Logarithmic singularity in the density four-point function of two-dimensional critical percolation in the bulk

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Abstract

We provide definitive proof of the logarithmic nature of the percolation conformal field theory in the bulk by showing that the four-point function of the density operator has a logarithmic divergence as two points collide and that the same divergence appears in the operator product expansion (OPE) of two density operators. The right hand side of the OPE contains two operators with the same scaling dimension, one of them multiplied by a term with a logarithmic singularity. Our method involves a probabilistic analysis of the percolation events contributing to the four-point function. It does not require algebraic considerations, nor taking the $Q \to 1$ limit of the Q-state Potts model, and is amenable to a rigorous mathematical formulation. The logarithmic divergence appears as a consequence of scale invariance combined with independence.

1 Introduction

In this paper, we give direct evidence of the logarithmic nature of the conformal field theory associated to two-dimensional critical percolation in the bulk. Percolation provides the simplest example of a purely geometric phase transition. In percolation and other geometric models, such as the Fortuin-Kasteleyn (FK) random cluster model, the focus is on connectivity properties. These are encoded in connection probabilities, which replace the spin correlation functions of the Ising and Potts models as fundamental quantities of interest. At the critical point, the large scale geometric properties of percolation are believed to be described by a conformal field theory (CFT). This belief is based on the assumption that, in the continuum limit, connection probabilities can be expressed in terms of CFT correlation functions. The latter assumption received strong support from the recent proof that, at the critical point, connection probabilities behave exactly like CFT correlation functions [Cam24].

If one postulates a CFT description of the large scale properties of critical percolation, then the relevant CFT must have central charge $c = 0$ because the percolation partition function is not sensitive to finite size effects [BCN86, Aff86]. In order to be nontrivial, a $c = 0$ CFT must be non-unitary, since the only unitary CFT with $c = 0$ does not admit any observables other than the identity field.

Lack of unitarity has serious physical implications, it can complicate the mathematical analysis and lead to the appearance of logarithmic singularities in some correlation functions [RS92, Sal92, Gur93] and in the operator product expansion (OPE) of certain fields [Gur99, GL02]. CFTs in which the correlators of basic fields can have logarithmic divergences at short distance are called logarithmic and have attracted considerable attention due to their role in the study of, e.g., the Wess-Zumino-Witten (WZW) model, the quantum Hall effect, disordered critical systems, self-avoiding polymers, percolation and the FK model.

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However, despite significant recent progress [SKZ07, DV10, CR13, PSVD13, Dot16, Dot20, HGSJS20, NR21, HS22], the field of logarithmic CFTs is considerably less developed than that of ordinary CFTs. In particular, not many explicit examples of correlation functions exhibiting a logarithmic singularity are known.

For critical percolation in finite domains or the upper half-plane (i.e., in the presence of a boundary), logarithmic terms were identified in the expected number of clusters crossing a rectangle [Mai03] and in some crossing probabilities that generalize Cardy's crossing formula [SKZ07, Sim13], and were conjectured to appear in a boundary four-point function [GV18].

In [VJS12], a logarithmic term was found in the full-plane two-point function of an operator which is not a pure scaling operator [1], but in general, the study of the percolation CFT in the bulk (i.e., on the full plane) is considered much harder than in the presence of a boundary due to the constraints of crossing symmetry and the absence of null states.

The logarithmic correlation functions in [GV18] and [VJS12] were found by taking the $Q \to 1$ limit of correlation functions calculated for the Q-state Potts model with $Q \neq 1$. Although this strategy is standard, it cannot be justified rigorously, at least at present, and the $Q \to 1$ limit is very delicate and remarkably subtle, as observed, for example, in [CR13] and [Dot20]. As an example of the subtlety involved in the limit, we note that the observables considered in [VJS12] cannot be defined for $Q = 1$, while the logarithmic term appears only in the limit $Q \to 1$.

In this paper, we provide a direct calculation, using only percolation observables and techniques, of the four-point function of the percolation density (spin) operator [2], which measures the cluster density. Its correlation functions are linear combinations of connection probabilities, the most fundamental percolation quantities. We will show that its four-point function has a logarithmic divergence, as two of the four points collide, and will argue that this singularity is present also in the OPE of two density operators, in such a way as to imply the presence, in the right hand side of the OPE, of two operators with the same scaling dimension. This is consistent with the idea that logarithmic CFTs are characterized by the presence of logarithmic pairs of operators with the same scaling dimension [Gur93]. Importantly, we do not have to assume the presence of two operators with the same scaling dimension, since this is revealed by our analysis of the four-point function. Indeed, we do not make any a priori assumption on the operator content of the percolation CFT and we do not use the $Q \to 1$ limit of the Q-state Potts model nor Coulomb gas techniques. As a consequence, our analysis is amenable to a rigorous mathematical formulation [CF24].

