t,s}^\sigma\|_{\mathcal{C}}^2 = \sup_{x \in \mathbb{R}^2} \|\partial^a \Psi_{\tau,\varepsilon;t,s}^{\pm,\varsigma}(x)\|_{\mathcal{B}}^2 \leq 2 \sup_{x \in \mathbb{R}^2} \sum_{\tilde{\varsigma} \in \mathbb{F}} \int_s^t \int_{\mathbb{T}_\tau^2} |\partial^a G_{\tau,\varepsilon;u}^{\pm,\tilde{\varsigma}}(x-y)|^2 dy du.$$

Using Lemma 5.3 and the fact that for all  $y \in \mathbb{T}_\tau^2 = (-1/(2\tau), 1/(2\tau))^2$  and  $n \in \mathbb{Z}^2$  such that  $|n|_\infty \geq 2$  it holds  $|y + n/\tau|_\infty \geq |y|_\infty + |n|_\infty/(2\tau)$  one shows that

$$\begin{aligned} \int_{\mathbb{T}_\tau^2} |\partial^a G_{\tau,\varepsilon;u}^{\pm,\tilde{\varsigma}}(x-y)|^2 dy &= \int_{\mathbb{T}_\tau^2} |\partial^a G_{\tau,\varepsilon;u}^{\pm,\tilde{\varsigma}}(y)|^2 dy \\ &\leq \sum_{n \in \mathbb{N}_0^2} \int_{\mathbb{T}_\tau^2} |\partial^a G_{\varepsilon;u}^{\pm,\tilde{\varsigma}}(y + n/\tau)|^2 dy \leq C u^{-2-2|a|}. \end{aligned}$$

As a result, we conclude that there exists  $C \in (0, \infty)$  such that for all  $\varepsilon \in [0, 1]$ ,  $t \in (0, 1]$ ,  $s \in (0, t]$  and  $a \in \mathbb{A}$ ,  $\sigma \in \mathbb{G}$  it holds

$$\|\partial^a \Psi_{\tau, \varepsilon; t, s}^\sigma\|_{\mathcal{C}} \leq C s^{-1/2-|a|}. \quad (6.3)$$

Next, we observe that

$$\begin{aligned} \|\partial^a \Psi_{t, s}^\sigma - \partial^a \Psi_{\varepsilon; t, s}^\sigma\|_{\mathcal{C}}^2 &= \sup_{x \in \mathbb{R}^2} \|\partial^a \Psi_{t, s}^{\pm, \varsigma}(x) - \partial^a \Psi_{\varepsilon; t, s}^{\pm, \varsigma}(x)\|_{\mathcal{B}}^2 \\ &\leq 2 \sup_{x \in \mathbb{R}^2} \sum_{\tilde{\varsigma} \in \mathbb{F}} \int_s^t \int_{\mathbb{R}^2} |\partial^a G_u^{\pm, \varsigma, \tilde{\varsigma}}(x-y) - \partial^a G_{\varepsilon; u}^{\pm, \varsigma, \tilde{\varsigma}}(x-y)|^2 dy du. \end{aligned}$$

The integrand on the RHS of the above equality vanishes identically if  $u \in [4\varepsilon, 1]$  by Remark 5.2 (C). As a result, by the bound (6.3) we obtain

$$\|\partial^a \Psi_{t, s}^\sigma - \partial^a \Psi_{\varepsilon; t, s}^\sigma\|_{\mathcal{C}} \leq C \lambda_{4\varepsilon}^\kappa \lambda_s^{-\kappa} s^{-1/2-|a|}. \quad (6.4)$$

Next, note that

$$\begin{aligned} \|\tilde{w}(\partial^a \Psi_{\varepsilon; t, s}^\sigma - \partial^a \Psi_{\tau, \varepsilon; t, s}^\sigma)\|_{\mathcal{C}}^2 &= \sup_{x \in \mathbb{R}^2} \tilde{w}(x)^2 \|\partial^a \Psi_{\varepsilon; t, s}^{\pm, \varsigma}(x) - \partial^a \Psi_{\tau, \varepsilon; t, s}^{\pm, \varsigma}(x)\|_{\mathcal{B}}^2 \\ &\leq 2 \sup_{x \in \mathbb{R}^2} \tilde{w}(x)^2 \sum_{\tilde{\varsigma} \in \mathbb{F}} \int_s^t \int_{\mathbb{R}^2} |\partial^a G_{\varepsilon; u}^{\pm, \varsigma, \tilde{\varsigma}}(x-y) - \partial^a G_{\tau, \varepsilon; u}^{\pm, \varsigma, \tilde{\varsigma}}(x-y) 1_{\mathbb{T}_\tau^2}(y)|^2 dy du, \end{aligned}$$

where  $1_{\mathbb{T}_\tau^2}$  is the characteristic function of the set  $\mathbb{T}_\tau^2 = (-1/(2\tau), 1/(2\tau))$ . Observe that  $\tilde{w}(x)^2 \leq \tilde{w}(y)^2 \tilde{w}(x-y)^{-2}$ . Consequently, it holds

$$\begin{aligned} \sup_{x \in \mathbb{R}^2} \tilde{w}(x)^2 \int_{\mathbb{R}^2} |\partial^a G_{\varepsilon; u}^{\pm, \varsigma, \tilde{\varsigma}}(x-y) - \partial^a G_{\tau, \varepsilon; u}^{\pm, \varsigma, \tilde{\varsigma}}(x-y) 1_{\mathbb{T}_\tau^2}(y)|^2 dy \\ \leq \sup_{x \in \mathbb{T}_{2\tau}^2} \int_{\mathbb{T}_\tau^2} |\partial^a G_{\varepsilon; u}^{\pm, \varsigma, \tilde{\varsigma}}(x-y) - \partial^a G_{\tau, \varepsilon; u}^{\pm, \varsigma, \tilde{\varsigma}}(x-y)|^2 dy \\ + \sup_{x \in \mathbb{R}^2 \setminus \mathbb{T}_{2\tau}^2} \tilde{w}(x)^2 \int_{\mathbb{T}_\tau^2} |\partial^a G_{\varepsilon; u}^{\pm, \varsigma, \tilde{\varsigma}}(x-y) - \partial^a G_{\tau, \varepsilon; u}^{\pm, \varsigma, \tilde{\varsigma}}(x-y)|^2 dy \\ + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2 \setminus \mathbb{T}_\tau^2} \tilde{w}(y)^2 \tilde{w}(x-y)^{-2} |\partial^a G_{\varepsilon; u}^{\pm, \varsigma, \tilde{\varsigma}}(x-y)|^2 dy. \end{aligned}$$

Using the fact that  $\tilde{w}(x) = (1 + |x|)^{-1/2}$  we obtain

$$\begin{aligned}
& \sup_{x \in \mathbb{R}^2} \tilde{w}(x)^2 \int_{\mathbb{R}^2} |\partial^a G_{\varepsilon;u}^{\pm;\varsigma,\tilde{\varsigma}}(x-y) - \partial^a G_{\tau,\varepsilon;u}^{\pm;\varsigma,\tilde{\varsigma}}(x-y) 1_{\mathbb{T}_\tau^2}(y)|^2 dy \\
& \leq \sup_{x \in \mathbb{T}_{2\tau}^2} \int_{\mathbb{T}_\tau^2} |\partial^a G_{\varepsilon;u}^{\pm;\varsigma,\tilde{\varsigma}}(x-y) - \partial^a G_{\tau,\varepsilon;u}^{\pm;\varsigma,\tilde{\varsigma}}(x-y)|^2 dy \\
& \quad + (2\tau) \int_{\mathbb{T}_\tau^2} |\partial^a G_{\varepsilon;u}^{\pm;\varsigma,\tilde{\varsigma}}(y) - \partial^a G_{\tau,\varepsilon;u}^{\pm;\varsigma,\tilde{\varsigma}}(y)|^2 dy \\
& \quad + \tau \int_{\mathbb{R}^2} \tilde{w}(y)^{-2} |\partial^a G_{\varepsilon;u}^{\pm;\varsigma,\tilde{\varsigma}}(y)|^2 dy.
\end{aligned}$$

Since for all  $x \in \mathbb{T}_{2\tau}^2$ ,  $y \in \mathbb{T}_\tau^2$  and  $n \in \mathbb{Z}^2 \setminus \{0\}$  it holds  $|x - y + n/\tau|_\infty \geq |n|_\infty/(4\tau)$  by Lemma 5.3 we obtain that there exist  $c, C \in (0, \infty)$  such that for all  $\varepsilon \in [0, 1]$ ,  $u \in (0, 1]$  and  $a \in \mathbb{A}$ ,  $\varsigma, \tilde{\varsigma} \in \mathbb{F}$  it holds

$$\sup_{x \in \mathbb{T}_{2\tau}^2} \sup_{y \in \mathbb{T}_\tau^2} |\partial^a G_{\varepsilon;u}^{\pm;\varsigma,\tilde{\varsigma}}(x-y) - \partial^a G_{\tau,\varepsilon;u}^{\pm;\varsigma,\tilde{\varsigma}}(x-y)| \leq C u^{-2-|a|} \exp(-c/\tau^{\zeta_b})$$

This implies the bound

$$\sup_{x \in \mathbb{T}_{2\tau}^2} \int_{\mathbb{T}_\tau^2} |\partial^a G_{\varepsilon;u}^{\pm;\varsigma,\tilde{\varsigma}}(x-y) - \partial^a G_{\tau,\varepsilon;u}^{\pm;\varsigma,\tilde{\varsigma}}(x-y)|^2 dy \leq C u^{-2-2|a|}$$

with a possibly different constant  $C \in (0, \infty)$ . Moreover, by Lemma 5.3 we have

$$\int_{\mathbb{R}^2} \tilde{w}(y)^{-2} |\partial^a G_{\varepsilon;u}^{\pm;\varsigma,\tilde{\varsigma}}(y)|^2 dy \leq C u^{-2-2|a|}.$$

Consequently, we obtain

$$\|\partial^a \Psi_{\varepsilon;t,s}^\sigma - \partial^a \Psi_{\tau,\varepsilon;t,s}^\sigma\|_{\mathcal{C}} \leq C \tau^{1/2} s^{-1/2-|a|}. \quad (6.5)$$

The lemma follows now from the bounds (6.3), (6.4), (6.5).  $\square$

## 7 Effective potential and Polchinski equation

In this section we introduce the notion of an effective potential and prove that a solution of the mild form of the Polchinski equation (7.3) satisfies the equation for an effective potential. A solution  $U_{\tau,\varepsilon;\bullet}$  of the Polchinski equation will be constructed in Sec. 14 in terms of the fixed point  $X_{\tau,\varepsilon;\bullet}$  of the map  $\mathbf{X}_{\tau,\varepsilon;\bullet}$  constructed in Sec. 13. Then by the results of this section such  $U_{\tau,\varepsilon;t}$  is an effective potential at the scale  $t$ . Using this fact in Sec. 15 we will show that  $U_{\tau,\varepsilon;t=0}$  is the generating functional of the connected amputated Schwinger functions.

**Lemma 7.1.** *Let  $\tau, \varepsilon \in (0, 1]$ . Suppose that  $\mathbf{E} \exp(U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon})) \neq 0$ . The interacting measure satisfies the equality*

$$\mu_{\tau, \varepsilon}(F) = \frac{\mathbf{E}(F(\Psi_{\tau, \varepsilon}) \exp(U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon})))}{\mathbf{E} \exp(U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon}))} \in \mathbb{C}$$

for all functionals  $F \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$ .

*Proof.* The lemma is a consequence of Def. 4.16 of the measure  $\mu_{\tau, \varepsilon}$  and Lemma 6.15.  $\square$

**Definition 7.2.** Let  $\tau, \varepsilon \in (0, 1]$  and  $U_{\tau, \varepsilon} \equiv U_\tau(g_{\tau, \varepsilon}, r_{\tau, \varepsilon}) \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  be defined as specified in Def. 4.16. We call a functional  $U_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  an effective potential at the spatial scale  $t \in [0, 1]$  (of the Gross-Neveu model with the parameters  $g_{\tau, \varepsilon}, r_{\tau, \varepsilon} \in \mathbb{R}$ ) if it satisfies the following equality

$$\exp(U_{\tau, \varepsilon; t}(\phi)) = \mathbf{E} \exp(U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon; t, 0} + \phi)) \in \mathcal{G}$$

for all  $\phi \in \mathcal{C}_{\tau, \varepsilon} \otimes \mathcal{G}^-$ , where  $\mathcal{C}_{\tau, \varepsilon} = (\text{Span}\{x \mapsto e^{ip \cdot x} \mid p \in \Lambda_{\tau, \varepsilon}\})^\mathbb{G} \otimes \mathcal{G}^- \subset C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes \mathcal{G}^-$  was introduced in Def. 4.5 and  $\mathcal{G}$  is an infinite-dimensional Grassmann algebra.

*Remark 7.3.* Note that for  $s \leq t$  it holds

$$\exp(U_{\tau, \varepsilon; s}(\Psi_{\tau, \varepsilon; t, s} + \phi)) = \mathbf{E}_s \exp(U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon; t, 0} + \phi)).$$

Consequently, for  $s \leq t$  an effective potential fulfills the following identity

$$\exp(U_{\tau, \varepsilon; t}(\phi)) = \mathbf{E} \exp(U_{\tau, \varepsilon; s}(\Psi_{\tau, \varepsilon; t, s} + \phi)).$$

*Remark 7.4.* For all  $\tau, \varepsilon, t \in (0, 1]$  we have

$$\dot{G}_{\tau, \varepsilon; t}(\bullet - \bullet) \in \mathcal{C}_{\tau, \varepsilon} \otimes \mathcal{C}_{\tau, \varepsilon} \subset C^\infty(\mathbb{R}^2 \times \mathbb{R}^2),$$

where the function  $G_{\tau, \varepsilon; t}(\bullet - \bullet) \in C(\mathbb{R}^2 \times \mathbb{R}^2)$  coincides with the map  $(x, y) \mapsto G_{\tau, \varepsilon; t}(x - y)$ .

**Lemma 7.5.** *Let  $\tau, \varepsilon \in (0, 1]$  and  $\mathcal{G}$  be an infinite-dimensional Grassmann algebra. Suppose that the function  $[0, 1] \ni t \mapsto U_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  satisfies the Polchinski flow equation*

$$\partial_t U_{\tau, \varepsilon; t}(\phi) = -\frac{1}{2} \langle D_\phi^2 U_{\tau, \varepsilon; t}(\phi), \dot{G}_{\tau, \varepsilon; t}(\bullet - \bullet) \rangle_\tau + \frac{1}{2} \langle D_\phi U_{\tau, \varepsilon; t}(\phi), \dot{G}_{\varepsilon; t} * D_\phi U_{\tau, \varepsilon; t}(\phi) \rangle_\tau \quad (7.1)$$

with the initial condition  $U_{\tau, \varepsilon; 0}(\phi) = U_{\tau, \varepsilon}(\phi)$  for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . Then for every  $t \in [0, 1]$  the functional  $U_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  is an effective potential at the spatial scale  $t$ .

*Proof.* Let us first observe that for every  $U \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  such that  $U(\phi) \in \mathcal{G}^+$  for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$  it holds

$$D_\phi \exp(U(\phi)) = \exp(U(\phi)) D_\phi U(\phi) \in \mathcal{S}'(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-,$$

$$D_\phi^2 \exp(U(\phi)) = \exp(U(\phi)) D_\phi^2 U(\phi) - \exp(U(\phi)) (D_\phi U(\phi) \otimes D_\phi U(\phi)) \in \mathcal{S}'(\mathbb{T}_\tau^4)^\mathbb{G}^2 \otimes_{\text{alg}} \mathcal{G}^+.$$



In consequence, the function  $[0, 1] \ni t \mapsto U_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  satisfies the Polchinski flow equation (7.1) for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$  if and only if it satisfies the equation

$$\partial_t \exp(U_{\tau, \varepsilon; t}(\phi)) = -\frac{1}{2} \langle D_\phi^2 \exp(U_{\tau, \varepsilon; t}(\phi)), \dot{G}_{\tau, \varepsilon; t}(\bullet - \bullet) \rangle_\tau$$

for all  $t \in (0, 1]$  and  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . Note that by Lemma 6.17 the equation

$$\partial_t \mathbf{E} \exp(U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon; t, 0} + \phi)) = -\frac{1}{2} \langle D_\phi^2 \mathbf{E} \exp(U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon; t, 0} + \phi)), \dot{G}_{\tau, \varepsilon; t}(\bullet - \bullet) \rangle_\tau$$

holds true for all  $t \in (0, 1]$  and  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . The statement follows now from Lemma 7.6.  $\square$

**Lemma 7.6.** *Let  $\tau, \varepsilon \in (0, 1]$  and  $\mathcal{G}$  be an infinite-dimensional Grassmann algebra. Suppose that the function  $[0, 1] \ni t \mapsto Z_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  satisfies the equation*

$$\partial_t Z_{\tau, \varepsilon; t}(\phi) = -\frac{1}{2} \langle D_\phi^2 Z_{\tau, \varepsilon; t}(\phi), \dot{G}_{\tau, \varepsilon; t}(\bullet - \bullet) \rangle_\tau \quad (7.2)$$

with the initial condition  $Z_{\tau, \varepsilon; 0}(\phi) = 0$  for all  $\phi \in \mathcal{C}_{\tau, \varepsilon} \otimes \mathcal{G}^-$ . Then  $Z_{\tau, \varepsilon; t}(\phi) = 0$  for all  $t \in [0, 1]$  and  $\phi \in \mathcal{C}_{\tau, \varepsilon} \otimes \mathcal{G}^-$ .

*Proof.* Recall that a functional  $Z_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  is a collection  $Z_{\tau, \varepsilon; t} = (Z_{\tau, \varepsilon; t}^m)_{m \in \mathbb{N}_0}$  such that  $Z_{\tau, \varepsilon; t}^0 \in \mathbb{C}$  and  $Z_{\tau, \varepsilon; t}^m \in \mathcal{S}'(\mathbb{T}_\tau^{2m})^{\mathbb{G}^m}$  is antisymmetric for  $m \in \mathbb{N}_+$ . Next, set  $m_{\tau, \varepsilon} := |\Lambda_{\tau, \varepsilon}| |\mathbb{G}| = \dim(\mathcal{C}_{\tau, \varepsilon})$  and let  $\{e_1, \dots, e_{m_{\tau, \varepsilon}}\}$  be a basis of  $\mathcal{C}_{\tau, \varepsilon}$ . For  $m \in \mathbb{N}_+$  and  $i_1, \dots, i_m \in \{1, \dots, m_{\tau, \varepsilon}\}$  define

$$Z_{\tau, \varepsilon; t}^{(i_1, \dots, i_m)} := \frac{1}{m!} \sum_{\pi \in \mathcal{P}_m} (-1)^{\text{sgn}(\pi)} \langle Z_{\tau, \varepsilon; t}^m, e_{\pi(i_1)} \otimes \dots \otimes e_{\pi(i_m)} \rangle \in \mathbb{C},$$

where  $\mathcal{P}_m$  is the set of permutations of  $\{1, \dots, m\}$ . Observe that  $Z_{\tau, \varepsilon; t}^{(i_1, \dots, i_m)}$  vanishes identically if  $m > m_{\tau, \varepsilon}$ . Hence, by Remark 7.4 Eq. (7.2) implies that the finite collection

$$\{Z_{\tau, \varepsilon; \bullet}^0\} \cup \{Z_{\tau, \varepsilon; \bullet}^{(i_1, \dots, i_m)} \mid m, i_1, \dots, i_m \in \{1, \dots, m_{\tau, \varepsilon}\}\}$$

satisfies a first order linear ODE with a trivial boundary condition. This proves the claim.  $\square$

**Lemma 7.7.** *Let  $\tau, \varepsilon \in (0, 1]$  and  $\mathcal{G}$  be an infinite-dimensional Grassmann algebra. Suppose that a continuous function  $[0, 1] \ni t \mapsto U_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  satisfies the integral form of the Polchinski equation*

$$U_{\tau, \varepsilon; t}(\phi) = \mathbf{E} U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon; t, 0} + \phi) + \frac{1}{2} \int_0^t \mathbf{E} \langle D_\phi U_{\tau, \varepsilon; s}(\Psi_{\tau, \varepsilon; t, s} + \phi), \dot{G}_{\varepsilon; s} * D_\phi U_{\tau, \varepsilon; s}(\Psi_{\tau, \varepsilon; t, s} + \phi) \rangle_\tau ds \quad (7.3)$$

for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . Then for every  $t \in [0, 1]$  the functional  $U_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  is an effective potential at the spatial scale  $t$ .

*Proof.* It holds

$$\begin{aligned}\partial_t U_{\tau,\varepsilon;t}(\phi) &= \partial_t \mathbf{E} U_{\tau,\varepsilon;0}(\Psi_{\tau,\varepsilon;t,0} + \phi) \\ &+ \frac{1}{2} \int_0^t \partial_t \mathbf{E} \langle D_\phi U_{\tau,\varepsilon;s}(\Psi_{\tau,\varepsilon;t,s} + \phi), \dot{G}_{\varepsilon;s} * D_\phi U_{\tau,\varepsilon;s}(\Psi_{\tau,\varepsilon;t,s} + \phi) \rangle_\tau ds \\ &+ \frac{1}{2} \langle D_\phi U_{\tau,\varepsilon;t}(\phi), \dot{G}_{\varepsilon;t} * D_\phi U_{\tau,\varepsilon;t}(\phi) \rangle_\tau.\end{aligned}$$

Lemma 6.17 implies that

$$\partial_t \mathbf{E} F(\Psi_{\tau,\varepsilon;t,s} + \phi) = -\frac{1}{2} \langle D_\phi^2 F(\Psi_{\tau,\varepsilon;t,s} + \phi), \dot{G}_{\tau,\varepsilon;t}(\bullet - \bullet) \rangle_\tau$$

for all  $F \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2))^\mathbb{G}$ . Hence, we obtain

$$\begin{aligned}\partial_t U_{\tau,\varepsilon;t}(\phi) &= -\frac{1}{2} \langle D_\phi^2 \mathbf{E} U_{\tau,\varepsilon;0}(\Psi_{\tau,\varepsilon;t,0} + \phi), \dot{G}_{\tau,\varepsilon;t}(\bullet - \bullet) \rangle_\tau \\ &- \frac{1}{4} \int_0^t \langle D_\phi^2 \mathbf{E} \langle D_\phi U_{\tau,\varepsilon;s}(\Psi_{\tau,\varepsilon;t,s} + \phi), \dot{G}_{\varepsilon;s} * D_\phi U_{\tau,\varepsilon;s}(\Psi_{\tau,\varepsilon;t,s} + \phi) \rangle_\tau, \dot{G}_{\tau,\varepsilon;t}(\bullet - \bullet) \rangle_\tau ds \\ &+ \frac{1}{2} \langle D_\phi U_{\tau,\varepsilon;t}(\phi), \dot{G}_{\varepsilon;t} * D_\phi U_{\tau,\varepsilon;t}(\phi) \rangle_\tau.\end{aligned}$$

Using Eq. (7.3) the sum of the first two terms on the RHS of the above equation can be rewritten as

$$-\frac{1}{2} \langle D_\phi^2 U_{\tau,\varepsilon;t}(\phi), \dot{G}_{\tau,\varepsilon;t}(\bullet - \bullet) \rangle_\tau.$$

This implies that the function  $[0, 1] \ni t \mapsto U_{\tau,\varepsilon;t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2))^\mathbb{G}$  satisfies the Polchinski flow equation (7.1) with the initial condition  $U_{\tau,\varepsilon;0}(\phi) = U_{\tau,\varepsilon}(\phi)$ . The statement follows now from Lemma 7.5.  $\square$

## 8 Symmetries

In this section we discuss the symmetries of the Gross-Neveu model that are instrumental in the decomposition of kernels of functionals into the local part and the remainder, which is defined in Sec. 11. As we mentioned in Sec. 2 the form of the local terms that appear in this decomposition is restricted by the symmetries of the kernels.

**Definition 8.1.** The jet prolongation of  $\varphi \in C^\infty(\mathbb{R}^2)^\mathbb{G}$  is defined by

$$\mathbf{J} : C^\infty(\mathbb{R}^2)^\mathbb{G} \rightarrow C^\infty(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}, \quad (\mathbf{J}\varphi)^{a,\sigma} \equiv \mathbf{J}\varphi^{a,\sigma} = \partial^a \varphi^\sigma, \quad a \in \mathbb{A}, \sigma \in \mathbb{G}.$$

**Definition 8.2** (Symmetries of plane). Let  $x \in \mathbb{R}^2$  and let  $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a matrix of a rotation around the origin or a reflection with respect to a line passing through the origin.

For every  $R = (R^{jk})_{j,k \in \{1,2\}}$  as above we choose a complex invertible  $2 \times 2$  matrix  $\gamma(R)$  such that

$$\gamma(R)^{-1} \gamma_j \gamma(R) = \sum_{k \in \{1,2\}} R^{jk} \gamma_k,$$

where  $\gamma_1, \gamma_2$  are the gamma matrices introduced in Def. 4.2. We set  $\Gamma(R) = \gamma(R)^{\oplus N}$ . For  $\varphi \equiv (\bar{\varphi}, \varphi) \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$  we define

$$\mathbf{T}(x, R)\varphi \equiv (\mathbf{T}(x, R)\bar{\varphi}, \mathbf{T}(x, R)\varphi) \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$$

by the equalities

$$(\mathbf{T}(x, R)\bar{\varphi})(y) := \Gamma(R^{-1})^t \bar{\varphi}(R^{-1}(y - x)), \quad (\mathbf{T}(x, R)\varphi)(y) := \Gamma(R)\varphi(R^{-1}(y - x))$$

for all  $y \in \mathbb{R}^2$ . Let  $m \in \mathbb{N}_+$ . We say that a Schwartz distribution  $V \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{G}^m}$  is invariant under the symmetries of the plane iff

$$\langle V, \varphi_1 \otimes \dots \otimes \varphi_m \rangle = \langle V, \mathbf{T}(x, R)\varphi_1 \otimes \dots \otimes \mathbf{T}(x, R)\varphi_m \rangle \quad (8.1)$$

for all  $\varphi_1, \dots, \varphi_m \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$ , all  $x \in \mathbb{R}^2$  and all  $R$  as above. The action of  $\mathbf{T}(x, R)$  on  $\mathcal{S}(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}$  is defined in such a way that

$$\mathbf{T}(x, R)(\mathbf{J}\varphi) := \mathbf{J}(\mathbf{T}(x, R)\varphi)$$

for all  $\varphi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$ , where  $\mathbf{J}$  is the jet prolongation introduced in Definition 8.1. We say that a Schwartz distribution  $V \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{A}^m \times \mathbb{G}^m}$  is invariant under the symmetries of the plane iff the condition (8.1) is satisfied for all  $\varphi_1, \dots, \varphi_m \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}$ , all  $x \in \mathbb{R}^2$  and all  $R$  as above.

*Remark 8.3.* It is possible to choose the assignment  $R \mapsto \gamma(R)$  such that  $(x, R) \mapsto \mathbf{T}(x, R)$  is a projective representation of the Euclidean group on  $\mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$ . The choice of the assignment  $R \mapsto \gamma(R)$  does not play a role in what follows.

**Definition 8.4** (Symmetries of torus). A Schwartz distribution  $V \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{A}^m \times \mathbb{G}^m}$  is invariant under the symmetries of the torus iff the condition (8.1) is satisfied for all  $x \in \mathbb{R}^2$  and all  $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that are a rotation by a multiple of  $\pi/2$  or a reflection with respect to the line  $x^1 = 0$  or  $x^2 = 0$ .

**Definition 8.5** (Internal rotations). Let  $\varphi \equiv (\bar{\varphi}, \varphi) \equiv (\varphi^-, \varphi^+) \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}$  and a permutation  $\pi \in \mathcal{P}_N$ . We define  $\mathbf{T}(\pi)\varphi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$  by the equality

$$(\mathbf{T}(\pi)\varphi)^{a, (\pm, n, \alpha)}(x) := \varphi^{a, (\pm, \pi(n), \alpha)}(x)$$

for all  $a \in \mathbb{A}$ ,  $n \in \{1, \dots, N\}$ ,  $\alpha \in \{1, 2\}$  and  $x \in \mathbb{R}^2$ . Let  $m \in \mathbb{N}_+$ . We say that a Schwartz distribution  $V \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{A}^m \times \mathbb{G}^m}$  is invariant under the internal symmetries iff

$$\langle V, \varphi_1 \otimes \dots \otimes \varphi_m \rangle = \langle V, \mathbf{T}(\pi)\varphi_1 \otimes \dots \otimes \mathbf{T}(\pi)\varphi_m \rangle$$

for all  $\pi \in \mathcal{P}_N$  and all  $\varphi_1, \dots, \varphi_m \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}$ .

**Definition 8.6** (Charge conjugation). For  $\varphi \equiv (\bar{\varphi}, \varphi) \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}$  we define we define  $\mathbf{C}\varphi \equiv (\mathbf{C}\bar{\varphi}, \mathbf{C}\varphi) \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$  by the equalities

$$(\mathbf{C}\bar{\varphi})^a(x) = \Gamma_2 \varphi^a(x), \quad (\mathbf{C}\varphi)^a(x) = \Gamma_2 \bar{\varphi}^a(x)$$

for  $a \in \mathbb{A}$ ,  $x \in \mathbb{R}^2$ . Let  $m \in \mathbb{N}_+$ . We say that a distribution  $V \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{A}^m \times \mathbb{G}^m}$  is invariant under the charge conjugation symmetry iff

$$\langle V, \varphi_1 \otimes \dots \otimes \varphi_m \rangle = \langle V, \mathbf{C}\varphi_1 \otimes \dots \otimes \mathbf{C}\varphi_m \rangle$$

for all  $\varphi_1, \dots, \varphi_m \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$ .

*Remark 8.7.* One shows that if  $m \in \mathbb{N}_+ \setminus 2\mathbb{N}_+$  and  $V \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{A}^m \times \mathbb{G}^m}$  is invariant under the charge conjugation symmetry, then  $V = 0$ .

*Remark 8.8.* The only local functionals of degree two in  $\psi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}$  with up to one derivative compatible with all of the above symmetries are of the form

$$\int_{\mathbb{R}^2} \bar{\psi}(x) \cdot \psi(x) dx, \quad \int_{\mathbb{R}^2} \bar{\psi}(x) \cdot ((\Gamma_1 \partial_1 + \Gamma_2 \partial_2) \psi)(x) dx. \quad (8.2)$$

The only local functionals of degree four in  $\psi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}$  without derivatives compatible with all of the above symmetries are of the form

$$\int_{\mathbb{R}^2} (\bar{\psi}(x) \cdot \psi(x))^2 dx, \quad \sum_{j \in \{1,2\}} \int_{\mathbb{R}^2} (\bar{\psi}(x) \cdot \Gamma_j \psi(x))^2 dx, \quad \int_{\mathbb{R}^2} (\bar{\psi}(x) \cdot \Gamma_1 \Gamma_2 \psi(x))^2 dx. \quad (8.3)$$

For the proof of the above claims we refer the reader to [MW73, Appendix] or [DY23, Appendix C].

## 9 Weights

In this section we define the weights that are used in the definitions of various norms and establish some of their properties. The use of weights growing stretched exponentially allows to prove stretched exponential decay of truncated correlations. This choice of the weights will also play a role in the proof of the estimates for the map  $\mathbf{R}$  presented in Sec. 11.

**Definition 9.1.** The diameter of the set of points  $\{x_1, \dots, x_m\} \subset \mathbb{R}^2$  is defined by

$$D(x_1, \dots, x_m) := \max_{i,j \in \{1, \dots, m\}} |x_i - x_j|.$$

Let  $\zeta := 4/5$ . For  $m \in \mathbb{N}_+$ ,  $\nu \in [0, 1/2]$  and  $t \in (0, 1]$  we define the weights  $w_\nu \in C(\mathbb{R}^2)$  and  $w_{t;\nu}^m \in C(\mathbb{R}^{2m})$  by  $w_\nu(x) = (1 + |x|)^{-\nu}$  and

$$w_{t;\nu}^m(x_1, \dots, x_m) := (1 + |x_1|)^{-\nu} (1 + D(x_1, \dots, x_m))^{1/2-\nu} \exp(t^{-\zeta} D(x_1, \dots, x_m)^\zeta).$$

We set  $\tilde{w} := w_{1/2}$ ,  $w_t^m := w_{t;0}^m$ ,  $\tilde{w}_t^m := w_{t;1/2}^m$  and  $\zeta_\star := -2(1/\zeta - 7/8) = -3/4$ .

*Remark 9.2.* Note that the upper index  $m \in \mathbb{N}_+$  of  $w_{t;\nu}^m$  does not denote the power.

*Remark 9.3.* We will frequently use the fact that  $\tilde{w}_t^m(x_1, \dots, x_m) \leq w_t^m(x_1, \dots, x_m)$  for all  $m \in \mathbb{N}_+$ ,  $t \in (0, 1]$  and  $x_1, \dots, x_m \in \mathbb{R}^2$ . The weights  $\tilde{w}_t^m$  will be used to study the infinite volume limit. Observe that the weight  $\tilde{w}$  appeared already in Lemma 6.19.

*Remark 9.4.* It holds  $1/\tilde{w}(x) \leq 1/\tilde{w}(y) 1/\tilde{w}(x-y)$  for all  $x, y \in \mathbb{R}^2$ .

**Lemma 9.5.** *There exists  $C \in (0, \infty)$  such that for all  $l \in \{0, 1\}$ ,  $\nu \in [0, 1/2]$ ,  $t \in (0, 1]$ ,  $s \in (0, t)$ ,  $k \in \{1, \dots, m\}$  and  $x_1, \dots, x_m, y_1, \dots, y_m, y, z \in \mathbb{R}^2$  it holds*

- (a)  $w_{t;\nu}^m(x_1, \dots, x_m) \leq w_{t;\nu}^{k+1}(x_1, \dots, x_k, y) w_t^2(y, z) w_t^{m-k+1}(z, x_{k+1}, \dots, x_m),$
- (b)  $w_{t;\nu}^m(x_m, \dots, x_1) \leq w_t^{m-k+1}(z, x_m, \dots, x_{k+1}) w_t^2(y, z) w_{t;\nu}^{k+1}(y, x_k, \dots, x_1),$
- (c)  $w_{t;\nu}^m(x_1, \dots, x_m) \leq w_{t;\nu}^m(y_1, \dots, y_m) w_t^2(y_1, x_1) \dots w_t^2(y_m, x_m),$
- (d)  $w_{t;\nu}^k(x_1, \dots, x_k) \leq w_{t;\nu}^m(x_1, \dots, x_k, x_{k+1}, \dots, x_m),$
- (e)  $w_{t;\nu}^k(x_1, \dots, x_k) \leq w_t^m(x_1, \dots, x_k, x_{k+1}, \dots, x_m) w_\nu(x_m),$
- (f)  $\int_0^1 (1-u)^l D(x_1, \dots, x_m)^{l+1} w_{t;\nu}^m(ux_1, \dots, ux_m) du \leq C s^{l+1} (1-s/t)^{\zeta^*} w_{s;\nu}^m(x_1, \dots, x_m).$

*Remark 9.6.* The above lemma will be only used with  $\nu \in \{0, 1/2\}$ .

*Proof.* Items (a) and (b) follow from the bounds

$$\begin{aligned} D(x_1, \dots, x_m) &\leq D(x_1, \dots, x_k, y) + |y - z| + D(z, x_{k+1}, \dots, x_m), \\ |y| &\leq |y - z| + D(z, x_{k+1}, \dots, x_m) + |x_m|, \end{aligned}$$

which are consequences of the triangle inequality, and the bounds

$$(a + b + c)^\mu \leq a^\mu + b^\mu + c^\mu, \quad (1 + a + b + c)^\mu \leq (1 + a)^\mu (1 + b)^\mu (1 + c)^\mu$$

valid for  $\mu \in (0, 1]$  and  $a, b, c \in [0, \infty)$ . Item (c) is proved along the same lines with the use of the bounds

$$D(x_1, \dots, x_m) \leq D(y_1, \dots, y_m) + |y_1 - x_1| + \dots + |y_m - x_m|, \quad |y_1| \leq |y_1 - x_1| + |x_1|.$$

To prove Item (d) one uses the fact that the function

$$[0, \infty) \ni d \mapsto (1 + d)^{1/2-\nu} \exp(t^{-\zeta} d^\zeta) \in \mathbb{R}$$

is monotonic. Item (e) follows from Item (d), the bound  $|x_m| \leq |x_1| + D(x_1, \dots, x_m)$  and the inequality  $(1 + a + b)^\mu \leq (1 + a)^\mu (1 + b)^\mu$  valid for  $\mu \in (0, 1]$  and  $a, b \in [0, \infty)$ . We proceed to the proof of Item (f). Let  $D(x_1, \dots, x_m) = d$ . Observe that it holds

$$\begin{aligned} d^{l+1} \frac{w_t^{m;\nu}(ux_1, \dots, ux_m)}{w_s^{m;\nu}(x_1, \dots, x_m)} &= \frac{(1 + |x_1|)^\nu}{(1 + u|x_1|)^\nu} \frac{(1 + ud)^{1/2-\nu}}{(1 + d)^{1/2-\nu}} d^{l+1} \exp((u^\zeta t^{-\zeta} - s^{-\zeta})d^\zeta) \\ &\leq u^{-\nu} d^{l+1} \exp((u^\zeta t^{-\zeta} - s^{-\zeta})d^\zeta) \leq \hat{C} u^{-\nu} (s^{-\zeta} - u^\zeta t^{-\zeta})^{-(l+1)/\zeta}, \end{aligned}$$

where  $\hat{C} \in (0, \infty)$  is such that  $d^{l+1} \exp(-d^\zeta r) \leq \hat{C} r^{-(l+1)/\zeta}$  for all  $l \in \{0, 1\}$ ,  $r \in (0, \infty)$  and  $\zeta \in (0, 1]$ . Consequently, it holds

$$\int_0^1 (1-u)^l D(x_1, \dots, x_m)^{l+1} \frac{w_t^{m;\nu}(ux_1, \dots, ux_m)}{w_s^{m;\nu}(x_1, \dots, x_m)} du \leq \hat{C} \int_0^1 \frac{s^{l+1} u^{-\nu} (1-u)^l}{(1-u^\zeta (s/t)^\zeta)^{(l+1)/\zeta}} du$$

Note that for  $0 < s < t \leq 1$  and  $0 < u \leq 1$  it holds  $1 - u^\zeta \geq \zeta(1-u)$  and

$$(1-u^\zeta (s/t)^\zeta)^{(l+1)/\zeta} \geq \zeta^{(l+1)/\zeta} (1-u(s/t))^{(l+1)/\zeta} \geq \zeta^{(l+1)/\zeta} (1-u)^{\zeta_\#(l)} (1-s/t)^{(l+1)(1/\zeta-7/8)},$$

where  $\zeta_\#(l) := (l+1)/\zeta - (l+1)(1/\zeta - 7/8) > 0$ . Since  $l \in \{0, 1\}$  and  $l - \zeta_\#(l) > -1$  the bound stated in Item (f) holds true with

$$C = \max_{l \in \{0, 1\}} \zeta^{-2/\zeta} \hat{C} \int_0^1 \frac{u^{-1/2} (1-u)^l}{(1-u)^{\zeta_\#(l)}} du < \infty.$$

This finishes the proof.  $\square$

## 10 Spaces of kernels and functionals

In this section we define the spaces of kernels and functionals in infinite volume that are used in Sec. 13 to solve the system of equations (2.24), (2.28), (2.20), (2.21), (2.22) introduced in Sec. 2 using the Banach fixed point theorem. We also define the map  $\mathbf{A}_{r,\varepsilon;t,s}$  and analyse its properties. We refer the reader to Sec. 2 for the motivation behind the definitions stated in this section.

