

COMPUTER-ASSISTED PROOFS OF EXISTENCE OF INVARIANT TORI IN QUASI-PERIODIC SYSTEMS VIA FOURIER METHODS

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Abstract The goal of this paper is to provide a methodology to prove existence of (fiberwise hyperbolic) real-analytic invariant tori in real-analytic quasi-periodic skew-product dynamical systems that present nearly-invariant tori of the same characteristics. The methodology is based on the application of a Newton-Kantorovich theorem whose hypotheses are tested using Fourier analysis methods for a numerical approximation of the parameterization of an invariant torus.

1 Introduction

The focus of this work is fixed on real-analytic skew-product dynamical systems with quasi-periodic forcing, for which we are interested in proving the existence of invariant tori close to approximately invariant tori (produced, possibly, by numerical methods). Following the methodology of a validation theorem as in [6, 10] (or also called a radii polynomial approach, see [15, 18]), applied over real-analytic objects, we will use the help of Fourier analysis to rigorously verify the hypotheses of a Newton-Kantorovich theorem. The Fourier methods in [5] avoid standard convolution techniques and can be applied to problems with non-polynomial non-linearities. See also [7, 16] for other applications. For another approach, based on the so-called Fourier models (than include a truncated Fourier series and a bound of the tails), see [6]. With such goal in mind, computations will be implemented with interval arithmetics on the computer, which

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will then validate our object of interest. Such procedure is what is called a Computer-Assisted Proof.

Our approach in this study involves the application of several techniques, including functional equations, Newton-Kantorovich theorems, and Fourier series analysis, to effectively manage the numerical aspects inherent in our subject matter. This combination of methods allows us to approximate and manipulate our data in a manner that is both accurate and adaptable. Importantly, the flexibility of this methodology means that it can be easily tailored to address a wide range of problems sharing similar characteristics. By employing these tools, we not only address the specific challenges encountered in our research but also lay the groundwork for potential applications in various other problem-solving scenarios.

The work will be divided in several parts, starting with the setting of the problem, followed by the validation theorem, where it will be shown how we can prove the existence of an invariant torus, a section about Fourier analysis and how it will help in our endeavor, and finally a section in which how the computer-assisted proof itself is brought to term is fully explained for fiberwise invariant tori in a quasi-periodically forced standard map [6].

2 Setting

2.1 Definitions

Following standard practice, let $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ be the real torus, and $\mathbb{T}_{\mathbb{C}}^d = \mathbb{T}^d + i\mathbb{R}^d$ be the complex torus. We denote a *complex strip* (in $\mathbb{T}_{\mathbb{C}}^d$) of width $\rho > 0$ by

$$\mathbb{T}_{\rho}^d = \{\theta \in \mathbb{T}_{\mathbb{C}}^d : \text{Im}|\theta_i| < \rho, i = 1, \dots, d\}.$$

We denote by $C^0(\bar{\mathbb{T}}_{\rho}^d, \mathbb{C}^m)$ the Banach space of continuous functions $u : \bar{\mathbb{T}}_{\rho}^d \rightarrow \mathbb{C}^m$ endowed with the norm

$$\|u\|_{\rho} = \sup_{\theta \in \bar{\mathbb{T}}_{\rho}^d} |u(\theta)|,$$

where $|\cdot|$ is the supremum norm in $\mathbb{C}^m$. We denote by $A(\bar{\mathbb{T}}_{\rho}^d, \mathbb{C}^m)$ the Banach space of continuous functions $u : \bar{\mathbb{T}}_{\rho}^d \rightarrow \mathbb{C}^m$, holomorphic on $\mathbb{T}_{\rho}^d$ and such that $u(\mathbb{T}^d) \subset \mathbb{R}^m$ (so f is real-analytic), endowed with the supremum norm. Notice that $A(\bar{\mathbb{T}}_{\rho}^d, \mathbb{C}^m) \subset C^0(\bar{\mathbb{T}}_{\rho}^d, \mathbb{C}^m)$, and that the inclusion is closed.

Consider the phase space an annulus $\mathcal{A}$ in $\mathbb{R}^n \times \mathbb{T}^d$, that is, an open set $\mathcal{A} \subset \mathbb{R}^n \times \mathbb{T}^d$ homotopic to $\mathcal{V} \times \mathbb{T}^d$, where $\mathcal{V} \subset \mathbb{R}^n$ is open. Let $\mathcal{B} \subset \mathbb{C}^n \times \mathbb{T}_{\mathbb{C}}^d$ be a complex neighborhood of the annulus $\mathcal{A}$.

2.2 Skew-Product Dynamical Systems

In this paper, we consider real-analytic *skew-product* dynamical systems

$$\begin{aligned} \hat{F} = (F, f) : \quad \mathcal{B} &\longrightarrow \mathbb{C}^n \times \mathbb{T}_{\mathbb{C}}^d \\ (z, \theta) &\longrightarrow (F(z, \theta), f(\theta)) \end{aligned} ,$$

where $\pi_{\mathbb{T}_{\mathbb{C}}^d}(\mathcal{B}) = \mathbb{T}_r^d$ for a given $r > 0$, and $F: \mathcal{B} \rightarrow \mathbb{C}^n$ and $f: \mathbb{T}_r^d \rightarrow \mathbb{T}_{\mathbb{C}}^d$ are real-analytic, and thus $F(\mathcal{A}) \subset \mathbb{R}^n$ and $f(\mathbb{T}^d) = \mathbb{T}^d$. More particularly, we will work with skew products defined over rotations with angle ω , that is, $f(\theta) = R_\omega(\theta) = \theta + \omega$. In such systems, if $\omega \in \mathbb{R}^d$ such that $\forall k \in \mathbb{Z}^d \setminus \{0\}$ and $\forall p \in \mathbb{Z}$, $k\omega \neq p$, then (F, R_ω) is called a quasi-periodic skew product. In any case $R_\omega(\mathbb{T}_r^d) = \mathbb{T}_r^d$. As mentioned before, similar cases with lower regularity conditions have been extensively studied in [6, 10].

The object of focus in this work are real-analytic tori, whose parameterization are given by $K \in A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$ with $0 < \rho < r$, which defines a real-analytic section of the bundle $\mathbb{C}^n \times \mathbb{T}_{\mathbb{C}}^d$, with graph $\mathcal{K}_\rho = \{(K(\theta), \theta) \mid \theta \in \bar{\mathbb{T}}_\rho^d\}$. We often abuse notation and refer to K as a torus, rather than a section or the parameterization of the torus $\mathcal{K}_\rho$. If the map K satisfies the functional equation

$$F(K(\theta - \omega), \theta - \omega) - K(\theta) = 0 \quad (2.1)$$

then the torus $\mathcal{K}_\rho$ is invariant under (F, R_ω) and its inner dynamics is the rigid rotation by ω .

The invariance equation (2.1) can be rewritten in functional terms. Let

$$\mathcal{B}^* = \{K \in A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n) \mid \forall \theta \in \bar{\mathbb{T}}_\rho^d, (K(\theta), \theta) \in \mathcal{B}\}.$$

Then let $E: \mathcal{B}^* \rightarrow A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$ be the operator defined as

$$E(K)(\theta) = F(K(\theta - \omega), \theta - \omega) - K(\theta),$$

then K is an invariant torus for (F, R_ω) if and only if

$$E(K)(\theta) = 0. \quad (2.2)$$

The differential of (2.2) is the linear operator $DE(K) : A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n) \rightarrow A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$ defined as

$$DE(K)\Delta(\theta) = D_z F(K(\theta - \omega), \theta - \omega)\Delta(\theta - \omega) - \Delta(\theta).$$

Notice that the nature of the solutions of (2.2), and hence of the bounded linear operator $DE(K)$, is strongly related to the dynamical properties of the linearized dynamics around K .

2.3 Transfer Operator and Hyperbolicity

The linear dynamics mentioned before is given by the vector bundle map

$$\begin{aligned} (M, R_\omega) : \mathbb{C}^n \times \bar{\mathbb{T}}_\rho^d &\longrightarrow \mathbb{C}^n \times \bar{\mathbb{T}}_\rho^d \\ (v, \theta) &\longrightarrow (M(\theta)v, \theta + \omega) \end{aligned},$$

where $M: \mathbb{T}_\rho^d \rightarrow \mathbb{C}^{n \times n}$ is the *transfer matrix* $M(\theta) = D_z F(K(\theta), \theta)$.

To this linear quasi-periodic skew-product we associate a functional object, the *transfer operator* $\mathcal{M}$, which is the bounded linear operator $\mathcal{M} : C^0(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n) \rightarrow C^0(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$ defined as

$$\mathcal{M}(\Delta)(\theta) = M(\theta - \omega)\Delta(\theta - \omega).$$

Notice that $A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$ is invariant under the action of the transfer operator.

We say a real-analytic torus K is fiberwise hyperbolic if the transfer operator $\mathcal{M}$ is hyperbolic on $A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$ (that is, its spectrum does not intersect the unit circle). This implies that $\mathcal{M} - \mathcal{I}$ is invertible in $A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$.

Remark 2.1 In [3], using [13], it is proved the spectral inclusion

$$\text{Spec}(\mathcal{M}, A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)) \subset \text{Spec}(\mathcal{M}, C^0(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)).$$

Hence, if $\mathcal{M}$ is hyperbolic as an operator in $C^0(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$ then it is hyperbolic as an operator in $A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$.

As it is well-known, hyperbolicity of (M, R_ω) has to do with the existence of stable and unstable bundles. In this work we will assume these bundles are trivializable (something that sometimes can be got using double-covering tricks).