2 Preliminary asymptotic analysis of the four-point function of the density operator

We start with a preliminary analysis of the four-point function. At the percolation critical point, the operator $\psi(z)$ that measures the cluster density at a point z has scaling dimension $2h_\psi = 5/48$ (see [Cam24]). This suggests the identification $\psi = \phi_{0,1/2}$, where $\phi_{0,1/2}$ is the $Q \to 1$ limit of the spin (magnetization) operator of the Q-state Potts model [DF84, Car84].

The four-point function $\langle \psi(z_1)\psi(z_2)\psi(z_3)\psi(z_4)\rangle$ can be written as a sum of four terms (see [Cam24]),

$$\begin{aligned}\langle \psi(z_1)\psi(z_2)\psi(z_3)\psi(z_4)\rangle &= P(z_1 \leftrightarrow z_2 \leftrightarrow z_3 \leftrightarrow z_4) + P(z_1 \leftrightarrow z_2 \leftrightarrow z_3 \leftrightarrow z_4) \\ &\quad + P(z_1 \leftrightarrow z_3 \not\leftrightarrow z_2 \leftrightarrow z_4) + P(z_1 \leftrightarrow z_4 \not\leftrightarrow z_2 \leftrightarrow z_3),\end{aligned} \tag{2.1}$$

where the first term corresponds to the "probability" that all points belong to the same cluster and the remaining terms correspond to the "probabilities" that two points belong to one cluster and the remaining two to a different cluster [3].

[1] Conformal covariance implies that the two-point function of a pure scaling operator is a pure power.

[2] Formally, this is the $Q \to 1$ limit of the spin field of the Q-state Potts model, but it can be defined without reference to the Potts model (see [Cam24]).

[3] The function P should be interpreted as the continuum limit of the renormalized lattice probability (the renormalization factor scales like the inverse of the lattice spacing to the power of $8h_\psi$) and can therefore be larger than 1.

If $z_1, z_2 \to z$, the leading behavior comes from the first two terms in the right hand side of (2.1) and is given by $|z_1 - z_2|^{-5/24}|z_3 - z_4|^{-5/24}$. The last two terms require that the clusters of z_1 and z_2 be different. Therefore, as $z_1, z_2 \to z$, they correspond to the insertion at z of an operator ϕ producing two distinct clusters, which is related to the four-leg operator with scaling dimension $5/4$, where the four legs are produced by the boundaries of the two clusters. Hence, the leading contribution of the last two terms is $|z_1 - z_2|^{-5/24}|z_1 - z_2|^{5/4}F(z, z_3, z_4)$, where $F(z, z_3, z_4) = |z - z_3|^{-5/4}|z - z_4|^{-5/4}|z_3 - z_4|^{25/24}$. As observed in [Dot16] and [Cam24], this suggests the OPE

$$\psi(z_1)\psi(z_2) = |z_1 - z_2|^{-5/24}\big(1 + C_{\psi\psi\phi}|z_1 - z_2|^{5/4}\phi(z) + \dots\big), \tag{2.2}$$

where $C_{\psi\psi\phi}$ is the structure constant appearing in the three-point function $\langle\psi(z_1)\psi(z_2)\phi(z_3)\rangle$ and the ellipsis denotes the contribution from other operators.

We will now show that a more careful analysis of the first two terms in the right hand side of (2.1) reveals the presence of a term that behaves like $|z_1 - z_2|^{-5/24}|z_1 - z_2|^{5/4}F(z, z_3, z_4)\big|\log|z_1 - z_2|\big|$.

3 Logarithmic singularity

In describing percolation events, we will use the standard terminology of open and closed clusters and paths. Given two subsets of the plane, A and B, we consider the following events:

- $z_1 \xleftrightarrow{A} z_2$: there is an open path between z_1 and z_2 contained in A,
- $z_1 \xleftrightarrow[B]{} z_2$: z_1, z_2 belong to the same open cluster but there is no open path fully contained in B,
- $z_1 \xleftrightarrow[B]{A} z_2$: there is an open path between z_1 and z_2 contained in A but no open path fully contained in B.