**Definition 10.1.** Let  $m, n \in \mathbb{N}_+$  and  $\mathcal{A}$  be a unital Banach algebra. We denote by  $\mathcal{S}(\mathbb{R}^{2m}, \mathcal{A})$  the space of Schwartz test functions valued in  $\mathcal{A}$  equipped with the usual family of semi-norms. The space of  $\mathcal{A}$ -valued Schwartz distributions is denoted by  $\mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})$  and by definition coincides with the space of continuous maps  $\mathcal{S}(\mathbb{R}^{2m}) \rightarrow \mathcal{A}$  equipped with the topology of pointwise convergence. We denote by

$$\langle \bullet, \bullet \rangle : \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A}) \times \mathcal{S}(\mathbb{R}^{2m}, \mathcal{A}) \rightarrow \mathcal{A}$$

the unique bilinear map such that  $\langle V, \varphi A \rangle := \langle V, \varphi \rangle A$  for all  $V \in \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^{2m})$  and  $A \in \mathcal{A}$ , where  $\langle V, \varphi \rangle$  is the standard pairing between a distribution and a test function. We denote by

$$\bullet \otimes \bullet : \mathcal{S}(\mathbb{R}^{2m}, \mathcal{A}) \times \mathcal{S}(\mathbb{R}^{2n}, \mathcal{A}) \rightarrow \mathcal{S}(\mathbb{R}^{m+n}, \mathcal{A})$$

the unique bilinear map such that  $\varphi A \otimes \psi B := (\varphi \otimes \psi) AB$  for all  $\varphi \in \mathcal{S}(\mathbb{R}^{2m})$ ,  $\psi \in \mathcal{S}(\mathbb{R}^{2n})$  and  $A, B \in \mathcal{A}$ , where  $\varphi \otimes \psi \in \mathcal{S}(\mathbb{R}^{2m+2n})$  is the standard tensor product of Schwartz functions. Let  $\mathbb{K}$  be a finite set. The maps

$$\begin{aligned} \langle \bullet, \bullet \rangle &: \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{K}} \times \mathcal{S}(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{K}} \rightarrow \mathcal{A}, \\ \bullet \otimes \bullet &: \mathcal{S}(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{K}^m} \times \mathcal{S}(\mathbb{R}^{2n}, \mathcal{A})^{\mathbb{K}^n} \rightarrow \mathcal{S}(\mathbb{R}^{2m+2n}, \mathcal{A})^{\mathbb{K}^{m+n}} \end{aligned}$$

are defined in analogy to Def. 3.7.

*Remark 10.2.* In what follows, we assume that  $\mathcal{G}$  is an infinite dimensional Grassmann algebra. It is important  $\mathcal{G}$  is infinite dimensional as we shall frequently use the fact that for every  $n \in \mathbb{N}_+$  there exist  $g_1, \dots, g_n \in \mathcal{G}$  such that  $g_1 \dots g_n \neq 0$ . We do not equip  $\mathcal{G}$  with any topology.

*Remark 10.3.* Let  $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$  be a graded unital algebra. We define  $\mathcal{A} \otimes_{\text{alg}} \mathcal{G}$  to be the algebraic graded tensor product of graded algebras  $\mathcal{A}$  and  $\mathcal{G}$ . In particular, the product in  $\mathcal{A} \otimes_{\text{alg}} \mathcal{G}$  satisfies the condition  $(A \otimes g)(B \otimes h) = (-1)^{\deg(g) \deg(B)} AB \otimes gh$  for all  $A, B \in \mathcal{A}$  and  $gh \in \mathcal{G}$  of definite parity. We identify  $A \in \mathcal{A}$  and  $g \in \mathcal{G}$  with

$$A \otimes 1 \in \mathcal{A} \otimes_{\text{alg}} \mathcal{G} \quad \text{and} \quad 1 \otimes g \in \mathcal{A} \otimes_{\text{alg}} \mathcal{G},$$

respectively. We use these identifications to make sense of  $A + g, Ag \in \mathcal{A} \otimes_{\text{alg}} \mathcal{G}$ . Let  $m \in \mathbb{N}_+$  and  $\mathbb{K}$  be a finite set. We define the paring

$$\langle \bullet, \bullet \rangle : \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{K}^m} \times (\mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{K}^m} \otimes_{\text{alg}} \mathcal{G}) \rightarrow \mathcal{A} \otimes_{\text{alg}} \mathcal{G}$$

as the unique bilinear map such that  $\langle V, \varphi \otimes g \rangle := \langle V, \varphi \rangle \otimes g$  for all  $V \in \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{K}}$ ,  $\varphi \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{K}}$  and  $g \in \mathcal{G}$ . We denote by

$$\bullet \otimes \bullet : (\mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{K}^m} \otimes_{\text{alg}} \mathcal{G}) \times (\mathcal{S}'(\mathbb{R}^{2n}, \mathcal{A})^{\mathbb{K}^n} \otimes_{\text{alg}} \mathcal{G}) \rightarrow \mathcal{S}'(\mathbb{R}^{2m+2n}, \mathcal{A})^{\mathbb{K}^{m+n}} \otimes_{\text{alg}} \mathcal{G}$$

the unique bilinear map such that  $(\varphi \otimes g) \otimes (\psi \otimes h) := (-1)^{\deg(g) \deg(\psi)} (\varphi \otimes \psi) \otimes gh$  for all  $\varphi \in \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{K}^m}$ ,  $\psi \in \mathcal{S}'(\mathbb{R}^{2n}, \mathcal{A})^{\mathbb{K}^n}$  and  $g, h \in \mathcal{G}$  of definite parity.

**Definition 10.4.** Let  $m \in \mathbb{N}_+$  and  $\mathcal{A}$  be a unital Banach algebra. We denote by  $\mathcal{P}_m$  the group of permutations of the set  $\{1, \dots, m\}$ . The antisymmetric part of a distribution  $V \in \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{A}^m \times \mathbb{G}^m}$  is the distribution  $\mathbf{S}V \in \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{A}^m \times \mathbb{G}^m}$  defined by

$$\langle \mathbf{S}V, \varphi_1 \otimes \dots \otimes \varphi_m \rangle = \frac{1}{m!} \sum_{\pi \in \mathcal{P}_m} (-1)^{\text{sgn}(\pi)} \langle V, \varphi_{\pi(1)} \otimes \dots \otimes \varphi_{\pi(m)} \rangle$$

for all  $\varphi_1, \dots, \varphi_m \in \mathcal{S}'(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}$ . We say that a distribution  $V \in \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{A}^m \times \mathbb{G}^m}$  is antisymmetric iff  $V = \mathbf{S}V$ .

*Remark 10.5.* Recall that  $\mathcal{B} = \mathcal{B}(\mathcal{H})$  is the algebra of bounded operators acting on the Hilbert space  $\mathcal{H}$  introduced in Def. 6.2

**Definition 10.6.** Let  $m \in \mathbb{N}_+$  and  $\mathcal{A}$  be a unital Banach subalgebra of  $\mathcal{B}$ . We denote by  $\mathcal{M}^m(\mathcal{A})$  the vector space of Schwartz distributions  $V \in \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})$  such that there exists a kernel  $V_K : \mathbb{R}^2 \times \text{Borel}(\mathbb{R}^{2(m-1)}) \rightarrow \mathcal{A}$  satisfying the following conditions:

- (1) for every  $A \in \text{Borel}(\mathbb{R}^{2(m-1)})$  the map  $x \mapsto V_K(x, A)$  is Borel measurable,
- (2) for every  $x \in \mathbb{R}^2$  the map  $A \mapsto V_K(x, A)$  is a Banach space valued measure,

(3) the following norm

$$\|V\|_{\mathcal{M}^m} := \sup_{x_1 \in \mathbb{R}^2} \int_{\mathbb{R}^{2(m-1)}} \|V_K(x_1, dx_2 \dots dx_m)\|_{\mathcal{B}}$$

is finite, where  $\|V_K(x_1, dx_2 \dots dx_m)\|_{\mathcal{B}}$  denotes the variation of  $V_K(x_1, dx_2 \dots dx_m)$ ,

(4) for all  $\varphi \in \mathcal{S}(\mathbb{R}^{2m})$  it holds

$$\langle V, \varphi \rangle = \int_{\mathbb{R}^{2m}} V_K(x_1, dx_2 \dots dx_m) \varphi(x_1, \dots, x_m) dx_1. \quad (10.1)$$

We write  $\mathcal{M}^m := \mathcal{M}^m(\mathbb{C})$ .

*Remark 10.7.* Note that it holds  $\mathcal{M}^m(\mathbb{C}) \subset \mathcal{M}^m(\mathcal{A}) \subset \mathcal{M}^m(\mathcal{B})$ .

*Remark 10.8.* Let  $K : \mathbb{R}^{2m} \rightarrow \mathcal{B}$  be measurable such that

$$\|K\|_{\mathcal{K}^m} := \sup_{x_1 \in \mathbb{R}^2} \int_{\mathbb{R}^{2(m-1)}} \|K(x_1, \dots, x_m)\|_{\mathcal{B}} dx_2 \dots dx_m < \infty.$$

Then  $V \in \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{B})$  defined by Eq. (10.1) with

$$V_K(x_1, dx_2 \dots dx_m) = K(x_1, \dots, x_m) dx_2 \dots dx_m$$

belongs to  $\mathcal{M}^m(\mathcal{B})$  and  $\|V\|_{\mathcal{M}^m} = \|K\|_{\mathcal{K}^m}$ .

*Remark 10.9.* Let  $\delta^{(m)} \in \mathcal{S}'(\mathbb{R}^{2m})$  be the Dirac measure on the diagonal, i.e.

$$\langle \delta^{(m)}, \varphi_1 \otimes \dots \otimes \varphi_m \rangle := \int_{\mathbb{R}^2} \varphi_1(x) \dots \varphi_m(x) dx$$

for all  $\varphi_1, \dots, \varphi_m \in \mathcal{S}(\mathbb{R}^2)$ . Then  $\delta^{(m)} \in \mathcal{M}^m$  and  $\|\delta^{(m)}\|_{\mathcal{M}^m} = 1$ .

*Remark 10.10.* It follows from the definition of the variation of the vector measure and the properties (0) and (4) of the conditional expected value introduced in Def. 6.7 that

$$\mathbf{E}_t V \in \mathcal{M}^m(\mathcal{F}_t), \quad \|\mathbf{E}_t V\|_{\mathcal{M}^m} \leq \|V\|_{\mathcal{M}^m}$$

for all  $V \in \mathcal{M}^m(\mathcal{F})$  and  $t \in [0, 1]$ .

*Remark 10.11.* Recall that  $w_1^m$  and  $\tilde{w}_1^m$  are weights introduced in Def. 9.1.

**Definition 10.12.** Let  $m \in \mathbb{N}_+$  and  $\mathcal{A}$  be a unital Banach subalgebra of  $\mathcal{B}$ . The vector space  $\mathcal{N}_{\#}^m(\mathcal{A}) \subset \mathcal{M}^m(\mathcal{A})^{\mathbb{A}^m \times \mathbb{G}^m}$  consists of Schwartz distributions

$$V \equiv (V^{a, \sigma})_{a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m} \in \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{A}^m \times \mathbb{G}^m}$$

such that the following norm

$$\|V\|_{\mathcal{N}^m} := \sum_{a \in \mathbb{A}^m} \sum_{\sigma \in \mathbb{G}^m} \|w_1^m V^{a, \sigma}\|_{\mathcal{M}^m}$$



is finite. We define the norm  $\|V\|_{\mathcal{N}^m}$  in an analogous way with  $w_1^m$  replaced by  $\tilde{w}_1^m$ . We denote by  $\mathcal{N}^m(\mathcal{A})$  the subspace of  $\mathcal{N}_\#^m(\mathcal{A})$  consisting of antisymmetric Schwartz distributions, cf. Def. 10.4. The vector space  $\mathcal{N}(\mathcal{A}) := \bigtimes_{m \in \mathbb{N}_+} \mathcal{N}^m(\mathcal{A})$  consists of tuples

$$V \equiv (V^m)_{m \in \mathbb{N}_+} \equiv (V^{m,a,\sigma})_{m \in \mathbb{N}_+, a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m}.$$

We endow  $\mathcal{N}(\mathcal{A})$  with the product topology. The subspace  $\mathcal{N}^{\text{fin}}(\mathcal{A}) \subset \mathcal{N}(\mathcal{A})$  consists of tuples  $V$  such that  $V^m = 0$  for all but finitely many  $m \in \mathbb{N}_+$ . We set  $\Pi^m V := V^m$  and  $\Pi^{m,a,\sigma} V := V^{m,a,\sigma}$ . For  $V \in \mathcal{N}(\mathcal{A})$  and  $k \in \mathbb{N}_+$  we define  $\Pi_k V \equiv ((\Pi_k V)^m)_{m \in \mathbb{N}_+} \in \mathcal{N}(\mathcal{A})$  by the equalities  $(\Pi_k V)^m := 0$  for all  $m \in \mathbb{N}_+ \setminus \{k\}$  and  $(\Pi_k V)^m := V^m$  for  $m = k$ . We also define  $\Pi_{>k} V \equiv ((\Pi_{>k} V)^m)_{m \in \mathbb{N}_+} \in \mathcal{N}(\mathcal{A})$  by the equalities  $(\Pi_{>k} V)^m := 0$  for all  $m \in \{1, \dots, k\}$  and  $(\Pi_{>k} V)^m := V^m$  for  $m \in \mathbb{N}_+ \setminus \{1, \dots, k\}$ . For  $V \in \mathcal{N}^{\text{fin}}(\mathcal{A})$  and  $\varphi \in \mathcal{S}(\mathbb{R}^2, \mathcal{A})^{\mathbb{A} \times \mathbb{G}} \otimes_{\text{alg}} \mathcal{G}$  we write

$$V[\varphi] := \sum_{m \in \mathbb{N}_+} \langle V^m, \varphi^{\otimes m} \rangle \in \mathcal{A} \otimes_{\text{alg}} \mathcal{G}.$$

For  $k \in \mathbb{N}_0$  we define  $D_\varphi^k V[\varphi]$  in analogy to Def. 3.14. We set  $\mathcal{N}_\#^m := \mathcal{N}_\#^m(\mathbb{C})$ ,  $\mathcal{N}^m := \mathcal{N}^m(\mathbb{C})$ ,  $\mathcal{N} := \mathcal{N}(\mathbb{C})$  and  $\mathcal{N}^{\text{fin}} := \mathcal{N}^{\text{fin}}(\mathbb{C})$ .

*Remark 10.13.* Let  $V \in \mathcal{N}_\#^m(\mathcal{A})$ . Since the norm  $\|V\|_{\mathcal{N}^m} < \infty$  involves a weight of stretched exponential growth it holds  $\mathcal{X}V^{a,\sigma} \in \mathcal{M}$  for all  $a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m$  and all translationally-invariant polynomials  $\mathcal{X} \in C^\infty(\mathbb{R}^{2m})$ .

*Remark 10.14.* Using the fact that  $V \in \mathcal{N}^m(\mathcal{A})$  is antisymmetric one shows that it is uniquely determined by the map

$$\mathcal{S}(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^- \ni \varphi \mapsto \langle V, \varphi^{\otimes m} \rangle \in \mathcal{A} \otimes_{\text{alg}} \mathcal{G}.$$

Actually, given  $\psi_1, \dots, \psi_m \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}$  we choose  $\varphi = \sum_{j=1}^m \psi_j g_j$ , where  $g_1, \dots, g_m \in \mathcal{G}$  are such that  $g = g_1 \dots g_m \neq 0$ . Then  $m! \langle V, \psi_1 \otimes \dots \otimes \psi_m \rangle \otimes g = \langle V, \varphi^{\otimes m} \rangle$ .

**Definition 10.15.** Let  $m \in \mathbb{N}_+$ ,  $\alpha \in (0, \infty)$ ,  $\beta, \gamma \in [0, \infty)$  and  $\rho_{\gamma, \kappa}(m) := \gamma + 2\kappa m$ . For continuous maps

$$(0, 1] \ni s \mapsto V_s^m \equiv (V_s^{m,a,\sigma})_{a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m} \in \mathcal{N}^m(\mathcal{B})$$

we define

$$\|V_\bullet^m\|_{\mathcal{V}^{m;\gamma}} := \sup_{s \in (0,1]} \sum_{a \in \mathbb{A}^m} \sum_{\sigma \in \mathbb{G}^m} \lambda_s^{-\rho_{\gamma, \kappa}(m)} s^{2-m/2-|a|} \|w_s^m V_s^{m,a,\sigma}\|_{\mathcal{M}^m}. \quad (10.2)$$

The vector space  $\mathcal{V}^{m;\gamma}$  consists of continuous maps

$$(0, 1] \ni s \mapsto V_s^m \equiv (V_s^{m,a,\sigma})_{a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m} \in \mathcal{N}^m \equiv \mathcal{N}^m(\mathbb{C})$$

such that  $\|V_\bullet^m\|_{\mathcal{V}^{m;\gamma}} < \infty$ . The vector space  $\mathcal{V}^\gamma$  consists of continuous maps

$$(0, 1] \ni s \mapsto V_s \in \mathcal{N} \equiv \mathcal{N}(\mathbb{C})$$

such that  $V_\bullet^m \equiv \Pi^m V_\bullet \in \mathcal{V}^{m;\gamma}$  for all  $m \in \mathbb{N}_+$ . The vector space  $\mathcal{V}^{\text{fin};\gamma}$  consists of continuous maps

$$(0, 1] \ni s \mapsto V_s \in \mathcal{N}^{\text{fin}} \equiv \mathcal{N}^{\text{fin}}(\mathbb{C})$$

such that  $V_\bullet^m \equiv \Pi^m V_\bullet \in \mathcal{V}^{m;\gamma}$  for all  $m \in \mathbb{N}_+$ . We denote by  $\mathcal{V}^{\alpha,\beta;\gamma}$  the closure of  $\mathcal{V}^{\text{fin};\gamma} \subset \mathcal{V}^\gamma$  with respect to the norm

$$\|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}} := \sup_{m \in \mathbb{N}_+} \alpha^m m^\beta \|V_\bullet^m\|_{\mathcal{V}^{m;\gamma}}. \quad (10.3)$$

For  $V_\bullet^m \in \mathcal{V}^{m;\gamma}$  we define  $\|V_\bullet^m\|_{\tilde{\mathcal{V}}^{m;\gamma}}$  in an analogous way to  $\|V_\bullet^m\|_{\mathcal{V}^{m;\gamma}}$  with  $w_s^m$  replaced by  $\tilde{w}_s^m$  in Eq. (10.2). For  $V_\bullet \in \mathcal{V}^{\alpha,\beta;\gamma}$  we define  $\|V_\bullet\|_{\tilde{\mathcal{V}}^{\alpha,\beta;\gamma}}$  in an analogous way to  $\|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$  with  $\|\bullet\|_{\mathcal{V}^{m;\gamma}}$  replaced by  $\|\bullet\|_{\tilde{\mathcal{V}}^{m;\gamma}}$  in Eq. (10.3).

*Remark 10.16.* Since  $\tilde{w}_t^m \leq w_t^m$  the norms  $\|\bullet\|_{\tilde{\mathcal{V}}^{m;\gamma}}$  and  $\|\bullet\|_{\tilde{\mathcal{V}}^{\alpha,\beta;\gamma}}$  are weaker than the norms  $\|\bullet\|_{\mathcal{V}^{m;\gamma}}$  and  $\|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$ , respectively. The norms  $\|\bullet\|_{\tilde{\mathcal{V}}^{m;\gamma}}$  and  $\|\bullet\|_{\tilde{\mathcal{V}}^{\alpha,\beta;\gamma}}$  will be used to study the infinite volume limit. Observe that we did not define the spaces  $\tilde{\mathcal{V}}^{m;\gamma}$  and  $\tilde{\mathcal{V}}^{\alpha,\beta;\gamma}$ .

*Remark 10.17.* Note that the weight  $\tilde{w}_t^m$  used in the definition of the norms  $\|\bullet\|_{\tilde{\mathcal{V}}^{m;\gamma}}$  and  $\|\bullet\|_{\tilde{\mathcal{V}}^{\alpha,\beta;\gamma}}$  is not invariant under permutations of its arguments. For a permutation  $\pi \in \mathcal{P}_m$  define  $\mathbf{S}_\pi \tilde{w}_t^m \in C(\mathbb{R}^{2m})$  by the equality  $(\mathbf{S}_\pi \tilde{w}_t^m)(x_1, \dots, x_m) := \tilde{w}_t^m(x_{\pi(1)}, \dots, x_{\pi(m)})$ . Using the fact that

$$V^m \equiv (V^{m,a,\sigma})_{a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m} \in \mathcal{N}^m(\mathcal{B}) \subset \mathcal{S}'(\mathbb{R}^{2m}, \mathcal{A})^{\mathbb{A}^m \times \mathbb{G}^m}$$

is antisymmetric one shows that

$$\sum_{a \in \mathbb{A}^m} \sum_{\sigma \in \mathbb{G}^m} \|(\mathbf{S}_\pi \tilde{w}_s^m) V^{m,a,\sigma}\|_{\mathcal{M}^m} = \sum_{a \in \mathbb{A}^m} \sum_{\sigma \in \mathbb{G}^m} \|\tilde{w}_s^m V^{m,a,\sigma}\|_{\mathcal{M}^m}.$$

*Remark 10.18.* For all  $\alpha, \tilde{\alpha} \in [1, \infty)$ ,  $\beta, \tilde{\beta}, \gamma, \tilde{\gamma} \in [0, \infty)$  and  $V_\bullet \in \mathcal{V}^{\alpha,\beta;\gamma}$  such that  $\tilde{\alpha} \leq \alpha$ ,  $\tilde{\beta} < \beta$ ,  $\tilde{\gamma} \leq \gamma$  it holds

$$\|V_\bullet\|_{\mathcal{V}^{\tilde{\alpha},\tilde{\beta};\tilde{\gamma}}} \leq \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}, \quad \|V_\bullet\|_{\tilde{\mathcal{V}}^{\tilde{\alpha},\tilde{\beta};\tilde{\gamma}}} \leq \|V_\bullet\|_{\tilde{\mathcal{V}}^{\alpha,\beta;\gamma}}.$$

*Remark 10.19.* For all  $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} \in [1, \infty)$ ,  $k \in \mathbb{N}_+$  there exists  $C \in (0, \infty)$  such that for all  $\gamma \in [0, \infty)$  and  $V_\bullet \in \mathcal{V}^{\alpha,\beta;\gamma}$  it holds

$$\|\Pi^k V_\bullet\|_{\mathcal{V}^{k;\gamma}} \leq C \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}, \quad \|\Pi_k V_\bullet\|_{\mathcal{V}^{\tilde{\alpha},\tilde{\beta};\gamma}} \leq C \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}.$$

**Lemma 10.20.** *For all  $m \in \mathbb{N}_+$  and  $\alpha \in (0, \infty)$ ,  $\beta, \gamma \in [0, \infty)$  the spaces  $(\mathcal{V}^{m;\gamma}, \|\bullet\|_{\mathcal{V}^{m;\gamma}})$  and  $(\mathcal{V}^{\alpha,\beta;\gamma}, \|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}})$  are Banach spaces.*

*Proof.* We first observe that for every  $m \in \mathbb{N}_+$  the space  $(\mathcal{M}^m(\mathbb{C}), \|\bullet\|_{\mathcal{M}^m})$  is a Banach space and given a Banach space  $\mathcal{T}$  the spaces  $C_b((0, 1], \mathcal{T})$  and  $l_0^\infty(\mathcal{T})$  are also Banach spaces. Using the fact that  $\lambda_s^{-\rho_{\gamma,\kappa}(m)} s^{2-m/2-|a|} w_s^m \neq 0$  for all  $s \in (0, 1]$ ,  $a \in \mathbb{A}^m$  we conclude that the space  $(\mathcal{V}^{m;\gamma}, \|\bullet\|_{\mathcal{V}^{m;\gamma}})$  is a Banach space. Similarly, since  $\alpha^m m^\beta \neq 0$  for all  $m \in \mathbb{N}_+$  the space  $(\mathcal{V}^{\alpha,\beta;\gamma}, \|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}})$  is a Banach space.  $\square$

*Remark 10.21.* Recall that  $\mathcal{F}$  is the Banach subalgebra of  $\mathcal{B} = \mathcal{B}(\mathcal{H})$  generated by the white noise, cf. Def. 6.2.

**Definition 10.22.** Let  $\tau, \varepsilon \in [0, 1]$  and  $t, s \in (0, 1]$ . The maps

$$\mathbf{A}_{\tau, \varepsilon; t, s}, \mathbf{C}_{\tau, \varepsilon; t, s} : \mathcal{N}^{\text{fin}}(\mathcal{F}) \rightarrow \mathcal{N}^{\text{fin}}(\mathcal{F})$$

are defined by the equalities

$$\mathbf{A}_{\tau, \varepsilon; t, s} V[\varphi] := V[\varphi + \mathbf{J}\Psi_{\tau, \varepsilon; t \vee s, s}] - V[\mathbf{J}\Psi_{\tau, \varepsilon; t \vee s, s}], \quad \varphi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^-,$$

$$\mathbf{C}_{\tau, \varepsilon; t, s} V := \mathbf{A}_{\tau, \varepsilon; t, s} V - V,$$

where  $\Psi_{\tau, \varepsilon; t, s}$  was introduced in Def. 6.11 and  $\mathbf{J}$  is the jet prolongation introduced in Def. 8.1. We also set  $\mathbf{A}_{\tau, \varepsilon; t, s}^m := \Pi^m \mathbf{A}_{\tau, \varepsilon; t, s}$ ,  $\mathbf{A}_{t, s}^{m, a} := \Pi^{m, a} \mathbf{A}_{\tau, \varepsilon; t, s}$  and  $\mathbf{A}_{t, s}^{m, a, \sigma} := \Pi^{m, a, \sigma} \mathbf{A}_{\tau, \varepsilon; t, s}$  and analogously for the map  $\mathbf{C}_{\tau, \varepsilon; t, s}$ . We omit  $\tau$  and  $\varepsilon$  if  $\tau = 0$  and  $\varepsilon = 0$ .

*Remark 10.23.* The equation defining the map  $\mathbf{A}_{\tau, \varepsilon; t, s}$  should be interpreted as

$$\langle \mathbf{A}_{\tau, \varepsilon; t, s}^m V, \varphi^{\otimes m} \rangle = \sum_{k \in \mathbb{N}_0} \frac{(m+k)!}{m!k!} \langle V^{m+k}, \varphi^{\otimes m} \otimes \mathbf{J}\Psi_{\tau, \varepsilon; t \vee s, s}^{\otimes k} \rangle \quad (10.4)$$

for all  $m \in \mathbb{N}_+$ . The map  $\mathbf{A}_{\tau, \varepsilon; t, s}$  is well defined thanks to Remark 10.14 and the fact that  $\|\mathbf{J}\Psi_{\tau, \varepsilon; t \vee s, s}^{a, \sigma}\|_{\mathcal{G}} < \infty$  for all  $\tau, \varepsilon \in [0, 1]$ ,  $t, s \in (0, 1]$  and  $a \in \mathbb{A}$ ,  $\sigma \in \mathbb{G}$ , which follows from Lemma 6.19.

*Remark 10.24.* Let  $t, s \in (0, 1]$  be such that  $t \geq s$  and let  $V \in \mathcal{N}^{\text{fin}}(\mathcal{F}_s)$ . Then we have

$$\mathbf{A}_{\tau, \varepsilon; t, s} V \in \mathcal{N}^{\text{fin}}(\mathcal{F}_t), \quad \mathbf{E}_t \mathbf{A}_{\tau, \varepsilon; t, s} V = \mathbf{E} \mathbf{A}_{\tau, \varepsilon; t, s} V, \quad \mathbf{E}_s \mathbf{A}_{\tau, \varepsilon; t, s} V = \mathbf{A}_{\tau, \varepsilon; t, s} \mathbf{E} V$$

by Def. 6.11 of  $\Psi_{\tau, \varepsilon; t, s}$  and the properties (1) and (3) of the conditional expected value introduced in Def. 6.7.

*Remark 10.25.* Let  $t, u, s \in (0, 1]$  be such that  $t \geq u \geq s$  and let  $V \in \mathcal{N}^{\text{fin}}(\mathbb{C})$ . Then we have

- (A)  $\mathbf{A}_{\tau, \varepsilon; t, s} V = \mathbf{A}_{\tau, \varepsilon; t, u} \mathbf{A}_{\tau, \varepsilon; u, s} V$ ,
- (B)  $\mathbf{A}_{\tau, \varepsilon; t, u} \mathbf{E} \mathbf{A}_{\tau, \varepsilon; u, s} V = \mathbf{E}_u \mathbf{A}_{\tau, \varepsilon; t, u} \mathbf{A}_{\tau, \varepsilon; u, s} V = \mathbf{E}_u \mathbf{A}_{\tau, \varepsilon; t, s} V$ ,
- (C)  $\mathbf{E} \mathbf{A}_{\tau, \varepsilon; t, u} \mathbf{E} \mathbf{A}_{\tau, \varepsilon; u, s} V = \mathbf{E} \mathbf{A}_{\tau, \varepsilon; t, s} V$

by Remark 10.24 and the tower property (2) of the conditional expected value introduced in Def. 6.7.

**Lemma 10.26.** *There exists  $\lambda_\star \in (0, 1]$  such that for all  $\lambda \in (0, \lambda_\star]$  the following is true. Let  $m \in \mathbb{N}_+$ ,  $\alpha, \beta \in [1, \infty)$ ,  $\gamma \in [0, \infty)$ ,  $\tau, \varepsilon \in [0, 1]$  and  $V_\bullet \in \mathcal{V}^{\text{fin}; \gamma}$ . For all  $t \in (0, 1]$  the map*

$$(0, 1] \ni s \mapsto \mathbf{A}_{\tau, \varepsilon; t, s}^m V_s \in \mathcal{N}^m(\mathcal{F}_{t, s}) \quad (10.5)$$

*is continuous and it holds*

- (A)  $\|s \mapsto \mathbf{A}_{\tau,\varepsilon;t,s} V_s\|_{\mathcal{V}^{\alpha/2,\beta;\gamma}} \leq \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}},$
- (B)  $\|s \mapsto \mathbf{C}_{\tau,\varepsilon;t,s} V_s\|_{\mathcal{V}^{\alpha/2,\beta-1;\gamma}} \leq \lambda^\kappa \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}},$
- (C)  $\|s \mapsto \mathbf{A}_{\tau,\varepsilon;t,s} V_s\|_{\tilde{\mathcal{V}}^{\alpha/2,\beta;\gamma}} \leq \|V_\bullet\|_{\tilde{\mathcal{V}}^{\alpha,\beta;\gamma}},$
- (D)  $\|s \mapsto \mathbf{C}_{\tau,\varepsilon;t,s} V_s\|_{\tilde{\mathcal{V}}^{\alpha/2,\beta-1;\gamma}} \leq \lambda^\kappa \|V_\bullet\|_{\tilde{\mathcal{V}}^{\alpha,\beta;\gamma}},$
- (E)  $\|s \mapsto (\mathbf{A}_{t,s} - \mathbf{A}_{\tau,\varepsilon;t,s}) V_s\|_{\tilde{\mathcal{V}}^{\alpha/2,\beta-1;\gamma}} \leq \lambda_{\tau\vee\varepsilon}^\kappa \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}.$

In particular the map (10.5) is well-defined for all  $V_\bullet \in \mathcal{V}^{\alpha,\beta;\gamma}$ .

*Remark 10.27.* Note that the parameters  $\alpha$  of the norms  $\mathcal{V}^{\alpha,\beta;\gamma}$ ,  $\tilde{\mathcal{V}}^{\alpha,\beta;\gamma}$  that appear on both sides of the bounds stated in the above lemma are different. As we discussed in Sec. 2 the bounds of this type appear to be false if the parameters  $\alpha$  of the norms are the same on both sides of the bounds. This is the main reason why we work with the space  $(\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}, \|\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}})$  introduced below. One of the consequences of the above lemma is the relation between the norms  $\|\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}}$  and  $\|\bullet\|_{\mathcal{V}^{2\alpha,\beta;\gamma}}$  stated in Remark 10.35.

*Proof.* First recall that  $\lambda_s \leq \lambda$  by Def. 2.2 and

$$\|\mathbf{J}\Psi_{\tau,\varepsilon;t,s}^{a,\sigma}\|_{\mathcal{C}} \leq \lambda^{-\kappa} s^{-1/2}, \quad \|\tilde{w}(\mathbf{J}\Psi_{t,s}^{a,\sigma} - \mathbf{J}\Psi_{\tau,\varepsilon;t,s}^{a,\sigma})\|_{\mathcal{C}} \leq \lambda^{-\kappa} \lambda_{\tau\vee\varepsilon}^\kappa \lambda_s^{-\kappa} s^{-1/2}$$

by Lemma 6.19. Noting that  $\rho_{\gamma,\kappa}(m) = \rho_{\gamma,\kappa}(m+k) - 2\kappa k$  and using Lemma 9.5 (d) we obtain

$$\begin{aligned} & \sum_{a \in \mathbb{A}^m} \sum_{\sigma \in \mathbb{G}^m} \lambda_s^{-\rho_{\gamma,\kappa}(m)} s^{2-m/2-|a|} \|w_s^m \mathbf{A}_{\tau,\varepsilon;t,s}^{m,a,\sigma} V_s\|_{\mathcal{M}^m} \\ & \leq \sum_{k \in \mathbb{N}_0} \sum_{a \in \mathbb{A}^{m+k}} \sum_{\sigma \in \mathbb{G}^{m+k}} \frac{(m+k)!}{m!k!} \lambda_s^{-\rho_{\gamma,\kappa}(m+k)+\kappa k} s^{2-m/2-k/2-|a|} \|w_s^{m+k} V_s^{m+k,a,\sigma}\|_{\mathcal{M}^{m+k}}. \end{aligned}$$

Using Definition 10.15 of the norms  $\|\bullet\|_{\mathcal{V}^{m+k;\gamma}}$  and  $\|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$  we estimate the expression on the RHS of the above bound by

$$\begin{aligned} \sum_{k \in \mathbb{N}_0} \frac{(m+k)!}{m!k!} \lambda^{\kappa k} \|V_\bullet^{m+k}\|_{\mathcal{V}^{m+k;\gamma}} & \leq \sum_{k \in \mathbb{N}_0} \frac{(m+k)!}{m!k!} \lambda^{\kappa k} \alpha^{-m-k} (m+k)^{-\beta} \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}} \\ & \leq \alpha^{-m} m^{-\beta} (1 - \lambda^\kappa)^{-m-1} \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}} \leq 2^m \alpha^{-m} m^{-\beta} \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}. \end{aligned}$$

The last bound above is true of  $\lambda^\kappa \leq 1/5$ . This proves the bound (A). Using analogous estimates as above with the sum over  $k \in \mathbb{N}_0$  replaced by the sum over  $k \in \mathbb{N}_+$  we obtain

$$\begin{aligned} & \sum_{a \in \mathbb{A}^m} \sum_{\sigma \in \mathbb{G}^m} \lambda_s^{-\rho_{\gamma,\kappa}(m)} s^{2-m/2-|a|} \|w_s^m (\mathbf{A}_{\tau,\varepsilon;t,s}^{m,a,\sigma} V_s - V_s^{m,a,\sigma})\|_{\mathcal{M}^m} \\ & \leq \alpha^{-m} (m+1)^{-\beta} ((1 - \lambda^\kappa)^{-m-1} - 1) \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}} \leq \lambda^\kappa 2^m \alpha^{-m} m^{1-\beta} \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}, \end{aligned}$$

where again the last bound above is true of  $\lambda^\kappa \leq 1/5$ . This implies the bound (B). To prove the bounds (C) and (D) we use exactly the same strategy but with the weight  $w_s^m$  replaced

by  $\tilde{w}_s^m$ . Let us turn to the proof of the bound (E). By Lemma 9.5 (e) we have

$$\begin{aligned} & \lambda^\kappa \lambda_{\tau \vee \varepsilon}^{-\kappa} \sum_{a \in \mathbb{A}^m} \sum_{\sigma \in \mathbb{G}^m} \lambda_s^{-\rho_{\gamma, \kappa}(m)} s^{2-m/2-|a|} \|\tilde{w}_s^m (\mathbf{A}_{t,s}^{m,a,\sigma} - \mathbf{A}_{\tau, \varepsilon; t, s}^{m,a,\sigma}) V_s\|_{\mathcal{M}^m} \\ & \leq \sum_{k \in \mathbb{N}_+} \sum_{a \in \mathbb{A}^{m+k}} \sum_{\sigma \in \mathbb{G}^{m+k}} \frac{(m+k)!}{m!(k-1)!} \lambda_s^{-\rho_{\gamma, \kappa}(m+k)+\kappa k} s^{2-m/2-k/2-|a|} \|w_s^{m+k} V_s^{m+k,a,\sigma}\|_{\mathcal{M}^{m+k}}. \end{aligned}$$

Using again Definition 10.15 of the norms  $\|\bullet\|_{\mathcal{V}^{m+k;\gamma}}$  and  $\|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$  we estimate the expression on the RHS of the above bound by

$$\begin{aligned} & \sum_{k \in \mathbb{N}_+} \frac{(m+k)!}{m!(k-1)!} \lambda^{\kappa k} \|V_\bullet^{m+k}\|_{\mathcal{V}^{m+k;\gamma}} \leq \sum_{k \in \mathbb{N}_+} \frac{(m+k)!}{m!(k-1)!} \lambda^{\kappa k} \alpha^{-m-k} (m+k)^{-\beta} \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}} \\ & \leq \lambda^\kappa (m+1) \alpha^{-m} (m+1)^{-\beta} (1-\lambda^\kappa)^{-m-2} \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}} \leq \lambda^\kappa 2^m \alpha^{-m} m^{1-\beta} \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}. \end{aligned}$$

The last bound above is true of  $\lambda^\kappa \leq 1/5$ . This proves the bound (E).  $\square$

**Definition 10.28.** We say that a distribution  $V \in \mathcal{N}^m \subset \mathcal{N}_\#^m$  is compatible with the flow of charge iff  $V = 0$  for  $m \in \mathbb{N}_+ \setminus 2\mathbb{N}_+$  and otherwise it holds  $V = \mathbf{S}W$  for some  $W \in \mathcal{N}_\#^m$  such that:

- (a)  $\langle W, \varphi_1 \otimes \dots \otimes \varphi_m \rangle = \langle W, \mathbf{C}\varphi_1 \otimes \mathbf{C}\varphi_2 \otimes \varphi_3 \otimes \dots \otimes \varphi_m \rangle$ ,
- (b)  $\langle W, \varphi_1 \otimes \dots \otimes \varphi_m \rangle = \langle W, \varphi_{2\pi(1)-1} \otimes \varphi_{2\pi(1)} \otimes \dots \otimes \varphi_{2\pi(m/2)-1} \otimes \varphi_{2\pi(m/2)} \rangle$ ,
- (c)  $\langle W, \varphi_1 \otimes \dots \otimes \varphi_m \rangle = -\langle W, \varphi_2 \otimes \varphi_1 \otimes \varphi_3 \otimes \dots \otimes \varphi_m \rangle$

for all  $\varphi_1, \dots, \varphi_m \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G}$  and  $\pi \in \mathcal{P}_{m/2}$ .