In particular, in the applications of this paper we will consider a real-analytic torus K to be fiberwise hyperbolic if there exists a continuous maps $P: \bar{\mathbb{T}}_\rho^d \rightarrow \mathbb{C}^{n \times n}$ and $\Lambda: \bar{\mathbb{T}}_\rho^d \rightarrow \mathbb{C}^{n \times n}$, real-analytic in $\mathbb{T}_\rho^d$, such that

$$P(\theta + \omega)^{-1} M(\theta) P(\theta) - \Lambda(\theta) = 0,$$

and

$$\Lambda(\theta) = \begin{pmatrix} \Lambda_s(\theta) & 0 \\ 0 & \Lambda_u(\theta) \end{pmatrix}$$

with each block $\Lambda_s(\theta) \in \mathbb{C}^{n_s \times n_s}$, $\Lambda_u(\theta) \in \mathbb{C}^{n_u \times n_u}$ satisfying $\|\Lambda_s(\theta)\|_\rho \leq \lambda_s < 1$ and $\|(\Lambda_u(\theta))^{-1}\|_\rho \leq \lambda_u < 1$ for some λ_s and λ_u , where the norms for matrices of real-analytic functions are naturally extended to

$$\|\Lambda\|_\rho = \max_{i=1, \dots, n} \sum_{j=1}^n \|\Lambda_{i,j}\|_\rho.$$

Remark 2.2 We have tailored the definition of hyperbolicity to the methods of this paper, especially regarding the reducibility to simple dynamics. In particular, the torus being hyperbolic means that under an appropriate linear change of variables (given by the map P), the transfer map is reducible into a block-diagonal form (given by the map Λ), with stable and unstable bundles. In such case, the first n_s columns of P represent the directions of the stable subbundles, and the remaining n_u columns represent the directions of the unstable subbundles.

Remark 2.3 From the previous constructs, one obtains the spectral gap condition

$$\text{Spec}(\mathcal{M}, A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)) \cap \{z \in \mathbb{C} \mid \lambda_s < |z| < \lambda_u\} = \emptyset$$

which implies the hyperbolicity of the transfer operator in $A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$.

Remark 2.4 In an abuse of notation we will refer to the norm of a linear operator (like the transfer operator $\mathcal{M}$) in $A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$ as

$$\|\mathcal{M}\|_\rho = \|\mathcal{M}\|_{A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)} = \|M\|_\rho.$$

Then, if $\mathcal{L}_s$ and $\mathcal{L}_u$ are the transfer operators associated to Λ_s and Λ_u respectively, we have

$$\|(\mathcal{L}_s - \mathcal{I})^{-1}\|_\rho \leq \frac{1}{1 - \lambda_s} \quad \text{and} \quad \|(\mathcal{L}_u - \mathcal{I})^{-1}\|_\rho \leq \frac{\lambda_u}{1 - \lambda_u}.$$

We emphasize that, since $DE(K) = \mathcal{M} - \mathcal{I}$, the hyperbolicity property of the transfer operator $\mathcal{M}$ implies the invertibility of $\mathcal{M} - \mathcal{I}$ and therefore of $DE(K)$ and hence the applicability of Newton's method to equation (2.2). For more details on the relation between the transfer operator and the hyperbolicity of the linear dynamics we refer to the reader to [3, 6].

3 Validation Results

In this section we present a contraction lemma, which shows existence of zeroes in differentiable maps between Banach spaces given approximate zeroes satisfying certain conditions. The main theorem will show afterwards how to apply it to our case.

3.1 A Newton-Kantorovich Lemma

First we formulate a Newton-Kantorovich lemma. This procedure is a standard on how objects are validated theoretically first, as seen in works as [6, 10, 15, 18] and many others that delve in computer-assisted methods.

Lemma 3.1 *Let $E : B_R(x_0) \subset X \rightarrow Y$ be a C^1 mapping where X, Y are Banach spaces and $x_0 \in X$, such that $DE(x_0) : X \rightarrow Y$ is bounded and invertible (with bounded inverse by the Banach Isomorphism Theorem). Let ε, σ be constants and $b : (0, R) \rightarrow \mathbb{R}_+$ a function such that*

- a) $\|E(x_0)\| \leq \varepsilon$;
- b) $\|DE(x_0)^{-1}\| \leq \sigma$;
- c) $\forall r < R, \forall x \in \bar{B}_r(x_0), \|DE(x) - DE(x_0)\| \leq b(r)\|x - x_0\|$.

Assume also that for a certain $r_* < R$

- 1) $\frac{1}{2}\sigma b(r_*)r_*^2 - r_* + \sigma\varepsilon \leq 0$;
- 2) $\sigma b(r_*)r_* < 1$.

Then there exists a unique $x_* \in \bar{B}_{r_*}(x_0)$ such that $E(x_*) = 0$. Moreover, $DE(x_*)$ is bounded and invertible, and

$$\|x_* - x_0\| \leq \frac{\sigma\varepsilon}{1 - \sigma b(r_*)r_*}, \quad \|DE(x_*)^{-1}\| \leq \frac{\sigma}{1 - \sigma b(r_*)\|x_* - x_0\|} \leq \frac{\sigma}{1 - \sigma b(r_*)r_*}.$$

Proof. We start by defining a map $T : B_R(x_0) \rightarrow X$ by $T(x) = x - DE(x_0)^{-1}E(x)$ such that the zeroes of E correspond to the fixed points of T (the iterates of T give a quasi-Newton method for the zeroes of E). The standard procedure is to apply Banach's Fixed Point Theorem. For that we search for conditions on $r < R$ to apply the theorem.

We can start by checking where does T map the ball $\bar{B}_r(x_0)$, so for any $x \in \bar{B}_r(x_0)$,

$$\begin{aligned}
& \|T(x) - x_0\| \\
&= \|x - x_0 - DE(x_0)^{-1}E(x)\| \\
&= \|x - x_0 - DE(x_0)^{-1}(E(x) - E(x_0) + E(x_0))\| \\
&= \|(x - x_0) - DE(x_0)^{-1} \int_0^1 DE(x_0 + t(x - x_0))dt (x - x_0) - DE(x_0)^{-1}E(x_0)\| \\
&= \sigma \frac{1}{2}b(r)\|x - x_0\|^2 + \sigma\varepsilon \leq \sigma \frac{1}{2}b(r)r^2 + \sigma\varepsilon
\end{aligned}$$

where the $b(r)\|x - x_0\|$ factor is given from c). Next we can compute the Lipschitz constant L_r of T . Let $x_1, x_2 \in \bar{B}_r(x_0)$

$$\begin{aligned}
& \|T(x_2) - T(x_1)\| \\
&= \|x_2 - x_1 - DE(x_0)^{-1}(E(x_2) - E(x_1))\| \\
&= \left\| -DE(x_0)^{-1} \int_0^1 (DE(x_1 + t(x_2 - x_1)) - DE(x_0))(x_2 - x_1)dt \right\| \\
&\leq \sigma \int_0^1 b(r)\|x_1 + t(x_2 - x_1) - x_0\|dt \|x_2 - x_1\| \\
&\leq \sigma b(r)r \|x_2 - x_1\|
\end{aligned}$$

which follows by hypothesis c). So the Lipschitz constant L_r of T in $\bar{B}_r(x_0)$ is $L_r = \sigma b(r)r$. Now, by hypotheses 1) and 2), there is a $r_* < R$ such that $\|T(x) - x_0\| \leq r_*$, and hence $T(\bar{B}_{r_*}(x_0)) \subset \bar{B}_{r_*}(x_0)$, and $L_{r_*} < 1$, which implies that T is a contraction and therefore $\exists!$ $x_* \in \bar{B}_{r_*}(x_0)$ such that $T(x_*) = x_*$ (that is, $E(x_*) = 0$). Furthermore, from the Banach Contraction Mapping Theorem we also get

$$\|x_* - x_0\| \leq \frac{\|T(x_0) - x_0\|}{1 - L_{r_*}} \leq \frac{\sigma\varepsilon}{1 - \sigma b(r_*)r_*}.$$

As for the last result we see

$$\begin{aligned}
DE(x_*) &= DE(x_0) + DE(x_*) - DE(x_0) \\
&= DE(x_0)(\text{Id} + DE(x_0)^{-1}(DE(x_*) - DE(x_0))) .
\end{aligned}$$

By hypothesis, $\|DE(x_0)^{-1}(DE(x_*) - DE(x_0))\| \leq \sigma b(r_*)r_* < 1$, which means that $\text{Id} + DE(x_0)^{-1}(DE(x_*) - DE(x_0))$ is invertible and the result follows. $\square$

Remark 3.2 In [15, 18] condition c) is slightly different, which makes 1) and 2) simplified in one condition

$$\sigma b(r_*)r_*^2 - r_* + \sigma + \varepsilon < 0. \quad (3.1)$$

Notice that, from hypotheses a), b), c), if (3.1) is satisfied, then 1) and 2) are also satisfied.

Remark 3.3 Notice that we can apply Lemma 3.1 for any $r_* \in (r_-, r_+)$ where

$$r_- = \inf\{r \in (0, R) \mid \frac{1}{2}\sigma b(r)r^2 - r + \sigma\varepsilon \leq 0 \text{ and } \sigma b(r)r < 1\},$$

$$r_+ = \sup\{r \in (0, R) \mid \frac{1}{2}\sigma b(r)r^2 - r + \sigma\varepsilon \leq 0 \text{ and } \sigma b(r)r < 1\}.$$

Hence, there is a unique solution x_* of $E(x) = 0$ in the ball $B_{r_+}(x_0)$, and moreover it is in $\bar{B}_{r_-}(x_0)$.

3.2 Validation Theorem

This is the a posteriori theorem we will use to perform computer-assisted proofs of existence of fiberwise hyperbolic invariant tori.