Now consider disks $B_n = \{z : |z - \frac{z_1+z_2}{2}| \leq 2^n|z_1 - z_2|\}$ for $n = 1, \dots, N$, where N is chosen so that $2^N \sim 1/|z_1 - z_2|$, that is, $N \sim -\log|z_1 - z_2|$. We assume that z_1 and z_2 are close to each other and that z_3 and z_4 are outside B_N (see Fig. 3.1). Letting $B_n^c = \mathbb{C} \setminus B_n$ denote the complement of B_n, the sum of the first two terms of (2.1) can be written as

$$\begin{aligned}
P(z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_4) = {} & P(z_1 \xleftrightarrow{B_1} z_2, z_3 \leftrightarrow z_4) \\
& + \sum_{n=2}^{N} P(z_1 \xleftrightarrow[B_{n-1}]{B_n} z_2, z_3 \leftrightarrow z_4) + P(z_1 \xleftrightarrow[B_N]{} z_2, z_3 \leftrightarrow z_4) \\
= {} & P\Big(z_1 \xleftrightarrow{B_1} z_2, z_3 \xleftrightarrow{B_1^c} z_4\Big) + P(z_1 \xleftrightarrow{B_1} z_2, z_3 \xleftrightarrow[B_1^c]{} z_4) \\
& + \sum_{n=2}^{N} \Big[P(z_1 \xleftrightarrow[B_{n-1}]{B_n} z_2, z_3 \xleftrightarrow{B_n^c} z_4) \\
& + P(z_1 \xleftrightarrow[B_{n-1}]{B_n} z_2, z_3 \xleftrightarrow[B_n^c]{} z_4)\Big] + P(z_1 \xleftrightarrow[B_N]{} z_2, z_3 \leftrightarrow z_4) \\
= {} & P(z_1 \xleftrightarrow{B_1} z_2)P(z_3 \xleftrightarrow{B_1^c} z_4) + P(z_1 \xleftrightarrow{B_1} z_2, z_3 \xleftrightarrow[B_1^c]{} z_4) \\
& + \sum_{n=2}^{N} \Big[P(z_1 \xleftrightarrow[B_{n-1}]{B_n} z_2)P(z_3 \xleftrightarrow{B_n^c} z_4) + P(z_1 \xleftrightarrow[B_{n-1}]{B_n} z_2, z_3 \xleftrightarrow[B_n^c]{} z_4)\Big] \\
& + P(z_1 \xleftrightarrow[B_N]{} z_2, z_3 \leftrightarrow z_4),
\end{aligned} \tag{3.1}$$

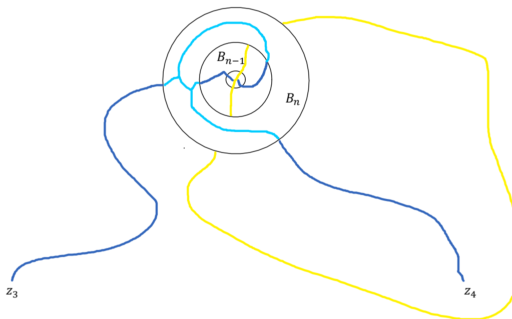

Figure 3.1: The event $\{z_1 \xleftrightarrow[B_{n-1}]{B_n} z_2, z_3 \xleftrightarrow[B_n^c]{} z_4\}$. Yellow lines (lighter shade) denote closed paths, (light) blue lines (darker shade) denote open paths. z_1 and z_2 are contained in the inner disk and not marked. They are not connected by an open path within the next disk, B_{n-1}, but are connected within the largest disk, B_n, with radius twice that of B_{n-1}. z_3 and z_4 are connected by an open path, but not outside B_n. The open paths connecting z_1, z_2 and z_3, z_4 can overlap within B_n. The number N of disks one can insert between z_1, z_2 and z_3, z_4 is of order $\log(1/|z_1 - z_2|)$.

where the last equality follows from the independence of the percolation events considered. Similarly, we have

$$\begin{aligned}P(z_1 \leftrightarrow z_2)P(z_3 \leftrightarrow z_4) &= P(z_1 \xleftrightarrow{B_1} z_2)P(z_3 \xleftrightarrow{B_1^c} z_4) + P(z_1 \xleftrightarrow{B_1} z_2)P(z_3 \xleftrightarrow[B_1^c]{} z_4) \\ &+ \sum_{n=2}^{N}\Big[P(z_1 \xleftrightarrow[B_{n-1}]{B_n} z_2)P(z_3 \xleftrightarrow{B_n^c} z_4) + P(z_1 \xleftrightarrow[B_{n-1}]{B_n} z_2)P(z_3 \xleftrightarrow[B_n^c]{} z_4)\Big] \\ &+ P(z_1 \xleftrightarrow[B_N]{} z_2)P(z_3 \leftrightarrow z_4).\end{aligned} \tag{3.2}$$