*Remark 10.29.* We say that  $V \in \mathcal{N}$  is invariant under a certain symmetry or compatible with the flow of charge iff  $V^m \in \mathcal{N}^m \subset \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{A}^m \times \mathbb{G}^m}$  is invariant under this symmetry or compatible with the flow of charge for all  $m \in \mathbb{N}_+$ .

*Remark 10.30.* The map  $\mathbf{C}$  was introduced in Def. 8.6. Note that if  $V \in \mathcal{N}^m$  is compatible with the flow of charge, then it is invariant under the charge conjugation. If  $m \in \{4, 6, \dots\}$ , then generically the reverse implication is false. Observe also that the second and the third of the functionals (8.3) are not compatible with the flow of charge. Hence, there are only three relevant or marginal local functionals invariant under all the symmetries listed in Sec. 8 and compatible with the flow of charge: the quadratic functionals (8.2) and the first of the quartic functionals (8.3). The reduction of the number of relevant or marginal local functionals that have to be investigated is the main reason behind introducing the notion of the compatibility with the flow of charge. See [GK85a, Sec. 5.B] for a simple argument showing that perturbative corrections to an effective potential are compatible with the flow of charge.

**Definition 10.31.** Let  $m \in \mathbb{N}_+$ . We denote by  $\mathcal{N}_+^m$  the subspace of  $\mathcal{N}^m$  consisting of Schwartz distributions invariant under the symmetries of the torus and the internal symmetries and compatible with the flow of charge. We denote by  $\mathcal{N}_0^m$  the subspace of  $\mathcal{N}_+^m$

consisting of Schwartz distributions invariant under the symmetries of the plane. The closures of  $\mathcal{N}_+^m$  and  $\mathcal{N}_0^m$  in  $\mathcal{N}^m$  are denoted by  $(\mathcal{N}_+^m)^c$  and  $(\mathcal{N}_0^m)^c$ , respectively. Let  $\text{sgn}(\tau) := 0$  if  $\tau = 0$  and  $\text{sgn}(\tau) := +$  if  $\tau \in (0, 1]$ . For  $\tau \in (0, 1]$  we set  $\mathcal{N}_\tau^m := \mathcal{N}_{\text{sgn}(\tau)}^m$  and

$$\begin{aligned}\mathcal{N}_\tau^{\text{fin}} &:= \{V \in \mathcal{N}^{\text{fin}} \mid \forall m \in \mathbb{N}_+ V^m \in \mathcal{N}_\tau^m\}, \\ \mathcal{V}_\tau^{\text{fin}; \gamma} &:= \{V_\bullet \in \mathcal{V}^{\text{fin}; \gamma} \mid \forall s \in (0, 1] V_s \in \mathcal{N}_\tau^{\text{fin}}\}.\end{aligned}$$

**Definition 10.32.** Let  $\alpha, \beta \in [1, \infty)$ ,  $\gamma \in [0, \infty)$  and  $\tau, \varepsilon \in [0, 1]$ . The vector space  $\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}$  is the closure of  $\mathcal{V}_\tau^{\text{fin}; \gamma} \subset \mathcal{V}^{\alpha, \beta; \gamma}$  with respect to the following norm

$$\|V_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}} := \sup_{t \in [0, 1]} \|s \mapsto \mathbf{A}_{\tau, \varepsilon; t} V_s\|_{\mathcal{V}^{\alpha, \beta; \gamma}}. \quad (10.6)$$

For  $V_\bullet \in \mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}$  we define  $\|V_\bullet\|_{\tilde{\mathcal{W}}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}}$  in an analogous way to  $\|V_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}}$  with  $\|\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma}}$  replaced by  $\|\bullet\|_{\tilde{\mathcal{V}}^{\alpha, \beta; \gamma}}$  in Eq. (10.6).

*Remark 10.33.* By Remark 10.16 the norm  $\|\bullet\|_{\tilde{\mathcal{W}}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}}$  is weaker than the norm  $\|\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}}$ . The former norm will be used to study the infinite volume limit.

*Remark 10.34.* Note that

$$\|V_\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma}} \leq \|V_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}}, \quad \|V_\bullet\|_{\tilde{\mathcal{V}}^{\alpha, \beta; \gamma}} \leq \|V_\bullet\|_{\tilde{\mathcal{W}}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}}.$$

Thus, in particular  $\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma} \subset \mathcal{V}^{\alpha, \beta; \gamma}$ .

*Remark 10.35.* Observe that by Lemma 10.26 (A), (C) it holds

$$\|V_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}} \leq \|V_\bullet\|_{\mathcal{V}^{2\alpha, \beta; \gamma}}, \quad \|V_\bullet\|_{\tilde{\mathcal{W}}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}} \leq \|V_\bullet\|_{\tilde{\mathcal{V}}^{2\alpha, \beta; \gamma}}.$$

*Remark 10.36.* Note that  $\mathcal{N}_\tau^m$ ,  $\tau \in [0, 1]$ , is not a closed subspace of  $\mathcal{N}^m$ . It is easy to see that the invariance under the symmetries of the torus/plane, the internal symmetries and the charge conjugation symmetry are preserved under limits. However, this is not the case for the compatibility with the flow of charge. Hence, in general, it need not be true that for  $V_\bullet \in \mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}$  it holds  $V_s^m \in \mathcal{N}_\tau^m$  for  $m \in \mathbb{N}_+$  and  $s \in (0, 1]$ . Nonetheless,  $V_s^m \in (\mathcal{N}_\tau^m)^c$  for all  $m \in \mathbb{N}_+$  and  $s \in (0, 1]$ .

**Lemma 10.37.** For all  $\alpha, \beta \in [1, \infty)$ ,  $\gamma \in [0, \infty)$  and  $\tau, \varepsilon \in [0, 1]$  the space  $(\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}, \|\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}})$  is a Banach space.

*Proof.* Let  $(^p V)_{p \in \mathbb{N}_+}$  be a Cauchy sequence in  $\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}$ . Then,  $(^p V)_{p \in \mathbb{N}_+}$  is a Cauchy sequence in  $\mathcal{V}^{\alpha, \beta; \gamma}$ . Consequently, there exists  $V \in \mathcal{V}^{\alpha, \beta; \gamma}$  such that

$$\lim_{p \rightarrow \infty} \|V_\bullet - (^p V)_\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma}} = 0.$$

By Lemma 10.26 (A) it holds

$$\|s \mapsto \mathbf{A}_{\tau, \varepsilon; t, s}^m ((^p V)_s - V_s)\|_{\mathcal{V}^m; \gamma} \leq 2^m \alpha^m m^{-\beta} \|(^p V)_\bullet - V_\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma}}.$$

Let  $\delta \in (0, \infty)$ . There exists  $P \in \mathbb{N}_+$  such that for all  $p, q \in \mathbb{N}_+$ ,  $p, q \geq P$  it holds

$$\|(pV)_\bullet - (qV)_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}} \leq \delta$$

and for every  $m \in \mathbb{N}_+$  there exists  $p(m) \in \mathbb{N}_+$ ,  $p(m) \geq P$ , such that

$$\|s \mapsto \mathbf{A}_{\tau, \varepsilon; t, s}^m((p(m)V)_s - V_s)\|_{\mathcal{V}^{m; \gamma}} \leq \alpha^{-m} m^{-\beta} \delta.$$

As a result, for all  $p \in \mathbb{N}_+$ ,  $p \geq P$  it holds

$$\begin{aligned} & \|s \mapsto \mathbf{A}_{\tau, \varepsilon; t, s}^m(V_s - (pV)_s)\|_{\mathcal{V}^{m; \gamma}} \\ & \leq \|s \mapsto \mathbf{A}_{\tau, \varepsilon; t, s}^m(V_s - (p(m)V)_s)\|_{\mathcal{V}^{m; \gamma}} + \|s \mapsto \mathbf{A}_{\tau, \varepsilon; t, s}^m((p(m)V)_s - (pV)_s)\|_{\mathcal{V}^{m; \gamma}} \\ & \leq \alpha^{-m} m^{-\beta} \delta + \alpha^{-m} m^{-\beta} \|(pV)_\bullet - (qV)_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}} \leq 2\alpha^{-m} m^{-\beta} \delta. \end{aligned}$$

Hence,  $V_\bullet \in \mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}$  and  $\lim_{p \rightarrow \infty} \|V_\bullet - (pV)_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}} = 0$ .  $\square$

## 11 Local part and remainder

In this section we introduce operators  $\mathbf{L}$ ,  $\mathbf{L}_\partial$  and  $\mathbf{R}$  and establish their most important properties. The operators  $\mathbf{L}$ ,  $\mathbf{L}_\partial$  and  $\mathbf{R}$  are used to decompose kernels of functionals into the local part and the remainder. As we discussed in Sec. 2 the above-mentioned decomposition is crucial for the solution of the renormalization problem.

**Definition 11.1.** The distributions

$$U^2 \equiv (U^{2, a, \sigma})_{a \in \mathbb{A}^2, \sigma \in \mathbb{G}^2}, U_\partial^2 \equiv (U_\partial^{2, a, \sigma})_{a \in \mathbb{A}^2, \sigma \in \mathbb{G}^2} \in \mathcal{N}_0^2 \subset \mathcal{S}'(\mathbb{R}^4)^{\mathbb{A}^2 \times \mathbb{G}^2},$$

such that for all  $a \in \mathbb{A}^2$  and  $\sigma \in \mathbb{G}^2$  it holds  $U^{2, a, \sigma} = 0$  unless  $|a| = 0$  and  $U_\partial^{2, a, \sigma} = 0$  unless  $|a| = 1$  are defined by the equalities

$$\begin{aligned} \langle U^2, (\mathbf{J}\psi)^{\otimes 2} \rangle &= \int_{\mathbb{R}^2} \bar{\psi}(x) \cdot \underline{\psi}(x) \, dx, \\ \langle U_\partial^2, (\mathbf{J}\psi)^{\otimes 2} \rangle &= \int_{\mathbb{R}^2} \bar{\psi}(x) \cdot ((\Gamma_1 \partial_1 + \Gamma_2 \partial_2) \underline{\psi})(x) \, dx \end{aligned} \tag{11.1}$$

for all  $\psi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^-$ , where  $\mathbf{J}$  is the jet prolongation introduced in Def. 8.1. The distribution

$$U^4 \equiv (U^{4, a, \sigma})_{a \in \mathbb{A}^4, \sigma \in \mathbb{G}^4} \in \mathcal{N}_0^4 \subset \mathcal{S}'(\mathbb{R}^8)^{\mathbb{A}^4 \times \mathbb{G}^4}$$

such that for all  $a \in \mathbb{A}^4$  and  $\sigma \in \mathbb{G}^4$  it holds  $U^{4, a, \sigma} = 0$  unless  $|a| = 0$  is defined by the equalities

$$\langle U^4, (\mathbf{J}\psi)^{\otimes 4} \rangle = \int_{\mathbb{R}^2} (\bar{\psi}(x) \cdot \underline{\psi}(x))^2 \, dx \tag{11.2}$$

for all  $\psi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^-$ . For  $g, r, z \in \mathbb{C}$  we define  $U(g, r, z) \in \mathcal{N}$  by the equation

$$U(g, r, z)[\psi] := g \langle U^4, \psi^{\otimes 4} \rangle + r \langle U^2, \psi^{\otimes 2} \rangle + z \langle U_{\partial}^2, \psi^{\otimes 2} \rangle$$

for all  $\psi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^-$ .

*Remark 11.2.* Note that  $U(g, r, z) = g U(1, 0, 0) + r U(0, 1, 0) + z U(0, 0, 1)$ .

*Remark 11.3.* Recall that by Def. 10.31 distributions belonging to the space  $\mathcal{N}_0^m \subset \mathcal{N}^m \subset \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{A}^m \times \mathbb{G}^m}$  are invariant under the symmetries of the plane and the internal symmetries and compatible with the flow of charge. Moreover, distributions belonging to  $\mathcal{N}^m$  are antisymmetric. Using the above-mentioned properties one proves that  $U^2, U_{\partial}^2, U^4$  are uniquely defined by the equalities stated in Def. 11.1. In particular, by the charge conjugation invariance and Remark 4.3 it holds

$$\langle U_{\partial}^2, \psi^{\otimes 2} \rangle = \frac{1}{2} \sum_{i=1}^2 \int_{\mathbb{R}^2} (\bar{\psi}^0(x) \cdot \Gamma_i \psi^{a_i}(x) - \bar{\psi}^{a_i}(x) \cdot \Gamma_i \psi^0(x)) dx$$

for all  $\psi = (\bar{\psi}^a, \psi^a)_{a \in \mathbb{A}} \in \mathcal{S}'(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^-$  and  $a_1 = (1, 0) \in \mathbb{A}$ ,  $a_2 = (0, 1) \in \mathbb{A}$ .

*Remark 11.4.* It holds

- if  $\sigma = ((-1, 1), (+1, 1)) \in \mathbb{G}^2$ , then  $U^{2, \sigma, 0} = \delta^{(2)}/2$ ,
- if  $a = ((0, 0), (1, 0)) \in \mathbb{A}^2$ ,  $\sigma = ((-1, 1), (+1, 2)) \in \mathbb{G}^2$ , then  $U^{2, \sigma, a} = \delta^{(2)}/4$ ,
- if  $\sigma = ((-1, 1), (+1, 1), (-1, 1), (+1, 1)) \in \mathbb{G}^4$ , then  $U^{4, \sigma, 0} = \delta^{(4)}/6$ ,

where  $\delta^{(m)} \in \mathcal{S}'(\mathbb{R}^{2m})$  is the Dirac measure on the diagonal introduced in Remark 10.9.

*Remark 11.5.* In the definitions stated below we use the fact that for every distribution  $V \in \mathcal{M}^m \subset \mathcal{S}'(\mathbb{R}^{2m})$  there exists a kernel  $V_K$  in the sense of Definition 10.6 such that Eq. (10.1) holds. Note that the maps introduced in the definitions below do not depend on the choice of the kernel  $V_K$  associated to a distribution  $V$ .

**Definition 11.6.** Let  $V^2 = (V^{2, a, \sigma})_{a \in \mathbb{A}^2, \sigma \in \mathbb{G}^2} \in \mathcal{N}^2$  and  $V^4 = (V^{4, a, \sigma})_{a \in \mathbb{A}^4, \sigma \in \mathbb{G}^4} \in \mathcal{N}^4$  be translationally invariant. We define  $\mathbf{L}V^2 \in \mathbb{C}$  by the equality

$$\mathbf{L}V^2 := 2 \int_{\mathbb{R}^2} V^{2, 0, \sigma}(x_1, dx_2)$$

with  $\sigma = ((-1, 1), (+1, 1)) \in \mathbb{G}^2$ . We define  $\mathbf{L}_{\partial}V^2 := \hat{\mathbf{L}}_{\partial}V^2 + \check{\mathbf{L}}_{\partial}V^2 \in \mathbb{C}$  by the equalities

$$\hat{\mathbf{L}}_{\partial}V^2 := 4 \int_{\mathbb{R}^2} (x_2 - x_1)^{a_1 + a_2} V^{2, 0, \sigma}(x_1, dx_2), \quad \check{\mathbf{L}}_{\partial}V^2 := 4 \int_{\mathbb{R}^2} V^{2, a, \sigma}(x_1, dx_2),$$

with  $a = (a_1, a_2) = ((0, 0), (1, 0)) \in \mathbb{A}^2$  and  $\sigma = ((-1, 1), (+1, 2)) \in \mathbb{G}^2$ . We define  $\mathbf{L}V^4 \in \mathbb{C}$  by the equality

$$\mathbf{L}V^4 := 6 \int_{\mathbb{R}^6} V^{4, 0, \sigma}(x_1, dx_2, dx_3, dx_4)$$

with  $\sigma = ((-1, 1), (+1, 1), (-1, 1), (+1, 1)) \in \mathbb{G}^4$ .



*Remark 11.7.* By translational invariance the above definitions of  $\mathbf{L}V^2, \mathbf{L}_\partial V^2, \mathbf{L}V^4$  do not depend on the choice of  $x_1 \in \mathbb{R}^2$ .

*Remark 11.8.* Note that the following equalities  $\mathbf{L}U^2 = 1, \mathbf{L}_\partial U_\partial^2 = 1, \mathbf{L}U^4 = 1$  hold true.

**Lemma 11.9.** *Let  $\nu \in [0, 1/2]$ . There exists  $C \in (0, \infty)$  such that for all  $t \in (0, 1]$  and all translationally invariant  $V^2 = (V^{2,a,\sigma})_{a \in \mathbb{A}^2, \sigma \in \mathbb{G}^2} \in \mathcal{N}^2, V^4 = (V^{4,a,\sigma})_{a \in \mathbb{A}^4, \sigma \in \mathbb{G}^4} \in \mathcal{N}^4$  it holds:*

- (A)  $|\mathbf{L}V^2| \leq \sup_{a \in \mathbb{A}^2} \sup_{\sigma \in \mathbb{G}^2} \|w_{t;\nu}^2 V^{2,a,\sigma}\|_{\mathcal{M}^2},$
- (B)  $|\mathbf{L}_\partial V^2| \leq C t \sup_{a \in \mathbb{A}^2} \sup_{\sigma \in \mathbb{G}^2} \|w_{t;\nu}^2 V^{2,a,\sigma}\|_{\mathcal{M}^2},$
- (C)  $|\mathbf{L}V^4| \leq \sup_{a \in \mathbb{A}^4} \sup_{\sigma \in \mathbb{G}^4} \|w_{t;\nu}^4 V^{4,a,\sigma}\|_{\mathcal{M}^4}.$

*Proof.* By Remark 11.7 we can set  $x_1 = 0$  in the equations defining  $\mathbf{L}V^2, \mathbf{L}_\partial V^2, \mathbf{L}V^4$ , which are given in Def. 11.6. The bounds follow now immediately from Def. 10.6 of the norm  $\|\bullet\|_{\mathcal{M}^m}$  and the fact that there exists  $C \in (0, \infty)$  such that  $|x_2 - x_1|/w_{t;\nu}^2(x_1, x_2) \leq C t$  for  $x_1 = 0$  and all  $x_2 \in \mathbb{R}^2$  and  $t \in (0, 1]$ .  $\square$

**Lemma 11.10.** *Let  $V^2 = (V^{2,a,\sigma})_{a \in \mathbb{A}^2, \sigma \in \mathbb{G}^2} \in (\mathcal{N}_+^2)^c, V^4 = (V^{4,a,\sigma})_{a \in \mathbb{A}^4, \sigma \in \mathbb{G}^4} \in (\mathcal{N}_+^4)^c$ . Then we have*

$$\mathbf{L}V^2, \hat{\mathbf{L}}_\partial V^2, \check{\mathbf{L}}_\partial V^2, \mathbf{L}_\partial V^4 \in \mathbb{R} \quad (11.3)$$

and the following equalities

$$\mathbf{L}V^2 \langle U^2, (\mathbf{J}\psi)^{\otimes 2} \rangle = \sum_{\sigma \in \mathbb{G}^2} \int_{\mathbb{R}^4} V^{2,0,\sigma}(x_1, dx_2) \psi^{\sigma_1}(x_1) \psi^{\sigma_2}(x_1) dx_1,$$

$$\begin{aligned} \hat{\mathbf{L}}_\partial V^2 \langle U_\partial^2, (\mathbf{J}\psi)^{\otimes 2} \rangle \\ = \sum_{\substack{a \in \mathbb{A}^2 \\ |a|=1}} \sum_{\sigma \in \mathbb{G}^2} \int_{\mathbb{R}^4} (x_2 - x_1)^{a_1+a_2} V^{2,0,\sigma}(x_1, dx_2) (\mathbf{J}\psi)^{a_1,\sigma_1}(x_1) (\mathbf{J}\psi)^{a_2,\sigma_2}(x_1) dx_1, \end{aligned}$$

$$\check{\mathbf{L}}_\partial V^2 \langle U_\partial^2, (\mathbf{J}\psi)^{\otimes 2} \rangle = \sum_{\substack{a \in \mathbb{A}^2 \\ |a|=1}} \sum_{\sigma \in \mathbb{G}^2} \int_{\mathbb{R}^4} V^{2,a,\sigma}(x_1, dx_2) (\mathbf{J}\psi)^{a_1,\sigma_1}(x_1) (\mathbf{J}\psi)^{a_2,\sigma_2}(x_1) dx_1,$$

$$\mathbf{L}V^4 \langle U^4, (\mathbf{J}\psi)^{\otimes 4} \rangle = \sum_{\sigma \in \mathbb{G}^4} \int_{\mathbb{R}^8} V^{4,0,\sigma}(x_1, dx_2, dx_3, dx_4) \psi^{\sigma_1}(x_1) \psi^{\sigma_2}(x_1) \psi^{\sigma_3}(x_1) \psi^{\sigma_4}(x_1) dx_1$$

hold true for all  $\psi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^-$ .

*Remark 11.11.* Recall that  $\mathbf{J}\psi^{a,\sigma} \equiv (\mathbf{J}\psi)^{a,\sigma}$  is the jet prolongation. In particular,  $\psi^\sigma = \mathbf{J}\psi^{0,\sigma}$ . The space  $\mathcal{N}_+^m$  was introduced in Def. 10.31 and  $(\mathcal{N}_+^m)^c$  denotes its closure in  $\mathcal{N}^m$ .

*Proof.* By Lemma 11.9 without loss of generality we can assume that  $V^2 \in \mathcal{N}_+^2$ ,  $V^4 \in \mathcal{N}_+^4$ . The conditions (11.3) follow from the charge conjugation invariance. It is clear that the equalities listed in the statement of the lemma hold for some distributions  $U^2, U_\partial^2, U^4$  such that: (1) they satisfy all of the conditions stated in Def. 11.1 possibly with the exception of the equalities (11.1) and (11.2), (2) are proportional to the Dirac measures on the diagonal, (3) are invariant under the symmetries of the torus and the internal rotations and (4) are compatible with the flow of the charge. By Remark 10.30 the conditions (2-4) imply that the equalities (11.1) and (11.2) hold up to a constant. To conclude it is enough to use the properties stated in Remark 11.4.  $\square$

**Definition 11.12.** Let

$$V^2 = (V^{2,a,\sigma})_{a \in \mathbb{A}^2, \sigma \in \mathbb{G}^2} \in \mathcal{N}^2 \subset \mathcal{S}'(\mathbb{R}^4)^{\mathbb{A}^2 \times \mathbb{G}^2}$$

be such that  $\mathcal{X}V^2 \in \mathcal{N}^2$  for all translationally invariant polynomials  $\mathcal{X} \in C^\infty(\mathbb{R}^4)$  of degree two. We define  $\hat{W}^2 \in \mathcal{N}_\#^2 \subset \mathcal{S}'(\mathbb{R}^4)^{\mathbb{A}^2 \times \mathbb{G}^2}$  and  $\check{W}^2 \in \mathcal{S}'(\mathbb{R}^4)^{\mathbb{A}^2 \times \mathbb{A}^2 \times \mathbb{G}^2}$  by Eq. (10.1) with

$$\hat{W}_K^{2,a,\sigma}(x_1, dx_2) := \frac{1}{a!} (x_1 - x_1)^{a_1} (x_2 - x_1)^{a_2} \int_0^1 (1-u) u^{-2} (V_K^{2,0,\sigma})(u^{-1}x_1, u^{-1}dx_2) du$$

and

$$\check{W}_K^{2,b,c,\sigma}(x_1, dx_2) := (x_1 - x_1)^{b_1} (x_2 - x_1)^{b_2} \int_0^1 u^{-2} (V_K^{2,c,\sigma})(u^{-1}x_1, u^{-1}dx_2) du$$

if  $|a| = 2$ ,  $|b| = |c| = 1$  and  $\hat{W}_K^{2,a,\sigma} = 0$ ,  $\check{W}_K^{2,b,c,\sigma} = 0$  otherwise. We define  $W^2 \in \mathcal{N}_\#^2$  by the equality

$$W^{2,a,\sigma} = \hat{W}^{2,a,\sigma} + \sum_{\substack{b,c \in \mathbb{A}^2 \\ b+c=a}} \check{W}^{2,b,c,\sigma}.$$

We define  $\mathbf{R}V^2 \equiv ((\mathbf{R}V^2)^{a,\sigma})_{a \in \mathbb{A}^2, \sigma \in \mathbb{G}^2} \in \mathcal{N}^2$  by the equalities

$$\begin{aligned} (\mathbf{R}V^2)^{a,\sigma} &= 0, & |a| &\leq 1, \\ (\mathbf{R}V^2)^{a,\sigma} &= V^{2,a,\sigma} + (\mathbf{S}W^2)^{a,\sigma}, & |a| &= 2, \\ (\mathbf{R}V^2)^{a,\sigma} &= V^{2,a,\sigma}, & |a| &\geq 3. \end{aligned}$$

Let

$$V^4 = (V^{4,a,\sigma})_{a \in \mathbb{A}^4, \sigma \in \mathbb{G}^4} \in \mathcal{N}^4 \subset \mathcal{S}'(\mathbb{R}^8)^{\mathbb{A}^4 \times \mathbb{G}^4}$$

be such that  $\mathcal{X}V^4 \in \mathcal{N}^4$  for all translationally invariant polynomials  $\mathcal{X} \in C^\infty(\mathbb{R}^8)$  of degree one. We define  $W^4 \in \mathcal{N}_\#^4 \subset \mathcal{S}'(\mathbb{R}^8)^{\mathbb{A}^4 \times \mathbb{G}^4}$  by Eq. (10.1) with

$$\begin{aligned} W_K^{4,a,\sigma}(x_1, dx_2, dx_3, dx_4) &:= (x_1 - x_1)^{a_1} (x_2 - x_1)^{a_2} (x_3 - x_1)^{a_3} (x_4 - x_1)^{a_4} \\ &\quad \times \int_0^1 u^{-6} (V_K^{4,0,\sigma})(u^{-1}x_1, u^{-1}dx_2, u^{-1}dx_3, u^{-1}dx_4) du \end{aligned}$$

if  $|a| = 1$  and  $W_K^{4,a,\sigma} = 0$  otherwise. We define  $\mathbf{R}V^4 \equiv ((\mathbf{R}V^4)^{a,\sigma})_{a \in \mathbb{A}^4, \sigma \in \mathbb{G}^4} \in \mathcal{N}^4$  by the equalities

$$\begin{aligned} (\mathbf{R}V^4)^{a,\sigma} &= 0, & |a| &= 0, \\ (\mathbf{R}V^4)^{a,\sigma} &= V^{4,a,\sigma} + (\mathbf{S}W^4)^{a,\sigma}, & |a| &= 1, \\ (\mathbf{R}V^4)^{a,\sigma} &= V^{4,a,\sigma}, & |a| &\geq 2. \end{aligned}$$

*Remark 11.13.* Recall that  $\mathbf{S}V$ , introduced in Def. 10.4, denotes the antisymmetric part of  $V$ . Note that  $\hat{W}_K^{2,a,\sigma} = 0$  unless  $a_1 = 0$ ,  $\check{W}_K^{2,b,c,\sigma} = 0$  unless  $b_1 = 0$  and  $W_K^{4,a,\sigma} = 0$  unless  $a_1 = 0$ .

*Remark 11.14.* It follows from Def. 9.1 of the weight  $w_s^m$  that if  $V_\bullet^2 \in \mathcal{V}^{2;\gamma}$ , then  $\mathcal{X}V_s^2 \in \mathcal{N}^2$  for all translationally invariant polynomials  $\mathcal{X} \in C^\infty(\mathbb{R}^4)$  and all  $s \in (0, 1)$  and if  $V_\bullet^4 \in \mathcal{V}^{4;\gamma}$ , then  $\mathcal{X}V_s^4 \in \mathcal{N}^4$  for all translationally invariant polynomials  $\mathcal{X} \in C^\infty(\mathbb{R}^8)$  and all  $s \in (0, 1)$ .

*Remark 11.15.* The map  $\mathbf{R}$  is compatible with the symmetries. More precisely, let  $\tau \in \{0, +\}$  and suppose that  $V^2 \in (\mathcal{N}_\tau^2)^c \subset \mathcal{N}^2$  is such that  $\mathcal{X}V^2 \in \mathcal{N}^2$  for all translationally invariant polynomials  $\mathcal{X} \in C^\infty(\mathbb{R}^4)$  of degree two and  $V^4 \in (\mathcal{N}_+^4)^c \subset \mathcal{N}^4$  is such that  $\mathcal{X}V^4 \in \mathcal{N}^4$  for all translationally invariant polynomials  $\mathcal{X} \in C^\infty(\mathbb{R}^8)$  of degree one. Then  $\mathbf{R}V^2 \in \mathcal{N}_\tau^2$  and  $\mathbf{R}V^4 \in \mathcal{N}_\tau^4$ .

**Lemma 11.16.** *There exists  $C \in (0, \infty)$  such that it holds*

$$\begin{aligned} \|w_{t;\nu}^2 (\mathbf{R}V^2)^{a,\sigma}\|_{\mathcal{M}^2} &\leq C (1 - s/t)^{\zeta_\star} \sup_{b \in \mathbb{A}^2} s^{|a|-|b|} \|w_{s;\nu}^2 V^{2,b,\sigma}\|_{\mathcal{M}^2}, \\ \|w_{t;\nu}^4 (\mathbf{R}V^4)^{a,\sigma}\|_{\mathcal{M}^4} &\leq C (1 - s/t)^{\zeta_\star} \sup_{b \in \mathbb{A}^4} s^{|a|-|b|} \|w_{s;\nu}^4 V^{4,b,\sigma}\|_{\mathcal{M}^4}. \end{aligned}$$

for all  $\nu \in [0, 1/2]$ ,  $t \in (0, 1]$ ,  $s \in (0, t)$  and all  $V^2 \in \mathcal{N}^2$ ,  $V^4 \in \mathcal{N}^4$  such that the RHS of the above bounds are finite.

*Remark 11.17.* Recall that  $\zeta_\star = -3/4$  was introduced in Def. 9.1. See also Lemma 9.5 (f).

*Proof.* Since by assumption  $V^2, V^4$  are antisymmetric it suffices to prove that for all  $a \in \mathbb{A}^2$ ,  $|a| = 2$ ,  $\sigma \in \mathbb{G}^2$  it holds

$$\|w_{t;\nu}^2 \hat{W}^{2,a,\sigma}\|_{\mathcal{M}^2} \leq C (1 - s/t)^{\zeta_\star} s^{|a|} \sup_{\sigma \in \mathbb{G}^2} \|w_{s;\nu}^2 V^{2,0,\sigma}\|_{\mathcal{M}^2},$$

for all  $b, c \in \mathbb{A}^2$ ,  $|b| = |c| = 1$ ,  $\sigma \in \mathbb{G}^2$  it holds

$$\|w_{t;\nu}^2 \check{W}^{2,b,c,\sigma}\|_{\mathcal{M}^2} \leq C (1 - s/t)^{\zeta_\star} s^{|b|} \sup_{\sigma \in \mathbb{G}^2} \|w_{s;\nu}^2 V^{2,c,\sigma}\|_{\mathcal{M}^2}$$

and for all  $a \in \mathbb{A}^4$ ,  $|a| = 1$ ,  $\sigma \in \mathbb{G}^4$  it holds

$$\|w_{t;\nu}^4 W^{4,a,\sigma}\|_{\mathcal{M}^4} \leq C (1 - s/t)^{\zeta_\star} s^{|a|} \sup_{\sigma \in \mathbb{G}^4} \|w_{s;\nu}^4 V^{4,0,\sigma}\|_{\mathcal{M}^4}.$$

The above estimates follow easily from Def. 11.12 and Lemma 9.5 (f).  $\square$

**Lemma 11.18.** *Let  $V^2 \in (\mathcal{N}_+^2)^c \subset \mathcal{N}^2$  be such that  $\mathcal{X}V^2 \in \mathcal{N}^2$  for all translationally invariant polynomials  $\mathcal{X} \in C^\infty(\mathbb{R}^4)$  of degree two and  $V^4 \in (\mathcal{N}_+^4)^c \subset \mathcal{N}^4$  be such that  $\mathcal{X}V^4 \in \mathcal{N}^4$  for all translationally invariant polynomials  $\mathcal{X} \in C^\infty(\mathbb{R}^8)$  of degree one. Then it holds*

$$\begin{aligned} \langle (\mathbf{R} + U^2 \mathbf{L} + U_\partial^2 \mathbf{L}_\partial) V^2, (\mathbf{J}\psi)^{\otimes 2} \rangle &= \langle V^2, (\mathbf{J}\psi)^{\otimes 2} \rangle, \\ \langle (\mathbf{R} + U^4 \mathbf{L}) V^4, (\mathbf{J}\psi)^{\otimes 4} \rangle &= \langle V^4, (\mathbf{J}\psi)^{\otimes 4} \rangle. \end{aligned}$$

*Proof.* By Lemmas 11.9 and 11.16 both sides of the identities listed in the statement of the lemma depend continuously on  $V^2 \in \mathcal{N}^2$ ,  $V^4 \in \mathcal{N}^4$ . Thus, without loss of generality we can assume that  $V^2 \in \mathcal{N}_+^2$ ,  $V^4 \in \mathcal{N}_+^4$ . The rest of the proof is an application of the Taylor theorem. We first note that the equality

$$\langle ((\mathbf{R} + U^2 \mathbf{L} + U_\partial^2 \mathbf{L}_\partial) V^2)^{a,\sigma}, (\mathbf{J}\psi)^{a_1,\sigma_1} \otimes (\mathbf{J}\psi)^{a_2,\sigma_2} \rangle = \langle V^{2,a,\sigma}, (\mathbf{J}\psi)^{a_1,\sigma_1} \otimes (\mathbf{J}\psi)^{a_2,\sigma_2} \rangle$$

holds for all  $a \in \mathbb{A}^2$ ,  $|a| \geq 2$ , and  $\sigma \in \mathbb{G}^2$  and the equality

$$\langle ((\mathbf{R} + U^4 \mathbf{L}) V^4)^{a,\sigma}, (\mathbf{J}\psi)^{a_1,\sigma_1} \otimes \dots \otimes (\mathbf{J}\psi)^{a_4,\sigma_4} \rangle = \langle V^{4,a,\sigma}, (\mathbf{J}\psi)^{a_1,\sigma_1} \otimes \dots \otimes (\mathbf{J}\psi)^{a_4,\sigma_4} \rangle$$

for all  $a \in \mathbb{A}^4$ ,  $|a| \geq 1$ , and  $\sigma \in \mathbb{G}^4$ . In general the above equalities are false but we will prove that the sums of both sides over  $a$  and  $\sigma$  coincide. This follows from Def. 11.12 of the map  $\mathbf{R}$  as well as Eq. (11.4), (11.5) and (11.6) established below. We first observe that by the Taylor theorem the following identity

$$\begin{aligned} \psi^{\sigma_1}(x_1) \psi^{\sigma_2}(x_2) &= \psi^{\sigma_1}(x_1) \psi^{\sigma_2}(x_1) + \sum_{\substack{a \in \mathbb{A}^2 \\ |a|=1}} (x_1 - x_1)^{a_1} (x_2 - x_1)^{a_2} (\mathbf{J}\psi)^{a_1,\sigma_1}(x_1) (\mathbf{J}\psi)^{a_2,\sigma_2}(x_1) \\ &+ \sum_{\substack{a \in \mathbb{A}^2 \\ |a|=2}} \frac{1}{a!} (x_1 - x_1)^{a_1} (x_2 - x_1)^{a_2} \int_0^1 (1-u) (\mathbf{J}\psi)^{a_1,\sigma_1}(x_1) (\mathbf{J}\psi)^{a_2,\sigma_2}(x_1 + u(x_2 - x_1)) du \end{aligned}$$

is true for  $\sigma \in \mathbb{G}^2$ . As a result, by Lemma 11.10 and Def. 11.12 of  $\hat{W}^{2,a,\sigma}$  we obtain

$$\begin{aligned} \sum_{\sigma \in \mathbb{G}^2} \langle V^{2,0,\sigma}, \psi^{\sigma_1} \otimes \psi^{\sigma_2} \rangle \\ = \sum_{a \in \mathbb{A}^2} \sum_{\sigma \in \mathbb{G}^2} \langle U^{2,a,\sigma} \mathbf{L} V^2 + U_\partial^{2,a,\sigma} \hat{\mathbf{L}}_\partial V^2 + \hat{W}^{2,a,\sigma}, (\mathbf{J}\psi)^{a_1,\sigma_1} \otimes (\mathbf{J}\psi)^{a_2,\sigma_2} \rangle. \end{aligned} \quad (11.4)$$

Next, we note that

$$\begin{aligned} (\mathbf{J}\psi)^{c_1,\sigma_1}(x_1) (\mathbf{J}\psi)^{c_2,\sigma_2}(x_2) &= (\mathbf{J}\psi)^{c_1,\sigma_1}(x_1) (\mathbf{J}\psi)^{c_2,\sigma_2}(x_1) \\ &+ \sum_{\substack{b \in \mathbb{A}^2 \\ |b|=1}} (x_1 - x_1)^{b_1} (x_2 - x_1)^{b_2} \int_0^1 (\mathbf{J}\psi)^{b_1+c_1,\sigma_1}(x_1) (\mathbf{J}\psi)^{b_2+c_2,\sigma_2}(x_1 + u(x_2 - x_1)) du. \end{aligned}$$

for  $c \in \mathbb{A}^2$  and  $\sigma \in \mathbb{G}^2$ . Consequently, by Lemma 11.10 and Def. 11.12 of  $\check{W}^{2,b,c,\sigma}$  we have

$$\begin{aligned} \sum_{\substack{c \in \mathbb{A}^2 \\ |c|=1}} \sum_{\sigma \in \mathbb{G}^2} \langle V^{2,c,\sigma}, (\mathbf{J}\psi)^{c_1,\sigma_1} \otimes (\mathbf{J}\psi)^{c_2,\sigma_2} \rangle &= \sum_{a \in \mathbb{A}^2} \sum_{\sigma \in \mathbb{G}^2} \langle U_{\partial}^{2,a,\sigma} \check{\mathbf{L}}_{\partial} V^2, (\mathbf{J}\psi)^{a_1,\sigma_1} \otimes (\mathbf{J}\psi)^{a_2,\sigma_2} \rangle \\ &+ \sum_{b,c \in \mathbb{A}^2} \sum_{\sigma \in \mathbb{G}^2} \langle \check{W}^{2,b,c,\sigma}, (\mathbf{J}\psi)^{b_1+c_1,\sigma_1} \otimes (\mathbf{J}\psi)^{b_2+c_2,\sigma_2} \rangle. \end{aligned} \quad (11.5)$$

Finally, using the identity

$$\prod_{j=1}^4 \psi^{\sigma_j}(x_j) = \prod_{j=1}^4 \psi^{\sigma_j}(x_1) + \sum_{\substack{a \in \mathbb{A}^4 \\ |a|=1}} \prod_{j=1}^4 (x_j - x_1)^{a_j} \int_0^1 \prod_{j=1}^4 (\mathbf{J}\psi)^{a_j,\sigma_j}(x_1 + u(x_j - x_1)) du,$$

Lemma 11.10 and Def. 11.12 of  $W^{4,a,\sigma}$  we get

$$\begin{aligned} \sum_{\sigma \in \mathbb{G}^4} \langle V^{4,0,\sigma}, \psi^{\sigma_1} \otimes \dots \otimes \psi^{\sigma_4} \rangle \\ = \sum_{a \in \mathbb{A}^4} \sum_{\sigma \in \mathbb{G}^4} \langle U^{4,a,\sigma} \mathbf{L} V^4 + W^{4,a,\sigma}, (\mathbf{J}\psi)^{a_1,\sigma_1} \otimes \dots \otimes (\mathbf{J}\psi)^{a_4,\sigma_4} \rangle. \end{aligned} \quad (11.6)$$

This finishes the proof.  $\square$

## 12 Useful maps

In this section we study properties of the operators that appear on the RHS of the equations (2.24), (2.28), (2.20), (2.21), (2.22) introduced in Sec. 2. Recall that our goal is to solve this system of equation by rewriting it as a fixed point equation of a certain map  $\mathbf{X}_{\tau,\varepsilon,\bullet}$ , which is defined in Sec. 13. The estimates we establish below play a crucial role in the proof that the map  $\mathbf{X}_{\tau,\varepsilon,\bullet}$  is well-defined and is a contraction.