Theorem 3.4 Assume that given a $\rho > 0$ we have a real-analytic torus $K_0 : \bar{\mathbb{T}}_\rho^d \rightarrow \mathbb{C}^n$. Let $D_{\rho,R} = \{(z, \theta) \mid z \in \mathbb{C}^n, \theta \in \bar{\mathbb{T}}_\rho^d, |z - K_0(\theta)| < R\}$ for a certain R and $F : D_{\rho,R} \rightarrow \mathbb{C}^n$ be a real-analytic map, defining a skew-product over the rotation $\omega \in \mathbb{R}^d$. Assume that

- a) $\|F(K_0(\theta - \omega), \theta - \omega) - K_0(\theta)\|_\rho \leq \varepsilon$;
- b) $M_0(\theta) = D_z F(K_0(\theta), \theta)$, the corresponding linear skew-product, is hyperbolic in $A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$ and $\|(\mathcal{M}_0 - \text{Id})^{-1}\|_{A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)} \leq \sigma$;
- c) $\forall r < R$ and $\forall (z, \theta) \in \bar{D}_{\rho,r}$, $|D_z F(z, \theta) - D_z F(K_0(\theta), \theta)| \leq b(r)|z - K_0(\theta)|$.

If moreover for a certain $r_* < R$

- 1) $\frac{1}{2}\sigma b(r_*)r_*^2 - r_* + \sigma\varepsilon \leq 0$;
- 2) $\sigma b(r_*)r_* < 1$,

then there exists a unique $K_* \in \bar{B}_{r_*}$ such that $E(K_*) = 0$. Moreover, the cocycle given by $M_*(\theta) = D_z F(K_*(\theta), \theta)$ is hyperbolic and the torus is fiberwise-hyperbolic.

Proof. It is only necessary to adapt the conditions to match those of Lemma 3.1. Define for $r \leq R$, $\bar{B}_r(K_0) = \{K \in A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n) \mid \|K - K_0\|_\rho \leq r\}$. Let $E : B_R(K_0) \subset A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n) \rightarrow A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$ be defined by $E(K_0)(\theta) := F(K_0(\theta - \omega), \theta - \omega) - K_0(\theta)$. Condition a) is straightforward. For condition b) it is easy to see that for $K \in B_R(K_0)$, $DE(K) : A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n) \rightarrow A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)$ is given by $DE(K)\Delta(\theta) = M(\theta - \omega)\Delta(\theta - \omega) - \Delta(\theta)$. Thus, $DE(K_0) = \mathcal{M}_0 - \mathcal{I}$. For condition c) we can just see that for any $K \in \bar{B}_r$,

$$\begin{aligned} & \|DE(K) - DE(K_0)\|_{A(\bar{\mathbb{T}}_\rho^d, \mathbb{C}^n)} \\ &= \|D_z F(K(\theta - \omega), \theta - \omega) - D_z F(K_0(\theta - \omega), \theta - \omega)\|_\rho \leq b(r)\|K - K_0\|_\rho. \end{aligned}$$

The result follows directly from applying Lemma 3.1. $\square$

Note that a similar theorem is proved in [10] in the C^r -category.

Remark 3.5 Following Remark 3.3 and Theorem 3.4, we can say that $\exists! K_* \in B_{r_+}$ such that $E(K_*) = 0$ and $\|K_* - K_0\| \leq r_-$.

3.3 Hyperbolicity Control

It is now natural then to wonder how to derive the invertibility of $DE(K_0) = \mathcal{M}_0 - \mathcal{I}$ from the hyperbolicity that we defined earlier. This amounts to calculating $(\mathcal{M}_0 - \mathcal{I})^{-1}v(\theta)$, that is, solving $(\mathcal{M}_0 - \mathcal{I})u(\theta) = v(\theta)$. For this purpose it is useful to write the reducibility properties of our M_0 in terms of the transfer operator.

In this paper we assume our approximately invariant torus K_0 to be approximately reducible in the sense we described in Section 2. In particular, we will see a method to produce bound b) of Theorem 3.4.

Lemma 3.6 *Assume that there exist a map $P_1: \mathbb{T}_\rho^d \rightarrow \mathbb{C}^{n \times n}$ whose approximate inverse is given by $P_2: \mathbb{T}_\rho^d \rightarrow \mathbb{C}^{n \times n}$, and a map $\Lambda: \mathbb{T}_\rho^d \rightarrow \mathbb{C}^{n \times n}$ such that*

$$\begin{aligned} E_{\text{red}}(\theta - \omega) &= P_2(\theta)M_0(\theta - \omega)P_1(\theta - \omega) - \Lambda(\theta - \omega), \\ E_{\text{inv}}(\theta) &= \text{Id} - P_2(\theta)P_1(\theta), \end{aligned}$$

where $\Lambda = \text{diag}(\Lambda_s, \Lambda_u)$ (see Section 2.3), such that $\|\Lambda_s\|_\rho \leq \lambda_s < 1$ and $\|\Lambda_u^{-1}\|_\rho \leq \lambda_u < 1$. Define

$$\lambda = \max \left(\lambda_s, 2 - \frac{1}{\lambda_u} \right).$$

Assume also that the error in reducibility, along with the error of invertibility of the map P_1 , is rather small;

- a) $\|E_{\text{red}}(\theta)\|_\rho = \varepsilon_1 < 1$;
- b) $\|E_{\text{inv}}(\theta)\|_\rho = \varepsilon_2 < 1$,

and moreover

- c) $\lambda + \varepsilon_1 + \varepsilon_2 < 1$.

Then $\|(\mathcal{M} - \mathcal{I})^{-1}\|_\rho \leq \sigma$ where

$$\sigma = \|P_1\|_\rho \frac{1}{1 - (\lambda + \varepsilon_1 + \varepsilon_2)} \|P_2\|_\rho.$$

Proof. From b) we can induce that both maps P_1 and P_2 are invertible. Since $E_{\text{inv}} = \text{Id} - P_2P_1$ and therefore $P_2P_1 = \text{Id} - E_{\text{inv}}$, using a Neumann series argument given b), both $\text{Id} - E_{\text{inv}}$ and P_2P_1 are invertible and

$$P_1^{-1} = (\text{Id} - E_{\text{inv}})^{-1}P_2, \quad P_2^{-1} = P_1(\text{Id} - E_{\text{inv}})^{-1}.$$

We need now to express the reducibility error equation in terms of their functionals, defined as follows

- $\mathcal{P}_1\Delta(\theta) = P_1(\theta)\Delta(\theta)$;
- $\mathcal{P}_2\Delta(\theta) = P_2(\theta)\Delta(\theta)$;
- $\mathcal{M}\Delta(\theta) = M(\theta - \omega)\Delta(\theta - \omega)$;
- $\mathcal{L}_s\Delta_s(\theta) = \Lambda_s(\theta - \omega)\Delta_s(\theta - \omega)$;
- $\mathcal{L}_u\Delta_u(\theta) = \Lambda_u(\theta - \omega)\Delta_u(\theta - \omega)$;
- $\mathcal{E}_{\text{red}}\Delta(\theta) = E_{\text{red}}(\theta - \omega)\Delta(\theta - \omega)$;
- $\mathcal{E}_{\text{inv}}\Delta(\theta) = E_{\text{inv}}(\theta)\Delta(\theta)$.

With this we can move from the cocycle notation to the transfer operator notation.

$$\begin{aligned}\mathcal{E}_{\text{red}} &= \mathcal{P}_2 \mathcal{M} \mathcal{P}_1 - \mathcal{L} \\ &= \mathcal{P}_2(\mathcal{M} - \mathcal{I})\mathcal{P}_1 + \mathcal{P}_2\mathcal{P}_1 - \mathcal{L} + \mathcal{I} - \mathcal{I} \\ &= \mathcal{P}_2(\mathcal{M} - \mathcal{I})\mathcal{P}_1 - \mathcal{E}_{\text{inv}} - (\mathcal{L} - \mathcal{I}),\end{aligned}$$

which implies

$$\mathcal{E}_{\text{inv}} + \mathcal{E}_{\text{red}} = \mathcal{P}_2(\mathcal{M} - \mathcal{I})\mathcal{P}_1 - (\mathcal{L} - \mathcal{I}),$$

from where

$$\begin{aligned}\mathcal{M} - \mathcal{I} &= \mathcal{P}_2^{-1}((\mathcal{L} - \mathcal{I}) + \mathcal{E}_{\text{inv}} + \mathcal{E}_{\text{red}})\mathcal{P}_1^{-1} \\ &= -\mathcal{P}_2^{-1}[\mathcal{I} - (\mathcal{L} + \mathcal{E}_{\text{red}} + \mathcal{E}_{\text{inv}})]\mathcal{P}_1^{-1}.\end{aligned}\tag{3.2}$$

Now, if $\|\Lambda_s\|_\rho \leq \lambda_s < 1$ and $\|\Lambda_u^{-1}\|_\rho \leq \lambda_u < 1$, by the Fixed Point Theorem, $\|(\mathcal{L}_s - \mathcal{I})^{-1}\|_\rho \leq \frac{1}{1-\lambda_s}$ and $\|(\mathcal{L}_u - \mathcal{I})^{-1}\|_\rho \leq \frac{\lambda_u}{1-\lambda_u}$ and therefore

$$\|(\mathcal{L} - \mathcal{I})^{-1}\|_\rho \leq \max\left(\frac{1}{1-\lambda_s}, \frac{\lambda_u}{1-\lambda_u}\right) = \frac{1}{1-\lambda}$$

given our definition of λ . By c) the whole equation (3.2) is invertible and

$$(\mathcal{M} - \mathcal{I})^{-1} = -\mathcal{P}_1[\mathcal{I} - (\mathcal{L} + \mathcal{E}_{\text{red}} + \mathcal{E}_{\text{inv}})]^{-1}\mathcal{P}_2.$$

Giving as a result

$$\|(\mathcal{M} - \mathcal{I})^{-1}\|_\rho \leq \|\mathcal{P}_1\|_\rho \frac{1}{1 - (\lambda + \varepsilon_1 + \varepsilon_2)} \|\mathcal{P}_2\|_\rho \leq \|P_1\|_\rho \frac{1}{1 - (\lambda + \varepsilon_1 + \varepsilon_2)} \|P_2\|_\rho.$$

□

4 Fourier Series Estimates

The heart of our computer-assisted methodology, as presented in this work, revolves around bounding the error that arises when approximating a periodic function using its discrete Fourier transform. This problem, which naturally arises in the field of approximation theory, has long been recognized for its significance. Motivated by the context of our present paper, we specifically tackle this problem for analytic functions. The estimates presented in this section have been borrowed from those provided in [5] and adapted for this specific scenario (such as our simplification in even sized grids or the 1D case implementation). Nonetheless, for the sake of completeness, we present here the most general and relevant results. For more details and complete proofs, refer to [5].

