θ-derivations on convolution algebras

M. Eisaei[1] and Gh. R. Moghimi [1*]
[1] Department of Mathematics, Payame Noor University (PNU), Tehran 19395-4697, Iran
mojdehessaei59@student.pnu.ac.ir
moghimimath@pnu.ac.ir

Abstract

In this paper, we investigate θ-derivations on Banach algebra $L_0^\infty(w)^*$. First, we study the range of them and prove the Singer-Wermer conjucture. We also give a characterization of the space of all θ-derivations on $L_0^\infty(w)^*$. Then, we prove automatic continuity and Posner's theorems for θ-derivations.

Keywords: Convolution algebras, θ-derivations, Singer-Wermer conjucture, automatic continuity, Posner's theorems.
2020 MSC: 47B47, 46H40, 16W25.

1 Introduction

Let A be a Banach algebra with the center $Z(A)$ and the right annihilator; i.e.,

$$Z(A) = \{a \in A : \quad ax = xa \quad \text{for all} \quad x \in A\}$$

and

$$\text{ran}(A) = \{r \in A : \quad ar = 0 \quad \text{for all} \quad a \in A\}.$$

Let $D : A \to A$ be a linear map and $k \in \mathbb{N}$. Then D is called *k-centralizing* if for every $m \in A$, we have

$$[D(m), m^k] := D(m)m^k - m^k D(m) \in Z(A).$$

*Corresponding author

In particular, if $[D(m), m^k] = 0$, then D is called *k-commuting.* Assume now that $\theta : A \to A$ is a homomorphism. Then D is called a *θ-derivation* if

$$D(mn) = D(m)\theta(n) + \theta(m)D(n)$$

for all $m, n \in A$. If θ is the identity map, then D is called a *derivation.*

Let us recall that a continuous function $w : [0, \infty) \to [1, \infty)$ is called a *weight function* if $w(0) = 1$ and for all $x, y \in [0, \infty)$

$$w(x + y) \leq w(x)w(y).$$

Let $L^1(w)$ be the Banach space of all Lebesgue measurable functions f on $[0, \infty)$ such that $wf \in L^1([0, \infty))$, the Banach algebra of all Lebesque integrable functions on $[0, \infty)$. It is well-known that $L^1(w)$ is a Banach algebra with the convolution product

$$\varphi * \psi(x) = \int_0^\infty \varphi(y)\psi(x - y)d(y), \quad (\varphi, \psi \in L^1([0, \infty))$$

and the norm

$$\|\varphi\|_w = \int_0^\infty w(x)|\varphi|(x)dx \quad (\varphi \in L^1([0, \infty));$$

see [4, 15]. Let also $L_0^\infty(w)$ be the Banach space of all Lebesgue measure functions f on $[0, \infty)$ such that

$$\lim_{x\to\infty} \text{ess sup}\left\{\frac{f(y)\chi_{(x,\infty)}(y)}{w(y)} : y \geq 0\right\} = 0,$$

where $\chi_{(x,\infty)}$ is the characteristic function of (x, ∞) on $[0, \infty)$. It is well-known that the dual of $L_0^\infty(w)$, represented by $L_0^\infty(w)^*$, is a Banach algebra with the first Arens product defined by

$$\langle mn, f\rangle = \langle m, nf\rangle,$$

where $\langle nf, \varphi\rangle = \langle n, f\varphi\rangle$, in which

$$f\varphi(x) = \int_0^\infty f(x+y)\varphi(y)dy$$

for all $m, n \in L_0^\infty(w)^*$, $f \in L_0^\infty(w)$, $\varphi \in L^1(w)$ and $x \geq 0$; see [7, 8, 11, 12]. Note that every element $\varphi \in L^1(w)$ can be regarded as an element of $L_0^\infty(w)^*$,

$$\langle \varphi, f\rangle = \int \varphi(x)f(x)dx$$

for all $f \in L_0^\infty(w)$. We denoted by $\Lambda(L_0^\infty(w)^*)$ the set of all right identities of $L_0^\infty(w)^*)$ with bounded one. For every $u \in \Lambda(L_0^\infty(w)^*)$ and $m \in L_0^\infty(w)^*$, we have $m - um \in \mathrm{ran}(L_0^\infty(w)^*)$ and

$$m = um + (m - um).$$

It follows that

$$L_0^\infty(w)^* = uL_0^\infty(w)^* \oplus \mathrm{ran}(L_0^\infty(w)^*).$$

One can prove that the radical of $L_0^\infty(w)^*$ is equal to $\mathrm{ran}(L_0^\infty(w)^*)$; see [13].