**Definition 12.1.** For  $\tau, \varepsilon \in [1, 0]$  and  $t \in (0, 1]$  we define

$$\dot{H}_{\tau,\varepsilon;t} \equiv (\dot{H}_{\tau,\varepsilon;t}^{a,\sigma})_{a \in \mathbb{A}^2, \sigma \in \mathbb{G}^2} \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)^{\mathbb{A}^2 \times \mathbb{G}^2}$$

by the formula

$$\dot{H}_{\tau,\varepsilon;t}^{a,\sigma}(x, y) := \partial_x^{a_1} \partial_y^{a_2} \dot{G}_{\tau,\varepsilon;t}^\sigma(x - y)$$

for all  $a = (a_1, a_2) \in \mathbb{A}^2 = \{0, 1, 2\}^2$ ,  $\sigma \in \mathbb{G}^2$  and  $x, y \in \mathbb{R}^2$ . We omit  $\tau$  if  $\tau = 0$  and we omit  $\varepsilon$  if  $\varepsilon = 0$ .

*Remark 12.2.* The function  $\dot{H}_{\tau,\varepsilon;t} \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)^{\mathbb{A}^2 \times \mathbb{G}^2}$  is antisymmetric and invariant under the symmetries of the torus, the internal symmetries and the charge conjugation symmetry. The proof of the invariance under the symmetries of the torus and the internal symmetries

is straightforward. To prove the invariance under the charge conjugation symmetry one uses Remark 4.3. If  $\tau = 0$ , then the above function is also invariant under the symmetries of the plane.

**Lemma 12.3.** *There exists  $\lambda_\star \in (0, 1]$  such that for all  $\lambda \in (0, \lambda_\star]$ ,  $\varepsilon \in [1, 0]$ ,  $t \in (0, 1]$ , and  $a \in \mathbb{A}^2$ ,  $\sigma \in \mathbb{G}^2$  it holds*

$$\|w_t^2 \dot{H}_{\varepsilon;t}^{a,\sigma}\|_{\mathcal{M}^2} \leq \lambda^{-\kappa} t^{-|a|}, \quad \|w_t^2 (\dot{H}_t^{a,\sigma} - \dot{H}_{\varepsilon;t}^{a,\sigma})\|_{\mathcal{M}^2} \leq \lambda^{-\kappa} \lambda_\varepsilon^\kappa \lambda_t^{-\kappa} t^{-|a|},$$

where the weight  $w_t^2$  was introduced in Def. 9.1. Moreover, the map  $(0, 1] \ni t \mapsto \dot{H}_{\varepsilon;t} \in \mathcal{N}^2$  is continuous for all  $\varepsilon \in [0, 1]$ .

*Proof.* Observe that  $w_t^2(x_1, x_2) = (1 + |x_1 - x_2|)^{1/2} \exp(t^{-\zeta}|x_1 - x_2|^\zeta)$  and  $\zeta = 4/5 < \zeta_b$ . As a result, by Lemma 5.3 and Remark 5.2 (E) there exists a universal constant  $C \in (0, \infty)$  such that  $\|w_t^2 \dot{H}_{\varepsilon;t}^{a,\sigma}\|_{\mathcal{M}^2} \leq C t^{-|a|}$ . This proves the first of the estimates stated in the lemma. The second of the estimates follows from the above bound and Remark 5.2 (C).  $\square$

**Definition 12.4.** Let  $m \in \mathbb{N}_+$ ,  $\varepsilon \in [0, 1]$ ,  $s \in (0, 1]$ . The map  $\mathbf{B}_{\varepsilon;s}^m : \mathcal{N}(\mathcal{F}) \rightarrow \mathcal{N}^m(\mathcal{F})$  is defined by

$$\langle \mathbf{B}_{\varepsilon;s}^m(V), \varphi^{\otimes m} \rangle := \frac{1}{2} \sum_{k=0}^m (-1)^{m-k} (k+1)(m-k+1) \langle V^{k+1} \otimes V^{m-k+1}, \varphi^{\otimes k} \otimes \dot{H}_{\varepsilon;s} \otimes \varphi^{\otimes(m-k)} \rangle$$

for all  $\varphi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^-$ . The map  $\mathbf{B}_{\varepsilon;s}^m : \mathcal{N}(\mathcal{F}) \times \mathcal{N}(\mathcal{F}) \rightarrow \mathcal{N}^m(\mathcal{F})$  is defined by

$$\mathbf{B}_{\varepsilon;s}^m(V, W) := (\mathbf{B}_{\varepsilon;s}^m(V + W, V + W) - \mathbf{B}_{\varepsilon;s}^m(V - W, V - W))/4.$$

The maps  $\mathbf{B}_{\varepsilon;s} : \mathcal{N} \rightarrow \mathcal{N}$ ,  $\mathbf{B}_{\varepsilon;s} : \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N}$  are uniquely defined by the condition  $\Pi^m \mathbf{B}_{\varepsilon;s} = \mathbf{B}_{\varepsilon;s}^m$  for all  $m \in \mathbb{N}_+$ . We also set  $\mathbf{B}_{\varepsilon;s}^{m,a} := \Pi^{m,a} \mathbf{B}_{\varepsilon;s}$  and  $\mathbf{B}_{\varepsilon;s}^{m,a,\sigma} := \Pi^{m,a,\sigma} \mathbf{B}_{\varepsilon;s}$ . We omit  $\varepsilon$  if  $\varepsilon = 0$ .

*Remark 12.5.* Using the antisymmetric property one shows that for  $V, W \in \mathcal{N}$  it holds

$$\mathbf{B}_{\varepsilon;s}(V, W)[\varphi] = \langle D_\varphi V[\varphi] \otimes D_\varphi W[\varphi], \dot{H}_{\varepsilon;s} \rangle - \langle D_\varphi V[0] \otimes D_\varphi W[0], \dot{H}_{\varepsilon;s} \rangle.$$

The above identity implies that

$$\begin{aligned} \mathbf{B}_{\varepsilon;s}(V, W)[\varphi + \mathbf{J}\Psi_{\tau,\varepsilon;t \vee s,s}] - \mathbf{B}_{\varepsilon;s}(V, W)[\mathbf{J}\Psi_{\tau,\varepsilon;t \vee s,s}] \\ = \langle D_\varphi V[\varphi + \mathbf{J}\Psi_{\tau,\varepsilon;t \vee s,s}] \otimes D_\varphi W[\varphi + \mathbf{J}\Psi_{\tau,\varepsilon;t \vee s,s}], \dot{H}_{\varepsilon;s} \rangle \\ - \langle D_\varphi V[\mathbf{J}\Psi_{\tau,\varepsilon;t \vee s,s}] \otimes D_\varphi W[\mathbf{J}\Psi_{\tau,\varepsilon;t \vee s,s}], \dot{H}_{\varepsilon;s} \rangle. \end{aligned}$$

Taking into account Def. 10.22 of the map  $\mathbf{A}_{\tau,\varepsilon;t,s}$  we obtain

$$\mathbf{A}_{\tau,\varepsilon;t,s} \mathbf{B}_{\varepsilon;s}(V, W) = \mathbf{B}_{\varepsilon;s}(\mathbf{A}_{\tau,\varepsilon;t,s} V, \mathbf{A}_{\tau,\varepsilon;t,s} W). \quad (12.1)$$

**Lemma 12.6.** *Let  $\alpha \in [1, \infty)$ ,  $\beta \in (2, \infty)$ ,  $\gamma_1, \gamma_2 \in [0, \infty)$ . There exists  $\lambda_\star \in (0, 1]$  such that for all  $\lambda \in (0, \lambda_\star]$ ,  $\tau, \varepsilon \in [0, 1]$  and  $V_\bullet \in \mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma_1}$ ,  $W_\bullet \in \mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma_2}$  the map  $s \mapsto s \mathbf{B}_{\varepsilon; s}(V_s, W_s)$  belongs to  $\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta-1; \gamma_1+\gamma_2}$  and for all  $V_\bullet \in \mathcal{V}^{\alpha, \beta; \gamma_1}$ ,  $W_\bullet \in \mathcal{V}^{\alpha, \beta; \gamma_2}$  it holds*

$$\begin{aligned} (A) \quad & \|s \mapsto s \mathbf{B}_{\varepsilon; s}(V_s, W_s)\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta-1; \gamma_1+\gamma_2}} \leq \lambda^\kappa \|V_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma_1}} \|W_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma_2}}, \\ (B) \quad & \|s \mapsto s \mathbf{B}_{\varepsilon; s}(V_s, W_s)\|_{\tilde{\mathcal{W}}_{\tau, \varepsilon}^{\alpha, \beta-1; \gamma_1+\gamma_2}} \leq \lambda^\kappa \|V_\bullet\|_{\tilde{\mathcal{W}}_{\tau, \varepsilon}^{\alpha, \beta; \gamma_1}} \|W_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma_2}}, \\ (C) \quad & \|s \mapsto s (\mathbf{B}_s(V_s, W_s) - \mathbf{B}_{\varepsilon; s}(V_s, W_s))\|_{\tilde{\mathcal{W}}_{\tau, \varepsilon}^{\alpha/2, \beta-1; \gamma_1+\gamma_2}} \leq \lambda_\varepsilon^\kappa \|V_\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma_1}} \|W_\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma_2}}. \end{aligned}$$

*Proof.* Let  $\nu \in \{0, 1/2\}$ ,  $s \in (0, 1]$  and  $V_s, W_s \in \mathcal{N}(\mathcal{F})$ . By Def. 12.4 of the map  $\mathbf{B}_{\varepsilon; s}$ , Def. 10.6 of the norm  $\mathcal{M}^m$ , Lemma 9.5 (a), (b) and Remark 10.17 we obtain

$$\begin{aligned} \sum_{a \in \mathbb{A}^m} \sum_{\sigma \in \mathbb{G}^m} \|w_{s; \nu}^m \mathbf{B}_{\varepsilon; s}^{m, a, \sigma}(V_s, W_s)\|_{\mathcal{M}^m} &\leq \sup_{a \in \mathbb{A}^2} \sup_{\sigma \in \mathbb{G}^2} \|w_s^2 \dot{H}_{\varepsilon; s}^{a, \sigma}\|_{\mathcal{M}^2} \\ &\times \sum_{k=0}^m (k+1)(m-k+1) \sum_{a \in \mathbb{A}^{k+1}} \sum_{\sigma \in \mathbb{G}^{k+1}} \|w_{s; \nu}^{k+1} V_s^{k+1, a, \sigma}\|_{\mathcal{M}^{k+1}} \\ &\times \sum_{a \in \mathbb{A}^{m-k+1}} \sum_{\sigma \in \mathbb{G}^{m-k+1}} \|w_s^{m-k+1} W_s^{m-k+1, a, \sigma}\|_{\mathcal{M}^{m-k+1}} \quad (12.2) \end{aligned}$$

and

$$\begin{aligned} \sum_{a \in \mathbb{A}^m} \sum_{\sigma \in \mathbb{G}^m} \|w_{s; \nu}^m (\mathbf{B}_s^{m, a, \sigma}(V_s, W_s) - \mathbf{B}_{\varepsilon; s}^{m, a, \sigma}(V_s, W_s))\|_{\mathcal{M}^m} &\leq \sup_{a \in \mathbb{A}^2} \sup_{\sigma \in \mathbb{G}^2} \|w_s^2 (\dot{H}_s^{a, \sigma} - \dot{H}_{\varepsilon; s}^{a, \sigma})\|_{\mathcal{M}^2} \\ &\times \sum_{k=0}^m (k+1)(m-k+1) \sum_{a \in \mathbb{A}^{k+1}} \sum_{\sigma \in \mathbb{G}^{k+1}} \|w_{s; \nu}^{k+1} V_s^{k+1, a, \sigma}\|_{\mathcal{M}^{k+1}} \\ &\times \sum_{a \in \mathbb{A}^{m-k+1}} \sum_{\sigma \in \mathbb{G}^{m-k+1}} \|w_s^{m-k+1} W_s^{m-k+1, a, \sigma}\|_{\mathcal{M}^{m-k+1}}. \quad (12.3) \end{aligned}$$

Note that

$$\rho_{\gamma_1+\gamma_2, \kappa}(m) = \rho_{\gamma_1, \kappa}(k+1) + \rho_{\gamma_2, \kappa}(m-k+1) - 4\kappa.$$

Moreover, observe that for all  $\beta \in (2, \infty)$  there exists  $\lambda_\star \in (0, 1]$  such that for all  $\alpha \in [1, \infty)$ ,  $m \in \mathbb{N}_+$  and  $\lambda \in (0, \lambda_\star]$  it holds

$$\sum_{k=0}^m \frac{\alpha^{-(k+1)}}{(k+1)^{\beta-1}} \frac{\alpha^{-(m-k+1)}}{(m-k+1)^{\beta-1}} \leq \lambda^{-\kappa} \frac{\alpha^{-m}}{m^{\beta-1}}.$$

Recall also that  $\lambda_s^\kappa \leq \lambda^\kappa$  for  $s \in (0, 1]$ . Consequently, by Def. 10.15 of the norm  $\|\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma}}$  and Lemma 12.3 the bounds (12.2) and (12.3) imply that

$$\begin{aligned} (A_0) \quad & \|s \mapsto s \mathbf{B}_{\varepsilon; s}(V_s, W_s)\|_{\mathcal{V}^{\alpha, \beta-1; \gamma_1+\gamma_2}} \leq \lambda^\kappa \|V_\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma_1}} \|W_\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma_2}}, \\ (B_0) \quad & \|s \mapsto s \mathbf{B}_{\varepsilon; s}(V_s, W_s)\|_{\tilde{\mathcal{V}}^{\alpha, \beta-1; \gamma_1+\gamma_2}} \leq \lambda^\kappa \|V_\bullet\|_{\tilde{\mathcal{V}}^{\alpha, \beta; \gamma_1}} \|W_\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma_2}}, \\ (C_0) \quad & \|s \mapsto s (\mathbf{B}_s(V_s, W_s) - \mathbf{B}_{\varepsilon; s}(V_s, W_s))\|_{\tilde{\mathcal{V}}^{\alpha, \beta-1; \gamma_1+\gamma_2}} \leq \lambda_\varepsilon^\kappa \|V_\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma_1}} \|W_\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma_2}}. \end{aligned}$$

The bounds (A) and (B) follow from Def. 10.32 of the norm  $\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}$ , the bounds (A<sub>0</sub>) and (B<sub>0</sub>) and the identity (12.1). The bound (C) is a consequence of the bound (C<sub>0</sub>) and Lemma 10.26 (C).

It remains to prove that the map  $s \mapsto s \mathbf{B}_{\varepsilon;s}(V_s, W_s)$  belongs to  $\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta-1;\gamma_1+\gamma_2}$ . By the bound (A) and the fact that  $\mathcal{V}_{\tau}^{\text{fin};\gamma}$  is a dense subset of  $\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}$  it suffices to establish the above claim for  $V_{\bullet} \in \mathcal{V}_{\tau}^{\text{fin};\gamma_1}$  and  $W_{\bullet} \in \mathcal{V}_{\tau}^{\text{fin};\gamma_2}$ . It follows from Lemma 12.8 that  $\mathbf{B}_{\varepsilon;s}(V_s, W_s) \in \mathcal{N}_{\tau}^{\text{fin}}$  for all  $s \in (0, 1]$ . Recall that the map  $(0, 1] \ni s \mapsto \dot{H}_{\varepsilon;s} \in \mathcal{N}^2$  is continuous by Lemma 12.3. Hence, by a bound similar to the bound (12.2) the map  $(0, 1] \ni s \mapsto \mathbf{B}_{\varepsilon;s}^m(V_s, W_s) \in \mathcal{N}^m$  is continuous. Using the above fact and the bound (A) we infer that the map  $s \mapsto s \mathbf{B}_{\varepsilon;s}(V_s, W_s)$  belongs to  $\mathcal{V}_{\tau}^{\text{fin};\gamma_1+\gamma_2} \subset \mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta-1;\gamma_1+\gamma_2}$ . This finishes the proof.  $\square$

**Lemma 12.7.** *Let  $\alpha \in [2, \infty)$ ,  $\beta \in [1, \infty)$ ,  $\gamma \in [0, \infty)$ . For  $\tau, \varepsilon \in [0, 1]$ ,  $u \in (0, 1]$  and  $V_{\bullet} \in \mathcal{V}^{\alpha,\beta;\gamma}$  we define  $\mathbf{I}_{\tau,\varepsilon;u} V_{\bullet} \equiv (\mathbf{I}_{\tau,\varepsilon;u}^m V_{\bullet})_{m \in \mathbb{N}_+} \in \mathcal{N}$  by the equalities  $\mathbf{I}_{\tau,\varepsilon;u}^m V_{\bullet} = 0$  for  $m \in \mathbb{N}_+ \setminus 2\mathbb{N}_+$ ,*

$$\mathbf{I}_{\tau,\varepsilon;u}^m V_{\bullet} := \int_0^u \mathbf{E} \mathbf{A}_{\tau,\varepsilon;u,s}^m V_s / s \, ds \in \mathcal{N}^m$$

for  $m \in \{6, 8, \dots\}$  and

$$\mathbf{I}_{\tau,\varepsilon;u}^m V_{\bullet} := \int_0^u \mathbf{R} \mathbf{E} \mathbf{A}_{\tau,\varepsilon;u,s}^m V_s / s \, ds \in \mathcal{N}^m$$

for  $m \in \{2, 4\}$  provided the integrands appearing above are absolutely integrable. We set  $\mathbf{I}_{\tau,\varepsilon;u}^{m,a,\sigma} V = \Pi^{m,a,\sigma} \mathbf{I}_{\tau,\varepsilon;u} V$ . Moreover, we omit  $\tau$  and  $\varepsilon$  if  $\tau = 0$  and  $\varepsilon = 0$ . There exists a constant  $C \in (0, \infty)$  such that for all  $\tau, \varepsilon \in [0, 1]$  and  $V_{\bullet} \in \mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}$  the map  $u \mapsto \mathbf{I}_{\tau,\varepsilon;u} V_{\bullet}$  belongs to  $\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta+1;\gamma}$  and for all  $\tau, \varepsilon \in [0, 1]$  and  $V_{\bullet} \in \mathcal{V}^{\alpha,\beta;\gamma}$  it holds

- (A)  $\|u \mapsto \mathbf{I}_{\tau,\varepsilon;u} V_{\bullet}\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta+1;\gamma}} \leq C \|V_{\bullet}\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}},$
- (B)  $\|u \mapsto \mathbf{I}_{\tau,\varepsilon;u} V_{\bullet}\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{\alpha,\beta+1;\gamma}} \leq C \|V_{\bullet}\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}},$
- (C)  $\|u \mapsto (\mathbf{I}_u - \mathbf{I}_{\tau,\varepsilon;u}) V_{\bullet}\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{\alpha/4,\beta;\gamma}} \leq C \lambda_{\tau \vee \varepsilon}^{\kappa} \|V_{\bullet}\|_{\mathcal{V}^{\alpha,\beta;\gamma}}.$

*Proof.* We shall prove the following bounds

- (A<sub>1</sub>)  $\|u \mapsto \Pi_k \mathbf{I}_{\tau,\varepsilon;u} V_{\bullet}\|_{\mathcal{V}^{\alpha,\beta;\gamma}} \leq C \|V_{\bullet}\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}},$
- (B<sub>1</sub>)  $\|u \mapsto \Pi_k \mathbf{I}_{\tau,\varepsilon;u} V_{\bullet}\|_{\tilde{\mathcal{V}}^{\alpha,\beta;\gamma}} \leq C \|V_{\bullet}\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}},$
- (C<sub>1</sub>)  $\|u \mapsto \Pi_k (\mathbf{I}_u - \mathbf{I}_{\tau,\varepsilon;u}) V_{\bullet}\|_{\tilde{\mathcal{V}}^{\alpha/2,\beta-1;\gamma}} \leq C \lambda_{\tau \vee \varepsilon}^{\kappa} \|V_{\bullet}\|_{\mathcal{V}^{\alpha,\beta;\gamma}}.$

with  $k \in \{2, 4\}$  and

- (A<sub>2</sub>)  $\|u \mapsto \Pi_{>4} \mathbf{I}_{\tau,\varepsilon;u} V_{\bullet}\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta+1;\gamma}} \leq C \|V_{\bullet}\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}},$
- (B<sub>2</sub>)  $\|u \mapsto \Pi_{>4} \mathbf{I}_{\tau,\varepsilon;u} V_{\bullet}\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{\alpha,\beta+1;\gamma}} \leq C \|V_{\bullet}\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}},$
- (C<sub>2</sub>)  $\|u \mapsto \Pi_{>4} (\mathbf{I}_u - \mathbf{I}_{\tau,\varepsilon;u}) V_{\bullet}\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{\alpha/4,\beta;\gamma}} \leq C \lambda_{\tau \vee \varepsilon}^{\kappa} \|V_{\bullet}\|_{\mathcal{V}^{\alpha,\beta;\gamma}}.$

Recall that the operators  $\Pi_k$  and  $\Pi_{>k}$  were introduced in Def. 10.12. Note that by Remarks 10.35 and 10.19 the bounds (A<sub>1</sub>), (B<sub>1</sub>), (C<sub>1</sub>) imply analogous bounds with the norms as in the bounds (A), (B), (C) and some universal constant  $C \in (0, \infty)$ . Hence, to prove the



lemma it is enough to establish the bounds (A<sub>1</sub>), (B<sub>1</sub>), (C<sub>1</sub>) and (A<sub>2</sub>), (B<sub>2</sub>), (C<sub>2</sub>). In order to verify the bound (A<sub>1</sub>) note that by Lemma 11.16 and Remark 10.10 for  $m \in \{2, 4\}$  it holds

$$\begin{aligned} \|w_u^m \mathbf{I}_{\tau, \varepsilon; u}^{m, a, \sigma} V_\bullet\|_{\mathcal{M}^m} &\leq \int_0^u s^{-1} \|w_u^m \mathbf{REA}_{\tau, \varepsilon; u, s}^{m, a, \sigma} V_s\|_{\mathcal{M}^m} ds \\ &\leq \sup_{b \in \mathbb{A}^m} \int_0^u s^{|a| - |b| - 1} (1 - s/u)^{\zeta_\star} \|w_s^m \mathbf{A}_{\tau, \varepsilon; u, s}^{m, b, \sigma} V_s\|_{\mathcal{M}^m} ds. \end{aligned}$$

By Def. 11.12 of the map  $\mathbf{R}$  it holds  $\|w_u^m \mathbf{I}_{\tau, \varepsilon; u}^{m, a, \sigma} V_\bullet\|_{\mathcal{M}^m} = 0$  if  $|a| \leq 1$  and  $m = 2$ , or  $|a| = 0$  and  $m = 4$ . Using Def. 10.32 and 10.15 as well as Lemma 2.4 (B) we obtain

$$\begin{aligned} \|w_u^m \mathbf{I}_{\tau, \varepsilon; u}^{m, a, \sigma} V_\bullet\|_{\mathcal{M}^m} &\leq \alpha^{-m} m^{-\beta} \|V_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}} \int_0^u \lambda_s^{\rho_\gamma, \kappa(m)} s^{|a| + m/2 - 3} (1 - s/u)^{\zeta_\star} ds \\ &\leq C \alpha^{-m} m^{-\beta} \lambda_u^{\rho_\gamma, \kappa(m)} u^{|a| + m/2 - 2} \|V_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}} \end{aligned}$$

with some universal constant  $C \in (0, \infty)$  for  $m = 2$  and  $|a| > 1$ , or  $m = 4$  and  $|a| > 0$ . This proves the bound (A<sub>1</sub>). The proof of the bound (B<sub>1</sub>) is the same only the weight  $w_s^m$  is replaced with the weight  $\tilde{w}_s^m$ . In order to verify the bound (C<sub>1</sub>) note that by Lemma 11.16 and Remark 10.10 for  $m \in \{2, 4\}$  it holds

$$\begin{aligned} \|w_u^m (\mathbf{I}_{\tau, \varepsilon; u}^{m, a, \sigma} - \mathbf{I}_u^{m, a, \sigma}) V_\bullet\|_{\mathcal{M}^m} &\leq \int_0^u s^{-1} \|w_u^m \mathbf{RE}(\mathbf{A}_{\tau, \varepsilon; u, s}^{m, a, \sigma} - \mathbf{A}_{u, s}^{m, a, \sigma}) V_s\|_{\mathcal{M}^m} ds \\ &\leq \sup_{b \in \mathbb{A}^m} \int_0^u s^{|a| - |b| - 1} (1 - s/u)^{\zeta_\star} \|w_s^m (\mathbf{A}_{\tau, \varepsilon; u, s}^{m, b, \sigma} - \mathbf{A}_{u, s}^{m, b, \sigma}) V_s\|_{\mathcal{M}^m} ds. \end{aligned}$$

By Def. 11.12 of the map  $\mathbf{R}$  it holds  $\|w_u^m (\mathbf{I}_{\tau, \varepsilon; u}^{m, a, \sigma} - \mathbf{I}_u^{m, a, \sigma}) V_\bullet\|_{\mathcal{M}^m} = 0$  if  $|a| \leq 1$  and  $m = 2$ , or  $|a| = 0$  and  $m = 4$ . Using Lemma 10.26 (E) and Def. 10.15 as well as Lemma 2.4 (B) we obtain

$$\begin{aligned} \|w_u^m (\mathbf{I}_{\tau, \varepsilon; u}^{m, a, \sigma} - \mathbf{I}_u^{m, a, \sigma}) V_\bullet\|_{\mathcal{M}^m} &\leq (\alpha/2)^{-m} m^{1-\beta} \|s \mapsto (\mathbf{A}_{\tau, \varepsilon; u, s} - \mathbf{A}_{u, s}) V_s\|_{\mathcal{V}^{\alpha/2, \beta-1; \gamma}} \int_0^u \lambda_s^{\rho_\gamma, \kappa(m)} s^{|a| + m/2 - 3} (1 - s/u)^{\zeta_\star} ds \\ &\leq C (\alpha/2)^{-m} m^{1-\beta} \lambda_u^{\rho_\gamma, \kappa(m)} u^{|a| + m/2 - 2} \|V_\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma}} \end{aligned}$$

with some universal constant  $C \in (0, \infty)$  for  $m = 2$  and  $|a| > 1$ , or  $m = 4$  and  $|a| > 0$ . This proves the bound (C<sub>1</sub>). Let us turn to the proof of the bounds (A<sub>2</sub>), (B<sub>2</sub>), (C<sub>2</sub>). By Def. 10.32 of the norm  $\|\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}}$  and the triangle inequality it suffices to show that

$$\begin{aligned} (\text{A}_3) \quad &\|u \mapsto \Pi_k \mathbf{A}_{\tau, \varepsilon; t, u} \Pi_{>4} \mathbf{I}_{\tau, \varepsilon; u} V_\bullet\|_{\mathcal{V}^{\alpha, \beta+1; \gamma}} \leq C \|V_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}}, \\ (\text{B}_3) \quad &\|u \mapsto \Pi_k \mathbf{A}_{\tau, \varepsilon; t, u} \Pi_{>4} \mathbf{I}_{\tau, \varepsilon; u} V_\bullet\|_{\tilde{\mathcal{V}}^{\alpha, \beta+1; \gamma}} \leq C \|V_\bullet\|_{\tilde{\mathcal{W}}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}}, \\ (\text{C}_3) \quad &\|u \mapsto \Pi_k \mathbf{A}_{\tau, \varepsilon; t, u} \Pi_{>4} (\mathbf{I}_u - \mathbf{I}_{\tau, \varepsilon; u}) V_\bullet\|_{\tilde{\mathcal{V}}^{\alpha/4, \beta; \gamma}} \leq C \lambda_{T \vee \varepsilon}^\kappa \|V_\bullet\|_{\mathcal{V}^{\alpha, \beta; \gamma}} \end{aligned}$$

for  $k \in \{1, 2, 3, 4\}$  and

$$(\text{A}_4) \quad \|u \mapsto \Pi_{>4} \mathbf{A}_{\tau, \varepsilon; t, u} \Pi_{>4} \mathbf{I}_{\tau, \varepsilon; u} V_\bullet\|_{\mathcal{V}^{\alpha, \beta+1; \gamma}} \leq C \|V_\bullet\|_{\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}},$$

(B<sub>4</sub>)  $\|u \mapsto \Pi_{>4} \mathbf{A}_{\tau,\varepsilon;t,u} \Pi_{>4} \mathbf{I}_{\tau,\varepsilon;u} V_\bullet\|_{\mathcal{V}^{\alpha,\beta+1;\gamma}} \leq C \|V_\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}},$   
(C<sub>4</sub>)  $\|u \mapsto \Pi_{>4} \mathbf{A}_{\tau,\varepsilon;t,u} \Pi_{>4} (\mathbf{I}_u - \mathbf{I}_{\tau,\varepsilon;u}) V_\bullet\|_{\mathcal{V}^{\alpha/4,\beta;\gamma}} \leq C \lambda_{\tau \vee \varepsilon}^\kappa \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$   
with some constant  $C \in (0, \infty)$  independent of  $\tau, \varepsilon, t \in [0, 1]$ . Actually, the bounds (A<sub>4</sub>), (B<sub>4</sub>), (C<sub>4</sub>) with  $t = u$  together with Lemma 10.26 (A), (C), (E) and Remark 10.19 imply the bounds (A<sub>3</sub>), (B<sub>3</sub>), (C<sub>3</sub>). Hence, it remains to prove the bounds (A<sub>4</sub>), (B<sub>4</sub>), (C<sub>4</sub>). Using Def. 10.22 of the map  $\mathbf{A}_{\tau,\varepsilon;t,u}$  and Remark 10.25 (B) we obtain

$$\begin{aligned} \Pi_{>4} \mathbf{A}_{\tau,\varepsilon;t,u} \Pi_{>4} \mathbf{I}_{\tau,\varepsilon;u} V_\bullet &= \Pi_{>4} \mathbf{A}_{\tau,\varepsilon;t,u} \mathbf{I}_{\tau,\varepsilon;u} V_\bullet \\ &= \Pi_{>4} \mathbf{A}_{\tau,\varepsilon;t,u} \mathbf{E} \int_0^u \mathbf{A}_{\tau,\varepsilon;u,s} V_s / s \, ds = \Pi_{>4} \mathbf{E}_u \int_0^u \mathbf{A}_{\tau,\varepsilon;t,s} V_s / s \, ds. \end{aligned}$$

As a result, using Def. 10.32 and 10.15 as well as Remark 10.10 and Lemma 2.4 (A) we obtain

$$\begin{aligned} \|w_u^m \mathbf{A}_{\tau,\varepsilon;t,u}^{m,a,\sigma} \mathbf{I}_{\tau,\varepsilon;u} V_\bullet\|_{\mathcal{M}^m} &\leq \int_0^u s^{-1} \|w_u^m \mathbf{A}_{\tau,\varepsilon;t,s}^{m,a,\sigma} V_s\|_{\mathcal{M}^m} \, ds \leq \int_0^u s^{-1} \|w_s^m \mathbf{A}_{\tau,\varepsilon;t,s}^{m,a,\sigma} V_s\|_{\mathcal{M}^m} \, ds \\ &\leq \alpha^{-m} m^{-\beta} \|V_\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}} \int_0^u \lambda_s^{\rho_{\gamma,\kappa}(m)} s^{|a|+m/2-3} \, ds \\ &\leq 10 \alpha^{-m} m^{-\beta-1} \lambda_u^{\rho_{\gamma,\kappa}(m)} u^{|a|+m/2-2} \|V_\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}} \end{aligned}$$

for  $m \in \{5, 6, \dots\}$ . This implies the bound (A<sub>4</sub>). The proof of the bound (B<sub>4</sub>) is the same only the weight  $w_s^m$  is replaced with the weight  $\tilde{w}_s^m$ . In order to verify the bound (C<sub>4</sub>) we first note that by Lemma 10.26 (C) it holds

$$\begin{aligned} \|u \mapsto \Pi_{>4} \mathbf{A}_{\tau,\varepsilon;t,u} \Pi_{>4} (\mathbf{I}_u - \mathbf{I}_{\tau,\varepsilon;u}) V_\bullet\|_{\mathcal{V}^{\alpha/4,\beta;\gamma}} &\leq \|u \mapsto \mathbf{A}_{\tau,\varepsilon;t,u} \Pi_{>4} (\mathbf{I}_u - \mathbf{I}_{\tau,\varepsilon;u}) V_\bullet\|_{\mathcal{V}^{\alpha/4,\beta;\gamma}} \\ &\leq \|u \mapsto \Pi_{>4} (\mathbf{I}_u - \mathbf{I}_{\tau,\varepsilon;u}) V_\bullet\|_{\mathcal{V}^{\alpha/2,\beta;\gamma}}. \end{aligned}$$

Using Def. 10.32 and 10.15 as well as Lemmas 10.26 (E) and 2.4 (A) we obtain

$$\begin{aligned} \|w_u^m (\mathbf{I}_u^m - \mathbf{I}_{\tau,\varepsilon;u}^m) V_\bullet\|_{\mathcal{M}^m} &\leq \int_0^u s^{-1} \|w_s^m (\mathbf{A}_{u,s}^{m,a,\sigma} - \mathbf{A}_{\tau,\varepsilon;u,s}^{m,a,\sigma}) V_s\|_{\mathcal{M}^m} \, ds \\ &\leq (\alpha/2)^{-m} m^{1-\beta} \|s \mapsto (\mathbf{A}_{u,s} - \mathbf{A}_{\tau,\varepsilon;u,s}) V_s\|_{\mathcal{V}_{\tau,\varepsilon}^{\alpha/2,\beta-1;\gamma}} \int_0^u \lambda_s^{\rho_{\gamma,\kappa}(m)} s^{|a|+m/2-3} \, ds \\ &\leq 10 (\alpha/2)^{-m} m^{-\beta} \lambda_{\tau \vee \varepsilon}^\kappa \lambda_u^{\rho_{\gamma,\kappa}(m)} u^{|a|+m/2-2} \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}} \end{aligned}$$

for  $m \in \{5, 6, \dots\}$ , which implies that

$$\|u \mapsto \Pi_{>4} (\mathbf{I}_u - \mathbf{I}_{\tau,\varepsilon;u}) V_\bullet\|_{\mathcal{V}^{\alpha/2,\beta;\gamma}} \leq C \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}.$$

This finishes the proof of the bound (C<sub>4</sub>).