4.1 Discretization of the Torus and Fourier Transforms

Given a real-analytic function $u : \mathbb{T}_\rho^d \rightarrow \mathbb{C}$, we consider its Fourier series

$$u(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{u}_k e^{2\pi i k \cdot \theta},$$

where the Fourier coefficients are given by the *Fourier transform* (FT)

$$\hat{u}_k = \int_{[0,1]^d} u(\theta) e^{-2\pi i k \cdot \theta} d\theta. \quad (4.1)$$

Remark 4.1 Notice that the Fourier coefficients satisfy the symmetry $u_k^* = u_{-k}$, where u_k^* denotes the complex conjugate of u_k . Such coefficients decrease exponentially fast, which will prove a useful property later on.

For $\rho < \hat{\rho}$, its *Fourier norm* is given by

$$\|u\|_{F,\rho} = \sum_{k \in \mathbb{Z}^d} |\hat{u}_k| e^{2\pi |k|_1 \rho} < \infty,$$

where $|k|_1 = \sum_{i=1}^d |k_i|$. We observe that $\|u\|_\rho \leq \|u\|_{F,\rho} < \infty$. Apart from using the usual supremum norm, in practical applications the Fourier norm can be very useful. As will be shown in Section 5, computing the supremum norm of an object evaluated over a complex torus can result tricky, whereas the Fourier norm provides a more suitable way of bounding such norm.

We consider a sample of points on the regular grid of size $N_F = (N_{F,1}, \dots, N_{F,d}) \in \mathbb{N}^d$

$$\theta_j := (\theta_{j_1}, \dots, \theta_{j_d}) = \left(\frac{j_1}{N_{F,1}}, \dots, \frac{j_d}{N_{F,d}} \right),$$

where $j = (j_1, \dots, j_d)$, with $0 \leq j_\ell < N_{F,\ell}$ and $1 \leq \ell \leq d$. This defines an d -dimensional sampling $\{u_j\}$, with $u_j = u(\theta_j)$. The total number of points is $N_D = N_{F,1} \cdots N_{F,d}$. The integrals in Equation (4.1) are approximated using the trapezoidal rule on the regular grid, obtaining the discrete Fourier transform (DFT)

$$\tilde{u}_k = \frac{1}{N_D} \sum_{0 \leq j < N_F} u_j e^{-2\pi i k \cdot \theta_j},$$

where the sum runs over integer subindices $j \in \mathbb{Z}^d$ such that $0 \leq j_\ell < N_{F,\ell}$ for $\ell = 1, \dots, d$. Notice that $\tilde{u}_k$ is periodic with respect to the components $k_1, \dots, k_d$ of k , with periods $N_{F,1}, \dots, N_{F,d}$, respectively. The periodic function u is approximated by the discrete Fourier approximation (which is an approximation of the truncated Fourier series);

$$\tilde{u}(\theta) = \sum_{k \in \mathcal{I}_{N_F}} \tilde{u}_k e^{2\pi i k \cdot \theta},$$

where $\mathcal{I}_{N_F}$ is the finite set of multi-indices given by

$$\mathcal{I}_{N_F} = \left\{ k \in \mathbb{Z}^d \mid -\frac{N_{F,\ell}}{2} \leq k_\ell < \frac{N_{F,\ell}}{2}, 1 \leq \ell \leq d \right\}.$$

Along this section we will use the standard notation $[x]$ for the integer part of x : $[x] = \min \{j \in \mathbb{Z} : x \leq j\}$.

Remark 4.2 As we have previously stated, the Fourier coefficients are symmetrical, that is, $\hat{u}_k^* = \hat{u}_{-k}$, which holds for the DFT coefficients as well. This presents a problem regarding the way the DFT is defined. See that since we are treating the real-analytic case,

our function u evaluated over the points of the grid will acquire real values, but depending on the parity of the size of the grid, $N_{F,\ell}$, for $0 \leq \ell \leq d$, the discrete approximation outside the grid will not. The reason behind this phenomenon lies on the fact that if $N_{F,\ell}$ is odd, due to the coefficients' symmetry, the resulting function will remain real, but if $N_{F,\ell}$ is even, then $N_{F,\ell} - 1$ is odd, which means that the term $-\left[\frac{N_{F,\ell}}{2}\right]$ of the sum, called the Nyquist term, will be unpaired. The lack of its symmetrical pair results on a complex function whose derivative will have the imaginary term i . This does not present a major issue since the Nyquist term will naturally be very small. Nonetheless, if it is desired to look for a way to express the function u in terms of its DFT without this little problem, one shall eliminate the Nyquist term. Since doing this doesn't come without a cost of breaking the grid-Fourier correspondence, we will pre-process our data such that the approximate K is taken such that its Nyquist term is already zero.

4.2 Error Estimates on the Approximation of Analytic Periodic Functions

Based on the focus of our paper, we work with real-analytic functions in spaces defined on a complex strip of the torus (see Section 2). However, the arguments presented can be adapted to other spaces as well. Our main goal is to accurately quantify the error between $\tilde{u}$ and u by using appropriate norms. Next, we state the main result of this section, that allows us to control the error between $\tilde{u}$ and u .

Theorem 4.3 *Let $u \in A(\bar{\mathbb{T}}_\rho^d, \mathbb{C})$, with $\hat{\rho} > 0$. Let $\tilde{u}$ be the discrete Fourier approximation of u in the regular grid of size $N_F = (N_{F,1}, \dots, N_{F,d}) \in \mathbb{N}^d$. Then*

$$\|\tilde{u} - u\|_\rho \leq C_{N_F}(\rho, \hat{\rho}) \|u\|_{\hat{\rho}},$$

for $0 \leq \rho < \hat{\rho}$, where $C_{N_F}(\rho, \hat{\rho}) = S_{N_F}^{*1}(\rho, \hat{\rho}) + S_{N_F}^{*2}(\rho, \hat{\rho}) + T_{N_F}(\rho, \hat{\rho})$ is given by

$$S_{N_F}^{*1}(\rho, \hat{\rho}) = \prod_{\ell=1}^d \frac{1}{1 - e^{-2\pi\hat{\rho}N_{F,\ell}}} \sum_{\substack{\sigma \in \{-1, 1\}^d \\ \sigma \neq (1, \dots, 1)}} \prod_{\ell=1}^d e^{(\sigma_\ell - 1)\pi\hat{\rho}N_{F,\ell}} \nu_\ell(\sigma_\ell\hat{\rho} - \rho),$$

$$S_{N_F}^{*2}(\rho, \hat{\rho}) = \prod_{\ell=1}^d \frac{1}{1 - e^{-2\pi\hat{\rho}N_{F,\ell}}} \left(1 - \prod_{\ell=1}^d (1 - e^{-2\pi\hat{\rho}N_{F,\ell}})\right) \prod_{\ell=1}^d \nu_\ell(\hat{\rho} - \rho)$$

and

$$T_{N_F}(\rho, \hat{\rho}) = \left(\frac{e^{2\pi(\hat{\rho}-\rho)} + 1}{e^{2\pi(\hat{\rho}-\rho)} - 1}\right)^d \left(1 - \prod_{\ell=1}^d (1 - \mu_\ell(\hat{\rho} - \rho) e^{-\pi(\hat{\rho}-\rho)N_{F,\ell}})\right),$$

with

$$\nu_\ell(\delta) = \frac{e^{2\pi\delta} + 1}{e^{2\pi\delta} - 1} (1 - \mu_\ell(\delta) e^{-\pi\delta N_{F,\ell}}) \quad \text{and} \quad \mu_\ell(\delta) = \begin{cases} 1 & \text{if } N_{F,\ell} \text{ is even,} \\ \frac{2e^{\pi\delta}}{e^{2\pi\delta} + 1} & \text{if } N_{F,\ell} \text{ is odd.} \end{cases}$$

Remark 4.4 *As remarked in [5],*

$$C_{N_F}(\rho, \hat{\rho}) \simeq O(e^{-\pi(\hat{\rho}-\rho) \min_{\ell} \{N_F, \ell\}}),$$

which implies that the error caused by DFT approximation should be very small if $\hat{\rho}$ is chosen appropriately. This means that working with a DFT approximated version of our function should provide very accurate results when properly implemented with interval arithmetics.