Derivations and θ-derivations were studied by several authors [1-3, 9, 10, 13, 14]. For example, derivations on $L_0^\infty(w)^*$ investigated in [13]. They proved that the range of a derivation on $L_0^\infty(w)^*$ is contained into $\mathrm{ran}(L_0^\infty(w)^*)$. They also showed that the zero map is the only k-centralizing derivation on $L_0^\infty(w)^*$.

In this paper, we investigate θ-derivations on $L_0^\infty(w)^*$. In the case where, θ is an isomorphism, we prove that the range of θ-derivations on $L_0^\infty(w)^*$ is contained into the radical of $L_0^\infty(w)^*$. If θ is also continuous, then D is continuous if and only if $D|_{\mathrm{ran}(L_0^\infty(w)^*)}$ is continuous. In this case, $D|_{uL_0^\infty(w)^*}$ is always continuous. Finally, we study Posner first and second theorems for θ-derivations on $L_0^\infty(w)^*$.

2 Main Results

Singer and Wermer [16] showed that the range of a continuous derivation on a commutative Banach algebra is a subset of its radical. They conjectured that the continuity requirement for the derivations can be removed. Thomas [17] proved the conjecture. In the sequal, we investigate this conjecture for θ-derivation on non-commutative Banach algebra $L_0^\infty(w)^*$.

Theorem 2.1. *Let θ be a homomorphism on $L_0^\infty(w)^*$ and D be a θ-derivation on $L_0^\infty(w)^*$. Then the following statements hold.*

(i) *D maps* $\mathrm{ran}(L_0^\infty(w)^*)$ *and* $\Lambda(L_0^\infty(w)^*)$ *into* $\mathrm{ran}(L_0^\infty(w)^*)$.

(ii) *If θ is an isomorphism, then D maps $L_0^\infty(w)^*$ into* $\mathrm{ran}(L_0^\infty(w)^*)$.

Proof. (i) First, note that if $u \in \Lambda(L_0^\infty(w)^*)$ and $r \in \mathrm{ran}(L_0^\infty(w)^*)$, then $\theta(r) \in \mathrm{ran}(L_0^\infty(w)^*)$ and

$$\theta(u) = u + r_0$$

for some $r_0 \in \mathrm{ran}(L_0^\infty(w)^*)$; see Lemma 2.1 of [5]. So for every $k \in L_0^\infty(w)^*$, we have

$$\begin{aligned} kD(r) &= k(u + r_0).D(r) \\ &= k\theta(u)D(r) \\ &= k[D(ur) - D(u)\theta(r)] = 0. \end{aligned}$$

This shows that $D(r) \in \mathrm{ran}(L_0^\infty(w)^*)$. We also have

$$\begin{aligned} D(uu) &= D(u)\theta(u) + \theta(u)D(u) \\ &= D(u) + \theta(u)D(u). \end{aligned}$$

Hence $\theta(u)D(u) = 0$. It follows that

$$kD(u) = k\theta(u)D(u) = 0$$

for all $k \in L_0^\infty(w)^*$. Therefore, $D(u) \in \operatorname{ran}(L_0^\infty(w)^*)$.

(ii) Let θ be an isomorphism. Then $\theta^{-1}D$ is a derivation on $L_0^\infty(w)^*$. By Theorem 2.1 of [5],

$$\theta^{-1}D(L_0^\infty(w)^*) \subseteq \operatorname{ran}(L_0^\infty(w)^*).$$

Thus

$$D(L_0^\infty(w)^*) \subseteq \theta(\operatorname{ran}(L_0^\infty(w)^*)) \subseteq \operatorname{ran}(L_0^\infty(w)^*),$$

as claimed. $\square$

A mapping $T : L_0^\infty(w)^* \to L_0^\infty(w)^*$ is called *spectrally bounded* if there exists $c \geq 0$ such that $r(T(m)) \leq \alpha r(m)$ for all $m \in L_0^\infty(w)^*$, where $r(m)$ denotes the spectral radius of m.