It remains to prove that the map  $u \mapsto \mathbf{I}_{\tau,\varepsilon;u} V_\bullet$  belongs to  $\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta+1;\gamma}$ . By the bound (A) and the fact that  $\mathcal{V}_\tau^{\text{fin};\gamma}$  is a dense subset of  $\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}$  it suffices to establish the above claim for  $V_\bullet \in \mathcal{V}_\tau^{\text{fin};\gamma}$ . It follows from Lemma 12.8 that  $\mathbf{I}_{\tau,\varepsilon;u} V_\bullet \in \mathcal{N}_\tau^{\text{fin}}$  for all  $u \in (0, 1]$ . By the bound (A) the map  $u \mapsto \mathbf{I}_{\tau,\varepsilon;u} V_\bullet$  belongs to  $\mathcal{V}_\tau^{\text{fin};\gamma} \subset \mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta+1;\gamma}$ . This finishes the proof.  $\square$

**Lemma 12.8.** *Let  $\tau, \varepsilon \in [0, 1]$  and  $t, s \in (0, 1]$ . If  $V \in \mathcal{N}_\tau^{\text{fin}}$ , then  $\mathbf{B}_{\varepsilon;t}(V) \in \mathcal{N}_\tau^{\text{fin}}$  and  $\mathbf{EA}_{\tau,\varepsilon;t,s}V \in \mathcal{N}_\tau^{\text{fin}}$ .*

*Proof.* We have to prove that if  $V \in \mathcal{N}^{\text{fin}}$  is invariant under the symmetries of the torus/plane and the internal symmetries and compatible with the flow of charge, then  $\mathbf{B}_{\varepsilon;t}(V)$  and  $\mathbf{EA}_{\tau,\varepsilon;t,s}V$  have the same properties. First, note that the invariance of  $\mathbf{B}_{\varepsilon;t}(V)$  and  $\mathbf{EA}_{\tau,\varepsilon;t,s}V$  under the symmetries of the torus/plane and the internal symmetries is a simple consequence of Def. 10.22 and 12.4 of the maps  $\mathbf{A}_{\tau,\varepsilon;t,s}$  and  $\mathbf{B}_{\varepsilon;t}$  as well as Remarks 12.2 and 6.14. The non-trivial part of the proof is the verification of the compatibility with the flow of charge of  $\mathbf{B}_{\varepsilon;t}(V)$  and  $\mathbf{EA}_{\tau,\varepsilon;t,s}V$ . For an argument based on the analysis of Feynman diagrams we refer the reader to [GK85a, Sec. 5.B]. Let us give a direct proof of the above claim. Since by assumption  $V \in \mathcal{N}^{\text{fin}}$  is compatible with the flow of charge for every  $m \in \mathbb{N}_+$  there exists  $W^m \in \mathcal{N}_\#^m$  such that  $V^m = \mathbf{S}W^m$  and  $W^m$  satisfies the properties listed in Def. 10.28. Our goal is to prove that:

- (1) for all  $m \in \mathbb{N}_+$  there exists  $U^m \in \mathcal{N}_\#^m$  such that  $\mathbf{B}_{\varepsilon;t}^m V = \mathbf{S}U^m$  and  $U^m$  satisfies the conditions listed in Def. 10.28 and
- (2) for all  $m \in \mathbb{N}_+$  there exists  $U^m \in \mathcal{N}_\#^m$  such that  $\mathbf{EA}_{\tau,\varepsilon;t,s}^m V = \mathbf{S}U^m$  and  $U^m$  satisfies the conditions listed in Def. 10.28.

To prove (1) is enough to observe that  $U^m := \mathbf{B}_{\varepsilon;t}^m(W)$  satisfies the above-mentioned conditions by Remark 12.2. The proof of (2) is more involved. Let us set  $\mathcal{N}_\#^{-1} = \mathcal{N}_\#^0 = \mathbb{C}$ . For  $n \in \mathbb{N}_+$  and  $i, j \in \mathbb{N}_+$  such that  $i < j$  we define the operators

$$\mathbf{D}_{\tau,\varepsilon;t,s}^{(i,j)}, \mathbf{D}_{\tau,\varepsilon;t,s}, \tilde{\mathbf{D}}_{\tau,\varepsilon;t,s} : \mathcal{N}_\#^n \rightarrow \mathcal{N}_\#^{n-2}$$

by the equalities  $\mathbf{D}_{\tau,\varepsilon;t,s}^{(i,j)}V = 0$  for  $j > n$ ,

$$\begin{aligned} & \langle \mathbf{D}_{\tau,\varepsilon;t,s}^{(i,j)} V, \varphi_1 \otimes \dots \otimes \varphi_n \rangle \\ & := \mathbf{E} \langle V, \varphi_1 \otimes \dots \otimes \varphi_{i-1} \otimes \mathbf{J}\Psi_{\tau,\varepsilon;t\vee s,s} \otimes \varphi_{i+1} \otimes \dots \otimes \varphi_{j-1} \otimes \mathbf{J}\Psi_{\tau,\varepsilon;t\vee s,s} \otimes \varphi_{j+1} \otimes \dots \otimes \varphi_n \rangle \end{aligned}$$

for  $j \leq n$  and

$$\mathbf{D}_{\tau,\varepsilon;t,s} := 2 \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \mathbf{D}_{\tau,\varepsilon;t,s}^{(i,j)}, \quad \tilde{\mathbf{D}}_{\tau,\varepsilon;t,s} := (n\mathbf{D}_{\tau,\varepsilon;t,s}^{(1,2)} + n(n-2)\mathbf{D}_{\tau,\varepsilon;t,s}^{(2,3)}).$$

Note that by Remark 6.14 and Def. 12.1 the expected value appearing in the equation defining  $\mathbf{D}_{\tau,\varepsilon;t,s}^{(i,j)}$  can be expressed in terms of the function  $H_{\tau,\varepsilon;s} - H_{\tau,\varepsilon;t\vee s}$ . We observe that for all  $V \in \mathcal{N}^{\text{fin}}$  and  $m \in \mathbb{N}_+$  it holds

$$\begin{aligned} \mathbf{EA}_{\tau,\varepsilon;t,s}V &= \exp(\mathbf{D}_{\tau,\varepsilon;t,s}/2)V = \exp(\tilde{\mathbf{D}}_{\tau,\varepsilon;t,s}/2)V, \\ \mathbf{EA}_{\tau,\varepsilon;t,s}^m V &= \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{D}_{\tau,\varepsilon;t,s}/2)^n V^{m+2n} = \sum_{n=0}^{\infty} \frac{1}{n!} (\tilde{\mathbf{D}}_{\tau,\varepsilon;t,s}/2)^n V^{m+2n}, \end{aligned}$$

where we used the fact that  $V^m$  is antisymmetric for all  $m \in \mathbb{N}_+$ . The series on the RHS of the second equation above are actually finite sums due to the assumption  $V \in \mathcal{N}^{\text{fin}}$ . For  $m \in \mathbb{N}_+$  we define

$$U^m := \sum_{n=0}^{\infty} \frac{1}{n!} (\tilde{\mathbf{D}}_{\tau,\varepsilon;t,s}/2)^n W^{m+2n}.$$

Recall that  $(W^m)_{m \in \mathbb{N}_+}$  satisfies the properties (a), (b), (c) stated in Def. 10.28. Using the properties (b) and (c) we prove that  $\mathbf{E}\mathbf{A}_{\tau,\varepsilon;t,s}^m V = \mathbf{S}U^m$  for all  $m \in \mathbb{N}_+$ . Using the property (a) of  $(W^m)_{m \in \mathbb{N}_+}$  and the invariance of  $H_{\tau,\varepsilon;s} - H_{\tau,\varepsilon;t \vee s}$  under the charge conjugation symmetry we show the property (a) of  $(U^m)_{m \in \mathbb{N}_+}$ . The properties (b) and (c) of  $(U^m)_{m \in \mathbb{N}_+}$  follow from the properties (b) and (c) of  $(W^m)_{m \in \mathbb{N}_+}$  and the fact that  $H_{\tau,\varepsilon;s} - H_{\tau,\varepsilon;t \vee s}$  is antisymmetric. This finishes the proof.  $\square$

### 13 Fixed point problem

In this section we construct a solution  $X_{\tau,\varepsilon;\bullet} \equiv (g_{\tau,\varepsilon;\bullet}, r_{\tau,\varepsilon;\bullet}, z_{\tau,\varepsilon;\bullet}, W_{\tau,\varepsilon;\bullet})$  of the system of equations (2.24), (2.28), (2.20), (2.21), (2.22) introduced in Sec. 2. To this end, we rewrite this system of equations as a fixed point equation of a certain map  $\mathbf{X}_{\tau,\varepsilon;\bullet}$  acting in a complete metric space  $\mathcal{Y}_{\tau,\varepsilon}$  and prove that the map  $\mathbf{X}_{\tau,\varepsilon;\bullet}$  is a contraction provided the parameter  $\lambda \in (0, 1]$  is sufficiently small. As argued in Sec. 2 and proved in Sec. 14 using  $X_{\tau,\varepsilon;\bullet}$  it is possible to construct a solution  $U_{\tau,\varepsilon;\bullet}$  of the flow equation (7.3).

**Definition 13.1.** For  $\tau, \varepsilon \in [0, 1]$  and

$$X_{\bullet} \equiv (g_{\bullet}, r_{\bullet}, z_{\bullet}, W_{\bullet}) \in C((0, 1], \mathbb{C}) \times C((0, 1], \mathbb{C}) \times C((0, 1], \mathbb{C}) \times \mathcal{W}_{\tau,\varepsilon}^{8,4;2-80\kappa}$$

we define

$$\begin{aligned} \|X_{\bullet}\|_{\mathcal{D}} &:= \sup_{t \in (0,1]} \lambda_t^{1+10\kappa} |g_t| + \sup_{t \in (0,1]} \lambda_t^{36\kappa-1} t |r_t| + \sup_{t \in (0,1]} \lambda_t^{36\kappa-1} |z_t|, \\ \|X_{\bullet}\|_{\mathcal{X}_{\tau,\varepsilon}} &:= \|X_{\bullet}\|_{\mathcal{D}} + \|W_{\bullet}\|_{\mathcal{W}_{\tau,\varepsilon}^{8,4;2-80\kappa}}, \quad \|X_{\bullet}\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}} := \|X_{\bullet}\|_{\mathcal{D}} + \|W_{\bullet}\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{2,3;2-80\kappa}}. \end{aligned}$$

By definition the vector space  $\mathcal{X}_{\tau,\varepsilon}$  over  $\mathbb{C}$  consists of maps  $X_{\bullet}$  of the above form such that  $\|X_{\bullet}\|_{\mathcal{X}_{\tau,\varepsilon}} < \infty$ . We also define the set

$$\mathcal{Y}_{\tau,\varepsilon} := \{X_{\bullet} \in \mathcal{X}_{\tau,\varepsilon} \mid \|X_{\bullet}\|_{\mathcal{X}_{\tau,\varepsilon}} \leq 1, \mathcal{P}_{\varepsilon}(X_{\bullet}) \geq \lambda^{\kappa}, \forall t \in (0,1] \text{ Im } g_t = \text{Im } r_t = \text{Im } z_t = 0\},$$

where  $\mathcal{P}_{\varepsilon}(X_{\bullet}) := \inf_{t \in (0,1]} \lambda_{\varepsilon \vee t} g_t$ . We omit  $\tau, \varepsilon$  if  $\tau = 0, \varepsilon = 0$ .

**Lemma 13.2.** For all  $\tau, \varepsilon \in [0, 1]$  the space  $(\mathcal{X}_{\tau,\varepsilon}, \|\bullet\|_{\mathcal{X}_{\tau,\varepsilon}})$  is a Banach space. Moreover, the set  $\mathcal{Y}_{\tau,\varepsilon}$  is a closed subset of  $\mathcal{X}_{\tau,\varepsilon}$ .

*Proof.* The statement follows from Lemma 10.37.  $\square$

**Lemma 13.3.** Let  $\tau, \varepsilon \in [0, 1]$ .

- (A) For all  $X_\bullet \in \mathcal{X}_{\tau,\varepsilon}$  it holds  $\|X_\bullet\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}} \leq \|X_\bullet\|_{\mathcal{X}_{\tau,\varepsilon}}$ .  
 (B) For all  $X_\bullet \in \mathcal{X}$  it holds  $\|X_\bullet\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}} \leq \|X_\bullet\|_{\mathcal{X}}$ .

*Proof.* Item (A) follows from Def. 10.32 and Remark 10.18. To prove Item (B) observe that

$$\|W_\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{2,3;2-80\kappa}} \leq \|W_\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{2,3;2-80\kappa}} \leq \|W_\bullet\|_{\mathcal{V}^{4,3;2-80\kappa}} \leq \|W_\bullet\|_{\mathcal{V}^{8,4;2-80\kappa}} \leq \|W_\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{8,4;2-80\kappa}}$$

by Remark 10.33, 10.35, 10.18 and 10.34, respectively.  $\square$

**Definition 13.4.** Let  $\tau, \varepsilon \in [0, 1]$ . For  $X_\bullet \equiv (g_\bullet, r_\bullet, z_\bullet, W_\bullet) \in \mathcal{Y}_{\tau,\varepsilon}$  and  $t \in (0, 1]$  define

$$\begin{aligned} \mathbf{W}_{\tau,\varepsilon;t}^m(X_\bullet) &:= \int_0^t \mathbf{EA}_{\tau,\varepsilon;t,s}^m \mathbf{B}_{\varepsilon;s}(V_s) ds \in \mathcal{N}^m, & m \in \mathbb{N}_+ \setminus \{2, 4\}, \\ \mathbf{W}_{\tau,\varepsilon;t}^m(X_\bullet) &:= \int_0^t \mathbf{REA}_{\tau,\varepsilon;1,s}^m \mathbf{B}_{\varepsilon;s}(V_s) ds - \mathbf{EC}_{\tau,\varepsilon;1,t}^m W_t \in \mathcal{N}^m, & m \in \{2, 4\}, \end{aligned}$$

as well as

$$\begin{aligned} \mathbf{g}_{\tau,\varepsilon;t}(X_\bullet) &:= \lambda^{-1} + \int_t^1 (g_s)^2 \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_s) ds \in \mathbb{R}, \\ \mathbf{r}_{\tau,\varepsilon;t}(X_\bullet) &:= -\mathbf{LEA}_{\tau,\varepsilon;1,t}^2 U(1/g_t, 0, 0) - \int_t^1 \mathbf{LEA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_s) ds \in \mathbb{R}, \\ \mathbf{z}_{\tau,\varepsilon;t}(X_\bullet) &:= \int_0^t \mathbf{L}_\partial \mathbf{EA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_s) ds \in \mathbb{R}, \end{aligned}$$

where

$$V_s \equiv V_s(X_\bullet) = U(1/g_s, r_s, z_s) + W_s \in \mathcal{N}, \quad s \in (0, 1].$$

For  $\lambda \in (0, 1]$  small enough the map  $\mathbf{X}_{\tau,\varepsilon;\bullet} : \mathcal{Y}_{\tau,\varepsilon} \rightarrow \mathcal{X}_{\tau,\varepsilon}$  is defined by

$$\mathbf{X}_{\tau,\varepsilon;\bullet} := \mathbf{g}_{\tau,\varepsilon;\bullet} \times \mathbf{r}_{\tau,\varepsilon;\bullet} \times \mathbf{z}_{\tau,\varepsilon;\bullet} \times \mathbf{W}_{\tau,\varepsilon;\bullet}, \quad \mathbf{W}_{\tau,\varepsilon;\bullet} := \bigtimes_{m \in \mathbb{N}_+} \mathbf{W}_{\tau,\varepsilon;\bullet}^m.$$

*Remark 13.5.* The fact that the map  $\mathbf{X}_{\tau,\varepsilon;\bullet}$  is well-defined is non-trivial and is a consequence of the estimates established below. We also point out that  $\mathbf{W}_{\tau,\varepsilon;t}^m(X_\bullet) = 0$  for  $m \in \mathbb{N}_+ \setminus 2\mathbb{N}_+$  by Remark 8.7.

*Remark 13.6.* We call the parameter  $g_{\tau,\varepsilon;t}$  the inverse of the effective coupling constant. Note that we fixed  $g_{\tau,\varepsilon;t}$  at unit scale  $t = 1$  to be  $g_{\tau,\varepsilon;t=1} := \lambda^{-1}$ , where  $\lambda \in (0, \lambda_\star]$  and  $\lambda_\star \in (0, 1]$  is a small constant.

*Remark 13.7.* The parameter  $r_{\tau,\varepsilon;t}$  is the correction to the effective mass due to the interaction. Note that we fixed  $r_{\tau,\varepsilon;t}$  at unit scale  $t = 1$  to be  $r_{\tau,\varepsilon;t=1} = 0$ . Recall that the free part of the action contains the unit mass term. Hence, the above condition implies that the effective mass at unit scale equals one.

*Remark 13.8.* The fact that  $\mathbf{g}_{\tau,\varepsilon;t}(X_\bullet), \mathbf{r}_{\tau,\varepsilon;t}(X_\bullet), \mathbf{z}_{\tau,\varepsilon;t}(X_\bullet) \in \mathbb{R}$  follows from Lemma 11.10 and the symmetry properties of functionals  $U(g, r, z)$  and  $W \in \mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}$ .

**Theorem 13.9.** *There exists  $\lambda_\star \in (0, 1]$  and  $C \in (0, \infty)$  such that for all  $\lambda \in (0, \lambda_\star]$ , all  $\tau, \varepsilon \in [0, 1]$  and all  $X_\bullet, Y_\bullet \in \mathcal{Y}_{\tau,\varepsilon}$ ,  $Z_\bullet \in \mathcal{Y}$  it holds:*

- (A)  $\|s \mapsto \mathbf{X}_{\tau,\varepsilon;s}(X_\bullet)\|_{\mathcal{X}_{\tau,\varepsilon}} \leq C \lambda^\kappa$  and  $\mathcal{P}_\varepsilon(s \mapsto \mathbf{X}_{\tau,\varepsilon;s}(X_\bullet)) \geq 1/C$ ,
- (B)  $\|s \mapsto (\mathbf{X}_{\tau,\varepsilon;s}(X_\bullet) - \mathbf{X}_{\tau,\varepsilon;s}(Y_\bullet))\|_{\mathcal{X}_{\tau,\varepsilon}} \leq C \lambda^\kappa \|X_\bullet - Y_\bullet\|_{\mathcal{X}_{\tau,\varepsilon}}$ ,
- (C)  $\|s \mapsto (\mathbf{X}_{\tau,\varepsilon;s}(X_\bullet) - \mathbf{X}_{\tau,\varepsilon;s}(Z_\bullet))\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}} \leq C \lambda^\kappa \|X_\bullet - Z_\bullet\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}}$ ,
- (D)  $\|s \mapsto (\mathbf{X}_s(Z_\bullet) - \mathbf{X}_{\tau,\varepsilon;s}(Z_\bullet))\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}} \leq C \lambda_{\tau \vee \varepsilon}^\kappa$ .

*Proof.* The theorem is an immediate consequence of Lemmas 13.18, 13.26, 13.27, 13.28.  $\square$

**Corollary 13.10.** *There exists  $\lambda_\star \in (0, 1]$  and  $C \in (0, \infty)$  such that for all  $\lambda \in (0, \lambda_\star]$  and all  $\tau, \varepsilon \in [0, 1]$  the map  $\mathbf{X}_{\tau,\varepsilon;\bullet} : \mathcal{Y}_{\tau,\varepsilon} \rightarrow \mathcal{Y}_{\tau,\varepsilon}$  is well-defined and has the unique fixed point denoted by  $X_{\tau,\varepsilon;\bullet} \equiv (g_{\tau,\varepsilon;\bullet}, r_{\tau,\varepsilon;\bullet}, z_{\tau,\varepsilon;\bullet}, W_{\tau,\varepsilon;\bullet})$  such that*

$$\|X_\bullet - X_{\tau,\varepsilon;\bullet}\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}} \leq C \lambda_{\tau \vee \varepsilon}^\kappa, \quad (13.1)$$

where  $X_\bullet := X_{\tau,\varepsilon;\bullet}$  with  $\tau = 0, \varepsilon = 0$ , and

$$\|V_{\tau,\varepsilon;1}^m\|_{\mathcal{N}^m} \leq C, \quad \|V_1^m - V_{\tau,\varepsilon;1}^m\|_{\tilde{\mathcal{N}}^m} \leq C \lambda_{\tau \vee \varepsilon}^\kappa, \quad (13.2)$$

where  $V_{\tau,\varepsilon;1} \equiv (V_{\tau,\varepsilon;1}^m)_{m \in \mathbb{N}_+} := U(1/g_{\tau,\varepsilon;1}, r_{\tau,\varepsilon;1}, z_{\tau,\varepsilon;1}) + W_{\tau,\varepsilon;1} \in \mathcal{N}$  for all  $\tau, \varepsilon \in [0, 1]$  and  $V_1 := V_{\tau,\varepsilon;1}$  with  $\tau = 0, \varepsilon = 0$  and the norms  $\|\cdot\|_{\mathcal{N}^m}, \|\cdot\|_{\tilde{\mathcal{N}}^m}$  were introduced in Def. 10.12.

*Proof.* The fact that for all sufficiently small  $\lambda \in (0, 1]$  the map  $\mathbf{X}_{\tau,\varepsilon;\bullet} : \mathcal{Y}_{\tau,\varepsilon} \rightarrow \mathcal{Y}_{\tau,\varepsilon}$  is well-defined and is a contraction follows from Theorem 13.9 and Remark 13.8. The existence and uniqueness of the fixed point  $X_{\tau,\varepsilon;\bullet}$  is then a consequence of Lemma 13.2 and the Banach fixed point theorem. In order to prove the bound (13.1) note that

$$X_\bullet - X_{\tau,\varepsilon;\bullet} = \mathbf{X}_\bullet(X_\bullet) - \mathbf{X}_{\tau,\varepsilon;\bullet}(X_{\tau,\varepsilon;\bullet}).$$

Consequently, by the triangle inequality we obtain

$$\|X_\bullet - X_{\tau,\varepsilon;\bullet}\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}} \leq \|\mathbf{X}_\bullet(X_\bullet) - \mathbf{X}_{\tau,\varepsilon;\bullet}(X_\bullet)\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}} + \|\mathbf{X}_{\tau,\varepsilon;\bullet}(X_\bullet) - \mathbf{X}_{\tau,\varepsilon;\bullet}(X_{\tau,\varepsilon;\bullet})\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}}.$$

Since  $X_\bullet \in \mathcal{Y}$ ,  $X_{\tau,\varepsilon;\bullet} \in \mathcal{Y}_{\tau,\varepsilon}$  by the bounds (C) and (D) stated in Theorem 13.9 it holds

$$\|\mathbf{X}_{\tau,\varepsilon;\bullet}(X_\bullet) - \mathbf{X}_{\tau,\varepsilon;\bullet}(X_{\tau,\varepsilon;\bullet})\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}} \leq C \lambda^\kappa \|X_\bullet - X_{\tau,\varepsilon;\bullet}\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}}$$

and

$$\|\mathbf{X}_\bullet(X_\bullet) - \mathbf{X}_{\tau,\varepsilon;\bullet}(X_\bullet)\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}} \leq C \lambda_{\tau \vee \varepsilon}^\kappa.$$

Consequently, we have

$$\|X_\bullet - X_{\tau,\varepsilon;\bullet}\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}} \leq C \lambda_{\tau \vee \varepsilon}^\kappa + C \lambda^\kappa \|X_\bullet - X_{\tau,\varepsilon;\bullet}\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}}.$$

This proves the bound (13.1). By Def. 13.1, Remarks 10.34, 10.18 and the fact that  $\lambda_{t=1} = \lambda$  we have

$$\begin{aligned} \lambda^{-3/4} |1/g_{\tau,\varepsilon;1}| + |r_{\tau,\varepsilon;1}| + |z_{\tau,\varepsilon;1}| + \|W_{\tau,\varepsilon;\bullet}\|_{\mathcal{V}^{1,0;0}} &\leq 2, \\ \lambda^{3/2} |g_1 - g_{\tau,\varepsilon;1}| + |r_1 - r_{\tau,\varepsilon;1}| + |z_1 - z_{\tau,\varepsilon;1}| + \|W_\bullet - W_{\tau,\varepsilon;\bullet}\|_{\tilde{\mathcal{V}}^{1,0;0}} &\leq C \lambda_{\tau \vee \varepsilon}^\kappa \end{aligned}$$

for all  $\tau, \varepsilon \in [0, 1]$ . As a result, in particular

$$|1/g_1 - 1/g_{\tau,\varepsilon;1}| \leq C \lambda_{\tau \vee \varepsilon}^\kappa$$

for all  $\tau, \varepsilon \in [0, 1]$ . The bounds (13.2) follow now from Def. 10.15, 11.1. This finishes the proof.  $\square$

**Definition 13.11.** For  $\varepsilon \in [0, 1]$  and  $t \in (0, 1]$  we define  $\theta_{\varepsilon;t} := \lambda_t / \lambda_{\varepsilon \vee t} \in [0, 1]$ .

**Lemma 13.12.** For all  $\tau, \varepsilon \in [0, 1]$  and

$$X_\bullet \equiv (g_\bullet, r_\bullet, z_\bullet, W_\bullet) \in \mathcal{Y}_{\tau,\varepsilon}, \quad Y_\bullet \equiv (\tilde{g}_\bullet, \tilde{r}_\bullet, \tilde{z}_\bullet, \tilde{W}_\bullet) \in \mathcal{Y}_{\tau,\varepsilon}, \quad Z \equiv (\hat{g}_\bullet, \hat{r}_\bullet, \hat{z}_\bullet, \hat{W}_\bullet) \in \mathcal{Y}$$

it holds

$$\begin{aligned} (A) \quad &\sup_{s \in (0,1]} \lambda_s^{12\kappa-1} |\theta_{\varepsilon;s}^2 / g_s| \leq 1, \\ (B) \quad &\sup_{s \in (0,1]} \lambda_s^{12\kappa-1} |\theta_{\varepsilon;s}^2 / g_s - \theta_{\varepsilon;s}^2 / \tilde{g}_s| \leq \|X_\bullet - Y_\bullet\|_{\mathcal{X}_{\tau,\varepsilon}}, \\ (C) \quad &\sup_{s \in (0,1]} \lambda_s^{12\kappa-1} |\theta_{\varepsilon;s}^2 / g_s - \theta_{\varepsilon;s}^2 / \hat{g}_s| \leq \|X_\bullet - Z_\bullet\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}}. \end{aligned}$$

*Proof.* First observe that  $\lambda_{\varepsilon \vee s} \theta_{\varepsilon;s}^2 \leq \lambda_s$ ,  $\lambda_{\varepsilon \vee s}^2 \theta_{\varepsilon;s}^2 \leq \lambda_s^2$  and  $\lambda_s \leq \lambda$ . The above bounds and Def. 13.1 imply that

$$\begin{aligned} (A_1) \quad &\sup_{s \in (0,1]} \lambda_s^{\kappa-1} |\theta_{\varepsilon;s}^2 / g_s| \leq \lambda_s^\kappa / \mathcal{P}_\varepsilon(X_\bullet), \\ (B_1) \quad &\sup_{s \in (0,1]} \lambda_s^{12\kappa-1} |\theta_{\varepsilon;s}^2 / g_s - \theta_{\varepsilon;s}^2 / \tilde{g}_s| \leq \lambda_s^\kappa / \mathcal{P}_\varepsilon(X_\bullet) \lambda_s^\kappa / \mathcal{P}_\varepsilon(Y_\bullet) \|X_\bullet - Y_\bullet\|_{\mathcal{P}}, \\ (C_1) \quad &\sup_{s \in (0,1]} \lambda_s^{12\kappa-1} |\theta_{\varepsilon;s}^2 / g_s - \theta_{\varepsilon;s}^2 / \hat{g}_s| \leq \lambda_s^\kappa / \mathcal{P}_\varepsilon(X_\bullet) \lambda_s^\kappa / \mathcal{P}(X_\bullet) \|X_\bullet - Z_\bullet\|_{\mathcal{P}}. \end{aligned}$$

The lemma follows now from Def. 13.1.  $\square$

**Lemma 13.13.** Let  $\alpha, \beta \in [1, \infty)$ ,  $\gamma \in [0, \infty)$ . There exists  $C \in (0, \infty)$  such that for all  $\tau, \varepsilon \in [0, 1]$  it holds:

$$\begin{aligned} (A) \quad &\|s \mapsto \lambda_s^{-\gamma-8\kappa} U(1, 0, 0)\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}} \leq C, \\ (B) \quad &\|s \mapsto \lambda_s^{-\gamma-4\kappa} s U(0, 1, 0)\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}} \leq C, \\ (C) \quad &\|s \mapsto \lambda_s^{-\gamma-4\kappa} U(0, 0, 1)\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}} \leq C. \end{aligned}$$

Moreover, analogous bounds with  $\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}$  replaced by  $\tilde{\mathcal{W}}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}$  are true.

*Proof.* From Def. 11.1 of the functional  $U(g, r, z)$  and Def. 10.15 of the norms  $\|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$  and  $\|\bullet\|_{\tilde{\mathcal{V}}^{\alpha,\beta;\gamma}}$  it follows that the bounds

$$\begin{aligned} (A_1) \quad &\|s \mapsto \lambda_s^{-\gamma-8\kappa} U(1, 0, 0)\|_{\mathcal{V}^{2\alpha,\beta;\gamma}} \leq C, \\ (B_1) \quad &\|s \mapsto \lambda_s^{-\gamma-4\kappa} s U(0, 1, 0)\|_{\mathcal{V}^{2\alpha,\beta;\gamma}} \leq C, \\ (C_1) \quad &\|s \mapsto \lambda_s^{-\gamma-4\kappa} U(0, 0, 1)\|_{\mathcal{V}^{2\alpha,\beta;\gamma}} \leq C \end{aligned}$$

as well as the bounds with  $\mathcal{V}^{2\alpha,\beta;\gamma}$  replaced by  $\tilde{\mathcal{V}}^{2\alpha,\beta;\gamma}$  hold true. The bounds (A), (B), (C) are consequences of the bounds (A<sub>1</sub>), (B<sub>1</sub>), (C<sub>1</sub>) and Remark 10.35.  $\square$

*Remark 13.14.* It follows immediately from Def. 10.15 of the norm  $\|\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}$  and Def. 10.32 of the norm  $\|\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}}$  that for all continuous functions  $h_\bullet : (0, 1] \rightarrow \mathbb{R}$  it holds

$$\|s \mapsto h_s V_s\|_{\mathcal{V}^{\alpha,\beta;\gamma}} \leq \|h_\bullet\|_\infty \|V_\bullet\|_{\mathcal{V}^{\alpha,\beta;\gamma}}, \quad \|s \mapsto h_s V_s\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}} \leq \|h_\bullet\|_\infty \|V_\bullet\|_{\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}},$$

where  $\|h_\bullet\|_\infty := \sup_{s \in (0,1]} |h_s|$ . The analogous bounds with  $\mathcal{V}^{\alpha,\beta;\gamma}$  replaced by  $\tilde{\mathcal{V}}^{\alpha,\beta;\gamma}$  and  $\mathcal{W}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}$  replaced by  $\tilde{\mathcal{W}}_{\tau,\varepsilon}^{\alpha,\beta;\gamma}$  hold true as well.

**Lemma 13.15.** For  $\tau, \varepsilon \in [0, 1]$ ,  $X_\bullet \equiv (g_\bullet, r_\bullet, z_\bullet, W_\bullet) \in \mathcal{X}_{\tau,\varepsilon}$  and  $s \in (0, 1]$  define

$$\mathbf{V}_{\varepsilon;s}^{(1)}(X_\bullet) := U(0, r_s, z_s), \quad \mathbf{V}_{\varepsilon;s}^{(2)}(X_\bullet) := U(\theta_{\varepsilon;s}^2/g_s, 0, 0), \quad \mathbf{V}_{\varepsilon;s}^{(3)}(X_\bullet) := W_s$$

and

$$\mathbf{V}_{\varepsilon;s}^{(0)}(X_\bullet) := \mathbf{V}_{\varepsilon;s}^{(1)}(X_\bullet) + \mathbf{V}_{\varepsilon;s}^{(2)}(X_\bullet) + \mathbf{V}_{\varepsilon;s}^{(3)}(X_\bullet) = U(\theta_{\varepsilon;s}^2/g_s, r_s, z_s) + W_s.$$

We omit  $\varepsilon$  if  $\varepsilon = 0$ . There exists  $\lambda_\star \in (0, 1]$  and  $C \in (0, \infty)$  such that for all  $\lambda \in (0, \lambda_\star]$ , all  $\tau, \varepsilon \in [0, 1]$ , all  $X_\bullet, Y_\bullet \in \mathcal{X}_{\tau,\varepsilon}$ ,  $Z_\bullet \in \mathcal{Y}$  and all  $i \in \{1, 2, 3\}$  it holds:

- (A)  $\|s \mapsto \mathbf{V}_{\varepsilon;s}^{(i)}(X_\bullet)\|_{\mathcal{W}_{\tau,\varepsilon}^{8,4;\gamma(i)}} \leq C$ ,
- (B)  $\|s \mapsto (\mathbf{V}_{\varepsilon;s}^{(i)}(X_\bullet) - \mathbf{V}_{\varepsilon;s}^{(i)}(Y_\bullet))\|_{\mathcal{W}_{\tau,\varepsilon}^{8,4;\gamma(i)}} \leq C \|X_\bullet - Y_\bullet\|_{\mathcal{X}_{\tau,\varepsilon}}$ ,
- (C)  $\|s \mapsto (\mathbf{V}_{\varepsilon;s}^{(i)}(X_\bullet) - \mathbf{V}_{\varepsilon;s}^{(i)}(Z_\bullet))\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{2,3;\gamma(i)}} \leq C \|X_\bullet - Z_\bullet\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}}$ ,

where  $\gamma(0) = \gamma(1) = 1 - 40\kappa$ ,  $\gamma(2) = 1 - 20\kappa$  and  $\gamma(3) = 2 - 80\kappa$ .

*Proof.* For  $i = 1$  the bounds (A), (B), (C) follow from Lemma 13.13 applied with  $\gamma = \gamma(1)$ , Remark 13.14 and Def. 13.1. For  $i = 2$  the bounds (A), (B), (C) follow from Lemma 13.13 applied with  $\gamma = \gamma(2)$ , Lemma 13.12, Remark 13.14 and Def. 13.1. For  $i = 3$  the bounds (A), (B), (C) follow immediately from Def. 13.1. For  $i = 0$  the bounds (A), (B), (C) are consequences of the bounds (A), (B), (C) with  $i \in \{1, 2, 3\}$  and the triangle inequality.  $\square$

**Lemma 13.16.** For  $\tau, \varepsilon \in [0, 1]$ ,  $X_\bullet \in \mathcal{X}_{\tau,\varepsilon}$ ,  $i, j \in \{0, 1, 2, 3\}$  and  $s \in (0, 1]$  define

$$\mathbf{G}_{\varepsilon;s}^{(i,j)}(X_\bullet) := \mathbf{B}_{\varepsilon;s}(\mathbf{V}_s^{(i)}(X_\bullet), \mathbf{V}_s^{(j)}(X_\bullet)), \quad \mathbf{G}_{\varepsilon;s}(X_\bullet) := \mathbf{G}_{\varepsilon;s}^{(0,0)}(X_\bullet),$$

where the maps  $\mathbf{V}_{\varepsilon;s}^{(i)}$ ,  $i \in \{0, 1, 2, 3\}$ , were introduced in Lemma 13.15. We omit  $\varepsilon$  if  $\varepsilon = 0$ . There exists  $\lambda_\star \in (0, 1]$  and  $C \in (0, \infty)$  such that for all  $\lambda \in (0, \lambda_\star]$ , all  $\tau, \varepsilon \in [0, 1]$ , all  $X_\bullet, Y_\bullet \in \mathcal{X}_{\tau,\varepsilon}$ ,  $Z_\bullet \in \mathcal{Y}$  and all  $i, j \in \{0, 1, 2, 3\}$  it holds:

- (A)  $\|s \mapsto s \mathbf{G}_{\varepsilon;s}^{(i,j)}(X_\bullet)\|_{\mathcal{W}_{\tau,\varepsilon}^{8,3;\gamma(i)+\gamma(j)}} \leq C \lambda^\kappa$ ,
- (B)  $\|s \mapsto s (\mathbf{G}_{\varepsilon;s}^{(i,j)}(X_\bullet) - \mathbf{G}_{\varepsilon;s}^{(i,j)}(Y_\bullet))\|_{\mathcal{W}_{\tau,\varepsilon}^{8,3;\gamma(i)+\gamma(j)}} \leq C \lambda^\kappa \|X_\bullet - Y_\bullet\|_{\mathcal{X}_{\tau,\varepsilon}}$ ,
- (C)  $\|s \mapsto s (\mathbf{G}_{\varepsilon;s}^{(i,j)}(X_\bullet) - \mathbf{G}_{\varepsilon;s}^{(i,j)}(Z_\bullet))\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{2,2;\gamma(i)+\gamma(j)}} \leq C \lambda^\kappa \|X_\bullet - Z_\bullet\|_{\tilde{\mathcal{X}}_{\tau,\varepsilon}}$ ,
- (D)  $\|s \mapsto s (\mathbf{G}_s^{(i,j)}(Z_\bullet) - \mathbf{G}_{\varepsilon;s}^{(i,j)}(Z_\bullet))\|_{\tilde{\mathcal{W}}_{\tau,\varepsilon}^{2,2;\gamma(i)+\gamma(j)}} \leq C \lambda_\varepsilon^\kappa$ ,



where  $\gamma(0) = \gamma(1) = 1 - 40\kappa$ ,  $\gamma(2) = 1 - 20\kappa$  and  $\gamma(3) = 2 - 80\kappa$ .