Remark 4.5 *Although Theorem 4.3 is presented in the case of a d -dimensional torus, the estimates can be greatly simplified when working with a 1-dimensional torus, as it will be our case. Notice also, that if we choose N_F to be even (see [5] for other scenarios), the estimates simplify further to give the terms*

$$\begin{aligned} S_{N_F}^{*1}(\rho, \hat{\rho}) &= \frac{e^{-2\pi\hat{\rho}N_F}}{1 - e^{-2\pi\hat{\rho}N_F}} \frac{e^{-2\pi(\hat{\rho}+\rho)} + 1}{e^{-2\pi(\hat{\rho}+\rho)} - 1} (1 - e^{\pi(\hat{\rho}+\rho)N_F}), \\ S_{N_F}^{*2}(\rho, \hat{\rho}) &= \frac{e^{-2\pi\hat{\rho}N_F}}{1 - e^{-2\pi\hat{\rho}N_F}} \frac{e^{2\pi(\hat{\rho}-\rho)} + 1}{e^{2\pi(\hat{\rho}-\rho)} - 1} (1 - e^{-\pi(\hat{\rho}-\rho)N_F}), \\ T_{N_F}(\rho, \hat{\rho}) &= \frac{e^{2\pi(\hat{\rho}-\rho)} + 1}{e^{2\pi(\hat{\rho}-\rho)} - 1} e^{-\pi(\hat{\rho}-\rho)N_F}. \end{aligned}$$

4.3 Results on Matrices of Periodic Functions

In this section we consider some extensions of Theorem 4.3 to deal with matrix functions $A \in A(\bar{\mathbb{T}}_{\hat{\rho}}^d, \mathbb{C}^{m_1 \times m_2})$. Our goal is to control the propagation of the error when we perform matrix operations. Specifically, we are interested in the study of products and inverses, but the ideas given below can be adapted to control other operations if necessary. The first result is obtained directly from Theorem 4.3:

Corollary 4.6 *Let us consider two matrix functions $A \in A(\bar{\mathbb{T}}_{\hat{\rho}}^d, \mathbb{C}^{m_1 \times m_2})$, and $B \in A(\bar{\mathbb{T}}_{\hat{\rho}}^d, \mathbb{C}^{m_2 \times m_3})$. We denote by AB the product matrix and $\widetilde{AB}$ the corresponding approximation given by DFT. Given a grid of size $N_F = (N_{F,1}, \dots, N_{F,n})$, we evaluate A and B in the grid, and we interpolate the points $AB(\theta_j) = A(\theta_j)B(\theta_j)$ to get $\widetilde{AB}$. Then, we have*

$$\|AB - \widetilde{AB}\|_{\rho} \leq C_{N_F}(\rho, \hat{\rho}) \|A\|_{\rho} \|B\|_{\hat{\rho}},$$

for every $0 \leq \rho < \hat{\rho}$, where $C_{N_F}(\rho, \hat{\rho})$ is given in Theorem 4.3.

Notice that Corollary 4.6 is useful to control the product of approximated objects. If $\tilde{A}$ and $\tilde{B}$ are the corresponding approximations of A and B given by DFT, then

$$\|\tilde{A}\tilde{B} - \widetilde{\tilde{A}\tilde{B}}\|_{\rho} \leq C_{N_F}(\rho, \hat{\rho}) \|\tilde{A}\|_{\hat{\rho}} \|\tilde{B}\|_{\hat{\rho}} \leq C_{N_F}(\rho, \hat{\rho}) \|\tilde{A}\|_{F, \hat{\rho}} \|\tilde{B}\|_{F, \hat{\rho}} \quad (4.2)$$

for every $0 \leq \rho < \hat{\rho}$. Notice that since $\tilde{A}$ and $\tilde{B}$ are Fourier series with finite support, then it is interesting to control Equation (4.2) using Fourier norms. The second result allows us to control the inverse of a matrix using the discrete Fourier approximation:

Corollary 4.7 *Let us consider a matrix function $A \in A(\mathbb{T}_{\hat{\rho}}^d, \mathbb{C}^{m \times m})$. Given a grid of size $N_F = (N_{F,1}, \dots, N_{F,n})$, we evaluate A in the grid and compute the inverses $X(\theta_j) = A(\theta_j)^{-1}$. Then, if $\tilde{X}$ is the corresponding discrete Fourier approximation associated to the sample $X(\theta_j)$, the error $\Gamma(\theta) = I_m - A(\theta)\tilde{X}(\theta)$ satisfies*

$$\|\Gamma\|_{\rho} \leq C_{N_F}(\rho, \hat{\rho}) \|A\|_{\hat{\rho}} \|\tilde{X}\|_{\hat{\rho}},$$

for $0 \leq \rho < \hat{\rho}$. Moreover, if $\|\Gamma\|_{\rho} < 1$, there exists an inverse $A^{-1} \in A(\mathbb{T}_{\rho}^d, \mathbb{C}^{m \times m})$ satisfying

$$\|A^{-1} - \tilde{X}\|_{\rho} \leq \frac{\|\tilde{X}\|_{\hat{\rho}} \|\Gamma\|_{\rho}}{1 - \|\Gamma\|_{\rho}}.$$

Furthermore,

$$\begin{aligned} \|A^{-1}\|_{\rho} &\leq \|A^{-1} - \tilde{X}\|_{\rho} + \|\tilde{X}\|_{\rho} \leq \frac{\|\tilde{X}\|_{\hat{\rho}} \|\Gamma\|_{\rho}}{1 - \|\Gamma\|_{\rho}} + \|\tilde{X}\|_{\rho} = \frac{\|\tilde{X}\|_{\rho}}{1 - \|\Gamma\|_{\rho}} \\ &\leq \frac{\|\tilde{X}\|_{\rho}}{1 - C_{N_F}(\rho, \hat{\rho}) \|A\|_{\hat{\rho}} \|\tilde{X}\|_{\hat{\rho}}}. \end{aligned} \quad (4.3)$$

5 Methodology

We introduce now the methodology with which we can calculate the corresponding error bounds when implementing the validation theorem, especially by using the Fourier approximation results previously presented.

A crucial initial step involves meticulously selecting the inputs for validation. In our study, we will work with a real-analytic one-dimensional torus, which will be directly inputted into the algorithm as a truncated Fourier series (although the following procedure can be trivially generalized for a d -dimensional torus, adjusting grid sizes and everything else accordingly). These inputs can either originate from data already structured in this form or be prepared through preprocessing for convenience. Notably, the torus and other inputs are derived using the reducibility algorithm outlined in [8, 10].

For the validation process, we employed the Radix-2 DIT Cooley-Tukey algorithm to compute the Fast Fourier Transform (FFT), necessitating grid sizes that are powers of 2. This choice in algorithm ensures efficient computation and scalability, aligning with the requirements of our work [4]. Thus, the Nyquist term is pre-set to 0. Note, though, that the presented formulas work for both N even and odd, with the main difference that for N odd there is no Nyquist term that needs to be taken care of.

In this manner $K_0(\theta) = \widetilde{K_0(\theta)}$, and has the form

$$K_0(\theta) = \sum_{k=-\lfloor \frac{N-1}{2} \rfloor}^{\lfloor \frac{N-1}{2} \rfloor} \tilde{K}_{0,k} e^{2\pi i k \theta}.$$

Which leads easily to $K_0(\theta + \omega)$,

$$K_0(\theta + \omega) = \sum_{k=-\lceil \frac{N-1}{2} \rceil}^{\lceil \frac{N-1}{2} \rceil} (\tilde{K}_{0,k} e^{2\pi i k \omega}) e^{2\pi i k \theta}.$$

In the same manner, $P_1 = \tilde{P}_1$, $P_2 = \tilde{P}_2$, $\Lambda_s = \tilde{\Lambda}_s$ and $\Lambda_u = \tilde{\Lambda}_u$ will be our real-analytic inputs in Fourier space with Nyquist term 0. Notice that since our inputs are already truncated Fourier series, there is no error committed when going to grid space and back via Fourier Transform (as the FFT). We will also keep the same notation as in Section 2 for our real-analytic map F but changing the support point of the torus to θ , as this will allow to more easily work with the application of rotations. Nonetheless, with the aim of applying the results from Section 4, F will be defined over a slightly thicker domain, namely $D_{\hat{\rho}, R}$ for a $\hat{\rho} > \rho$.

5.1 Boxes Method

When it comes to rigorous numerics, interval arithmetic is usually the standard way to proceed. This will allow us to enclose small values that go beyond the computer's representation capabilities in an interval, easing then its manipulation. In this manner, and using the aforementioned inputs, we can compute the bounds that appear from K_0 in Theorem 3.4. Such bounds require us to compute ρ -norms, or supremum norms on a complex neighborhood of the base torus (or more practically, the grid). This can result troublesome, as we would need to evaluate non-truncated objects like, for example, $F(K_0(\theta), \theta)$ over a complex neighborhood of our grid using the computer. In some cases we can easily estimate such norm by raw bounding using the Fourier norm. But in other cases in which we may not have the desired object in Fourier space, we have to resort to another method. The Boxes Method allows us to extend the grid to the complex space by creating complex boxes $C_j = \{\theta_j + \varphi \mid |\operatorname{Re} \varphi| \leq \frac{1}{2N}, |\operatorname{Im} \varphi| \leq \rho\}$ (notice that the choice of ρ will depend on the context). These boxes are easily created in the computer as they are the cartesian product of two intervals (the real part and the imaginary part). For the computation of the norm, and following the previous example, first we will have to calculate the image of such boxes through K before applying F . Thus, we first need to calculate $K_0(\theta + \varphi)$. Notice that this is no more than rotating the torus as we have done before, but this time the rotation is complex. Again, this is easier done in Fourier series form given the symmetry of the coefficients thanks to the real-analyticity of K_0 :

$$K_0(\theta + \varphi) = \sum_{k=-\lceil \frac{N-1}{2} \rceil}^{\lceil \frac{N-1}{2} \rceil} (\tilde{K}_{0,k} e^{2\pi i k \varphi}) e^{2\pi i k \theta} = \sum_{k=-\lceil \frac{N-1}{2} \rceil}^{\lceil \frac{N-1}{2} \rceil} \left(\frac{\tilde{K}_{0,k}}{e^{2\pi i k \operatorname{Im} \varphi}} e^{2\pi i k \operatorname{Re} \varphi} \right) e^{2\pi i k \theta}.$$

We just have left to go back to grid space via IFFT. With this new torus, we can calculate $F(K(C_j), C_j)$ for each j . The norm of the object will then consist of the supremum out of all the new boxes.