Corollary 2.2. *Let θ be a homomorphism on $L_0^\infty(w)^*$. Then the following statements hold.*

(i) *The product of two θ-derivations on $L_0^\infty(w)^*$ is a θ-derivation on* $\operatorname{ran}(L_0^\infty(w)^*)$.

(ii) *Every θ-derivation on* $\operatorname{ran}(L_0^\infty(w)^*)$ *is spectrally bounded.*

In the next result, we investigate the automatic continuity of θ-derivation on $L_0^\infty(w)^*$; see [6] for the automatic continuity of derivation on commutative semisimple Banach algebras.

Proposition 2.3. *Let θ be a continuous isomorphism on $L_0^\infty(w)^*$ and D be a θ-derivation on $L_0^\infty(w)^*$. Then the following statements hold.*

(i) *$D|_{uL_0^\infty(w)^*}$ is always continuous for all $u \in \Lambda(L_0^\infty(w)^*)$.*

(ii) *D is continuous if and only if $D|_{\operatorname{ran}(L_0^\infty(w)^*)}$ is continuous.*

Proof. Let θ be a continuous isomorphism on $L_0^\infty(w)^*$ and D be a θ-derivation on $L_0^\infty(w)^*$. Then D maps $L_0^\infty(w)^*$ into $\operatorname{ran}(L_0^\infty(w)^*)$ and so

$$\begin{aligned} D(um) &= D(u)\theta(m) + \theta(u)D(m) \\ &= D(u)\theta(m) = D(u)\theta(u)\theta(m) \\ &= D(u)\theta(um) \end{aligned}$$

for all $m \in L_0^\infty(w)^*$ and $u \in \Lambda(L_0^\infty(w)^*)$. Thus

$$\begin{aligned} \|D(um)\| &= \|D(u)\theta(um)\| \\ &\leq \|D(u)\|\|\theta(um)\| \\ &\leq \|D(u)\|\|\theta\|\|um\|. \end{aligned}$$

For (ii), let D be continuous on $\operatorname{ran}(L_0^\infty(w)^*)$. Then there exist $c_1, c_2 \geq 0$ such that for every $m \in L_0^\infty(w)^*$, $u \in \Lambda(L_0^\infty(w)^*)$ and $r \in \operatorname{ran}(L_0^\infty(w)^*)$,

$$\|D(r)\| \leq c_1\|r\| \quad \text{and} \quad \|D(um)\| \leq c_2\|um\|.$$

Assume that $m \in L_0^\infty(w)^*$. Then

$$m = um + r,$$

where $r = m - um \in \operatorname{ran}(L_0^\infty(w)^*)$. So by (i), we obtain

$$\begin{aligned} \|D(m)\| &= \|D(um) + D(r)\| \\ &\leq \|D(um)\| + \|D(r)\| \\ &\leq c_2\|um\| + c_1\|r\| \\ &\leq c_2\|u\|\|m\| + c_1(\|m\| + \|u\|\|m\|) \\ &= (c_2 + 2c_1)\|m\|. \end{aligned}$$

That is, D is continuous. The converse is clear. $\square$

Let $\mathcal{A}$ be a Banach algebra and A be a closed subalgebra of $\mathcal{A}$. We denote by $Der(\mathcal{A}, A)$ the space of all θ-derivations from $\mathcal{A}$ into A, where θ is a homomorphism on $\mathcal{A}$. We also denote by $\mathcal{B}(\mathcal{A}, A)$ the space of all bounded linear operators from $\mathcal{A}$ into A. We write $Der(\mathcal{A}) := Der(\mathcal{A}, A)$ and $\mathcal{B}(\mathcal{A}) := \mathcal{B}(\mathcal{A}, A)$.