*Remark 13.17.* Observe that for  $X_\bullet \equiv (g_\bullet, r_\bullet, z_\bullet, W_\bullet) \in \mathcal{X}_{\tau, \varepsilon}$  it holds

$$\begin{aligned} \mathbf{B}_{\varepsilon; s}(U(1/g_s, r_s, z_s) + W_s) &= \mathbf{G}_{\varepsilon; s}(X_\bullet) = \mathbf{G}_{\varepsilon; s}^{(0,0)}(X_\bullet) \\ &= \mathbf{G}_{\varepsilon; s}^{(1,1)}(X_\bullet) + 2\mathbf{G}_{\varepsilon; s}^{(1,2)}(X_\bullet) + 2\mathbf{G}_{\varepsilon; s}^{(1,3)}(X_\bullet) + \mathbf{G}_{\varepsilon; s}^{(2,2)}(X_\bullet) + 2\mathbf{G}_{\varepsilon; s}^{(2,3)}(X_\bullet) + \mathbf{G}_{\varepsilon; s}^{(3,3)}(X_\bullet). \end{aligned}$$

*Proof.* Note that since  $\dot{H}_{\varepsilon; s} = 0$  if  $s \in (0, \varepsilon]$  it holds  $\mathbf{B}_{\varepsilon; s}(\bullet) = 0$  if  $s \in (0, \varepsilon]$ . Consequently, since  $\theta_{\varepsilon; s}^2 = 1$  if  $s \in (0, \varepsilon]$  we obtain

$$\mathbf{G}_{\varepsilon; s}^{(i,j)}(X_\bullet) := \mathbf{B}_{\varepsilon; s}(\mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet), \mathbf{V}_{\varepsilon; s}^{(j)}(X_\bullet))$$

and

$$\mathbf{G}_s^{(i,j)}(Z_\bullet) - \mathbf{G}_{\varepsilon; s}^{(i,j)}(Z_\bullet) = \mathbf{B}_s(\mathbf{V}_s^{(i)}(Z_\bullet), \mathbf{V}_s^{(j)}(Z_\bullet)) - \mathbf{B}_{\varepsilon; s}(\mathbf{V}_s^{(i)}(Z_\bullet), \mathbf{V}_s^{(j)}(Z_\bullet)).$$

Moreover, it holds

$$\begin{aligned} \mathbf{G}_{\varepsilon; s}^{(i,j)}(X_\bullet) - \mathbf{G}_{\varepsilon; s}^{(i,j)}(Y_\bullet) &= \mathbf{B}_{\varepsilon; s}(\mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet), \mathbf{V}_{\varepsilon; s}^{(j)}(X_\bullet) - \mathbf{V}_{\varepsilon; s}^{(j)}(Y_\bullet)) \\ &\quad + \mathbf{B}_{\varepsilon; s}(\mathbf{V}_{\varepsilon; s}^{(j)}(Y_\bullet), \mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet) - \mathbf{V}_{\varepsilon; s}^{(i)}(Y_\bullet)) \end{aligned}$$

and

$$\begin{aligned} \mathbf{G}_{\varepsilon; s}^{(i,j)}(X_\bullet) - \mathbf{G}_{\varepsilon; s}^{(i,j)}(Z_\bullet) &= \mathbf{B}_{\varepsilon; s}(\mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet), \mathbf{V}_{\varepsilon; s}^{(j)}(X_\bullet) - \mathbf{V}_{\varepsilon; s}^{(j)}(Z_\bullet)) \\ &\quad + \mathbf{B}_{\varepsilon; s}(\mathbf{V}_{\varepsilon; s}^{(j)}(Z_\bullet), \mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet) - \mathbf{V}_{\varepsilon; s}^{(i)}(Z_\bullet)). \end{aligned}$$

The application of Lemma 12.6 with  $\gamma_1 = \gamma(i)$  and  $\gamma_2 = \gamma(j)$  yields

$$\begin{aligned} \text{(A)} \quad &\|s \mapsto s \mathbf{G}_{\varepsilon; s}^{(i,j)}(X_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 3; \gamma(i) + \gamma(j)}} \leq \lambda^\kappa \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 4; \gamma(i)}} \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(j)}(X_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 4; \gamma(j)}}, \\ \text{(B)} \quad &\|s \mapsto s (\mathbf{G}_{\varepsilon; s}^{(i,j)}(X_\bullet) - \mathbf{G}_{\varepsilon; s}^{(i,j)}(Y_\bullet))\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 3; \gamma(i) + \gamma(j)}} \\ &\leq \lambda^\kappa \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 4; \gamma(i)}} \|s \mapsto (\mathbf{V}_{\varepsilon; s}^{(j)}(X_\bullet) - \mathbf{V}_{\varepsilon; s}^{(j)}(Y_\bullet))\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 4; \gamma(j)}} \\ &\quad + \lambda^\kappa \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(j)}(Y_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 4; \gamma(j)}} \|s \mapsto (\mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet) - \mathbf{V}_{\varepsilon; s}^{(i)}(Y_\bullet))\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 4; \gamma(i)}}, \\ \text{(C)} \quad &\|s \mapsto s (\mathbf{G}_{\varepsilon; s}^{(i,j)}(X_\bullet) - \mathbf{G}_{\varepsilon; s}^{(i,j)}(Z_\bullet))\|_{\mathcal{W}_{\tau, \varepsilon}^{2, 2; \gamma(i) + \gamma(j)}} \\ &\leq \lambda^\kappa \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{2, 3; \gamma(i)}} \|s \mapsto (\mathbf{V}_{\varepsilon; s}^{(j)}(X_\bullet) - \mathbf{V}_{\varepsilon; s}^{(j)}(Z_\bullet))\|_{\mathcal{W}_{\tau, \varepsilon}^{2, 3; \gamma(j)}} \\ &\quad + \lambda^\kappa \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(j)}(Z_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{2, 3; \gamma(j)}} \|s \mapsto (\mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet) - \mathbf{V}_{\varepsilon; s}^{(i)}(Z_\bullet))\|_{\mathcal{W}_{\tau, \varepsilon}^{2, 3; \gamma(i)}}, \\ \text{(D)} \quad &\|s \mapsto s (\mathbf{G}_s^{(i,j)}(Z_\bullet) - \mathbf{G}_{\varepsilon; s}^{(i,j)}(Z_\bullet))\|_{\mathcal{W}_{\tau, \varepsilon}^{2, 2; \gamma(i) + \gamma(j)}} \\ &\leq \lambda_\varepsilon^\kappa \|s \mapsto \mathbf{V}_s^{(i)}(Z_\bullet)\|_{\mathcal{V}^{4, 3; \gamma(i)}} \|s \mapsto \mathbf{V}_s^{(j)}(Z_\bullet)\|_{\mathcal{V}^{4, 3; \gamma(j)}}. \end{aligned}$$

Next, observe that

$$\begin{aligned} \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{2, 3; \gamma(i)}} &\leq \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet)\|_{\mathcal{V}^{4, 3; \gamma(i)}} \leq \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(i)}(X_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 4; \gamma(i)}}, \\ \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(i)}(Z_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{2, 3; \gamma(i)}} &\leq \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(i)}(Z_\bullet)\|_{\mathcal{V}^{4, 3; \gamma(i)}} \leq \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(i)}(Z_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 4; \gamma(i)}}, \\ \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(i)}(Z_\bullet)\|_{\mathcal{V}^{4, 3; \gamma(i)}} &\leq \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(i)}(Z_\bullet)\|_{\mathcal{V}^{8, 4; \gamma(i)}} \leq \|s \mapsto \mathbf{V}_{\varepsilon; s}^{(i)}(Z_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 4; \gamma(i)}} \end{aligned}$$

by Remarks 10.18, 10.34 and 10.35. The statement of the lemma is now an immediate consequence of Lemma 13.15.  $\square$

**Lemma 13.18.** *There exists  $\lambda_\star \in (0, 1]$  and  $C \in (0, \infty)$  such that for all  $\lambda \in (0, \lambda_\star]$ , all  $\tau, \varepsilon \in [0, 1]$  and all  $X_\bullet, Y_\bullet \in \mathcal{Y}_{\tau, \varepsilon}$ ,  $Z_\bullet \in \mathcal{Y}$  it holds:*

- (A)  $\|s \mapsto \mathbf{W}_{\tau, \varepsilon; s}(X_\bullet)\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 4; 2-80\kappa}} \leq C \lambda^\kappa,$
- (B)  $\|s \mapsto (\mathbf{W}_{\tau, \varepsilon; s}(X_\bullet) - \mathbf{W}_{\tau, \varepsilon; s}(Y_\bullet))\|_{\mathcal{W}_{\tau, \varepsilon}^{8, 4; 2-80\kappa}} \leq C \lambda^\kappa \|X_\bullet - Y_\bullet\|_{\mathcal{X}_{\tau, \varepsilon}},$
- (C)  $\|s \mapsto (\mathbf{W}_{\tau, \varepsilon; s}(X_\bullet) - \mathbf{W}_{\tau, \varepsilon; s}(Z_\bullet))\|_{\mathcal{W}_{\tau, \varepsilon}^{2, 3; 2-80\kappa}} \leq C \lambda^\kappa \|X_\bullet - Z_\bullet\|_{\tilde{\mathcal{X}}_{\tau, \varepsilon}},$
- (D)  $\|s \mapsto (\mathbf{W}_s(Z_\bullet) - \mathbf{W}_{\tau, \varepsilon; s}(Z_\bullet))\|_{\mathcal{W}_{\tau, \varepsilon}^{2, 3; 2-80\kappa}} \leq C \lambda_{\tau \vee \varepsilon}^\kappa.$

*Proof.* For  $\tau, \varepsilon \in [0, 1]$ ,  $X_\bullet \equiv (g_\bullet, r_\bullet, z_\bullet, W_\bullet) \in \mathcal{X}_{\tau, \varepsilon}$  and  $s \in (0, 1]$  we define

$$\mathbf{H}_{\tau, \varepsilon; s}(X_\bullet) := (\mathbf{H}_{\tau, \varepsilon; s}^m(X_\bullet))_{m \in \mathbb{N}_+} \in \mathcal{N}$$

by the equality

$$\mathbf{H}_{\tau, \varepsilon; s}(X_\bullet) := W_s - \mathbf{E}\mathbf{A}_{\tau, \varepsilon; 1, s}W_s = -\mathbf{E}\mathbf{C}_{\tau, \varepsilon; 1, s}W_s.$$

Recalling that the maps  $\mathbf{I}_{\tau, \varepsilon; s}$  and  $\mathbf{G}_{\tau, \varepsilon; s}$  were defined in Lemmas 12.7 and 13.16, respectively, and using the notation introduced in Def. 10.12 we obtain

$$\mathbf{W}_{\tau, \varepsilon; s}(X_\bullet) := \mathbf{W}_{\tau, \varepsilon; s}^{(1)}(X_\bullet) + \mathbf{W}_{\tau, \varepsilon; s}^{(2)}(X_\bullet),$$

where

$$\mathbf{W}_{\tau, \varepsilon; s}^{(1)}(X_\bullet) := \mathbf{I}_{\tau, \varepsilon; s}(u \mapsto u \mathbf{G}_{\varepsilon; u}(X_\bullet)), \quad \mathbf{W}_{\tau, \varepsilon; s}^{(2)}(X_\bullet) := (\Pi_2 + \Pi_4)\mathbf{H}_{\tau, \varepsilon; s}(X_\bullet).$$

Observe that

$$\begin{aligned} \mathbf{W}_{\tau, \varepsilon; s}^{(1)}(X_\bullet) - \mathbf{W}_{\tau, \varepsilon; s}^{(1)}(Y_\bullet) &= \mathbf{I}_{\tau, \varepsilon; s}(u \mapsto u (\mathbf{G}_{\varepsilon; u}(X_\bullet) - \mathbf{G}_{\varepsilon; u}(Y_\bullet))), \\ \mathbf{W}_{\tau, \varepsilon; s}^{(1)}(X_\bullet) - \mathbf{W}_{\tau, \varepsilon; s}^{(1)}(Z_\bullet) &= \mathbf{I}_{\tau, \varepsilon; s}(u \mapsto u (\mathbf{G}_{\varepsilon; u}(X_\bullet) - \mathbf{G}_{\varepsilon; u}(Z_\bullet))), \\ \mathbf{W}_s^{(1)}(Z_\bullet) - \mathbf{W}_{\tau, \varepsilon; s}^{(1)}(Z_\bullet) &= (\mathbf{I}_s - \mathbf{I}_{\tau, \varepsilon; s})(u \mapsto u \mathbf{G}_u(Z_\bullet)) + \mathbf{I}_{\tau, \varepsilon; s}(u \mapsto u (\mathbf{G}_u(Z_\bullet) - \mathbf{G}_{\varepsilon; u}(Z_\bullet))). \end{aligned}$$

It follows now from Lemma 12.7 and Lemma 13.16 that the bounds (A), (B), (C) with  $\mathbf{W}_{\tau, \varepsilon; s}(X_\bullet)$  replaced by  $\mathbf{W}_{\tau, \varepsilon; s}^{(1)}(X_\bullet)$  hold true. Consequently, by the triangle inequality to complete the proof of the lemma it suffices to establish the bounds (A), (B), (C) with  $\mathbf{W}_{\tau, \varepsilon; s}(X_\bullet)$  replaced by  $\mathbf{W}_{\tau, \varepsilon; s}^{(2)}(X_\bullet)$ . By Remarks 10.35 and 10.19 the latter bounds are implied by the following bounds:

- (A<sub>1</sub>)  $\|s \mapsto \mathbf{H}_{\tau, \varepsilon; s}(X_\bullet)\|_{\mathcal{V}^{4, 3; 2-80\kappa}} \leq C \lambda^\kappa,$
- (B<sub>1</sub>)  $\|s \mapsto (\mathbf{H}_{\tau, \varepsilon; s}(X_\bullet) - \mathbf{H}_{\tau, \varepsilon; s}(Y_\bullet))\|_{\mathcal{V}^{4, 3; 2-80\kappa}} \leq C \lambda^\kappa \|X_\bullet - Y_\bullet\|_{\mathcal{X}_{\tau, \varepsilon}},$
- (C<sub>1</sub>)  $\|s \mapsto (\mathbf{H}_{\tau, \varepsilon; s}(X_\bullet) - \mathbf{H}_{\tau, \varepsilon; s}(Z_\bullet))\|_{\tilde{\mathcal{V}}^{1, 2; 2-80\kappa}} \leq C \lambda^\kappa \|X_\bullet - Z_\bullet\|_{\tilde{\mathcal{X}}_{\tau, \varepsilon}},$
- (D<sub>1</sub>)  $\|s \mapsto (\mathbf{H}_s(Z_\bullet) - \mathbf{H}_{\tau, \varepsilon; s}(Z_\bullet))\|_{\tilde{\mathcal{V}}^{1, 2; 2-80\kappa}} \leq C \lambda_{\tau \vee \varepsilon}^\kappa.$

The above bounds follow from Lemma 10.26 (B) and (D), Remark 10.10 and Def. 13.1. This finishes the proof.  $\square$

**Lemma 13.19.** Let  $\gamma \in [0, \infty)$ . For  $\tau, \varepsilon \in [0, 1]$ ,  $W_\bullet \in \mathcal{W}_{\tau, \varepsilon}^{8,3;\gamma}$  and  $s \in (0, 1]$  define

$$\begin{aligned} \mathbf{c}_{\tau, \varepsilon; s}^{m,0}(W_\bullet) &:= \mathbf{LEA}_{\tau, \varepsilon; 1, s}^m W_s, \quad m \in \{2, 4\}, \\ \mathbf{c}_{\tau, \varepsilon; s}^{2,1}(W_\bullet) &:= \mathbf{L}_\partial \mathbf{EA}_{\tau, \varepsilon; 1, s}^2 W_s. \end{aligned}$$

There exists  $\lambda_\star \in (0, 1]$  and  $C \in (0, \infty)$  such that for all  $\lambda \in (0, \lambda_\star]$ , all  $\tau, \varepsilon \in [0, 1]$ ,  $t \in (0, 1]$ ,  $W_\bullet, \tilde{W}_\bullet \in \mathcal{W}_{\tau, \varepsilon}^{8,3;\gamma}$ ,  $\hat{W}_\bullet \in \mathcal{W}^{8,3;\gamma}$  and all  $(m, i) \in \{(2, 0), (4, 0), (2, 1)\}$  it holds:

$$\begin{aligned} (A) \quad & \lambda_t^{-\gamma-2\kappa m} t^{3-m/2-i} |\mathbf{c}_{\tau, \varepsilon; t}^{(m,i)}(W_\bullet)| \leq C \|s \mapsto s W_s\|_{\mathcal{W}_{\tau, \varepsilon}^{8,3;\gamma}}, \\ (B) \quad & \lambda_t^{-\gamma-2\kappa m} t^{3-m/2-i} |\mathbf{c}_{\tau, \varepsilon; t}^{(m,i)}(W_\bullet) - \mathbf{c}_{\tau, \varepsilon; t}^{(m,i)}(\tilde{W}_\bullet)| \leq C \|s \mapsto s (W_s - \tilde{W}_s)\|_{\mathcal{W}_{\tau, \varepsilon}^{8,3;\gamma}}, \\ (C) \quad & \lambda_t^{-\gamma-2\kappa m} t^{3-m/2-i} |\mathbf{c}_{\tau, \varepsilon; t}^{(m,i)}(W_\bullet) - \mathbf{c}_{\tau, \varepsilon; t}^{(m,i)}(\hat{W}_\bullet)| \leq C \|s \mapsto s (W_s - \hat{W}_s)\|_{\mathcal{W}_{\tau, \varepsilon}^{2,2;\gamma}}, \\ (D) \quad & \lambda_t^{-\gamma-2\kappa m} t^{3-m/2-i} |\mathbf{c}_t^{(m,i)}(\hat{W}_\bullet) - \mathbf{c}_{\tau, \varepsilon; t}^{(m,i)}(\hat{W}_\bullet)| \leq C \lambda_{\tau \vee \varepsilon}^\kappa \|s \mapsto s \hat{W}_s\|_{\mathcal{W}_{\tau, \varepsilon}^{8,3;\gamma}}. \end{aligned}$$

*Remark 13.20.* Actually, since the map  $\mathbf{c}_{\tau, \varepsilon; t}^{(m,i)}$  is linear Item (B) of the above lemma follows immediately from Item (A).

*Proof.* By Remark 10.19 and Lemma 10.26 (A), (C), (E) there exists  $C \in (0, \infty)$  such that

$$\begin{aligned} (A_1) \quad & \|s \mapsto s \mathbf{A}_{\tau, \varepsilon; 1, s}^m W_s\|_{\mathcal{V}^{m;\gamma}} \leq C \|s \mapsto s W_s\|_{\mathcal{W}_{\tau, \varepsilon}^{8,3;\gamma}}, \\ (B_1) \quad & \|s \mapsto s (\mathbf{A}_{\tau, \varepsilon; 1, s}^m W_s - \mathbf{A}_{\tau, \varepsilon; 1, s}^m \tilde{W}_s)\|_{\mathcal{V}^{m;\gamma}} \leq C \|s \mapsto s (W_s - \tilde{W}_s)\|_{\mathcal{W}_{\tau, \varepsilon}^{8,3;\gamma}}, \\ (C_1) \quad & \|s \mapsto s (\mathbf{A}_{\tau, \varepsilon; 1, s}^m W_s - \mathbf{A}_{\tau, \varepsilon; 1, s}^m \hat{W}_s)\|_{\mathcal{V}^{m;\gamma}} \leq C \|s \mapsto s (W_s - \hat{W}_s)\|_{\mathcal{W}_{\tau, \varepsilon}^{2,2;\gamma}}, \\ (D_1) \quad & \|s \mapsto s (\mathbf{A}_{1, s}^m \hat{W}_s - \mathbf{A}_{\tau, \varepsilon; 1, s}^m \hat{W}_s)\|_{\mathcal{V}^{m;\gamma}} \leq C \lambda_{\tau \vee \varepsilon}^\kappa \|s \mapsto s \hat{W}_s\|_{\mathcal{W}_{\tau, \varepsilon}^{8,3;\gamma}} \end{aligned}$$

for all  $\tau, \varepsilon \in [0, 1]$  and  $m \in \{2, 4\}$ . The bounds (A), (B), (C), (D) follow now from Def. 10.15 of the norm  $\|\cdot\|_{\mathcal{V}^{m;\gamma}}$ , Remark 10.10 and Lemma 11.9. This completes the proof.  $\square$

**Lemma 13.21.** For  $\tau, \varepsilon \in [0, 1]$ ,  $X_\bullet \equiv (g_\bullet, r_\bullet, z_\bullet, W_\bullet) \in \mathcal{X}_{\tau, \varepsilon}$  and  $s \in (0, 1]$  define

$$\begin{aligned} \mathbf{a}_{\tau, \varepsilon; s}^{(1)}(X_\bullet) &:= \mathbf{LEA}_{\tau, \varepsilon; 1, s}^4 \mathbf{B}_{\varepsilon; s}(U(1/g_s, 0, 0)), \\ \mathbf{a}_{\tau, \varepsilon; s}^{(2)}(X_\bullet) &:= \mathbf{LEA}_{\tau, \varepsilon; 1, s}^4 \mathbf{B}_{\varepsilon; s}(U(2/g_s, r_s, z_s), U(0, r_s, z_s)), \\ \mathbf{a}_{\tau, \varepsilon; s}^{(3)}(X_\bullet) &:= \mathbf{LEA}_{\tau, \varepsilon; 1, s}^4 \mathbf{B}_{\varepsilon; s}(2U(1/g_s, r_s, z_s) + W_s, W_s) \end{aligned}$$

and

$$\begin{aligned} \mathbf{a}_{\tau, \varepsilon; s}^{(4)}(X_\bullet) &:= \mathbf{L}_\partial \mathbf{EA}_{\tau, \varepsilon; 1, s}^2 \mathbf{B}_{\varepsilon; s}(U(1/g_s, 0, 0)), \\ \mathbf{a}_{\tau, \varepsilon; s}^{(5)}(X_\bullet) &:= \mathbf{L}_\partial \mathbf{EA}_{\tau, \varepsilon; 1, s}^2 \mathbf{B}_{\varepsilon; s}(U(2/g_s, r_s, z_s), U(0, r_s, z_s)), \\ \mathbf{a}_{\tau, \varepsilon; s}^{(6)}(X_\bullet) &:= \mathbf{L}_\partial \mathbf{EA}_{\tau, \varepsilon; 1, s}^2 \mathbf{B}_{\varepsilon; s}(2U(1/g_s, r_s, z_s) + W_s, W_s). \end{aligned}$$

There exists  $\lambda_\star \in (0, 1]$  and  $C \in (0, \infty)$  such that for all  $\lambda \in (0, \lambda_\star]$ , all  $\tau, \varepsilon \in [0, 1]$ , all  $t \in (0, 1]$ , all  $X_\bullet, Y_\bullet \in \mathcal{X}_{\tau, \varepsilon}$ ,  $Z_\bullet \in \mathcal{Z}$  and all  $i \in \{1, \dots, 6\}$  it holds:

$$\begin{aligned} (A) \quad & \lambda_t^{-\gamma(i)} t |\mathbf{a}_{\tau, \varepsilon; t}^{(i)}(X_\bullet)| \leq C \lambda^\kappa, \\ (B) \quad & \lambda_t^{-\gamma(i)} t |\mathbf{a}_{\tau, \varepsilon; t}^{(i)}(X_\bullet) - \mathbf{a}_{\tau, \varepsilon; t}^{(i)}(Y_\bullet)| \leq C \lambda^\kappa \|X_\bullet - Y_\bullet\|_{\mathcal{X}}, \\ (C) \quad & \lambda_t^{-\gamma(i)} t |\mathbf{a}_{\tau, \varepsilon; t}^{(i)}(X_\bullet) - \mathbf{a}_{\tau, \varepsilon; t}^{(i)}(Z_\bullet)| \leq C \lambda^\kappa \|X_\bullet - Z_\bullet\|_{\mathcal{Z}}, \end{aligned}$$

$$(D) \quad \lambda_t^{-\gamma(i)} t |\mathbf{a}_t^{(i)}(Z_\bullet) - \mathbf{a}_{\tau,\varepsilon;t}^{(i)}(Z_\bullet)| \leq C \lambda_{\tau \vee \varepsilon}^\kappa,$$

where  $\gamma(1) = 2 - 32\kappa$ ,  $\gamma(2) = \gamma(3) = 3 - 108\kappa$ ,  $\gamma(4) = 2 - 36\kappa$ ,  $\gamma(5) = \gamma(6) = 3 - 112\kappa$ .

*Remark 13.22.* Observe that it holds

$$\begin{aligned} (g_s)^2 \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_s) &= (g_s)^2 \mathbf{a}_{\tau,\varepsilon;s}^{(1)}(X_\bullet) + (g_s)^2 \mathbf{a}_{\tau,\varepsilon;s}^{(2)}(X_\bullet) + (g_s)^2 \mathbf{a}_{\tau,\varepsilon;s}^{(3)}(X_\bullet), \\ \mathbf{L}_\partial \mathbf{EA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_s) &= \mathbf{a}_{\tau,\varepsilon;s}^{(4)}(X_\bullet) + \mathbf{a}_{\tau,\varepsilon;s}^{(5)}(X_\bullet) + \mathbf{a}_{\tau,\varepsilon;s}^{(6)}(X_\bullet), \end{aligned} \quad (13.3)$$

where  $V_s = U(1/g_s, r_s, z_s) + W_s$  and  $X_\bullet \equiv (g_\bullet, r_\bullet, z_\bullet, W_\bullet)$ . Note that the expressions

$$(g_s)^2 \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_s), \quad \mathbf{LEA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_s), \quad \mathbf{L}_\partial \mathbf{EA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_s)$$

appear in Def. 13.4 of the maps  $\mathbf{g}_{\tau,\varepsilon;\bullet}$ ,  $\mathbf{r}_{\tau,\varepsilon;\bullet}$  and  $\mathbf{z}_{\tau,\varepsilon;\bullet}$ . The second of the above expressions will be estimated directly in Lemma 13.27 using Lemmas 13.19 and 13.16. One could also estimate directly the remaining two expressions without using the decompositions (13.3). However, estimates obtained in that way are not strong enough for our purposes as they do not take into account the fact that  $\mathbf{a}_{\tau,\varepsilon;s}^{(2)}(X_\bullet) = 0$  and  $\mathbf{a}_{\tau,\varepsilon;s}^{(6)}(X_\bullet) = 0$  for  $s \in (0, 1/2)$ , which we prove below. Let us also mention that we will not use at all the bounds for  $\mathbf{a}_{\tau,\varepsilon;s}^{(1)}(X_\bullet)$  stated in the above lemma. Instead we will estimate directly  $(g_s)^2 \mathbf{a}_{\tau,\varepsilon;s}^{(1)}(X_\bullet) = f_{\tau,\varepsilon;s}$ , where  $f_{\tau,\varepsilon;s}$  is introduced in Def. 13.23. Note that  $f_{\tau,\varepsilon;s}$  does not depend on  $X_\bullet$ .

*Proof.* First observe that for  $X_\bullet \equiv (g_\bullet, r_\bullet, z_\bullet, W_\bullet) \in \mathcal{X}_{\tau,\varepsilon}$  it holds

$$\begin{aligned} \mathbf{B}_{\varepsilon;s}(U(1/g_s, 0, 0)) &= \mathbf{G}_{\tau,\varepsilon}^{(2,2)}(X_\bullet), \\ \mathbf{B}_{\varepsilon;s}(U(2/g_s, r_s, z_s), U(0, r_s, z_s)) &= \mathbf{G}_{\tau,\varepsilon}^{(1,1)}(X_\bullet) + 2 \mathbf{G}_{\tau,\varepsilon}^{(1,2)}(X_\bullet), \\ \mathbf{B}_{\varepsilon;s}(2U(1/g_s, r_s, z_s) + W_s, W_s) &= 2 \mathbf{G}_{\tau,\varepsilon}^{(1,3)}(X_\bullet) + 2 \mathbf{G}_{\tau,\varepsilon}^{(2,3)}(X_\bullet) + \mathbf{G}_{\tau,\varepsilon}^{(3,3)}(X_\bullet), \end{aligned}$$

where the maps  $\mathbf{G}_{\tau,\varepsilon}^{(i,j)}$  were introduced in Lemma 13.16. Using the fact that  $\int_{\mathbb{R}^2} \dot{G}_{\varepsilon;s}(x) dx = 0$  for  $s \in (0, 1/2)$  by Remark 5.2 (D) as well as Def. 10.22, 12.4 and 11.1 one shows that

$$\begin{aligned} \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(U(2/g_s, r_s, z_s), U(0, r_s, z_s)) &= 0, \\ \mathbf{L}_\partial \mathbf{EA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(U(2/g_s, r_s, z_s), U(0, r_s, z_s)) &= 0 \end{aligned}$$

for  $s \in (0, 1/2)$ . Recall that  $\theta_{1/2;t} = \lambda_t / \lambda_{1/2 \vee t}$ . Since  $\theta_{1/2;s} = 1$  for  $s \in [1/2, 1]$  we obtain

$$\mathbf{a}_{\tau,\varepsilon;s}^{(2)}(X_\bullet) = \theta_{1/2;s}^{1-40\kappa} \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(U(2/g_s, r_s, z_s), U(0, r_s, z_s)) \quad (13.4)$$

and

$$\mathbf{a}_{\tau,\varepsilon;s}^{(6)}(X_\bullet) = \theta_{1/2;s}^{1-40\kappa} \mathbf{L}_\partial \mathbf{EA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(U(2/g_s, r_s, z_s), U(0, r_s, z_s)). \quad (13.5)$$

To prove the bounds (A), (B), (C), (D) in the cases  $i \in \{1, \dots, 6\}$  we use the following arguments, respectively.

- (1) We apply Lemma 13.19 with  $m = 4$ ,  $i = 0$  and  $\gamma = 2 - 40\kappa$  and Lemma 13.16 with  $(i, j) = (2, 2)$ .

- (2) We apply Lemma 13.19 with  $m = 4$ ,  $i = 0$  and  $\gamma = 2 - 80\kappa$  and Lemma 13.16 with  $(i, j) \in \{(1, 1), (1, 2)\}$  and use the identity (13.4).
- (3) We apply Lemma 13.19 with  $m = 4$ ,  $i = 0$  and  $\gamma = 3 - 120\kappa$  and Lemma 13.16 with  $(i, j) \in \{(1, 3), (2, 3), (3, 3)\}$ .
- (4) We apply Lemma 13.19 with  $m = 2$ ,  $i = 1$  and  $\gamma = 2 - 40\kappa$  and Lemma 13.16 with  $(i, j) = (2, 2)$ .
- (5) We apply Lemma 13.19 with  $m = 2$ ,  $i = 1$  and  $\gamma = 2 - 80\kappa$  and Lemma 13.16 with  $(i, j) \in \{(1, 1), (1, 2)\}$  and use the identity (13.5).
- (6) We apply Lemma 13.19 with  $m = 2$ ,  $i = 1$  and  $\gamma = 3 - 120\kappa$  and Lemma 13.16 with  $(i, j) \in \{(1, 3), (2, 3), (3, 3)\}$ .

This finishes the proof.  $\square$

**Definition 13.23.** For  $\tau, \varepsilon \in [0, 1]$  and  $t \in (0, 1]$  we define

$$f_{\tau, \varepsilon; t} := \lambda^{-1} + \int_t^1 \mathbf{LEA}_{\tau, \varepsilon; 1, s}^4 \mathbf{B}_{\varepsilon; s}(U(1, 0, 0)) ds, \quad h_{\tau, \varepsilon; t} := \mathbf{LEA}_{\tau, \varepsilon; 1, t}^2 U(1, 0, 0).$$

*Remark 13.24.* For  $X_\bullet \equiv (g_\bullet, r_\bullet, z_\bullet, W_\bullet) \in \mathcal{X}_{\tau, \varepsilon}$  it holds

$$\mathbf{g}_{\tau, \varepsilon; t}(X_\bullet) = f_{\tau, \varepsilon; t} + \int_t^1 (g_s)^2 (\mathbf{LEA}_{\tau, \varepsilon; 1, s}^4 \mathbf{B}_{\varepsilon; s}(V_s) - \mathbf{LEA}_{\tau, \varepsilon; 1, s}^4 \mathbf{B}_{\varepsilon; s}(U(1/g_s, 0, 0))) ds$$

and

$$\mathbf{r}_{\tau, \varepsilon; t}(X_\bullet) = -h_{\tau, \varepsilon; t}/g_t - \int_t^1 \mathbf{LEA}_{\tau, \varepsilon; 1, s}^2 \mathbf{B}_{\varepsilon; s}(V_s(X_\bullet)) ds,$$

where  $V_s(X_\bullet) = U(1/g_s, r_s, z_s) + W_s$ .

**Lemma 13.25.** *There exists  $\lambda_\star \in (0, 1]$  and  $C \in (0, \infty)$  such that for all  $\lambda \in (0, \lambda_\star]$  it holds*

- (A)  $\lambda_{\varepsilon \vee t}^{-1}/C \leq f_{\tau, \varepsilon; t} \leq C \lambda_t^{-1}$ ,
- (B)  $|f_t - f_{\tau, \varepsilon; t}| \leq C \lambda_{\varepsilon \vee \tau}^\kappa \lambda_t^{-1-8\kappa}$ ,
- (C)  $|h_{\tau, \varepsilon; t}| \leq C \lambda_t^{-4\kappa} t^{-1}$ ,
- (D)  $|h_t - h_{\tau, \varepsilon; t}| \leq C \lambda_{\varepsilon \vee \tau}^\kappa \lambda_t^{-4\kappa} t^{-1}$

for all  $\tau, \varepsilon \in [0, 1]$  and  $t \in (0, 1]$ .

*Proof.* Let us first note that  $\mathbf{A}_{\tau, \varepsilon; s, 1}^4 \mathbf{B}_{\varepsilon; s}(U(1, 0, 0))$  coincides with the RHS of Eq. (10.4) with  $V = \mathbf{B}_{\varepsilon; s}(U(1, 0, 0))$ . Since  $\mathbf{B}_{\varepsilon; s}^{4+k}(U(1, 0, 0)) = 0$  for all  $k \in \mathbb{N}_0$  but  $k = 2$  in the case at hand only the term  $k = 2$  contributes to the sum over  $k \in \mathbb{N}_0$  on the RHS of Eq. (10.4). Consequently, taking into account Eq. (6.1) and Def. 12.4 we conclude that  $\mathbf{LEA}_{\tau, \varepsilon; 1, s}^4 \mathbf{B}_{\varepsilon; s}(U(1, 0, 0))$  can be represented in terms of the propagators  $H_{\tau, \varepsilon; s}$  and  $\dot{H}_{\varepsilon; s}$  introduced in Def. 12.1, which are defined in terms of  $G_{\tau, \varepsilon; s}$  and  $\dot{G}_{\varepsilon; s}$ . More specifically,  $\mathbf{LEA}_{\tau, \varepsilon; 1, s}^4 \mathbf{B}_{\varepsilon; s}(U(1, 0, 0))$  is a linear combination of one-loop Feynman diagrams with two quartic vertices, one of which is integrated over  $\mathbb{R}^2$ . The following types of diagrams appear: (1) diagrams with the vertices connected by two edges representing the propagators

$\dot{G}_{\varepsilon;s}$  and  $G_{\tau,\varepsilon;s}$ , respectively, and (2) diagrams with the vertices connected by a single edge representing the propagator  $\dot{G}_{\varepsilon;s}$  and with a self-contraction of one of the vertices by the propagator  $G_{\tau,\varepsilon;s}$ . Note that for  $s \in (0, 1/2)$  the diagrams of the type (2) vanish identically since  $\int_{\mathbb{R}^2} \dot{G}_{\varepsilon;s}(x) dx = 0$  for  $s \in (0, 1/2)$  by Remark 5.2 (D). For  $s \in [1/2, 1]$  both of the above-mentioned diagrams contribute to  $\mathbf{LEA}_{\tau,\varepsilon;s,1}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0))$ . However, in this regime both propagators  $\dot{G}_{\varepsilon;s}$  and  $G_{\tau,\varepsilon;s}$  have the UV cutoff of order one and by elementary estimates  $\mathbf{LEA}_{\tau,\varepsilon;s,1}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0))$  is uniformly bounded for all  $\tau, \varepsilon \in [0, 1]$  and  $s \in [1/2, 1]$ . Thus, we can restrict attention to the diagrams of the type (1) and  $s \in (0, 1/2]$ .

Taking into account all possible contractions of the fields represented by edges of the diagrams of the type (1) we obtain that for  $s \in (0, 1/2]$  and  $\tau \in (0, 1]$  it holds

$$\mathbf{LEA}_{\tau,\varepsilon;s,1}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0)) = 8(N-1) \tau^2 \sum_{p \in (2\pi\tau\mathbb{Z})^2} \frac{\vartheta(2\varepsilon\omega(p))^2 \vartheta(s\omega(p)) \dot{\vartheta}(s\omega(p)) \omega(p) (1 - |p|^2)}{(1 + |p|^2)^2}$$

whereas for  $s \in (0, 1/2]$  and  $\tau = 0$  (recall that we omit  $\tau$  in the notation if  $\tau = 0$ ) it holds

$$\mathbf{LEA}_{\varepsilon;s,1}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0)) = \frac{8(N-1)}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\vartheta(2\varepsilon\omega(p))^2 \vartheta(s\omega(p)) \dot{\vartheta}(s\omega(p)) \omega(p) (1 - |p|^2)}{(1 + |p|^2)^2} dp.$$

As a result, there exists  $C \in (0, \infty)$  such that

$$|\mathbf{LEA}_{\varepsilon;s,1}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0)) - \mathbf{LEA}_{\tau,\varepsilon;s,1}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0))| \leq C(1 - \log s)$$

for all  $\tau, \varepsilon \in [0, 1]$  and  $s \in (0, 1/2]$ . This and the first paragraph of the proof implies that there exists  $C \in (0, \infty)$  such that

$$|f_{\tau,\varepsilon;t} - f_{\varepsilon;t}| \leq C \quad (13.6)$$

for all  $\tau, \varepsilon \in [0, 1]$  and  $t \in (0, 1]$ . By the Lebesgue dominated convergence theorem it holds

$$\lim_{s \searrow 0} s \mathbf{LEA}_{s,1}^4 \mathbf{B}_s(U(1, 0, 0)) = -\frac{\beta_2}{\pi} \int_{\mathbb{R}^2} \frac{\vartheta(|p|) \dot{\vartheta}(|p|)}{|p|} dp = -2\beta_2 \int_0^\infty \vartheta(|p|) \dot{\vartheta}(|p|) d|p| = \beta_2.$$

Next, observe that

$$\begin{aligned} & \mathbf{LEA}_{s,1}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0)) - \beta_2/s \\ &= \frac{\beta_2}{\pi s} \int_{\mathbb{R}^2} \left( \frac{\vartheta((s^2 + |p|^2)^{1/2}) \dot{\vartheta}((s^2 + |p|^2)^{1/2}) (s^2 - |p|^2)}{(s^2 + |p|^2)^{3/2}} + \frac{\vartheta(|p|) \dot{\vartheta}(|p|)}{|p|} \right) dp. \end{aligned}$$

Hence, we conclude that there exists  $C \in (0, \infty)$  such that it holds

$$|\mathbf{LEA}_{\varepsilon;s,1}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0)) - \beta_2/s| \leq C$$

for all  $\varepsilon \in [0, 1]$  and  $s \in (4\varepsilon, 1]$ . Consequently,

$$|f_{\varepsilon;t} - \lambda_t^{-1}| \leq \int_t^1 |\mathbf{LEA}_{\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0)) - \beta_2/s| \leq C \quad (13.7)$$

for all  $\varepsilon \in [0, 1]$  and  $t \in [4\varepsilon, 1]$ . Using the bound

$$\begin{aligned} & |\mathbf{LEA}_{\varepsilon;s,1}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0))| \\ & \leq \frac{\beta_2}{\pi s} \int_{\mathbb{R}^2} \frac{\vartheta(2\varepsilon(1 + |p|^2/s^2)^{1/2})^2 \vartheta((s^2 + |p|^2)^{1/2}) |\dot{\vartheta}((s^2 + |p|^2)^{1/2})| (s^2 - |p|^2)}{(s^2 + |p|^2)^{3/2}} dp \end{aligned}$$

we show that there exists  $C \in (0, \infty)$  such that

$$|\mathbf{LEA}_{\varepsilon;s,1}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0))| \leq C/(\varepsilon \vee s)$$

for all  $\varepsilon \in [0, 1]$  and  $s \in (0, 1]$ . Hence,

$$|f_{\varepsilon;t} - f_{\varepsilon;4\varepsilon}| \leq \int_t^{4\varepsilon} |\mathbf{LEA}_{\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0))| ds \leq 4C \quad (13.8)$$

for all  $\varepsilon \in [0, 1]$  and  $t \in (0, 4\varepsilon]$ . Combining the bounds (13.6), (13.7) and (13.8) we obtain that there exists  $\lambda_\star \in (0, 1]$  such that

$$|f_{\tau,\varepsilon;t} - \lambda_{\varepsilon\vee t}^{-1}| \leq \lambda_\star^{-1}/2 \leq \lambda^{-1}/2 \leq \lambda_{\varepsilon\vee t}^{-1}/2$$

for all  $\lambda \in (0, \lambda_\star]$ ,  $\tau, \varepsilon \in [0, 1]$  and  $s \in (0, 1]$ . This proves the bound (A).