5.2 The Invariance Error Bound

For the first bound, we can write the error produced in the invariance equation as

$$E(K_0)(\theta) = F(K_0(\theta), \theta) - K_0(\theta + \omega) .$$

Since the Fourier series of $F(K_0(\theta), \theta)$ is an infinite sum, we would like to approximate it by a finite sum where, in an abuse of notation, $\tilde{F}_{0,k}$ will indicate the k th Fourier coefficient of $F(K_0(\theta), \theta)$;

$$\widetilde{F(K_0(\theta), \theta)} = \sum_{k=-\lfloor \frac{N}{2} \rfloor}^{\lfloor \frac{N-1}{2} \rfloor} \tilde{F}_{0,k} e^{2\pi i k \theta} .$$

This allows us to split the computation of the error into two parts:

$$\|E(K_0)(\theta)\|_\rho \leq \|F(K_0(\theta), \theta) - \widetilde{F(K_0(\theta), \theta)}\|_\rho + \|\widetilde{F(K_0(\theta), \theta)} - K_0(\theta + \omega)\|_\rho \leq \varepsilon ,$$

where the last term of the sum can be easily bounded by the Fourier norm:

$$\|\widetilde{F(K_0(\theta), \theta)} - K_0(\theta + \omega)\|_\rho \leq \|\widetilde{F(K_0(\theta), \theta)} - K_0(\theta + \omega)\|_{F,\rho},$$

since it's a difference of truncated Fourier series and hence a truncated Fourier series with differences in the coefficients. For the first term and for a given $\hat{\rho} > \rho$ such that $\mathcal{K}_{\hat{\rho}} \subset D_{\hat{\rho},R}$, where recall $D_{\hat{\rho},R}$ is the domain of F , using Theorem 4.3 we have

$$\|F(K_0(\theta), \theta) - \widetilde{F(K_0(\theta), \theta)}\|_\rho \leq C_N(\rho, \hat{\rho}) \|F(K_0(\theta), \theta)\|_{\hat{\rho}} ,$$

where the last term will require the Boxes Method described before in Subsection 5.1 and a suitable choice for $\hat{\rho}$.

5.3 The Reducibility and Invertibility Error Bounds

The next bound to be computed is the reducibility error bound, where the reducibility error is given by

$$E_{\text{red}}(\theta) = P_2(\theta + \omega)M_0(\theta)P_1(\theta) - \Lambda(\theta) ,$$

with $M_0(\theta) = D_z F(K_0(\theta), \theta)$. Recall that we already have P_1 , P_2 , Λ_s and Λ_u (and therefore λ) as inputs. Using Corollary 4.6, the norm of the reducibility error is then

$$\begin{aligned} \|E_{\text{red}}(\theta)\|_\rho &= \|P_2(\theta + \omega)M_0(\theta)P_1(\theta) - \Lambda(\theta)\|_\rho \\ &\leq \|P_2(\theta + \omega)M_0(\theta)P_1(\theta) - \widetilde{P_2(\theta + \omega)M_0(\theta)P_1(\theta)}\|_\rho \\ &\quad + \|\widetilde{P_2(\theta + \omega)M_0(\theta)P_1(\theta)} - \Lambda(\theta)\|_\rho \\ &\leq C_N(\rho, \hat{\rho})\|P_2(\theta)\|_{\hat{\rho}}\|M_0(\theta)\|_{\hat{\rho}}\|P_1(\theta)\|_{\hat{\rho}} \\ &\quad + \|\widetilde{P_2(\theta + \omega)M_0(\theta)P_1(\theta)} - \Lambda(\theta)\|_{F,\rho} \\ &= \varepsilon_1, \end{aligned}$$

where the last term remains the same since it is the norm of the difference of a matrix of truncated Fourier series, and a constant matrix (given that $\Lambda(\theta)$ is an input).

Remark 5.1 Every time that a result of Theorem 4.3 is used (such as the derived Corollaries), a certain $\hat{\rho} > \rho$ is chosen. This means that for every computation of a bound we do in which we use those results, a different $\hat{\rho}$ can be chosen. However, we will play with ρ and $\hat{\rho}$ in order to find the optimal value of $C_N(\rho, \hat{\rho})$, and once found, it will be used in every calculation, meaning that we will keep the same choice of $\hat{\rho}$ throughout.

For the purpose of finding the σ bound in the validation Theorem, we require the value of λ . In our case, the Λ matrix will be a constant matrix as explained in Section 2 and therefore the value λ is given. However, this is not always the case, as other types of fixed points may lead to non-constant $\Lambda(\theta)$, requiring then a way to estimate a λ value. For such purpose (and although it is not necessary for our case), we illustrate here how to find such a value. Notice that

$$\|(\Lambda_u)^{-1}\|_\rho \leq \|(\Lambda_u)^{-1} - \widetilde{(\Lambda_u)^{-1}}\|_\rho + \|\widetilde{(\Lambda_u)^{-1}}\|_\rho \leq \|(\Lambda_u)^{-1} - \widetilde{(\Lambda_u)^{-1}}\|_\rho + \|\widetilde{(\Lambda_u)^{-1}}\|_{F,\rho}.$$

By Corollary 4.7,

$$\|(\Lambda_u)^{-1} - \widetilde{(\Lambda_u)^{-1}}\|_\rho \leq \frac{\|\widetilde{(\Lambda_u)^{-1}}\|_{\hat{\rho}} \|\Gamma\|_\rho}{1 - \|\Gamma\|_\rho}$$

where $\Gamma(\theta) = \text{Id}_{n_U} - \Lambda_u(\theta) \widetilde{(\Lambda_u(\theta))}^{-1}$ as used in Corollary 4.7, with

$$\|\Gamma\|_\rho \leq C_N(\rho, \hat{\rho}) \|\Lambda_u\|_{\hat{\rho}} \|\widetilde{(\Lambda_u)^{-1}}\|_{\hat{\rho}}.$$

All together

$$\|(\Lambda_u)^{-1} - \widetilde{(\Lambda_u)^{-1}}\|_\rho \leq \frac{C_N(\rho, \hat{\rho}) \|\Lambda_u\|_{F,\hat{\rho}} \|\widetilde{(\Lambda_u)^{-1}}\|_{F,\hat{\rho}}^2}{1 - C_N(\rho, \hat{\rho}) \|\Lambda_u\|_{F,\hat{\rho}} \|\widetilde{(\Lambda_u)^{-1}}\|_{F,\hat{\rho}}}.$$

The inequality holds due to the fact that Λ_u is an input and already a matrix of truncated Fourier series. Thus, following formula (4.3), we can take

$$\|(\Lambda_u)^{-1}\|_\rho \leq \frac{\|\widetilde{(\Lambda_u)^{-1}}\|_{F,\hat{\rho}}}{1 - C_N(\rho, \hat{\rho}) \|\Lambda_u\|_{F,\hat{\rho}} \|\widetilde{(\Lambda_u)^{-1}}\|_{F,\hat{\rho}}} = \lambda_u.$$

Next, we seek a bound for the invertibility error. Its norm can be computed using the same procedure:

$$\begin{aligned} \|E_{\text{inv}}(\theta)\|_\rho &\leq \|P_2(\theta)P_1(\theta) - \widetilde{P_2(\theta)P_1(\theta)}\|_\rho + \|\widetilde{P_2(\theta)P_1(\theta)} - \text{Id}\|_\rho \\ &\leq C_N(\rho, \hat{\rho}) \|P_2(\theta)\|_{\hat{\rho}} \|P_1(\theta)\|_{\hat{\rho}} + \|\widetilde{P_2(\theta)P_1(\theta)} - \text{Id}\|_{F,\rho} \\ &= \varepsilon_2. \end{aligned}$$

5.4 Computation of $b(r)$

We only have left the last condition of Theorem 3.4. Recall that condition c) required $\forall(z, \theta) \in \bar{D}_{\rho,r}$, $\|D_z F(z, \theta) - D_z F(K_0(\theta), \theta)\|_\rho \leq b(r)|z - K_0(\theta)|$. This means that we need a way of computing such $b(r)$. Notice that for all $\theta \in \bar{\mathbb{T}}_\rho$

$$\begin{aligned}
& \|D_z F(z, \theta) - D_z F(K_0(\theta), \theta)\|_\rho \\
&= \left\| \int_0^1 \frac{d}{dt} D_z F(K_0(\theta) + t(z - K_0(\theta)), \theta) dt \right\|_\rho \\
&\leq \int_0^1 |D_z^2 F(K_0(\theta) + t(z - K_0(\theta)), \theta)| dt \cdot |z - K_0(\theta)| \\
&\leq \sup_{\substack{\theta \in \bar{\mathbb{T}}_\rho \\ |y - K_0(\theta)| \leq r}} |D_z^2 F(y, \theta)| \cdot |z - K_0(\theta)|.
\end{aligned}$$

So we can take

$$b(r) = \max_{i=1, \dots, n} \sup_{(y, \theta) \in \bar{D}_{\rho, R}} |D_z^2 F^i(y, \theta)|,$$

where F^i represents the i th component of F , as the maximum of the norms of each component of F as bilinear maps. In fact, we compute $b(R)$, as $b(r) \leq b(R)$. The second derivative of F is then evaluated over points in the domain. Such domain are the points R -close to the approximate torus over a ρ -thick complex base torus, $\bar{\mathbb{T}}_\rho$. This means that in our case we will have to apply the Boxes Method to compute the second derivative, that is, thickening the torus by R in all directions after thickening it by ρ in the complex direction.

6 Algorithm and Results

The computations explained in Section 5 have been implemented in C++ using interval arithmetics. Two different classes have been created for such purpose, *ComplexInterval*, which has two members, an interval representing the real part and an interval representing the imaginary part of a complex value, and *RealInterval*, which comprises of a single member, an interval. Methods for each class have been implemented such that basic operations and operator overloads are covered, easing the way into building more complex functions as the FFT.