Comparing (3.1) and (3.2), we see that

$$\begin{aligned}P(z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_4) &= P(z_1 \leftrightarrow z_2)P(z_3 \leftrightarrow z_4) \\ &+ \Big[P(z_3 \xleftrightarrow[B_1^c]{} z_4 | z_1 \xleftrightarrow{B_1} z_2) - P(z_3 \xleftrightarrow[B_1^c]{} z_4)\Big]P(z_1 \xleftrightarrow{B_1} z_2) \\ &+ \sum_{n=2}^{N}\Big[P(z_1 \xleftrightarrow[B_{n-1}]{} z_2, z_3 \xleftrightarrow[B_n^c]{} z_4 | z_1 \xleftrightarrow{B_n} z_2) \\ &- P(z_1 \xleftrightarrow[B_{n-1}]{} z_2 | z_1 \xleftrightarrow{B_n} z_2)P(z_3 \xleftrightarrow[B_n^c]{} z_4)\Big]P(z_1 \xleftrightarrow{B_n} z_2) \\ &+ \Big[P(z_1 \xleftrightarrow[B_N]{} z_2, z_3 \leftrightarrow z_4 | z_1 \leftrightarrow z_2) \\ &- P(z_1 \xleftrightarrow[B_N]{} z_2 | z_1 \leftrightarrow z_2)P(z_3 \leftrightarrow z_4)\Big]P(z_1 \leftrightarrow z_2).\end{aligned} \tag{3.3}$$

The leading behavior comes from the term $P(z_1 \leftrightarrow z_2)P(z_3 \leftrightarrow z_4) \sim |z_1 - z_2|^{-5/24}|z_3 - z_4|^{-5/24}$.

To understand the behavior of the next term, observe that the event $z_3 \xleftrightarrow[B_1^c]{} z_4$ implies that, in B_1^c, z_3 and z_4 are connected to ∂B_1 by two disjoint open clusters which are separated by a closed cluster. We denote this event by $\mathcal{F}(z_3, z_4; B_1^c)$. Since $\mathcal{F}(z_3, z_4; B_1^c)$ means that the annulus $B_N \setminus B_1$ is crossed by two open paths (connecting z_3 and z_4 to ∂B_1) and two closed paths (which must be present to separate the open clusters supporting the two open paths), the four-arm exponent [ADA99, SW01] implies that

$P(\mathcal{F}(z_3,z_4;B_1^c)) \sim |z_1-z_2|^{5/4}$. Conditioning on $\mathcal{F}(z_3,z_4;B_1^c)$, we can write

$$
\begin{aligned}
&\big[P(z_3 \underset{B_1^c}{\longleftrightarrow} z_4 | z_1 \overset{B_1}{\longleftrightarrow} z_2) - P(z_3 \underset{B_1^c}{\longleftrightarrow} z_4)\big] P(z_1 \overset{B_1}{\longleftrightarrow} z_2) \\
&\quad = \big[P(z_3 \underset{B_1^c}{\longleftrightarrow} z_4 | z_1 \overset{B_1}{\longleftrightarrow} z_2, \mathcal{F}(z_3,z_4;B_1^c)) - P(z_3 \underset{B_1^c}{\longleftrightarrow} z_4 | \mathcal{F}(z_3,z_4;B_1^c))\big] P(z_1 \overset{B_1}{\longleftrightarrow} z_2) P(\mathcal{F}(z_3,z_4;B_1^c)),
\end{aligned}
\tag{3.4}
$$

where $P(z_1 \overset{B_1}{\longleftrightarrow} z_2)P(\mathcal{F}(z_3,z_4;B_1^c)) \sim |z_1-z_2|^{-5/24}|z_1-z_2|^{5/4}$. In a configuration such that $\mathcal{F}(z_3,z_4;B_1^c)$ occurs, the event $z_3 \underset{B_1^c}{\longleftrightarrow} z_4$ reduces to the event that the clusters of z_3 and z_4, which are disjoint when restricted to B_1^c, are connected inside B_1. Conditioning on $z_1 \overset{B_1}{\longleftrightarrow} z_2$ makes it easier for such an event to happen, so we can conclude that

$$
\big[P(z_3 \underset{B_1^c}{\longleftrightarrow} z_4 | z_1 \overset{B_1}{\longleftrightarrow} z_2) - P(z_3 \underset{B_1^c}{\longleftrightarrow} z_4)\big] P(z_1 \overset{B_1}{\longleftrightarrow} z_2) \sim |z_1-z_2|^{-5/24}|z_1-z_2|^{5/4}. \tag{3.5}
$$