To prove the bound (B) first note that

$$\begin{aligned} & |\mathbf{LEA}_{1,s}^4 \mathbf{B}_s(U(1, 0, 0)) - \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0))| \\ & \leq |\mathbf{c}_s^{4,0}(\mathbf{B}_s(U(1, 0, 0))) - \mathbf{c}_{\tau,\varepsilon;s}^{4,0}(\mathbf{B}_s(U(1, 0, 0)))| \\ & \quad + |\mathbf{c}_{\tau,\varepsilon;s}^{4,0}(\mathbf{B}_s(U(1, 0, 0))) - \mathbf{c}_{\tau,\varepsilon;s}^{4,0}(\mathbf{B}_{\varepsilon;s}(U(1, 0, 0)))| \end{aligned}$$

where the map  $\mathbf{c}_{\tau,\varepsilon;t}^{4,0}$  was introduced in Lemma 13.19. By Lemma 12.6 (A), (C) applied with  $\gamma_1 = \gamma_2 = -8\kappa$  and Lemma 13.13 applied with  $\gamma = -8\kappa$  there exists  $C \in (0, \infty)$  such that

$$\|\mathbf{B}_s(U(1, 0, 0))\|_{\mathcal{W}_{\tau,\varepsilon}^{8,3;-16\kappa}} \leq C, \quad \|\mathbf{B}_s(U(1, 0, 0)) - \mathbf{B}_{\varepsilon;s}(U(1, 0, 0))\|_{\mathcal{W}_{\tau,\varepsilon}^{2,2;-16\kappa}} \leq C \lambda_\varepsilon^\kappa$$

for all  $\tau, \varepsilon \in [0, 1]$  and  $s \in (0, 1]$ . Consequently, by Lemma 13.19 (B), (C), (D) applied with  $m = 4$ ,  $i = 0$  and  $\gamma = -16\kappa$  there exists  $C \in (0, \infty)$  such that

$$|\mathbf{LEA}_{1,s}^4 \mathbf{B}_s(U(1, 0, 0)) - \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(U(1, 0, 0))| \leq C \lambda_{\tau\vee\varepsilon}^\kappa \lambda_s^{-8\kappa} s^{-1}$$

for all  $\tau, \varepsilon \in [0, 1]$  and  $s \in (0, 1]$ . The bound (B) follows now from Lemma 2.4 (E) applied with  $\rho = -1 - 8\kappa$ .

Finally, let us turn to the proof of the bounds (C), (D). Observe that

$$h_{\tau,\varepsilon;t} = \mathbf{LEA}_{\tau,\varepsilon;1,t}^2 U(1, 0, 0) = t \mathbf{c}_{\tau,\varepsilon;t}^{2,0}(s \mapsto s^{-1} U(1, 0, 0)),$$

where the map  $\mathbf{c}_{\tau,\varepsilon;t}^{2,0}$  was introduced in Lemma 13.19. The bounds (C), (D) follow now from Lemma 13.19 (A), (D) applied with  $m = 2$ ,  $i = 0$  and  $\gamma = -8\kappa$  as well as Lemma 13.13 (A) applied with  $\gamma = -8\kappa$ . This finishes the proof.  $\square$

**Lemma 13.26.** *There exists  $\lambda_\star \in (0, 1]$  and  $C \in (0, \infty)$  such that for all  $\lambda \in (0, \lambda_\star]$ , all  $\tau, \varepsilon \in [0, 1]$ , all  $t \in (0, 1]$  and all  $X_\bullet, Y_\bullet \in \mathcal{Y}_{\tau, \varepsilon}$ ,  $Z_\bullet \in \mathcal{Y}$  it holds:*

- (A)  $\lambda_t^{1+10\kappa} |\mathbf{g}_{\tau, \varepsilon; t}(X_\bullet)| \leq C \lambda^\kappa$ ,
- (B)  $\lambda_t^{1+10\kappa} |\mathbf{g}_{\tau, \varepsilon; t}(X_\bullet) - \mathbf{g}_{\tau, \varepsilon; t}(Y_\bullet)| \leq C \lambda^\kappa \|X_\bullet - Y_\bullet\|_{\mathcal{X}}$ .
- (C)  $\lambda_t^{1+10\kappa} |\mathbf{g}_{\tau, \varepsilon; t}(X_\bullet) - \mathbf{g}_{\tau, \varepsilon; t}(Z_\bullet)| \leq C \lambda^\kappa \|X_\bullet - Z_\bullet\|_{\mathcal{X}^\sim}$ ,
- (D)  $\lambda_t^{1+10\kappa} |\mathbf{g}_t(Z_\bullet) - \mathbf{g}_{\tau, \varepsilon; t}(Z_\bullet)| \leq C \lambda_{\tau \vee \varepsilon}^\kappa$ ,
- (E)  $\lambda_{\varepsilon \vee t} \mathbf{g}_{\tau, \varepsilon; t}(X_\bullet) \geq 1/C$ .

*Proof.* For  $\tau, \varepsilon \in [0, 1]$ ,  $X_\bullet \equiv (g_\bullet, r_\bullet, z_\bullet, W_\bullet) \in \mathcal{X}_{\tau, \varepsilon}$  and  $s \in (0, 1]$  define

$$\mathbf{g}_{\tau, \varepsilon; s}^{(1)}(X_\bullet) := (g_s)^2 (\mathbf{a}_{\tau, \varepsilon; s}^{(2)}(X_\bullet) + \mathbf{a}_{\tau, \varepsilon; s}^{(3)}(X_\bullet)), \quad \mathbf{g}_{\tau, \varepsilon; t}^{(0)}(X_\bullet) := \int_t^1 \mathbf{g}_{\tau, \varepsilon; s}^{(1)}(X_\bullet) ds,$$

where the maps  $\mathbf{a}_{\tau, \varepsilon; s}^{(2)}$  and  $\mathbf{a}_{\tau, \varepsilon; s}^{(3)}$  were introduced in Lemma 13.21. Note that it holds

$$\mathbf{g}_{\tau, \varepsilon; t}(X_\bullet) = f_{\tau, \varepsilon; t} + \mathbf{g}_{\tau, \varepsilon; t}^{(0)}(X_\bullet).$$

We shall prove that for  $i \in \{0, 1\}$  it holds

- (A<sub>1</sub>)  $\lambda_t^{-\gamma-i} t^i |\mathbf{g}_{\tau, \varepsilon; t}^{(i)}(X_\bullet)| \leq C \lambda^\kappa$ ,
- (B<sub>1</sub>)  $\lambda_t^{-\gamma-i} t^i |\mathbf{g}_{\tau, \varepsilon; t}^{(i)}(X_\bullet) - \mathbf{g}_{\tau, \varepsilon; t}^{(i)}(Y_\bullet)| \leq C \lambda^\kappa \|X_\bullet - Y_\bullet\|_{\mathcal{X}}$ .
- (C<sub>1</sub>)  $\lambda_t^{-\gamma-i} t^i |\mathbf{g}_{\tau, \varepsilon; t}^{(i)}(X_\bullet) - \mathbf{g}_{\tau, \varepsilon; t}^{(i)}(Z_\bullet)| \leq C \lambda^\kappa \|X_\bullet - Z_\bullet\|_{\mathcal{X}^\sim}$ ,
- (D<sub>1</sub>)  $\lambda_t^{-\gamma-i} t^i |\mathbf{g}_t^{(i)}(Z_\bullet) - \mathbf{g}_{\tau, \varepsilon; t}^{(i)}(Z_\bullet)| \leq C \lambda_{\tau \vee \varepsilon}^\kappa$

for all  $\tau, \varepsilon \in [0, 1]$  and  $t \in (0, 1]$ , where  $\gamma = -128\kappa$ . For  $i = 1$  the above bounds follow from Lemma 13.21 and Def. 13.1. To prove the above bounds for  $i = 0$  we use the bounds with  $i = 1$  and Lemma 2.4 (E) applied with  $\rho = \gamma$ . Note that by Lemma 13.25 (A), (B) it holds

$$\lambda_t^{1+10\kappa} |f_{\tau, \varepsilon; t}| \leq C \lambda_t^{10\kappa} \leq C \lambda^\kappa, \quad \lambda_t^{1+10\kappa} |f_t - f_{\tau, \varepsilon; t}| \leq C \lambda_{\varepsilon \vee \tau}^\kappa \lambda_t^{2\kappa} \leq C \lambda_{\varepsilon \vee \tau}^\kappa.$$

The bounds (A), (B), (C), (D) follow from the above bounds and the bounds (A<sub>1</sub>), (B<sub>1</sub>), (C<sub>1</sub>), (D<sub>1</sub>) with  $i = 0$ .

To prove the bound (E) first note that since  $\dot{H}_{\varepsilon; s} = 0$  if  $s < \varepsilon$  it holds  $B_{\varepsilon; s} = 0$  and  $\mathbf{g}_{\tau, \varepsilon; s}^{(1)}(X_\bullet) = 0$  if  $s < \varepsilon$ . Consequently,  $\mathbf{g}_{\tau, \varepsilon; t}^{(0)}(X_\bullet) = \mathbf{g}_{\tau, \varepsilon; \varepsilon}^{(0)}(X_\bullet)$  if  $t < \varepsilon$ . Recall also that by Lemma 13.25 (A) it holds  $f_{\tau, \varepsilon; t} \geq \lambda_{\varepsilon \vee t}^{-1}/C$ . Consequently, there exists  $C \in (0, \infty)$  such that it holds

$$\begin{aligned} \mathbf{g}_{\tau, \varepsilon; t}(X_\bullet) &\geq \lambda_{\varepsilon \vee t}^{-1}/C - C \lambda^\kappa \lambda_{\varepsilon \vee t}^\gamma = \lambda_{\varepsilon \vee t}^{-1} (1/C - C \lambda^\kappa \lambda_{\varepsilon \vee t}^{1-128\kappa}) \\ &\geq \lambda_{\varepsilon \vee t}^{-1} (1/C - C \lambda^{1-127\kappa}) \geq \lambda_{\varepsilon \vee t}^{-1}/(2C) \end{aligned}$$

for all  $\lambda \in (0, \lambda_\star]$ ,  $\tau, \varepsilon \in [0, 1]$  and  $t \in (0, 1]$  provided  $\lambda_\star \in (0, 1]$  is small enough. This proves the bound (E) and completes the proof of the lemma.  $\square$

**Lemma 13.27.** *There exists  $\lambda_\star \in (0, 1]$  and  $C \in (0, \infty)$  such that for all  $\lambda \in (0, \lambda_\star]$ , all  $\tau, \varepsilon \in [0, 1]$ , all  $t \in (0, 1]$  and all  $X_\bullet, Y_\bullet \in \mathcal{Y}_{\tau, \varepsilon}$ ,  $Z_\bullet \in \mathcal{Y}$  it holds:*



- (A)  $\lambda_t^{36\kappa-1} t |\mathbf{r}_{\tau,\varepsilon;t}(X_\bullet)| \leq C \lambda^\kappa,$
- (B)  $\lambda_t^{36\kappa-1} t |\mathbf{r}_{\tau,\varepsilon;t}(X_\bullet) - \mathbf{r}_{\tau,\varepsilon;t}(Y_\bullet)| \leq C \lambda^\kappa \|X_\bullet - Y_\bullet\|_{\mathcal{X}},$
- (C)  $\lambda_t^{36\kappa-1} t |\mathbf{r}_{\tau,\varepsilon;t}(X_\bullet) - \mathbf{r}_{\tau,\varepsilon;t}(Z_\bullet)| \leq C \lambda^\kappa \|X_\bullet - Z_\bullet\|_{\mathcal{Z}},$
- (D)  $\lambda_t^{36\kappa-1} t |\mathbf{r}_t(Z_\bullet) - \mathbf{r}_{\tau,\varepsilon;t}(Z_\bullet)| \leq C \lambda_{\tau \vee \varepsilon}^\kappa.$

*Proof.* We first observe that for all  $\tau, \varepsilon \in [0, 1]$ ,  $X_\bullet \equiv (g_\bullet, r_\bullet, z_\bullet, W_\bullet) \in \mathcal{X}_{\tau,\varepsilon}$  and  $s \in (0, 1]$  it holds

$$\mathbf{r}_{\tau,\varepsilon;t}(X_\bullet) = \mathbf{r}_{\tau,\varepsilon;t}^{(1)}(X_\bullet) + \mathbf{r}_{\tau,\varepsilon;t}^{(2)}(X_\bullet),$$

where

$$\mathbf{r}_{\tau,\varepsilon;t}^{(1)}(X_\bullet) := - \int_t^1 \mathbf{r}_{\tau,\varepsilon;s}^{(3)}(X_\bullet) ds, \quad \mathbf{r}_{\tau,\varepsilon;t}^{(2)}(X_\bullet) := -h_{\tau,\varepsilon;t}/g_t$$

and

$$\mathbf{r}_{\tau,\varepsilon;t}^{(3)}(X_\bullet) := \mathbf{L} \mathbf{E} \mathbf{A}_{\tau,\varepsilon;1,t}^2 \mathbf{G}_{\varepsilon;t}(X_\bullet) = \mathbf{c}_{\tau,\varepsilon;t}^{2,0}(\mathbf{G}_{\varepsilon;t}(X_\bullet)).$$

Recall that the map  $\mathbf{G}_{\varepsilon;s}$  was introduced in Lemma 13.16, the maps  $\mathbf{a}_{\tau,\varepsilon;s}^{(1)}$ ,  $\mathbf{a}_{\tau,\varepsilon;s}^{(2)}$  and  $\mathbf{a}_{\tau,\varepsilon;s}^{(3)}$  were introduced in Lemma 13.21, the map  $\mathbf{c}_{\tau,\varepsilon;s}^{2,0}$  was introduced in Lemma 13.19 and  $h_{\tau,\varepsilon;t}$  was introduced in Def. 13.23. We shall prove that for  $i \in \{1, 2, 3\}$  it holds

- (A<sub>1</sub>)  $\lambda_t^{-\gamma(i)} t^{-\varrho(i)} |\mathbf{r}_{\tau,\varepsilon;t}^{(i)}(X_\bullet)| \leq C \lambda^\kappa,$
- (B<sub>1</sub>)  $\lambda_t^{-\gamma(i)} t^{-\varrho(i)} |\mathbf{r}_{\tau,\varepsilon;t}^{(i)}(X_\bullet) - \mathbf{r}_{\tau,\varepsilon;t}^{(i)}(Y_\bullet)| \leq C \lambda^\kappa \|X_\bullet - Y_\bullet\|_{\mathcal{X}},$
- (C<sub>1</sub>)  $\lambda_t^{-\gamma(i)} t^{-\varrho(i)} |\mathbf{r}_{\tau,\varepsilon;t}^{(i)}(X_\bullet) - \mathbf{r}_{\tau,\varepsilon;t}^{(i)}(Z_\bullet)| \leq C \lambda^\kappa \|X_\bullet - Z_\bullet\|_{\mathcal{Z}},$
- (D<sub>1</sub>)  $\lambda_t^{-\gamma(i)} t^{-\varrho(i)} |\mathbf{r}_t^{(i)}(Z_\bullet) - \mathbf{r}_{\tau,\varepsilon;t}^{(i)}(Z_\bullet)| \leq C \lambda_{\tau \vee \varepsilon}^\kappa,$

where  $\gamma(1) = 2 - 76\kappa$ ,  $\varrho(1) = -1$ ,  $\gamma(2) = 1 - 16\kappa$ ,  $\varrho(2) = -1$  and  $\gamma(3) = 1 - 76\kappa$ ,  $\varrho(3) = -2$ . Note that the above bounds with  $i \in \{1, 2\}$  imply immediately the bounds (A), (B), (C), (D). Actually, even stronger bounds with  $\lambda_t^{36\kappa-1}$  replaced by  $\lambda_t^{16\kappa-1}$  hold true.

It remains to prove the bounds (A<sub>1</sub>), (B<sub>1</sub>), (C<sub>1</sub>), (D<sub>1</sub>). Application of Lemma 13.19 with  $m = 2$ ,  $i = 0$  and  $\gamma = 2 - 80\kappa$  and Lemma 13.16 with  $(i, j) = (0, 0)$  yields the bounds (A<sub>1</sub>), (B<sub>1</sub>), (C<sub>1</sub>), (D<sub>1</sub>) with  $i = 3$ . Consequently, the bounds (A<sub>1</sub>), (B<sub>1</sub>), (C<sub>1</sub>), (D<sub>1</sub>) with  $i = 1$  follow from Lemma 2.4 (C) applied with  $\varrho = -1$  and  $\rho = 2 - 76\kappa$ . To show the bounds (A<sub>1</sub>), (B<sub>1</sub>), (C<sub>1</sub>), (D<sub>1</sub>) with  $i = 2$  we use Lemma 13.25 (C), (D) and Lemma 13.12.  $\square$

**Lemma 13.28.** *There exists  $\lambda_\star \in (0, 1]$  and  $C \in (0, \infty)$  such that for all  $\lambda \in (0, \lambda_\star]$ , all  $\tau, \varepsilon \in [0, 1]$ , all  $t \in (0, 1]$  and all  $X_\bullet, Y_\bullet \in \mathcal{X}_{\tau,\varepsilon}$ ,  $Z_\bullet \in \mathcal{Z}$  it holds:*

- (A)  $\lambda_t^{36\kappa-1} |\mathbf{z}_{\tau,\varepsilon;t}(X_\bullet)| \leq C \lambda^\kappa,$
- (B)  $\lambda_t^{36\kappa-1} |\mathbf{z}_{\tau,\varepsilon;t}(X_\bullet) - \mathbf{z}_{\tau,\varepsilon;t}(Y_\bullet)| \leq C \lambda^\kappa \|X_\bullet - Y_\bullet\|_{\mathcal{X}},$
- (C)  $\lambda_t^{36\kappa-1} |\mathbf{z}_{\tau,\varepsilon;t}(X_\bullet) - \mathbf{z}_{\tau,\varepsilon;t}(Z_\bullet)| \leq C \lambda^\kappa \|X_\bullet - Z_\bullet\|_{\mathcal{Z}},$
- (D)  $\lambda_t^{36\kappa-1} |\mathbf{z}_t(Z_\bullet) - \mathbf{z}_{\tau,\varepsilon;t}(Z_\bullet)| \leq C \lambda_{\tau \vee \varepsilon}^\kappa.$

*Proof.* It holds

$$\mathbf{z}_{\tau,\varepsilon;t}(X_\bullet) := \int_0^t (\mathbf{a}_{\tau,\varepsilon;s}^{(4)}(X_\bullet) + \mathbf{a}_{\tau,\varepsilon;s}^{(5)}(X_\bullet) + \mathbf{a}_{\tau,\varepsilon;s}^{(6)}(X_\bullet)) ds,$$

where the maps  $\mathbf{a}_{\tau,\varepsilon;s}^{(4)}$ ,  $\mathbf{a}_{\tau,\varepsilon;s}^{(5)}$  and  $\mathbf{a}_{\tau,\varepsilon;s}^{(6)}$  were introduced in Lemma 13.21. The bounds (A), (B), (C), (D) follow from Lemma 13.21 and Lemma 2.4 (D) applied with  $\rho = 1 - 36\kappa$ .  $\square$

## 14 Relation to Polchinski equation

In this section we construct a solution  $U_{\tau,\varepsilon;\bullet}$  of the flow equation (7.3) using the fixed point  $X_{\tau,\varepsilon;\bullet}$  of the map  $\mathbf{X}_{\tau,\varepsilon;\bullet}$  defined in Sec. 13. The strategy of the construction was discussed in Sec. 2. We first construct a solution  $V_{\tau,\varepsilon;\bullet}$  of the flow equation (14.2) and subsequently use it to construct a solution  $U_{\tau,\varepsilon;\bullet}$  of the flow equation (7.3).

**Theorem 14.1.** *Let  $\tau, \varepsilon \in (0, 1]$  and  $\lambda \in (0, \lambda_\star]$ , where  $\lambda_\star$  is as in Corollary 13.10. Let*

$$(0, 1] \ni t \mapsto X_{\tau,\varepsilon;t} \equiv (g_{\tau,\varepsilon;t}, r_{\tau,\varepsilon;t}, z_{\tau,\varepsilon;t}, W_{\tau,\varepsilon;t}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{N}$$

*be the fixed point of the map  $\mathbf{X}_{\tau,\varepsilon;\bullet} : \mathcal{Y}_{\tau,\varepsilon} \rightarrow \mathcal{Y}_{\tau,\varepsilon}$ , introduced in Def. 13.4. Moreover, set*

$$g_{\tau,\varepsilon;0} := g_{\tau,\varepsilon;\varepsilon}, \quad r_{\tau,\varepsilon;0} := r_{\tau,\varepsilon;\varepsilon}, \quad z_{\tau,\varepsilon;0} := 0, \quad W_{\tau,\varepsilon;0} := 0.$$

*For  $t \in [0, 1]$  define  $V_{\tau,\varepsilon;t} = (V_{\tau,\varepsilon;t}^m)_{m \in \mathbb{N}_+} \in \mathcal{N}$  by the equation*

$$V_{\tau,\varepsilon;t}[\varphi] := U(1/g_{\tau,\varepsilon;t}, r_{\tau,\varepsilon;t}, z_{\tau,\varepsilon;t})[\varphi] + W_{\tau,\varepsilon;t}[\varphi] \quad (14.1)$$

*for all  $\varphi \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . Then it holds*

$$\langle V_{\tau,\varepsilon;t}^m, (\mathbf{J}\varphi)^{\otimes m} \rangle = \langle \mathbf{E}\mathbf{A}_{\tau,\varepsilon;t,0}^m V_{\tau,\varepsilon;0}, (\mathbf{J}\varphi)^{\otimes m} \rangle + \int_0^t \langle \mathbf{E}\mathbf{A}_{\tau,\varepsilon;t,s}^m \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}), (\mathbf{J}\varphi)^{\otimes m} \rangle ds \quad (14.2)$$

*for all  $t \in (0, 1]$ ,  $m \in \mathbb{N}_+$  and  $\varphi \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ .*

**Remark 14.2.** Since  $\dot{G}_{\tau,\varepsilon;t} = 0$  for  $t \in (0, \varepsilon]$  by Remark 5.2 (A), it follows from Def. 12.4 that  $\mathbf{B}_{\varepsilon;t}(\bullet) = 0$  for all  $t \in (0, \varepsilon]$ . Consequently, by Def. 13.4 we have

$$V_{\tau,\varepsilon;0} = V_{\tau,\varepsilon;t}, \quad g_{\tau,\varepsilon;0} = g_{\tau,\varepsilon;t}, \quad r_{\tau,\varepsilon;0} = r_{\tau,\varepsilon;t}, \quad z_{\tau,\varepsilon;0} = z_{\tau,\varepsilon;t}, \quad W_{\tau,\varepsilon;0} = W_{\tau,\varepsilon;t}$$

for all  $t \in (0, \varepsilon]$ , where  $(g_{\tau,\varepsilon;\bullet}, r_{\tau,\varepsilon;\bullet}, z_{\tau,\varepsilon;\bullet}, W_{\tau,\varepsilon;\bullet})$  is the fixed point of  $\mathbf{X}_{\tau,\varepsilon;\bullet}$ .

*Proof.* First note that Eq. (14.1) is equivalent to the system of equations

$$\begin{aligned} V_{\tau,\varepsilon;t}^m &= W_{\tau,\varepsilon;t}^m, & m \in \mathbb{N}_+ \setminus \{2, 4\}, \\ V_{\tau,\varepsilon;t}^4 &= U^4 1/g_{\tau,\varepsilon;t} + W_{\tau,\varepsilon;t}^4, \\ V_{\tau,\varepsilon;t}^2 &= U^2 r_{\tau,\varepsilon;t} + U_\partial^2 z_{\tau,\varepsilon;t} + W_{\tau,\varepsilon;t}^2, \end{aligned}$$

where  $U^4, U^2, U_\partial^2$  were introduced in Def. 11.1. Observe that the fixed point of the map  $\mathbf{X}_{\tau,\varepsilon;\bullet} : \mathcal{Y}_{\tau,\varepsilon} \rightarrow \mathcal{Y}_{\tau,\varepsilon}$ , introduced in Def. 13.4, is equivalent to the system of Eqs. (14.3), (14.4)

and (14.5) presented below. These equations imply in particular that

$$\begin{aligned} V_{\tau,\varepsilon;0}^m &= 0, & m \in \mathbb{N}_+ \setminus \{2, 4\}, \\ V_{\tau,\varepsilon;0}^4 &= U^4 1/g_{\tau,\varepsilon;0}, \\ V_{\tau,\varepsilon;0}^2 &= U^2 r_{\tau,\varepsilon;0}. \end{aligned}$$

As a result, it holds

$$\begin{aligned} \mathbf{EA}_{\tau,\varepsilon;t,0}^m V_{\tau,\varepsilon;0} &= 0, & m \in \mathbb{N}_+ \setminus \{2, 4\}, \\ \mathbf{EA}_{\tau,\varepsilon;t,0}^4 V_{\tau,\varepsilon;0} &= U^4 1/g_{\tau,\varepsilon;0}, \\ \mathbf{EA}_{\tau,\varepsilon;t,0}^2 V_{\tau,\varepsilon;0} &= U^2 r_{\tau,\varepsilon;0} + 1/g_{\tau,\varepsilon;0} \mathbf{EA}_{\tau,\varepsilon;t,0}^2 U(1, 0, 0). \end{aligned}$$

If  $m \in \mathbb{N}_+ \setminus \{2, 4\}$ , then the first term on the RHS of Eq. (14.2) vanishes identically and  $V_{\tau,\varepsilon;t}^m = W_{\tau,\varepsilon;t}^m$ . Consequently, for  $m \in \mathbb{N}_+ \setminus \{2, 4\}$  Eq. (14.2) follows immediately from the fixed point equation

$$W_{\tau,\varepsilon;t}^m(X_\bullet) = \int_0^t \mathbf{EA}_{\tau,\varepsilon;t,s}^m \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) \, ds. \quad (14.3)$$

Let us turn to the proof of Eq. (14.2) for  $m = 4$ . By the fixed point equation we obtain

$$\begin{aligned} W_{\tau,\varepsilon;t}^4 &= \int_0^t \mathbf{REA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) \, ds - \mathbf{EC}_{\tau,\varepsilon;1,t}^4 W_{\tau,\varepsilon;t}, \\ g_{\tau,\varepsilon;t} &= \lambda^{-1} + \int_t^1 (g_{\tau,\varepsilon;s})^2 \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) \, ds. \end{aligned} \quad (14.4)$$

In particular, the function  $(0, 1] \ni t \mapsto g_{\tau,\varepsilon;t} \in \mathbb{R}$  is continuously differentiable and

$$\partial_t(1/g_{\tau,\varepsilon;t}) = -(\partial_t g_{\tau,\varepsilon;t})/(g_{\tau,\varepsilon;t})^2 = \mathbf{LEA}_{\tau,\varepsilon;1,t}^4 \mathbf{B}_{\varepsilon;t}(V_{\tau,\varepsilon;t}).$$

Consequently, we obtain

$$1/g_{\tau,\varepsilon;t} = 1/g_{\tau,\varepsilon;0} + \int_0^t \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) \, ds.$$

Next, we observe that by Lemma 11.18 it holds

$$\begin{aligned} \langle U^4 \mathbf{LEA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) + \mathbf{REA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}), (\mathbf{J}\varphi)^{\otimes 4} \rangle \\ = \langle \mathbf{EA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}), (\mathbf{J}\varphi)^{\otimes 4} \rangle. \end{aligned}$$

We also note that it follows from Def. 10.22 of the map  $\mathbf{C}_{\tau,\varepsilon;s,t}$  that  $\mathbf{EC}_{\tau,\varepsilon;1,t}^4 W_{\tau,\varepsilon;t}$  depends only on  $W_{\tau,\varepsilon;t}^m$  with  $m > 4$ . Hence, taking into account Eq. (14.3) we obtain

$$\mathbf{EC}_{\tau,\varepsilon;1,t}^4 W_{\tau,\varepsilon;t} = \mathbf{EC}_{\tau,\varepsilon;1,t}^4 \int_0^t \mathbf{EA}_{\tau,\varepsilon;t,s} \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) \, ds.$$

Consequently, by Remark 10.25 (C) and the equality  $\mathbf{C}_{\tau,\varepsilon;1,t}V = \mathbf{A}_{\tau,\varepsilon;1,t}V - V$  we get

$$\mathbf{EC}_{\tau,\varepsilon;1,t}^4 W_{\tau,\varepsilon;t} = \int_0^t \mathbf{EA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds - \int_0^t \mathbf{EA}_{\tau,\varepsilon;t,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds.$$

Using the above identities we arrive at

$$\begin{aligned} \langle V_{\tau,\varepsilon;t}^4, (\mathbf{J}\varphi)^{\otimes 4} \rangle &= \langle U^4 1/g_{\tau,\varepsilon;t}, (\mathbf{J}\varphi)^{\otimes 4} \rangle + \langle W_{\tau,\varepsilon;t}^4, (\mathbf{J}\varphi)^{\otimes 4} \rangle \\ &= \langle U^4 1/g_{\tau,\varepsilon;0}, (\mathbf{J}\varphi)^{\otimes 4} \rangle + \int_0^t \langle \mathbf{EA}_{\tau,\varepsilon;1,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}), (\mathbf{J}\varphi)^{\otimes 4} \rangle ds - \langle \mathbf{EC}_{\tau,\varepsilon;1,t}^4 W_{\tau,\varepsilon;t}, (\mathbf{J}\varphi)^{\otimes 4} \rangle \\ &= \langle \mathbf{EA}_{\tau,\varepsilon;t,0}^4 V_{\tau,\varepsilon;0}, (\mathbf{J}\varphi)^{\otimes 4} \rangle + \int_0^t \langle \mathbf{EA}_{\tau,\varepsilon;t,s}^4 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}), (\mathbf{J}\varphi)^{\otimes 4} \rangle ds. \end{aligned}$$

This proves Eq. (14.2) for  $m = 4$ . It remains to show that Eq. (14.2) holds true for  $m = 2$ . Observe that by the fixed point equation we obtain

$$\begin{aligned} W_{\tau,\varepsilon;t}^2 &= \int_0^t \mathbf{REA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds - \mathbf{EC}_{\tau,\varepsilon;1,t}^2 W_{\tau,\varepsilon;t}, \\ r_{\tau,\varepsilon;t} &= -\mathbf{LEA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;t}, 0, 0) - \int_t^1 \mathbf{LEA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds, \\ z_{\tau,\varepsilon;t} &= \int_0^t \mathbf{L}_\partial \mathbf{EA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds. \end{aligned} \quad (14.5)$$

The second of the above equations implies

$$r_{\tau,\varepsilon;0} = -\mathbf{LEA}_{\tau,\varepsilon;1,0}^2 U(1/g_{\tau,\varepsilon;0}, 0, 0) - \int_0^1 \mathbf{LEA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds,$$

and

$$\begin{aligned} r_{\tau,\varepsilon;t} &= r_{\tau,\varepsilon;0} + \mathbf{LEA}_{\tau,\varepsilon;t,0}^2 U(1/g_{\tau,\varepsilon;0}, 0, 0) + \mathbf{LEA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;0} - 1/g_{\tau,\varepsilon;t}, 0, 0) \\ &\quad + \int_0^t \mathbf{LEA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) ds, \end{aligned} \quad (14.6)$$

where we used the identity

$$\mathbf{EA}_{\tau,\varepsilon;1,0}^2 U(1/g_{\tau,\varepsilon;0}, 0, 0) = \mathbf{EA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;0}, 0, 0) + \mathbf{EA}_{\tau,\varepsilon;t,0}^2 U(1/g_{\tau,\varepsilon;0}, 0, 0).$$

Using Def. 10.22 of the maps  $\mathbf{A}_{\tau,\varepsilon;s,t}$  and  $\mathbf{C}_{\tau,\varepsilon;s,t}$  and Remark 10.25 (C) we show that

$$\begin{aligned} \mathbf{EC}_{\tau,\varepsilon;1,t}^2 W_{\tau,\varepsilon;t} &= \mathbf{EC}_{\tau,\varepsilon;1,t}^2 V_{\tau,\varepsilon;t} - \mathbf{EC}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;t}, r_{\tau,\varepsilon;t}, z_{\tau,\varepsilon;t}) \\ &= \mathbf{EC}_{\tau,\varepsilon;1,t}^2 V_{\tau,\varepsilon;t} - \mathbf{EA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;t}, 0, 0) \end{aligned} \quad (14.7)$$

and

$$\begin{aligned} \mathbf{EC}_{\tau,\varepsilon;1,t}^2 \mathbf{EA}_{\tau,\varepsilon;t,0} V_{\tau,\varepsilon;0} &= \mathbf{EC}_{\tau,\varepsilon;1,t}^2 \mathbf{EA}_{\tau,\varepsilon;t,0} U(1/g_{\tau,\varepsilon;0}, 0, 0) \\ &= (\mathbf{EA}_{\tau,\varepsilon;1,0}^2 - \mathbf{EA}_{\tau,\varepsilon;t,0}^2) U(1/g_{\tau,\varepsilon;0}, 0, 0) = \mathbf{EA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;0}, 0, 0). \end{aligned} \quad (14.8)$$

Moreover, we observe that  $\mathbf{EC}_{\tau,\varepsilon;1,t}^2 V_{\tau,\varepsilon;t}$  depends only on  $V_{\tau,\varepsilon;t}^m$  with  $m > 2$ . Hence, by Eq. (14.2) with  $m > 2$ , which has already been proved to be true, and Eq. (14.8) we obtain

$$\begin{aligned} \langle \mathbf{EC}_{\tau,\varepsilon;1,t}^2 V_{\tau,\varepsilon;t}, (\mathbf{J}\varphi)^{\otimes 2} \rangle &= \langle \mathbf{EA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;0}, 0, 0), (\mathbf{J}\varphi)^{\otimes 2} \rangle \\ &\quad + \int_0^t \langle \mathbf{EC}_{\tau,\varepsilon;1,t}^2 \mathbf{EA}_{\tau,\varepsilon;t,s} \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}), (\mathbf{J}\varphi)^{\otimes 2} \rangle ds. \end{aligned}$$

Consequently, by Remark 10.25 (C) and Eq. (14.7) it holds

$$\begin{aligned} \langle \mathbf{EC}_{\tau,\varepsilon;1,t}^2 W_{\tau,\varepsilon;t}, (\mathbf{J}\varphi)^{\otimes 2} \rangle &= \langle \mathbf{EA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;0} - 1/g_{\tau,\varepsilon;t}, 0, 0), (\mathbf{J}\varphi)^{\otimes 2} \rangle \\ &\quad + \int_0^t \langle (\mathbf{EA}_{\tau,\varepsilon;1,s}^2 - \mathbf{EA}_{\tau,\varepsilon;t,s}^2) \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}), (\mathbf{J}\varphi)^{\otimes 2} \rangle ds. \end{aligned}$$

Using Lemma 11.18 and the fact that  $\mathbf{REA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;0} - 1/g_{\tau,\varepsilon;t}, 0, 0) = 0$  as well as  $\mathbf{L}_{\partial} \mathbf{EA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;0} - 1/g_{\tau,\varepsilon;t}, 0, 0) = 0$  we obtain

$$\begin{aligned} \langle \mathbf{EC}_{\tau,\varepsilon;1,t}^2 W_{\tau,\varepsilon;t}, (\mathbf{J}\varphi)^{\otimes 2} \rangle &= \langle U^2 \mathbf{LEA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;0} - 1/g_{\tau,\varepsilon;t}, 0, 0), (\mathbf{J}\varphi)^{\otimes 2} \rangle \\ &\quad + \int_0^t \langle (\mathbf{EA}_{\tau,\varepsilon;1,s}^2 - \mathbf{EA}_{\tau,\varepsilon;t,s}^2) \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}), (\mathbf{J}\varphi)^{\otimes 2} \rangle ds. \end{aligned} \quad (14.9)$$

Next, we observe that by Lemma 11.18 it holds

$$\begin{aligned} \langle U^2 \mathbf{LEA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) + U_{\partial}^2 \mathbf{L}_{\partial} \mathbf{EA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}) + \mathbf{REA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}), (\mathbf{J}\varphi)^{\otimes 2} \rangle \\ = \langle \mathbf{EA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}), (\mathbf{J}\varphi)^{\otimes 2} \rangle. \end{aligned}$$

Hence, by the first and third of Eqs. (14.5) and Eq. (14.6) we obtain

$$\begin{aligned} \langle V_{\tau,\varepsilon;t}^2, (\mathbf{J}\varphi)^{\otimes 2} \rangle &= \langle U^2 r_{\tau,\varepsilon;t} + U_{\partial}^2 z_{\tau,\varepsilon;t} + W_{\tau,\varepsilon;t}^2, (\mathbf{J}\varphi)^{\otimes 2} \rangle \\ &= \langle U^2 r_{\tau,\varepsilon;0} + U^2 \mathbf{LEA}_{\tau,\varepsilon;t,0}^2 U(1/g_{\tau,\varepsilon;0}, 0, 0) + U^2 \mathbf{LEA}_{\tau,\varepsilon;1,t}^2 U(1/g_{\tau,\varepsilon;0} - 1/g_{\tau,\varepsilon;t}, 0, 0), (\mathbf{J}\varphi)^{\otimes 2} \rangle \\ &\quad + \int_0^t \langle \mathbf{EA}_{\tau,\varepsilon;1,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}), (\mathbf{J}\varphi)^{\otimes 2} \rangle ds - \langle \mathbf{EC}_{\tau,\varepsilon;1,t}^2 W_{\tau,\varepsilon;t}, (\mathbf{J}\varphi)^{\otimes 2} \rangle. \end{aligned}$$

Consequently, by Eq. (14.9) we have

$$\begin{aligned} \langle V_{\tau,\varepsilon;t}^2, (\mathbf{J}\varphi)^{\otimes 2} \rangle &= \langle U^2 r_{\tau,\varepsilon;0} + U^2 \mathbf{LEA}_{\tau,\varepsilon;t,0}^2 U(1/g_{\tau,\varepsilon;0}, 0, 0), (\mathbf{J}\varphi)^{\otimes 2} \rangle \\ &\quad + \int_0^t \langle \mathbf{EA}_{\tau,\varepsilon;t,s}^2 \mathbf{B}_{\varepsilon;s}(V_{\tau,\varepsilon;s}), (\mathbf{J}\varphi)^{\otimes 2} \rangle ds. \end{aligned}$$

Hence, Eq. (14.2) is true for  $m = 2$ . This finishes the proof.  $\square$

**Corollary 14.3.** Let  $\tau, \varepsilon \in (0, 1]$  and  $\lambda \in (0, \lambda_\star]$ , where  $\lambda_\star$  is as in Corollary 13.10. For  $t \in [0, 1]$  define  $V_{\tau, \varepsilon; t} = (V_{\tau, \varepsilon; t}^m)_{m \in \mathbb{N}_+} \in \mathcal{N}$  as in the statement of Theorem 14.1. For  $m \in \mathbb{N}_+$  and  $t \in [0, 1]$  define the antisymmetric distribution  $U_{\tau, \varepsilon; t}^m \in \mathcal{S}'(\mathbb{T}_\tau^{2m})^{\mathbb{G}^m}$  by the equality

$$\langle U_{\tau, \varepsilon; t}^m, \phi^{\otimes m} \rangle_\tau = \langle V_{\tau, \varepsilon; t}^m, \mathbf{J}(\chi_\tau \phi) \otimes (\mathbf{J}\phi)^{\otimes(m-1)} \rangle \quad (14.10)$$

for all  $\phi \in C^\infty(\mathbb{T}_\tau^2) \otimes_{\text{alg}} \mathcal{G}^-$ . For  $t \in [0, 1]$  define the functional  $\tilde{U}_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^{\mathbb{G}})$  by the equality

$$\tilde{U}_{\tau, \varepsilon; t}(\phi) := \sum_{m \in \mathbb{N}_+} \langle U_{\tau, \varepsilon; t}^m, \phi^{\otimes m} \rangle_\tau \in \mathcal{G}, \quad \phi \in C^\infty(\mathbb{T}_\tau^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^-.$$

For  $t \in [0, 1]$  define  $U^0 \in \mathbb{C}$  by the equality

$$U_{\tau, \varepsilon; t}^0 = \mathbf{E}U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon; t, 0}) + \frac{1}{2} \int_0^t \mathbf{E} \langle D_\phi \tilde{U}_{\tau, \varepsilon; s}(\Psi_{\tau, \varepsilon; t, s}), \dot{G}_{\varepsilon; s} * D_\phi \tilde{U}_{\tau, \varepsilon; s}(\Psi_{\tau, \varepsilon; t, s}) \rangle_\tau ds. \quad (14.11)$$

For  $t \in [0, 1]$  define the functional  $U_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^{\mathbb{G}})$  by the equality

$$U_{\tau, \varepsilon; t}(\phi) := \sum_{m \in \mathbb{N}_0} \langle U_{\tau, \varepsilon; t}^m, \phi^{\otimes m} \rangle_\tau \in \mathcal{G}, \quad \phi \in C^\infty(\mathbb{T}_\tau^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^-.$$

Finally, define the functional  $U_{\tau, \varepsilon} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^{\mathbb{G}})$  as specified in Def. 4.16 with  $g_{\tau, \varepsilon} := g_{\tau, \varepsilon; 0}$  and  $r_{\tau, \varepsilon} := r_{\tau, \varepsilon; 0}$ , where  $g_{\tau, \varepsilon; 0}$  and  $r_{\tau, \varepsilon; 0}$  are as in the statement of Theorem 14.1. Then the map  $[0, 1] \ni t \mapsto U_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^{\mathbb{G}})$  is continuous and is a solution of the integral form of the Polchinski equation

$$\begin{aligned} U_{\tau, \varepsilon; t}(\phi) &= \mathbf{E}U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon; t, 0} + \phi) \\ &\quad + \frac{1}{2} \int_0^t \mathbf{E} \langle D_\phi U_{\tau, \varepsilon; s}(\Psi_{\tau, \varepsilon; t, s} + \phi), \dot{G}_{\varepsilon; s} * D_\phi U_{\tau, \varepsilon; s}(\Psi_{\tau, \varepsilon; t, s} + \phi) \rangle_\tau ds \end{aligned}$$

for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^-$ .