Theorem 2.4. *Let θ be an isometrically isomorphism on $L_0^\infty(w)^*$ and $u \in \Lambda(L_0^\infty(w)^*)$. Then*

$$\mathrm{Der}(L_0^\infty(w)^*) = \mathrm{Der}(L_0^\infty(w)^*, uL_0^\infty(w)^*) \oplus \mathrm{Der}(L_0^\infty(w)^*, \mathrm{ran}(L_0^\infty(w)^*))$$

Proof. Let $D \in Der(L_0^\infty(w)^*)$. We define $d : L_0^\infty(w)^* \to uL_0^\infty(w)^*$ and $T : L_0^\infty(w)^* \to L_0^\infty(w)^*$ by

$$d(m) = D(um) \quad \text{and} \quad T(m) = D(m - um)$$

for all $m \in L_0^\infty(w)^*$. Note that

$$\begin{aligned} D(m) &= D(um + (m - um)) \\ &= D(um) + D(m - um) \\ &= d(m) + T(m) \end{aligned}$$

for all $m \in L_0^\infty(w)^*$. Let $m_1, m_2 \in L_0^\infty(w)^*$. Since θ is an isometrically isomorphism, $\theta(u) = u$. Thus

$$\begin{aligned} d(m_1 m_2) &= D(um_1 um_2) \\ &= D(um_1)\theta(um_2) + \theta(um_1)D(um_2) \\ &= D(um_1)\theta(u)\theta(m_2) + \theta(u)\theta(m_1)D(um_2) \\ &= D(um_1)\theta(m_2) + \theta(m_1)D(um_2) \\ &= d(m_1)\theta(m_2) + \theta(m_1)d(m_2). \end{aligned}$$

Hence $d \in Der(L_0^\infty(w)^*, uL_0^\infty(w)^*)$. On the other hand,

$$uL_0^\infty(w)^* \cap \operatorname{ran}(L_0^\infty(w)^*) = \{0\}.$$

These facts prove the result. $\square$

In the following, let $C_0(w)$ be the Banach space of all complex-valued continuous functions f on $[0, \infty)$ such that f/w vanishes at infinity.

Proposition 2.5. *Let θ be a homomorphism on $L_0^\infty(w)^*$, $D : L_0^\infty(w)^* \to \operatorname{ran}(L_0^\infty(w)^*)$ be a θ-derivation and $m \in L_0^\infty(w)^*$. If $D(m)$ is positive, then $D(m) = 0$.*

Proof. Let $m \in L_0^\infty(w)^*$ and $D(m)$ be positive. Then

$$\|D(m)\| = \|D(m)|_{C_0(w)}\| = 0;$$

see [8]. $\square$

Corollary 2.6. *Let θ be an isomorphism on $L_0^\infty(w)^*$ and D be a θ-derivation on $L_0^\infty(w)^*$. If $m \in L_0^\infty(w)^*$ such that is positive, then $D(m) = 0$.*

Proof. Let θ be an isomorphism on $L_0^\infty(w)^*$. It follows from Theorem 2.1 that D maps $L_0^\infty(w)^*$ into $\operatorname{ran}(L_0^\infty(w)^*)$. Now, apply Proposition 2.5. $\square$

Theorem 2.7. *Let θ be a homomorphism on $L_0^\infty(w)^*$ and D be a θ-derivation on $L_0^\infty(w)^*$. Then D is k-centralizing if and only if D is k-commuting. In this case, if θ is an isometrically isomorphism, then D maps $L_0^\infty(w)^*$ into $uL_0^\infty(w)^*$ and $D(m) = \theta(u)D(m)$ for all $m \in L_0^\infty(w)^*$ and $u \in \Lambda(L_0^\infty(w)^*)$.*

Proof. Let D be k-centralizing. Then $[D(m), m^k] \in Z(L_0^\infty(w)^*)$ for all $m \in$

$L_0^\infty(w)^*$. Let $u \in \Lambda(L_0^\infty(w)^*)$. Then

$$\begin{aligned} [D(m), m^k] &= [D(m), m^k]u \\ &= u[D(m), m^k] \\ &= uD(m)m^k - um^k D(m) \\ &= 0. \end{aligned}$$