For $n > 1$ (see Fig. 3.1), the event $z_3 \underset{B_n^c}{\longleftrightarrow} z_4$ is analogous to $z_3 \underset{B_1^c}{\longleftrightarrow} z_4$ and induces the event $\mathcal{F}(z_3,z_4;B_n^c)$ such that $P(\mathcal{F}(z_3,z_4;B_n^c)) \sim \big(2^n|z_1-z_2|\big)^{5/4}$. Furthermore, the event $z_1 \underset{B_{n-1}}{\longleftrightarrow} z_2$ implies that, inside B_{n-1}, z_1 and z_2 are connected to ∂B_{n-1} by two disjoint open clusters, separated by a closed cluster. We denote this event by $\mathcal{F}(z_1,z_2;B_{n-1})$. Since $\mathcal{F}(z_1,z_2;B_{n-1})$ implies that the annulus $B_{n-1} \setminus B_1$ is crossed by two disjoint open paths and two disjoint closed paths, the four-arm exponent gives $P(\mathcal{F}(z_1,z_2;B_{n-1})) \sim \big((1/2)^{n-2}\big)^{5/4}$. Using the fact that B_{n-1} and B_n^c are disjoint, we see that

$$
P(\mathcal{F}(z_1,z_2;B_{n-1}),\mathcal{F}(z_3,z_4;B_n^c)) = P(\mathcal{F}(z_1,z_2;B_{n-1}))P(\mathcal{F}(z_3,z_4;B_n^c)) \sim |z_1-z_2|^{5/4}, \tag{3.6}
$$

independently of n. With this, using arguments analogous to those above, we can write

$$
\begin{aligned}
&\Big[P(z_1 \underset{B_{n-1}}{\longleftrightarrow} z_2, z_3 \underset{B_n^c}{\longleftrightarrow} z_4 | z_1 \overset{B_n}{\longleftrightarrow} z_2) - P(z_1 \underset{B_{n-1}}{\longleftrightarrow} z_2 | z_1 \overset{B_n}{\longleftrightarrow} z_2) P(z_3 \underset{B_n^c}{\longleftrightarrow} z_4)\Big] P(z_1 \overset{B_n}{\longleftrightarrow} z_2) \\
&\quad = \Big[P(z_1 \underset{B_{n-1}}{\longleftrightarrow} z_2, z_3 \underset{B_n^c}{\longleftrightarrow} z_4 | z_1 \overset{B_n}{\longleftrightarrow} z_2, \mathcal{F}(z_1,z_2;B_{n-1}), \mathcal{F}(z_3,z_4;B_n^c)) \\
&\qquad - P(z_1 \underset{B_{n-1}}{\longleftrightarrow} z_2 | z_1 \overset{B_n}{\longleftrightarrow} z_2, \mathcal{F}(z_1,z_2;B_{n-1})) P(z_3 \underset{B_n^c}{\longleftrightarrow} z_4 | \mathcal{F}(z_3,z_4;B_n^c))\Big] \\
&\qquad\quad P(z_1 \overset{B_n}{\longleftrightarrow} z_2) P(\mathcal{F}(z_1,z_2;B_{n-1})) P(\mathcal{F}(z_3,z_4;B_n^c)) \\
&\qquad \sim g_n(z_1,z_2,z_3,z_4)|z_1-z_2|^{-5/24}|z_1-z_2|^{5/4},
\end{aligned}
\tag{3.7}
$$

where we introduced the notation

$$
g_n(z_1,z_2,z_3,z_4) = P(z_3 \underset{B_n^c}{\longleftrightarrow} z_4 | z_1 \underset{B_{n-1}}{\overset{B_n}{\longleftrightarrow}} z_2, \mathcal{F}(z_3,z_4;B_n^c)) - P(z_3 \underset{B_n^c}{\longleftrightarrow} z_4 | \mathcal{F}(z_3,z_4;B_n^c)). \tag{3.8}
$$