*Proof.* Using Theorem 14.1 and the fact that the distributions  $V_{\tau, \varepsilon; t}^m$ ,  $\mathbf{E}\mathbf{A}_{\tau, \varepsilon; t, 0}^m V_{\tau, \varepsilon; 0}$  and  $\mathbf{E}\mathbf{A}_{\tau, \varepsilon; t, s}^m \mathbf{B}_{\varepsilon; s}(V_{\tau, \varepsilon; s})$  are antisymmetric we obtain

$$\langle V_{\tau, \varepsilon; t}^m, \mathbf{J}^{\otimes m} \psi \rangle = \langle \mathbf{E}\mathbf{A}_{\tau, \varepsilon; t, 0}^m V_{\tau, \varepsilon; 0}, \mathbf{J}^{\otimes m} \psi \rangle + \int_0^t \langle \mathbf{E}\mathbf{A}_{\tau, \varepsilon; t, s}^m \mathbf{B}_{\varepsilon; s}(V_{\tau, \varepsilon; s}), \mathbf{J}^{\otimes m} \psi \rangle ds$$

for all  $t \in (0, 1]$ ,  $m \in \mathbb{N}_+$  and  $\psi \in \mathcal{S}(\mathbb{R}^{2m})^{\mathbb{G}^m}$ . Hence, by Def. 10.22 and 12.4 of the maps  $\mathbf{A}_{\tau, \varepsilon; t, s}^m$  and  $\mathbf{B}_{\varepsilon; s}$  we have

$$\begin{aligned} U_{\tau, \varepsilon; t}(\phi) &= \mathbf{E}U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon; t, 0} + \phi) \\ &\quad + \frac{1}{2} \int_0^t \mathbf{E} \langle D_\phi U_{\tau, \varepsilon; s}(\Psi_{\tau, \varepsilon; t, s} + \phi), \dot{G}_{\varepsilon; s} * D_\phi U_{\tau, \varepsilon; s}(\Psi_{\tau, \varepsilon; t, s} + \phi) \rangle_\tau ds \end{aligned}$$

for all  $t \in (0, 1)$  and  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . Finally, to prove the continuity of the map  $[0, 1] \ni t \mapsto U_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^\mathbb{G})$  we observe that the map is constant for on  $[0, \varepsilon]$  and use its definition in terms of the fixed point of the map  $\mathbf{X}_{\tau, \varepsilon; \bullet} : \mathcal{Y}_{\tau, \varepsilon} \rightarrow \mathcal{Y}_{\tau, \varepsilon}$  as well as Def. 13.1 of the set  $\mathcal{Y}_{\tau, \varepsilon}$  and Def. 10.32 of the space  $\mathcal{W}_{\tau, \varepsilon}^{\alpha, \beta; \gamma}$ . This finishes the proof.  $\square$

*Remark 14.4.* In the presence of the UV and IR cutoffs  $\tau, \varepsilon \in (0, 1]$  it is straightforward to show that  $U_{\tau, \varepsilon; t}^0 \in \mathbb{C}$  given by Eq. (14.11) is well-defined for all  $t \in [0, 1]$ . Consequently, for the choice of parameters  $g_{\tau, \varepsilon}$  and  $r_{\tau, \varepsilon}$  as in the above corollary it holds

$$\mathbf{E} \exp(U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon; t, 0})) = \exp(U_{\tau, \varepsilon; t}(0)) = \exp(U_{\tau, \varepsilon; t}^0) \neq 0.$$

In particular,

$$\mathbf{E} \exp(U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon})) = \exp(U_{\tau, \varepsilon; t=1}(0)) = \exp(U_{\tau, \varepsilon; t=1}^0) \neq 0.$$

## 15 Convergence of Schwinger functions

In this section we establish a relation between an effective potential  $U_{\tau, \varepsilon; t}$  at the scale  $t = 1$  and the generating functional of the Schwinger functions and complete the proof of Theorem 1.1. Recall that the effective potential  $U_{\tau, \varepsilon; t}$  was constructed in Sec. 14 with the use of the fixed point  $X_{\tau, \varepsilon; \bullet}$  of the map  $\mathbf{X}_{\tau, \varepsilon; \bullet}$  constructed in Sec. 13.

**Lemma 15.1.** *Let  $\tau, \varepsilon \in (0, 1]$ . Suppose that  $\mathbf{E} \exp(U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon})) \neq 0$ . The following equality*

$$\mu_{\tau, \varepsilon}(\exp(\langle \bullet, \phi \rangle_\tau)) = \exp(\langle \phi, G_{\tau, \varepsilon} * \phi \rangle_\tau / 2 + U_{\tau, \varepsilon; 1}(G_{\tau, \varepsilon} * \phi) - U_{\tau, \varepsilon; 1}(0))$$

*holds for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ , where the interacting measure  $\mu_{\tau, \varepsilon}$  was introduced in Def. 4.16 and  $U_{\tau, \varepsilon; 1}$  is an effective potential at unit scale introduced in Def. 7.2.*

*Proof.* First observe that by Def. 4.16 of the measure  $\mu_{\tau, \varepsilon}$  it holds

$$\mu_{\tau, \varepsilon}(\exp(\langle \bullet, \phi \rangle_\tau)) = \frac{\int \exp(-A_\tau(\psi_{\tau, \varepsilon}) + U_{\tau, \varepsilon}(\vartheta_\varepsilon * \psi_{\tau, \varepsilon}) + \langle \psi_{\tau, \varepsilon}, \vartheta_\varepsilon * \phi \rangle_\tau) d\psi_{\tau, \varepsilon}}{\int \exp(-A_\tau(\psi_{\tau, \varepsilon}) + U_{\tau, \varepsilon}(\vartheta_\varepsilon * \psi_{\tau, \varepsilon})) d\psi_{\tau, \varepsilon}}$$

for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . Using Def. 4.16 of the free action  $A_\tau$  one proves the following identity

$$\langle \psi, \phi \rangle_\tau - A_\tau(\psi) = \frac{1}{2} \langle G * \phi, \phi \rangle_\tau - A_\tau(\psi - G * \phi)$$

for all  $\psi, \phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . Consequently, using the equalities

$$\vartheta_\varepsilon * G * \vartheta_\varepsilon = G_\varepsilon, \quad G_\varepsilon * \phi = G_{\tau, \varepsilon} * \phi$$

we obtain

$$\begin{aligned} & \mu_{\tau, \varepsilon}(\exp(\langle \bullet, \phi \rangle_\tau)) \\ &= \exp(\langle \phi, G_{\tau, \varepsilon} * \phi \rangle_\tau / 2) \frac{\int \exp(-A_\tau(\psi_{\tau, \varepsilon} - G * \vartheta_\varepsilon * \phi) + U_{\tau, \varepsilon}(\vartheta_\varepsilon * \psi_{\tau, \varepsilon})) d\psi_{\tau, \varepsilon}}{\int \exp(-A_\tau(\psi_{\tau, \varepsilon}) + U_{\tau, \varepsilon}(\vartheta_\varepsilon * \psi_{\tau, \varepsilon})) d\psi_{\tau, \varepsilon}} \end{aligned}$$

for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . Note that since  $\mathbf{F}_\tau \mathbf{P}_\tau \vartheta_\varepsilon \subset \Lambda_{\tau,\varepsilon}$  it holds

$$G * \vartheta_\varepsilon * \phi(x) = \tau^2 \sum_{p \in \Lambda_{\tau,\varepsilon}} (\mathbf{F}_\tau G * \vartheta_\varepsilon * \phi)(p) e^{ip \cdot x} dp$$

and  $G * \vartheta_\varepsilon * \phi \in \mathcal{C}_{\tau,\varepsilon}$  for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ , where  $\mathcal{C}_{\tau,\varepsilon}$  was introduced in Def. 4.5. Taking advantage of the invariance under translation of the Berezin integral, which was stated in Lemma 4.12, we show the following equality

$$\begin{aligned} & \mu_{\tau,\varepsilon}(\exp(\langle \bullet, \phi \rangle_\tau)) \\ &= \exp(\langle \phi, G_{\tau,\varepsilon} *_\tau \phi \rangle_\tau / 2) \frac{\int \exp(-A_\tau(\psi_{\tau,\varepsilon}) + U_{\tau,\varepsilon}(\vartheta_\varepsilon * (\psi_{\tau,\varepsilon} + G * \vartheta_\varepsilon * \phi))) d\psi_{\tau,\varepsilon}}{\int \exp(-A_\tau(\psi_{\tau,\varepsilon}) + U_{\tau,\varepsilon}(\vartheta_\varepsilon * \psi_{\tau,\varepsilon})) d\psi_{\tau,\varepsilon}}. \end{aligned}$$

The above equality can be rewritten as

$$\begin{aligned} & \mu_{\tau,\varepsilon}(\exp(\langle \bullet, \phi \rangle_\tau)) \\ &= \exp(\langle \phi, G_{\tau,\varepsilon} *_\tau \phi \rangle_\tau / 2) \frac{\int \exp(-A_\tau(\psi_{\tau,\varepsilon}) + U_{\tau,\varepsilon}(\vartheta_\varepsilon * \psi_{\tau,\varepsilon} + G_{\tau,\varepsilon} *_\tau \phi)) d\psi_{\tau,\varepsilon}}{\int \exp(-A_\tau(\psi_{\tau,\varepsilon}) + U_{\tau,\varepsilon}(\vartheta_\varepsilon * \psi_{\tau,\varepsilon})) d\psi_{\tau,\varepsilon}} \end{aligned}$$

for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . Using Lemma 6.15 we arrive at

$$\mu_{\tau,\varepsilon}(\exp(\langle \bullet, \phi \rangle_\tau)) = \exp(\langle \phi, G_{\tau,\varepsilon} *_\tau \phi \rangle_\tau / 2) \frac{\mathbf{E}(\exp(U_{\tau,\varepsilon}(\Psi_{\tau,\varepsilon} + G_{\tau,\varepsilon} *_\tau \phi)))}{\mathbf{E}(\exp(U_{\tau,\varepsilon}(\Psi_{\tau,\varepsilon})))}.$$

The statement follows now from the fact that  $G_{\tau,\varepsilon} *_\tau \phi \in \mathcal{C}_{\tau,\varepsilon}$  for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ ,  $\Psi_{\tau,\varepsilon} = \Psi_{\tau,\varepsilon;1,0}$  and Def. 7.2 of an effective potential.  $\square$

**Definition 15.2.** Let  $\tau, \varepsilon \in (0, 1]$ . Suppose that  $\mathbf{E} \exp(U_{\tau,\varepsilon}(\Psi_{\tau,\varepsilon})) \neq 0$ . We call

$$\mathcal{S}(\mathbb{R}^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^- \ni \varphi \mapsto Z_{\tau,\varepsilon}[\varphi] := \mu_{\tau,\varepsilon}(\exp(\langle \bullet, \varphi \rangle)) \in \mathcal{G}$$

the generating functional. The Schwinger function of order  $m \in \mathbb{N}_+$  is defined by the equation

$$S_{\tau,\varepsilon}^m := D_\varphi^m Z_{\tau,\varepsilon}[\varphi] \big|_{\varphi=0} \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{G}^m}.$$

The truncated Schwinger function of order  $m \in \mathbb{N}_+$  is defined by the equation

$$T_{\tau,\varepsilon}^m := D_\varphi^m \log(Z_{\tau,\varepsilon}[\varphi]) \big|_{\varphi=0} \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{G}^m}.$$

*Remark 15.3.* The Schwinger function can be expressed in terms of the truncated Schwinger functions using the formula

$$\langle S_{\tau,\varepsilon}^m, \varphi^{\otimes m} \rangle = \sum_{\pi \in \Pi_m} \prod_{S \in \pi} \langle T_{\tau,\varepsilon}^{|S|}, \varphi^{\otimes |S|} \rangle$$

valid for all  $\varphi \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ , where  $\Pi_m$  denotes the set of partitions of the set  $\{1, \dots, m\}$  into disjoint subsets whose union coincides with  $\{1, \dots, m\}$  and  $|S|$  denotes the number of elements of a set  $S$ .



**Theorem 15.4.** Fix  $\lambda \in (0, \lambda_\star]$ , where  $\lambda_\star$  is as in Corollary 13.10. For all  $\tau, \varepsilon \in [0, 1]$  let

$$(0, 1] \ni t \mapsto X_{\tau, \varepsilon; t} \equiv (g_{\tau, \varepsilon; t}, r_{\tau, \varepsilon; t}, z_{\tau, \varepsilon; t}, W_{\tau, \varepsilon; t}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{N}$$

be the fixed point of the map  $\mathbf{X}_{\tau, \varepsilon; \bullet} : \mathcal{Y}_{\tau, \varepsilon} \rightarrow \mathcal{Y}_{\tau, \varepsilon}$ , which was introduced in Def. 13.4. Moreover, set

$$g_{\tau, \varepsilon} := g_{\tau, \varepsilon; \varepsilon}, \quad r_{\tau, \varepsilon} := r_{\tau, \varepsilon; \varepsilon},$$

Suppose that for all  $\tau, \varepsilon \in (0, 1]$  the interacting measure  $\mu_{\tau, \varepsilon}$  is defined as specified in Def. 4.16 with the above choice of the parameters  $g_{\tau, \varepsilon}$  and  $r_{\tau, \varepsilon}$ . Then for all  $m \in \mathbb{N}_+$  there exist  $S^m, T^m \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{G}^m}$  such that:

- (A)  $\lim_{\tau, \varepsilon \searrow 0} \langle T_{\tau, \varepsilon}^m, \varphi \rangle = \langle T^m, \varphi \rangle$  for all  $\varphi \in \mathcal{S}(\mathbb{R}^{2m})^{\mathbb{G}^m}$ ,
- (B)  $\lim_{\tau, \varepsilon \searrow 0} \langle S_{\tau, \varepsilon}^m, \varphi \rangle = \langle S^m, \varphi \rangle$  for all  $\varphi \in \mathcal{S}(\mathbb{R}^{2m})^{\mathbb{G}^m}$ ,
- (C)  $S^m, T^m$  are invariant under the symmetries of the plane in the sense of Def 8.2,
- (D)  $T^m$  has the properties stated in Items (C) and (D) of Theorem 1.1.
- (E)  $\lim_{|x| \rightarrow \infty} \exp(|x|^{1/2}) |\langle T^m, \varphi_x \otimes \psi \rangle| = 0$  for all  $n \in \{1, \dots, m-1\}$ ,  $\varphi \in C_c^\infty(\mathbb{R}^{2n})^{\mathbb{G}^n}$ ,  $\psi \in C_c^\infty(\mathbb{R}^{2(m-n)})^{\mathbb{G}^{m-n}}$ , where for  $x \in \mathbb{R}^2$  we define  $\varphi_x \in C_c^\infty(\mathbb{R}^{2n})^{\mathbb{G}^n}$  by the equality  $\varphi_x(y_1, \dots, y_n) := \varphi(y_1 - x, \dots, y_n - x)$  for all  $y_1, \dots, y_n \in \mathbb{R}^2$ .

*Remark 15.5.* Note that  $g_{\tau, \varepsilon} = g_{\tau, \varepsilon; t}$ ,  $r_{\tau, \varepsilon} = r_{\tau, \varepsilon; t}$  for all  $t \in (0, \varepsilon]$  by Remark 14.2.

*Remark 15.6.* The exponent 1/2 in Item (C) of the above theorem has no significance. With some extra work, it is possible to prove exponential decay of the truncated correlations. To this end, one has to use a decomposition  $(G_{\varepsilon, t})_{t \in [0, 1]}$  of the covariance  $G_\varepsilon$  at  $\varepsilon = 0$ , which in contrast to the decomposition introduced in Sec. 5, has exponential decay.

*Proof.* First, note that by Corollary 13.10 for all  $\tau, \varepsilon \in [0, 1]$  the map  $\mathbf{X}_{\tau, \varepsilon; \bullet} : \mathcal{Y}_{\tau, \varepsilon} \rightarrow \mathcal{Y}_{\tau, \varepsilon}$  is well-defined and has a unique fixed point  $X_{\tau, \varepsilon; \bullet} \equiv (g_{\tau, \varepsilon; \bullet}, r_{\tau, \varepsilon; \bullet}, z_{\tau, \varepsilon; \bullet}, W_{\tau, \varepsilon; \bullet}) \in \mathcal{Y}_{\tau, \varepsilon}$ . For all  $\tau, \varepsilon \in [0, 1]$  and  $t \in (0, 1]$  we define  $V_{\tau, \varepsilon; t} = (V_{\tau, \varepsilon; t}^m)_{m \in \mathbb{N}_+} \in \mathcal{N}$  by the equation

$$V_{\tau, \varepsilon; t}[\varphi] := U(1/g_{\tau, \varepsilon; t}, r_{\tau, \varepsilon; t}, z_{\tau, \varepsilon; t})[\varphi] + W_{\tau, \varepsilon; t}[\varphi] \quad (15.1)$$

for all  $\varphi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^-$ . Recall that we omit  $\tau$  if  $\tau = 0$  and we omit  $\varepsilon$  if  $\varepsilon = 0$ . Moreover, for all  $\tau, \varepsilon \in (0, 1]$  and  $t \in [0, 1]$  we define  $U_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^{\mathbb{G}})$  as in Corollary 14.3. Then by Corollary 14.3 and Lemma 7.7 for all  $t \in [0, 1]$  the functional  $U_{\tau, \varepsilon; t} \in \mathcal{N}(C^\infty(\mathbb{T}_\tau^2)^{\mathbb{G}})$  is an effective potential at scale  $t$ . Since by Remark 14.4 we have  $\mathbf{E} \exp(U_{\tau, \varepsilon}(\Psi_{\tau, \varepsilon})) \neq 0$  using Lemma 15.1 we conclude that

$$\mu_{\tau, \varepsilon}(\exp(\langle \bullet, \phi \rangle_\tau)) = \exp(\langle \phi, G_{\tau, \varepsilon} *_\tau \phi \rangle_\tau / 2 + U_{\tau, \varepsilon; 1}(G_{\tau, \varepsilon} *_\tau \phi) - U_{\tau, \varepsilon; 1}(0)) \quad (15.2)$$

for all  $\phi \in C^\infty(\mathbb{T}_\tau^2)^{\mathbb{G}} \otimes_{\text{alg}} \mathcal{G}^-$  and  $\tau, \varepsilon \in (0, 1]$ .

Let us prove Items (A) and (B). We observe that it suffices to show that for all  $m \in \mathbb{N}_+$  there exist  $S^m, T^m \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{G}^m}$  such that:

- (A')  $\lim_{\tau, \varepsilon \searrow 0} \langle T_{\tau, \varepsilon}^m, \varphi_1 \otimes \dots \otimes \varphi_m \rangle = \langle T^m, \varphi_1 \otimes \dots \otimes \varphi_m \rangle$  for all  $\varphi_1, \dots, \varphi_m \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$ ,

(B')  $\lim_{\tau, \varepsilon \searrow 0} \langle S_{\tau, \varepsilon}^m, \varphi_1 \otimes \dots \otimes \varphi_m \rangle = \langle S^m, \varphi_1 \otimes \dots \otimes \varphi_m \rangle$  for all  $\varphi_1, \dots, \varphi_m \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G}$ .  
 By Remark 15.3 Item (B') follows from Item (A'). We proceed to the proof of Item (A'). For  $\varphi \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$  we apply Eq. (15.2) with  $\phi = \mathbf{P}_\tau \varphi \in C^\infty(\mathbb{T}_\tau^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$  to obtain

$$\log(Z_{\tau, \varepsilon}[\varphi]) = \log(\mu_{\tau, \varepsilon}(\exp(\langle \bullet, \varphi \rangle))) = \langle \varphi, G_{\tau, \varepsilon} * \varphi \rangle / 2 + U_{\tau, \varepsilon; 1}(G_{\tau, \varepsilon} * \varphi) - U_{\tau, \varepsilon; 1}(0)$$

for  $\tau, \varepsilon \in (0, 1]$ . Note that by the translational invariance of  $V_{\tau, \varepsilon; t}^m$  Eq. (14.10) implies that

$$\langle U_{\tau, \varepsilon; t}^m, (G_{\tau, \varepsilon} * \varphi)^{\otimes m} \rangle_\tau = \langle V_{\tau, \varepsilon; t}^m, \mathbf{J}(G_\varepsilon * \varphi) \otimes (\mathbf{J}(G_{\tau, \varepsilon} * \varphi))^{\otimes(m-1)} \rangle.$$

for all  $m \in \mathbb{N}_+$ ,  $\tau, \varepsilon, t \in (0, 1]$  and  $\varphi \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . Consequently, using Def. 15.2 we arrive at

$$\begin{aligned} \langle T_{\tau, \varepsilon}^2, \varphi^{\otimes 2} \rangle &:= \langle \varphi, G_{\tau, \varepsilon} * \varphi \rangle + 2 \langle V_{\tau, \varepsilon; 1}^2, \mathbf{J}(G_\varepsilon * \varphi) \otimes (\mathbf{J}(G_{\tau, \varepsilon} * \varphi)) \rangle, \\ \langle T_{\tau, \varepsilon}^m, \varphi^{\otimes m} \rangle &:= m! \langle V_{\tau, \varepsilon; 1}^m, \mathbf{J}(G_\varepsilon * \varphi) \otimes (\mathbf{J}(G_{\tau, \varepsilon} * \varphi))^{\otimes(m-1)} \rangle, \quad m \in \mathbb{N}_+ \setminus \{2\}, \end{aligned}$$

for all  $\tau, \varepsilon \in (0, 1]$  and  $\varphi \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ . For  $m \in \mathbb{N}_+$  we define an antisymmetric distribution  $T^m \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{G}^m}$  by the equalities

$$\begin{aligned} \langle T^2, \varphi^{\otimes 2} \rangle &:= \langle \varphi, G * \varphi \rangle + 2 \langle V_1^2, (\mathbf{J}(G * \varphi))^{\otimes 2} \rangle, \\ \langle T^m, \varphi^{\otimes m} \rangle &:= m! \langle V_1^m, (\mathbf{J}(G * \varphi))^{\otimes m} \rangle, \quad m \in \mathbb{N}_+ \setminus \{2\}, \end{aligned}$$

for all  $\varphi \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G} \otimes_{\text{alg}} \mathcal{G}^-$ , where  $V_1 = (V_1^m)_{m \in \mathbb{N}_+} \in \mathcal{N}$  is defined by Eq. (15.1) with  $\tau = 0$ ,  $\varepsilon = 0$  and  $t = 1$ . Since by Lemma 15.10

$$\lim_{\tau, \varepsilon \searrow 0} \langle \varphi_1, G_{\tau, \varepsilon} * \varphi_2 \rangle = \langle \varphi_1, G * \varphi_2 \rangle$$

for all  $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G}$  in order to prove Item (A') it suffices to show that

$$\begin{aligned} \lim_{\tau, \varepsilon \searrow 0} \langle V_{\tau, \varepsilon; 1}^m, \mathbf{J}(G_\varepsilon * \varphi_1) \otimes \mathbf{J}(G_{\tau, \varepsilon} * \varphi_2) \otimes \dots \otimes \mathbf{J}(G_{\tau, \varepsilon} * \varphi_m) \rangle \\ = \langle V_1^m, \mathbf{J}(G * \varphi_1) \otimes \dots \otimes \mathbf{J}(G * \varphi_m) \rangle \end{aligned}$$

for all  $m \in \mathbb{N}_+$  and  $\varphi_1, \dots, \varphi_m \in \mathcal{S}(\mathbb{R}^2)^\mathbb{G}$ . To this end, we observe that

$$\begin{aligned} &|\langle V_1^m, \mathbf{J}(G * \varphi_1) \otimes \dots \otimes \mathbf{J}(G * \varphi_m) \rangle - \langle V_{\tau, \varepsilon; 1}^m, \mathbf{J}(G_\varepsilon * \varphi_1) \otimes \mathbf{J}(G_{\tau, \varepsilon} * \varphi_2) \otimes \dots \otimes \mathbf{J}(G_{\tau, \varepsilon} * \varphi_m) \rangle| \\ &\leq |\langle V_{\tau, \varepsilon; 1}^m, \mathbf{J}(G * \varphi_1) \otimes \dots \otimes \mathbf{J}(G * \varphi_m) - \mathbf{J}(G_\varepsilon * \varphi_1) \otimes \mathbf{J}(G_{\tau, \varepsilon} * \varphi_2) \otimes \dots \otimes \mathbf{J}(G_{\tau, \varepsilon} * \varphi_m) \rangle| \\ &\quad + |\langle V_1^m - V_{\tau, \varepsilon; 1}^m, \mathbf{J}(G * \varphi_1) \otimes \dots \otimes \mathbf{J}(G * \varphi_m) \rangle| \end{aligned}$$

By Remark 15.8 the above expression is bounded by

$$\|V_{\tau, \varepsilon; 1}^m\|_{\mathcal{N}^m} (A + B_{\tau, \varepsilon})^{m-1} B_{\tau, \varepsilon} + \|V_1^m - V_{\tau, \varepsilon; 1}^m\|_{\mathcal{N}^m} A^m,$$

where

$$A := \|\mathbf{J}(G * \varphi_1) / \tilde{w}\|_{L^1(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}} \vee \|\mathbf{J}(G * \varphi_2)\|_{L^\infty(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}} \vee \dots \vee \|\mathbf{J}(G * \varphi_m)\|_{L^\infty(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}}$$

and

$$B_{\tau,\varepsilon} := \|\mathbf{J}((G - G_\varepsilon) * \varphi_1)/\tilde{w}\|_{L^1(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}} \\ + \|\mathbf{J}((G - G_{\tau,\varepsilon}) * \varphi_2)\|_{L^\infty(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}} + \dots + \|\mathbf{J}((G - G_{\tau,\varepsilon}) * \varphi_m)\|_{L^\infty(\mathbb{R}^2)^{\mathbb{A} \times \mathbb{G}}}.$$

We used above the notation introduced in Remark 15.7 below. Item (A') follows now from Lemma 15.10 and the bounds for  $\|V_{\tau,\varepsilon;1}^m\|_{\mathcal{N}^m} \leq \|V_{\tau,\varepsilon;1}^m\|_{\mathcal{N}^m}$  and  $\|V_1^m - V_{\tau,\varepsilon;1}^m\|_{\mathcal{N}^m}$  established in Corollary 13.10. As argued above, this proves Item (B') as well as Items (A) and (B).

Let us turn to the proof of Item (C). Note that  $V_1 \in \mathcal{N}$  defined by Eq. (15.1) is invariant under the symmetries of the plane by Def. 11.1, 13.1 and 10.32. This together with the fact that the kernel  $G$  is invariant under the symmetries of the plane implies Item (C).

Item (D) is an immediate consequence of the definition of  $T^m \in \mathcal{S}'(\mathbb{R}^{2m})^{\mathbb{G}^m}$  given above, the properties of  $V_1 = (V_1^m)_{m \in \mathbb{N}_+} \in \mathcal{N}$ , the fact that  $\int_{\mathbb{R}^2} G(x) dx = 1$  and Def. 13.4.

In order to prove Item (E) it is enough to show that

$$\lim_{x \rightarrow \infty} \exp(|x|^{1/2}) |\langle \varphi_x, G * \psi \rangle| = 0$$

for all  $\varphi \in C_c^\infty(\mathbb{R}^2)^{\mathbb{G}}$  and  $\psi \in C_c^\infty(\mathbb{R}^2)^{\mathbb{G}}$  and

$$\lim_{x \rightarrow \infty} \exp(|x|^{1/2}) |\langle V_1^m, \mathbf{J}^{\otimes m}(G^{\otimes m} * (\varphi_x \otimes \psi)) \rangle| = 0$$

for all  $m \in \mathbb{N}_+$ ,  $n \in \{1, \dots, m-1\}$ ,  $\varphi \in C_c^\infty(\mathbb{R}^{2n})^{\mathbb{G}^n}$ ,  $\psi \in C_c^\infty(\mathbb{R}^{2(m-n)})^{\mathbb{G}^{m-n}}$ . Both of the above equalities follow from Remarks 15.8 and 15.9 and the bounds for  $\|V_1^m\|_{\mathcal{N}^m} \leq \|V_1^m\|_{\mathcal{N}^m}$  established in Corollary 13.10. This finishes the proof.  $\square$

*Remark 15.7.* Given  $p \in [1, \infty]$  and a finite tuple  $\varphi = (\varphi^k)_{k \in \mathbb{K}}$  of measurable functions over  $\mathbb{R}^2$  we define  $\|\varphi\|_{L^p(\mathbb{R}^2)^{\mathbb{K}}} := \sum_{k \in \mathbb{K}} \|\varphi^k\|_{L^p(\mathbb{R}^2)}$ .

*Remark 15.8.* Let  $m \in \mathbb{N}_+$  and  $V = (V^{a,\sigma})_{a \in \mathbb{A}^m, \sigma \in \mathbb{G}^m} \in \mathcal{N}^m$ . Recall that

$$\|V\|_{\mathcal{N}^m} = \sum_{a \in \mathbb{A}^m} \sum_{\sigma \in \mathbb{G}^m} \|\tilde{w}_1^m V^{a,\sigma}\|_{\mathcal{M}^m} \leq \|V\|_{\mathcal{N}^m},$$

where

$$\tilde{w}_1^m(x_1, \dots, x_m) = \tilde{w}(x_1) \exp(D(x_1, \dots, x_m)^\zeta), \quad \tilde{w}(x) = (1 + |x|)^{-1/2}, \quad \zeta = 4/5,$$

for all  $x, x_1, \dots, x_m \in \mathbb{R}^2$ . The following bound

$$|\langle V, \varphi \rangle| \leq \|V\|_{\mathcal{N}^m} \sup_{a \in \mathbb{A}^m} \sup_{\sigma \in \mathbb{G}^m} \sup_{x_2, \dots, x_m \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\varphi^{a,\sigma}(x_1, \dots, x_m)|}{\tilde{w}_1^m(x_1, \dots, x_m)} dx_1$$

is true for all  $\varphi \in \mathcal{S}(\mathbb{R}^{2m})^{\mathbb{A}^m \times \mathbb{G}^m}$ .

*Remark 15.9.* By Lemma 5.3 and Remark 5.2 (E), (F) there exists  $C \in (0, \infty)$  such that

$$\|x \mapsto (1 + |x|)^{1/2} \exp(|x|^\zeta) G(x)\|_{L^1(\mathbb{R}^2)^{\mathbb{G}^2}} \leq C, \quad \zeta = 4/5.$$

Consequently, by Lemma 9.5 (c) the following bound

$$\|G^{\otimes m} * V\|_{\mathcal{N}^m} \leq C^m \|V\|_{\mathcal{N}^m}$$

holds for all  $m \in \mathbb{N}_+$  and  $V \in \mathcal{N}^m$ .

**Lemma 15.10.** *The following equalities*

$$\lim_{\varepsilon \searrow 0} \|((G - G_\varepsilon) * \varphi) / \tilde{w}\|_{L^1(\mathbb{R}^2)^{\mathbb{G}}} = 0, \quad \lim_{\tau, \varepsilon \searrow 0} \|(G - G_{\tau, \varepsilon}) * \varphi\|_{L^\infty(\mathbb{R}^2)^{\mathbb{G}}} = 0$$

hold for all  $\varphi \in \mathcal{S}(\mathbb{R}^2)^{\mathbb{G}}$ .

*Proof.* By Remark 9.4 we have

$$\|((G - G_\varepsilon) * \varphi) / \tilde{w}\|_{L^1(\mathbb{R}^2)^{\mathbb{G}}} \leq \|(G - G_\varepsilon) / \tilde{w}\|_{L^1(\mathbb{R}^2)^{\mathbb{G}^2}} \|\varphi / \tilde{w}\|_{L^1(\mathbb{R}^2)^{\mathbb{G}}}$$

Moreover, by elementary estimates we obtain

$$\begin{aligned} \|(G - G_{\tau, \varepsilon}) * \varphi\|_{L^\infty(\mathbb{R}^2)^{\mathbb{G}}} &\leq \|(G - G_\varepsilon) * \varphi\|_{L^\infty(\mathbb{R}^2)^{\mathbb{G}}} + \|(G_\varepsilon - G_{\tau, \varepsilon}) * \varphi\|_{L^\infty(\mathbb{R}^2)^{\mathbb{G}}} \\ &\leq \|(G - G_\varepsilon) * \varphi\|_{L^\infty(\mathbb{R}^2)^{\mathbb{G}}} + \|G_\varepsilon * (\varphi - \mathbf{P}_\tau \varphi)\|_{L^\infty(\mathbb{R}^2)^{\mathbb{G}}} \\ &\leq \|G - G_\varepsilon\|_{L^1(\mathbb{R}^2)^{\mathbb{G}^2}} \|\varphi\|_{L^\infty(\mathbb{R}^2)^{\mathbb{G}}} + \|G_\varepsilon\|_{L^1(\mathbb{R}^2)^{\mathbb{G}^2}} \|\varphi - \mathbf{P}_\tau \varphi\|_{L^\infty(\mathbb{R}^2)^{\mathbb{G}}} \end{aligned}$$

Note that by Lemma 5.3 and Remark 5.2 (E) there exists  $C \in (0, \infty)$  such that

$$\|\dot{G}_{\varepsilon; t} / \tilde{w}\|_{L^1(\mathbb{R}^2)^{\mathbb{G}^2}} \leq C$$

for all  $\varepsilon \in [0, 1]$  and  $t \in (0, 1]$ . Consequently, by Remark 5.2 (F) we obtain

$$\|G_\varepsilon / \tilde{w}\|_{L^1(\mathbb{R}^2)^{\mathbb{G}^2}} \leq C, \quad \|G - G_\varepsilon\|_{L^1(\mathbb{R}^2)^{\mathbb{G}^2}} \leq \|(G - G_\varepsilon) / \tilde{w}\|_{L^1(\mathbb{R}^2)^{\mathbb{G}^2}} \leq 4\varepsilon C$$

for all  $\varepsilon \in [0, 1]$ . This finishes the proof.  $\square$

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