Thus D is k-commuting. The converse is trivial. Since $D(u) \in \mathrm{ran}(L_0^\infty(w)^*)$, we have

$$D(u) = [D(u), u^k] = 0. \tag{2.1}$$

For every $r \in \mathrm{ran}(L_0^\infty(w)^*)$, we have $r + u = (r + u)^k$. Thus

$$\begin{aligned} 0 &= [D(r+u), (r+u)^k] \\ &= [D(r), r+u] \\ &= D(r)u - uD(r) \\ &= D(r), \end{aligned}$$

because $D(r) \subseteq \mathrm{ran}(L_0^\infty(w)^*)$. Hence $D(r) = 0$. Therefore, by (2.1) we obtain

$$\begin{aligned} D(m) &= D(um) + D(m - um) \\ &= D(um) = D(u)\theta(m) + \theta(u)D(m) \\ &= \theta(u)D(m). \end{aligned}$$

To complete the proof, we only recall that by Lemma 2.1 of [5], $\theta(u) \in \Lambda(L_0^\infty(w)^*)$. $\square$

References

[1] M. H. Ahmadi Gandomani and M. J. Mehdipour, Generalized derivations on some convolution algebras. Aequationes Math. 92 (2018), no. 2, 223–241.

[2] M. H. Ahmadi Gandomani and M. J. Mehdipour, Symmetric bi-derivations and their generalizations on group algebras. Filomat 35 (2021), no. 4, 1233–1240.

[3] M. Bresar and M. Mathieu, Derivations mapping into the radical. III. J. Funct. Anal. 133 (1995), no. 1, 21–29.

[4] H. G. Dales and A. T. Lau, The second duals of Beurling algebras, Mem. Amer. Math. Soc., 177 (2005), no. 836, 1–191.

[5] M. Eisaei, M. J. Mehdipour and Gh. R. Moghimi, Left θ-derivations on weighted convolution algebras, arXiv:2403.16036.

[6] B. E. Johnson, Continuity of derivations on commutative algebras. Amer. J. Math. 91 (1969), 1–10.

[7] A. T. Lau and J. Pym, Concerning the second dual of the group algebra of a locally compact group. J. London Math. Soc. (2) 41 (1990), no. 3, 445–460.

[8] S. Maghsoudi, R. Nasr-Isfahani and A. Rejali, Arens multiplication on Banach algebras related to locally compact semigroups. Math. Nachr. 281 (2008), no. 10, 1495–1510.

[9] M. Mathieu and G. J. Murphy, Derivations mapping into the radical. Arch. Math. (Basel) 57 (1991), no. 5, 469–474.

[10] M. Mathieu and V. Runde, Derivations mapping into the radical. II. Bull. London Math. Soc. 24 (1992), no. 5, 485–487.

[11] M. J. Mehdipour and Gh. R. Moghimi,, The existence of nonzero compact right multipliers and Arens regularity of weighted Banach algebras. Rocky Mountain J. Math. 52 (2022), no. 6, 2101–2112.

[12] M. J. Mehdipour and R. Nasr-Isfahani, Weakly compact multipliers on Banach algebras related to a locally compact group. Acta Math. Hungar. 127 (2010), no. 3, 195–206.

[13] M. J. Mehdipour and Z. Saeedi, Derivations on convolution algebras. Bull. Korean Math. Soc. 52 (2015), no. 4, 1123–1132.

[14] E. C. Posner, Derivations in prime rings. Proc. Amer. Math. Soc. 8 (1957), 1093–1100.

[15] H. Reiter and J. D. Stegeman, Classical Harmonic Analysis and Locally Compact Groups, Second edition, London Mathematical Society Monographs, New Series, 22, The Clarendon Press, Oxford University Press, New York, 2000.

[16] I. M. Singer and J. Wermer, Derivations on commutative normed algebras. Math. Ann. 129 (1955), 260–264.

[17] M. Thomas, The image of a derivation is contained in the radical. Ann. of Math. (2) 128 (1988), no. 3, 435–460.