From scale invariance and the fact that different connectivity events inside $B_n \setminus B_{n-1}$ can use the same open paths and therefore "help each other" (see Fig. 3.1), one can deduce [CF24] that $g_n(z_1,z_2,z_3,z_4)$ is positive and bounded away from zero for all n of order $-\log|z_1-z_2|$, for which the disks B_n and B_{n-1} are macroscopic. Therefore,

$$
\begin{aligned}
&\sum_{n=2}^{N} \big[P(z_1 \underset{B_{n-1}}{\longleftrightarrow} z_2, z_3 \underset{B_n^c}{\longleftrightarrow} z_4 | z_1 \overset{B_n}{\longleftrightarrow} z_2) - P(z_1 \underset{B_{n-1}}{\longleftrightarrow} z_2 | z_1 \overset{B_n}{\longleftrightarrow} z_2) P(z_3 \underset{B_n^c}{\longleftrightarrow} z_4)\big] P(z_1 \overset{B_n}{\longleftrightarrow} z_2) \\
&\quad \sim g(z_1,z_2,z_3,z_4)|z_1-z_2|^{-5/24}|z_1-z_2|^{5/4}\big|\log|z_1-z_2|\big|,
\end{aligned}
\tag{3.9}
$$

for some $g(z_1, z_2, z_3, z_4) > 0$, which shows that $P(z_1 \longleftrightarrow z_2, z_3 \longleftrightarrow z_4)$ contains a term with a logarithmic divergence as $|z_1 - z_2| \to 0$. Similar considerations imply that the remaining term in the expression (3.3) of $P(z_1 \longleftrightarrow z_2, z_3 \longleftrightarrow z_4)$ behaves like

$$\begin{aligned}
&\big[P(z_1 \underset{B_N}{\longleftrightarrow} z_2, z_3 \leftrightarrow z_4 | z_1 \leftrightarrow z_2) - P(z_1 \underset{B_N}{\longleftrightarrow} z_2 | z_1 \leftrightarrow z_2) P(z_3 \leftrightarrow z_4)\big] P(z_1 \leftrightarrow z_2) \\
&\quad = \big[P(z_1 \underset{B_N}{\longleftrightarrow} z_2, z_3 \leftrightarrow z_4 | z_1 \leftrightarrow z_2, \mathcal{F}(z_1, z_2; B_N)) \\
&\qquad - P(z_1 \underset{B_N}{\longleftrightarrow} z_2 | z_1 \leftrightarrow z_2, \mathcal{F}(z_1, z_2; B_N)) P(z_3 \leftrightarrow z_4)\big] \\
&\qquad P(z_1 \leftrightarrow z_2) P(\mathcal{F}(z_1, z_2; B_N)) \\
&\qquad \sim |z_1 - z_2|^{-5/24} |z_1 - z_2|^{5/4}.
\end{aligned} \tag{3.10}$$

We now write (2.1) as

$$\begin{aligned}
\langle \psi(z_1)\psi(z_2)\psi(z_3)\psi(z_4)\rangle &= P(z_1 \leftrightarrow z_2) P(z_3 \leftrightarrow z_4) + G(z_1, z_2, z_3, z_4) \\
&\quad + P(z_1 \leftrightarrow z_3 \not\leftrightarrow z_2 \leftrightarrow z_4) + P(z_1 \leftrightarrow z_4 \not\leftrightarrow z_2 \leftrightarrow z_3),
\end{aligned} \tag{3.11}$$

where

$$G(z_1, z_2, z_3, z_4) = P(z_1 \longleftrightarrow z_2, z_3 \longleftrightarrow z_4) - P(z_1 \longleftrightarrow z_2) P(z_3 \longleftrightarrow z_4) \tag{3.12}$$

involves the insertion of four density operators and therefore [4], if f is a Möbius transformation,

$$G(f(z_1), f(z_2), f(z_3), f(z_4)) = \prod_{j=1}^{4} |f'(z_j)|^{-5/48} G(z_1, z_2, z_3, z_4). \tag{3.13}$$

Combined with (3.3), (3.5), (3.9) and (3.10), this implies that, when $z_1, z_2 \to z$,

$$G(z_1, z_2, z_3, z_4) \sim -F(z, z_3, z_4) |z_1 - z_2|^{-5/24} |z_1 - z_2|^{5/4} \log |x|, \tag{3.14}$$

where $F(z, z_3, z_4) = |z - z_3|^{-5/4} |z - z_4|^{-5/4} |z_3 - z_4|^{25/24}$, the cross-ratio

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} \tag{3.15}$$

is invariant under Möbius transformations, and $\log |x| \sim \log |z_1 - z_2|$.