6.1 Case of Study

The objects that will serve as starting points for the validation are approximate tori computed through a reducibility method (see [10] for more details). In this study we take f to be an irrational rotation as mentioned and F as the perturbed Standard Map. Thus, $\hat{F}$ is $(F, f) : \mathbb{C}^2 \times \mathbb{T}_\mathbb{C} \rightarrow \mathbb{C}^2 \times \mathbb{T}_\mathbb{C}$ given by

$$\begin{cases} f(\theta) = \theta + \omega, \\ F^1(x, y) = x + y - \frac{\kappa}{2\pi} \sin(2\pi x) - \epsilon \sin(2\pi\theta), \\ F^2(x, y) = y - \frac{\kappa}{2\pi} \sin(2\pi x) - \epsilon \sin(2\pi\theta). \end{cases} \quad (6.1)$$

For a given rotation $\omega \in \mathbb{R} \setminus \mathbb{Q}$, and fixed $\kappa, \epsilon \in \mathbb{R}$, we search for a real-analytic parameterization of a torus $K : \bar{\mathbb{T}}_\rho \rightarrow \mathbb{C}^2$, joint with changes of variable matrices $P_2, P_1 : \bar{\mathbb{T}}_\rho \rightarrow \mathbb{C}^{2 \times 2}$ that reduce the linearized dynamics to $\Lambda : \bar{\mathbb{T}}_\rho \rightarrow \mathbb{C}^{2 \times 2}$ such that

$\Lambda = \text{diag}(\Lambda_s, \Lambda_u)$ is constant, and $\lambda_s = |\Lambda_s|$ and $\lambda_u = |\Lambda_u^{-1}|$. That is, we plan to solve the equations

- $F(K(\theta - \omega), \theta - \omega) - K(\theta) = 0$,
- $DF(K(\theta - \omega), \theta - \omega)P_1(\theta - \omega) - P_1(\theta)\Lambda = 0$,
- $\text{Id} - P_1(\theta)P_2(\theta) = 0$.

These are the equations that of course lead to E , E_{red} and E_{inv} when applied to an approximation K_0 .

As for the rotation value, we take the famous value $\omega = \frac{1}{2}(\sqrt{5} - 1)$. Fixing $\kappa \in (0, 4)$, for $\epsilon = 0$, $K_0(\theta) = (\frac{1}{2}, 0)$ is a fiberwise hyperbolic invariant torus. For $\epsilon > 0$ small enough, the torus maintains its hyperbolicity up to a certain critical value ϵ_c , where the linear dynamics of the invariant torus are no longer reducible to diagonal (this is known as the phenomenon of bundle collapse, implying the destruction of the invariant curve, see [9, 10, 11]). Hence, the validation is possible, in principle, for $\epsilon \in [0, \epsilon_c]$.

In our case, we fixed $\kappa = 1.3$ such that the invariant tori start to lose hyperbolicity as ϵ approaches a critical value of $\epsilon_c \approx 1.2352755$ as explained in [6, 8], which means that the validation will get more nuanced and probably require a finer tuning of the parameters such as grid size, width of the complex band and so on. This can be easily seen on Figure 1, as the graph of the torus starts to fractalize as it approaches the critical value. These are the tori that are validated in the following section.

6.2 Validation Examples

As mentioned in Section 5, our inputs for the validation are K_0 , P_1 , P_2 and Λ . These inputs have been computed with double precision and therefore the validation will be restrained to the maximum precision of double types. It is also worth noting that when we look at the Fourier coefficients of the inputs, we see that after the initial decay of the coefficients, the values get very small. However, when computing Fourier norms, all those values are being multiplied by big exponentials (depending also on N) which can amplify that noise and make it significant. That is why it may be a good idea to cut those noise values off and keep only a reasonable neighborhood of the first coefficients. This will of course shrink the grid size but ensure our bounds are not dirtied up by unnecessary noise.

The first term to find then depending on such inputs is ρ , which, after a choice of $\hat{\rho}$, allows us to compute $C_N(\rho, \hat{\rho})$. We can find a good upper bound estimate for which we can take the value of ρ . Notice how the modulus of the Fourier coefficients of our input torus K_0 decays at an exponential rate of $2\pi N\rho^*$ (times a constant). Therefore we can perform an exponential regression on such coefficients to find an upper bound ρ^* of the ρ we can use for our validation. In Figure 2 we can see how the slope ρ^* of the black lines fit the coefficient decay. In a similar way we can see in Figure 3 that the upper bound for fitting the coefficients of the noisier P_1 is going to be higher than for K_0 .

The first idea here is to choose a $\rho \leq \rho^*$ and a $\hat{\rho} > \rho$ such that $C_N(\rho, \hat{\rho})$ becomes as small as possible. Of course we cannot choose any values as the choice of $\hat{\rho}$ will also have an impact on some objects' norms, just like with ρ . Nonetheless we still require it to be small enough such that the error bounds that are computed afterwards are also small

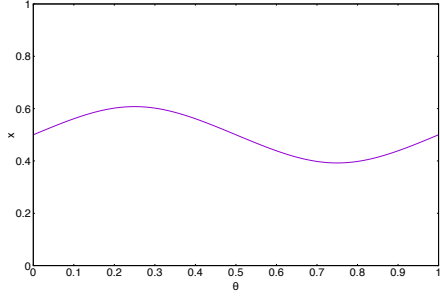
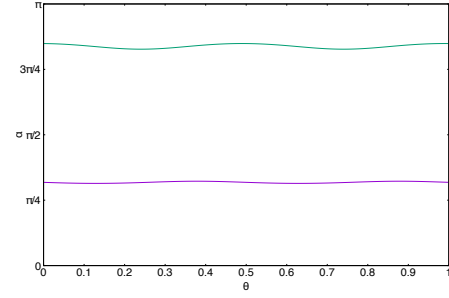
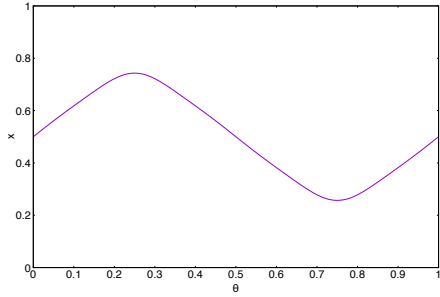
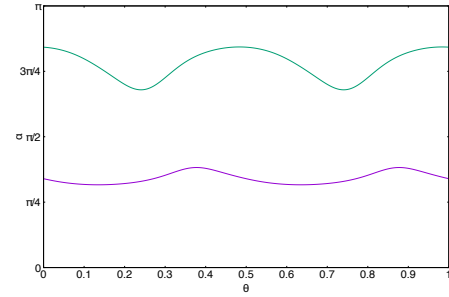
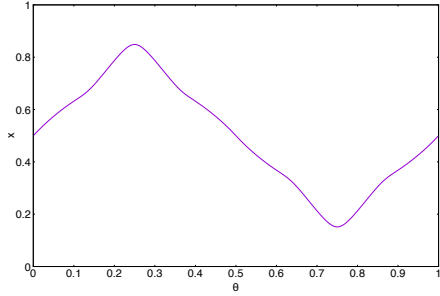
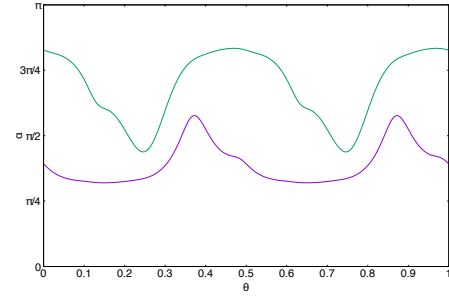
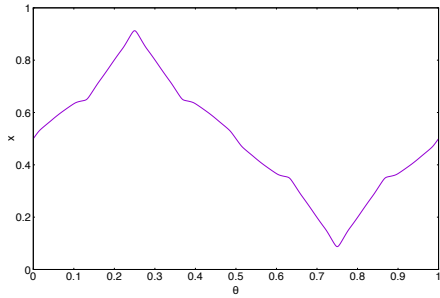
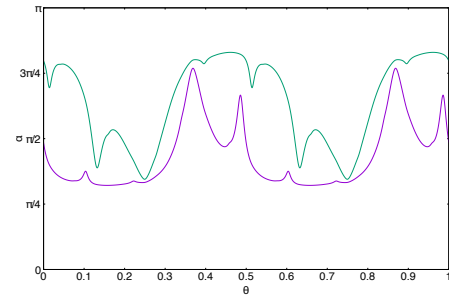
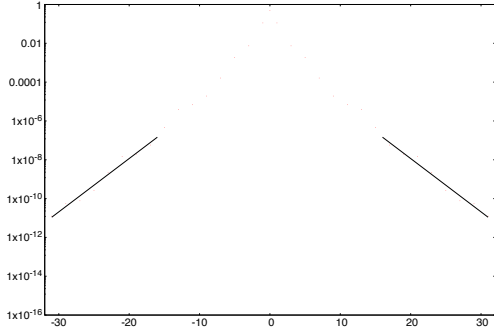
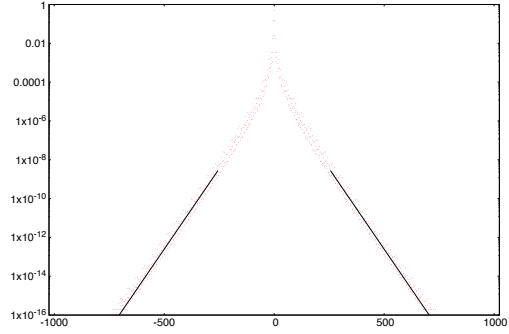
(A) x-projection of the torus, $\epsilon = 0.5$.(B) Invariant subbundles, $\epsilon = 0.5$.(C) x-projection of the torus, $\epsilon = 1$.(D) Invariant subbundles, $\epsilon = 1$.(E) x-projection of the torus, $\epsilon = 1.2$.(F) Invariant subbundles, $\epsilon = 1.2$.(G) x-projection of the torus, $\epsilon = 1.2342$.(H) Invariant subbundles, $\epsilon = 1.2342$.

FIGURE 1. Invariant tori (left) and their projectivized invariant subbundles (right) for several values of ϵ for the Standard Map (6.1). On the left is displayed the x-projection of the invariant tori and on the right the angle (α) between the stable and unstable subbundles and the semiaxis $x > 0$.