Combining (3.14) with (2.2) gives, asymptotically as $z_1, z_2 \to z$,

$$\langle \psi(z_1)\psi(z_2)\psi(z_3)\psi(z_4)\rangle \sim |z_1 - z_2|^{-5/24} \Big[|z_3 - z_4|^{-5/24} + |z_1 - z_2|^{5/4} F(z, z_3, z_4) \big(C_0 - C_L \log |x|\big) \Big] \tag{3.16}$$

for some finite, positive constants C_0 and C_L.

This is consistent with the following logarithmic OPE:

$$\psi(z_1)\psi(z_2) = |z_1 - z_2|^{-5/24} \Big[1 + |z_1 - z_2|^{5/4} \Big(C_{\psi\psi\hat{\phi}} \hat{\phi}(z) - C_{\psi\psi\phi} \log |z_1 - z_2| \phi(z) \Big) + \dots \Big], \tag{3.17}$$

where $\hat{\phi}$ and ϕ are two distinct fields with the same scaling dimension, $2h = 5/4$, such that

$$\begin{aligned}
\langle \hat{\phi}(z)\psi(z_3)\psi(z_4)\rangle &= \Big(C_{\psi\psi\hat{\phi}} - \frac{C^2_{\psi\psi\phi}}{C_{\psi\psi\hat{\phi}}} \log \Big(\frac{|z_3 - z_4|}{|z - z_4||z_3 - z|} \Big) \Big) F(z, z_3, z_4), \\
\langle \phi(z)\psi(z_3)\psi(z_4)\rangle &= C_{\psi\psi\phi} F(z, z_3, z_4)
\end{aligned} \tag{3.18}$$

and $C_0 = C^2_{\psi\psi\hat{\phi}}$, $C_L = C^2_{\psi\psi\phi}$.

[4] One can see this, for example, using arguments from [Cam24].

4 Conclusions

We have shown that the four-point function of the density (spin) operator in the CFT corresponding to critical percolation in the bulk has a logarithmic singularity as two of the points collide. The same singularity appears in the OPE of two density operators.

Our analysis differs significantly from previous work on percolation and is amenable to a rigorous mathematical formulation [CF24]. It also provides a new perspective and an analytic explanation, involving a splitting of scales, for the presence of a logarithmic divergence. The latter appears as a consequence of having $c = 0$, which manifests itself in the independence of percolation, combined with scale invariance. More precisely, the four-point function under consideration can be written as a sum of terms which essentially correspond to scaled versions of the following percolation event:

1. there is an $r > 0$ such that, within the disk $B(r)$ of radius r centered at z, the cluster is split in two disjoint components (so that, within $B(r)$, the event is equivalent to the insertion of a four-leg operator),

2. the two components "hook up" when the annulus $B(2r) \setminus B(r)$ is included.

The probabilities of these events, for different values of r over a range of scales of order $\log \frac{1}{|z_1 - z_2|}$, are of the same order because of scale invariance. The logarithm in (3.16), and consequently in the OPE (3.17), corresponds to the number of annuli $B(2r) \setminus B(r)$ one can insert in the space between z_1, z_2 and z_3, z_4 starting from the disk of radius $r = 2|z_1 - z_2|$ centered at $(z_1 + z_2)/2$ and doubling the radius at each step. We believe that this mechanism is quite general and can explain other logarithmic singularities, at least in the context of critical percolation [CF24].

It seems natural to identify the new fields ϕ and $\hat{\phi}$ that appear in (3.17) with the fields discussed in Section 4.2 of [Car13], namely, ϕ with the energy operator and $\hat{\phi}$ with the so-called two-cluster operator.

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References

[ADA99] Michael Aizenman, Bertrand Duplantier, and Amnon Aharony. Path-crossing exponents and the external perimeter in 2D percolation. *Phys. Rev. Lett.*, 83:1359–1362, Aug 1999.

[Aff86] Ian Affleck. Universal term in the free energy at a critical point and the conformal anomaly. *Phys. Rev. Lett.*, 56:746–748, Feb 1986.

[BCN86] H. W. J. Blöte, John L. Cardy, and M. P. Nightingale. Conformal invariance, the central charge, and universal finite-size amplitudes at criticality. *Phys. Rev. Lett.*, 56:742–745, Feb 1986.

[Cam24] Federico Camia. Conformal covariance of connection probabilities and fields in 2D critical percolation. *Comm. Pure Appl. Math.*, 77(3):2138–2176, 2024.

[Car84] John L. Cardy. Conformal invariance and surface critical behavior. *Nuclear Physics B*, 240(4):514–532, 1984.

[Car13] John L. Cardy. Logarithmic conformal field theories as limits of ordinary CFTs and some physical applications. *J. Phys. A: Math. Theor.*, 46(49):494001, Nov 2013.

[CF24] Federico Camia and Yu Feng. Conformally covariant probabilities, operator product expansions, and logarithmic correlations in two-dimensional critical percolation. *In preparation*, 2024.

[CR13] Thomas Creutzig and David Ridout. Logarithmic conformal field theory: beyond an introduction. *J. Phys. A: Math. Theor.*, 46(49):494006, Nov 2013.

[DF84] Vl.S. Dotsenko and V.A. Fateev. Conformal algebra and multipoint correlation functions in 2d statistical models. *Nucl. Phys. B.*, 240(3):312–348, 1984.

[Dot16] Vladimir S. Dotsenko. Correlation function of four spins in the percolation model. *Nucl. Phys. B.*, 911:712–743, 2016.

[Dot20] Vladimir S. Dotsenko. Four spins correlation function of the q states Potts model, for general values of q. Its percolation model limit $q \to 1$. *Nucl. Phys. B.*, 953:114973, 2020.

[DV10] Gesualdo Delfino and Jacopo Viti. On three-point connectivity in two-dimensional percolation. *J. Phys. A: Math. Theor.*, 44(3):032001, Dec 2010.

[GL02] Victor Gurarie and Andreas W W Ludwig. Conformal algebras of two-dimensional disordered systems. *J. Phys. A: Math. Gen.*, 35(27):L377–L384, Jun 2002.

[Gur93] V. Gurarie. Logarithmic operators in conformal field theory. *Nucl. Phys. B.*, 410(3):535–549, 1993.

[Gur99] V. Gurarie. c-Theorem for disordered systems. *Nucl. Phys. B.*, 546(3):765–778, 1999.

[GV18] Giacomo Gori and Jacopo Viti. Four-point boundary connectivities in critical two-dimensional percolation from conformal invariance. *J. High Energy Phys.*, (12):1–28, 2018.

[HGSJS20] Yifei He, Linnea Grans-Samuelsson, Jesper Lykke Jacobsen, and Hubert Saleur. Geometrical four-point functions in the two-dimensional critical Q-state Potts model: connections with the RSOS models. *J. High Energ. Phys.*, 2020(5):1–58, 2020.

[HS22] Yifei He and Hubert Saleur. A note on the identity module in $c = 0$ CFTs. *SciPost Phys.*, 12(3):100, 2022.

[Mai03] Robert S. Maier. On crossing event formulas in critical two-dimensional percolation. *J. Statist. Phys.*, 111(5-6):1027–1048, 2003.

[NR21] Rongvoram Nivesvivat and Sylvain Ribault. Logarithmic CFT at generic central charge: from Liouville theory to the Q-state Potts model. *SciPost Phys.*, 10(1):021, 2021.

[PSVD13] M. Picco, R. Santachiara, J. Viti, and G. Delfino. Connectivities of Potts Fortuin–Kasteleyn clusters and time-like Liouville correlator. *Nucl. Phys. B.*, 875(3):719–737, 2013.

[RS92] L. Rozansky and H. Saleur. Quantum field theory for the multi-variable Alexander-Conway polynomial. *Nucl. Phys. B.*, 376(3):461–509, 1992.

[Sal92] H. Saleur. Polymers and percolation in two dimensions and twisted n = 2 supersymmetry. *Nucl. Phys. B.*, 382(3):486–531, 1992.

[Sim13] Jacob J. H. Simmons. Logarithmic operator intervals in the boundary theory of critical percolation. *J. Phys. A: Math. Theor.*, 46(49):494015, 30, 2013.

[SKZ07] Jacob J H Simmons, Peter Kleban, and Robert M Ziff. Percolation crossing formulae and conformal field theory. *J. Phys. A: Math. Theor.*, 40(31):F771, Jul 2007.

[SW01] Stanislav Smirnov and Wendelin Werner. Critical exponents for two-dimensional percolation. *Math. Res. Lett.*, 8(5-6):729–744, 2001.

[VJS12] Romain Vasseur, Jesper Lykke Jacobsen, and Hubert Saleur. Logarithmic observables in critical percolation. *J. Stat. Mech. Theor. Exp.*, 2012(07):L07001, Jul 2012.