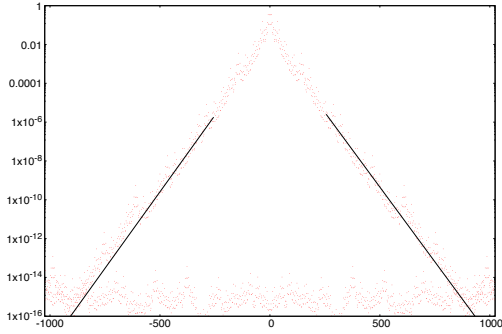


(A) Fourier coefficients and the fitted curve for $\rho^* \approx 0.1$ at $\epsilon = 1$.

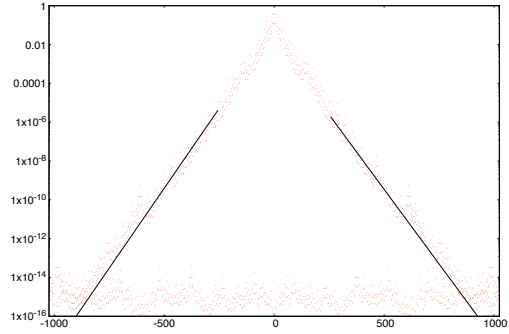


(B) Fourier coefficients and the fitted curve for $\rho^* \approx 0.006$ at $\epsilon = 1.2342$.

FIGURE 2. Fourier coefficients in a logarithmic scale of the x component of the torus for several values of ϵ (red) and a regression line of parameter ρ^* fitting the coefficients (black).



(A) Fourier coefficients and the fitted curve for the stable subbundle.



(B) Fourier coefficients and the fitted curve for the unstable subbundle.

FIGURE 3. Fourier coefficients in a logarithmic scale of the stable subbundle (left) and the unstable subbundle (right) of the torus for $\epsilon = 1.2342$ (red) and a regression line of parameter $\rho^* \approx 0.005$ fitting the coefficients (black).

enough to guarantee hyperbolicity. Next, we can compute the invariance, reducibility and invertibility error bounds, applying the Boxes Method as explained in Section 5 when necessary. With such bounds we can check whether the hyperbolicity condition is satisfied (hence the need for small bounds and values such as $C_N(\rho, \hat{\rho})$), and if so, calculate the hyperbolicity bound σ . For visualization purposes, we have chosen the same values for the parameters $N, \rho, \hat{\rho}$ and R to execute the validation for a couple values of ϵ . What can be clearly seen across Tables 1 and 2 is that, under the same parameters, all the error bounds increase as the parameter ϵ increases. This shows how the torus loses invariance and reducibility the closer we get to the critical value of ϵ , which in turn shows also how the torus is losing its hyperbolicity, as σ is rapidly increasing, doubling when

going just from $\epsilon = 0.5$ to $\epsilon = 1$. Although not as significant, a similar thing happens to the bound $b(R)$.

Term	Bound
$C_N(\rho, \hat{\rho})$	$9.1889223103814460684374498900346 \times 10^{-8}$
$\ E\ $	$\varepsilon = 1.2828980268004448607526103424455 \times 10^{-7}$
$\ E_{\text{red}}\ $	$\varepsilon_1 = 9.9384120231971379614823261022261 \times 10^{-7}$
$\ E_{\text{inv}}\ $	$\varepsilon_2 = 2.6492816206529570277935091994725 \times 10^{-7}$
$\ (\mathcal{M} - \mathcal{I})^{-1}\ $	$\sigma = 3.5047863969274353212557286917070$
$b(r)$	$b(R) = 6.2726860980665081720689004716316$
$\ P_1\ $	$1.5338805731759423288761342467093$
$\ P_2\ $	$1.4687982979744499328293125411365$
$\ \Lambda_s\ $	$\lambda_s = 0.357175$
$\ \Lambda_u\ $	$\lambda_u^{-1} = 2.79975$
$\ \Lambda\ $	$\lambda = 0.357175$

TABLE 1. Validation values for $\kappa = 1.3$, $\epsilon = 0.5$, $N = 64$, $\rho = 10^{-2}$, $\hat{\rho} = 10^{-1}$, $R = 1.5 \times 10^{-2}$, $r_- = 5.18586 \times 10^{-7}$ and $r_+ = 1.49999 \times 10^{-2}$.

Term	Bound
$C_N(\rho, \hat{\rho})$	$9.1889223103814460684374498900346 \times 10^{-8}$
$\ E\ $	$\varepsilon = 6.9886143393896989222160867344917 \times 10^{-7}$
$\ E_{\text{red}}\ $	$\varepsilon_1 = 3.2633447245349282716855911598803 \times 10^{-4}$
$\ E_{\text{inv}}\ $	$\varepsilon_2 = 1.0925073901570803841407006747087 \times 10^{-5}$
$\ (\mathcal{M} - \mathcal{I})^{-1}\ $	$\sigma = 7.6699450817858685652216386803508$
$b(r)$	$b(R) = 8.8552159445544865852315598397991$
$\ P_1\ $	$2.0186640836165030040458099350296$
$\ P_2\ $	$2.1000421847920570152253401199118$
$\ \Lambda_s\ $	$\lambda_s = 0.44695$
$\ \Lambda_u\ $	$\lambda_u^{-1} = 2.23739$
$\ \Lambda\ $	$\lambda = 0.44695$

TABLE 2. Validation values for $\kappa = 1.3$, $\epsilon = 1$, $N = 64$, $\rho = 10^{-2}$, $\hat{\rho} = 10^{-1}$, $R = 1.5 \times 10^{-2}$, $r_- = 5.40288 \times 10^{-6}$ and $r_+ = 1.47234 \times 10^{-2}$.

It is worth mentioning that although the validation is successful for such cases, the value for r_- gets also bigger the bigger the ϵ . This is because, since our estimates are getting worse, the capacity to ensure uniqueness of a truly invariant torus gets impaired, and therefore the radius of the tube centered in the invariant torus for which we can ensure uniqueness gets smaller (which relates to r_- getting bigger, as the radii at which the Banach Fixed Point Theorem ensures uniqueness).

What is also interesting is the values of the norms of P_1 and P_2 . Recall that, in this case, the first column of P_1 represents the direction of the stable subbundle and the second column represents the direction of the unstable subbundle. This means that, although the validated torus is still hyperbolic for such values of ϵ , the closer we get to the critical value of the parameter, the more variable the fiber becomes (for more information about this phenomenon, we refer the reader to [8]). This means that in order to bring each point of the torus of each fiber to the same coordinate system, we will require a more radical change of variables, represented by the matrix P_1 , which implies the norm of P_1 (and hence of P_2) getting progressively larger. This is the effect of the bundle collapse described before and observed in Figures 1a, 1d, 1f and 1h. This can become problematic as some of the bounds (such as the reducibility and invertibility error bounds) depend on the norms of P_1 and P_2 . One would be inclined to think that reducing $\hat{\rho}$ would be a solution to this problem, since evaluating the maps P_1 and P_2 over a narrower band would yield a smaller supremum. However, reducing $\hat{\rho}$ comes with a cost, and that is that $\rho - \hat{\rho}$ becomes smaller and therefore the controlling term $C_N(\rho, \hat{\rho})$ gets bigger. This would still cause the bound to remain big, and that is why the validation gets more intricate the closer we get to the critical value ϵ_c .

Term	Bound
$C_N(\rho, \hat{\rho})$	$9.8707312171782026833836734540706 \times 10^{-8}$
$\ E\ $	$\varepsilon = 2.9326180147973081272937376851241 \times 10^{-7}$
$\ E_{\text{red}}\ $	$\varepsilon_1 = 2.0734245146625048668499036868668 \times 10^{-4}$
$\ E_{\text{inv}}\ $	$\varepsilon_2 = 5.6118904929900122124863469654664 \times 10^{-5}$
$\ (\mathcal{M} - \mathcal{I})^{-1}\ $	$\sigma = 3.6213041171848072135862902799712 \times 10^2$
$b(r)$	$b(R) = 8.4382948067235647008416493688900$
$\ P_1\ $	$1.0471274358468112823731630281721 \times 10$
$\ P_2\ $	$1.1319070929210905461944029023272 \times 10$
$\ \Lambda_s\ $	$\lambda_s = 0.672437$
$\ \Lambda_u\ $	$\lambda_u^{-1} = 1.48713$
$\ \Lambda\ $	$\lambda = 0.672437$

TABLE 3. Validation values for $\kappa = 1.3$, $\epsilon = 1.2342$, $N = 2048$, $\rho = 7 \times 10^{-4}$, $\hat{\rho} = 4 \times 10^{-3}$, $R = 1.5 \times 10^{-2}$, $r_- = 1.33381 \times 10^{-4}$ and $r_+ = 3.27249 \times 10^{-4}$.

It is at this point that we have to start playing around a bit with N , ρ and $\hat{\rho}$ to satisfy the validation conditions, such as reducing $\hat{\rho}$ enough such that the norms of the P matrices are small enough but not too much such that $C_N(\rho, \hat{\rho})$ gets too big. For that it is also helpful to increase the amount of Fourier modes N and play around with the other parameters. This can be done in two different ways; first, our input can already be processed in such a way that the grid size is already bigger accounting for this situation, and second we can manually increase N by increasing the tail of the Fourier coefficients, that is, filling with zeroes. Having a larger N allows for more generous values of ρ and $\hat{\rho}$ which permits a choice of ρ that is closer to the fitting value ρ^* .

This behavior can be observed in Table 3. The parameter values used in Tables 1 and 2 won't work anymore for $\epsilon = 1.2342$, so we have to fine tune them. The first thing is to increase the amount of Fourier modes to 2048, and then find ρ and $\hat{\rho}$ accordingly. With these values it can be seen that although the validation is successful, the hyperbolicity bound σ is substantially bigger than for other values of ϵ essentially because the norms of the change of variables P_1 and P_2 increase, while the rates of contraction and expansion remain relatively far from 1. This is the signature of the bundle collapse leading to the destruction of a saddle-type invariant torus described in [9, 11] (see [1, 2, 12] for other scenarios).

The code used to perform the proofs is available at [17] and uses the interval arithmetic package MPFI [14]. The plots have been generated by gnuplot using the outputs of the same code.

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